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Portfolio Selection with Skewness:
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Abstract
The main aim of this contribution is to compare existing and newly developed techniques for geometrically representing mean-variance-skewness portfolio frontiers based on the rather widely adapted methodology of polynomial goal programming (PGP) on the one hand and the more recent approach based on the shortage function on the other hand. Moreover, we explain the working of these different methodologies in detail and provide graphical illustrations. Inspired by these illustrations, we prove a generalization of the well-known two fund separation theorem from traditional mean-variance portfolio theory.

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KEYWORDS: shortage function, PGP, efficient frontier, mean-variance, mean-skewness, mean-variance-skewness

Warning: Please print the figures in color for evaluation purposes.

1 Introduction
The limitations of modern portfolio theory trading off risk and expected return are meanwhile well-documented. A host of empirical studies rejects the hypothesis that portfolio returns can be characterized by normal distributions. Furthermore, there is ample evidence that investors do care about higher moments of return distributions. A recent study evaluating the out-of-sample performance of a variety of sample-based mean-variance (MV) portfolio models designed to reduce
the effect of estimation error reveals that none of these methods consistently outperforms a naive portfolio diversification rule (see DeMiguel, Garlappi, and Uppal (2009)). Nevertheless, a continuous stream of new proposals aims at improving the traditional MV portfolio model: for instance, Roman, Darby-Dowman, and Mitra (2007) combine two risk measures (i.e., variance and Conditional Value-at-Risk (CVaR)) and transform this multi-objective problem into a single objective problem, or one can regularize (i.e., stabilize) the MV optimization problem by considering it as a constrained least-squares regression problem, by adding a penalty term proportional to the sum of the absolute values of the portfolio weights (see Brodie, Daubechies, De Mol, Giannone, and Loris (2009)).

However, discontent with the MV model has alternatively led to an enormous stream of proposals to include, e.g., the skewness or (more rarely) higher order moments into portfolio theory. Limiting our discussion to mean-variance-skewness (MVS) portfolio optimization models, a variety of articles have over the years offered alternative approaches. Examples of a primal approach are found in Wang and Xia (2002), who determine MVS portfolios via a multiobjective programming approach, or in Athayde and Flores (2004), who determine analytical solutions characterizing the MVS portfolio frontier by minimizing the variance for given mean and skewness while assuming a risk-free asset and shorting. There are plenty of other recent research lines. Li, Qin, and Kar (2010), for example, develop a fuzzy mean-variance-skewness model as well as some variations. As another instance, Konno, Shirakawa, and Yamazaki (1993) formulate a general portfolio optimization problem maximizing skewness subject to fixed expected return and variance constraints, whereby both the quadratic and cubic terms are linearly approximated to yield a mean-absolute deviation-skewness model. Boyle and Ding (2005) pick up from there and check the effect of only using a linear approximation for the cubic term. Notice that a lot of these contributions tend to solve the MVS portfolio problem by privileging one or two of the objectives at the cost of the other(s). Starting from specifications of the indirect MVS utility function, dual approaches search for optimal portfolios via preference parameters reflecting attitudes towards risk and skewness (Jondeau and Rockinger (2006) and Harvey, Liechty, Liechty, and Müller (2010) are recent utility-based studies).

It is fair to say that so far no consensus seems to have emerged about a general approach to multi-moment portfolio models. While especially these primal approaches should ideally be somehow equivalent, we are unaware of any comparative study in the literature. In this context, this contribution is -to the best of our knowledge- the first attempt to develop a comparison between two primal MVS portfolio optimization approaches.

On the one hand, the seminal article by Lai (1991) developed a MVS portfolio optimization model under the assumptions of shorting and the availability of a risk-free asset (see, e.g., Chunhachinda, Dandapani, Hamid, and Prakash (1997) for an explicit list of assumptions). This article started a burgeoning literature whereby portfolio optimization is conceived as a multiple goal programming problem. In particular, in what is nowadays commonly referred to as the polynomial

\[ \text{1} \text{ An overview of alternative MCDA models in portfolio analysis is found in Spronk, Steuer, and Zopounidis (2005) or Steuer, Qi, and Hirschberger (2008).} \]
goal programming (PGP) approach, one attempts to find a compromise between several goals by minimizing some appropriate polynomial penalty function. As traditionally conceived, this results in simultaneously maximizing return and skewness for a given unit portfolio risk.\(^2\)

This Lai (1991) methodology has become a rather popular vehicle for empirical research looking at skewness persistence in a variety of international markets (e.g., Chunhachinda, Dandapani, Hamid, and Prakash (1997) or Sun and Yan (2003)), emerging markets (e.g., Canela and Colloza (2007)), or other markets known to follow non-normal distributions (like commodity trading advisors (CTAs), the Collateralized Fund Obligation (CFO) Equity Tranche, hedge funds, or funds of hedge funds (FOFs): see Aboul-Enein, Dionne, and Papageorgiou (2011), Anson (2006), Anson, Ho, and Silberstein (2007), Elkaim and Papageorgiou (2006), or Davies, Kat, and Lu (2009)). A variety of methodological refinements have been adopted to the basic model: trade-offs among lower partial moments (Chen and Shia (2007), trade-off between return and Value-at-Risk (see Chen (2008)), index tracking (see Wu, Chou, Yang, and Ong (2007)), among others. Furthermore, the sensitivity of results to intervalling and the nature of annualization of returns is now better understood (see Chang, Dupoyet, and Prakash (2008a) or Chang, Dupoyet, and Prakash (2008b)). A slight generalization of the basic Lai (1991) framework is found in Leung, Daouk, and Chen (2001). It maintains a risk-free asset but dispenses with the unit variance constraint and instead introduces a variance goal. This results in a three instead of a two dimensional goal programming problem, whereby the PGP objective function follows the computational form of the Minkowski distance. Davies, Kat, and Lu (2009) explicitly add the kurtosis as a fourth objective and introduce a normalization of the deviation variables.\(^3\) These are the main empirical applications and extensions currently known to us. In terms of the number of publications, in our count Lai (1991) is currently the most widely applied primal MVS portfolio model around in the literature.

On the other hand, the shortage function approach has been developed by Briec, Kerstens, and Lesourd (2004) in MV and by Briec, Kerstens, and Jokung (2007) in MVS, inspired by the introduction of the same function in production theory by Luenberger (1995). It basically provides a theoretical framework for a new approach initially proposed in the investment literature by Cantaluppi and Hug (2000) to directly measure portfolio performance by reference to its maximum potential on the (ex-ante or ex-post) portfolio frontier. Recently, the general case of a shortage function covering an arbitrary number of moments has been realized in Briec and Kerstens (2010). A theoretical advantage is that a firm link is established between the shortage function on the one hand and mixed risk aversion utility functions (representing preferences for odd moments and

\(^2\)MCDA has found a widespread application in the financial sector (see the surveys in, e.g., Kosmidou and Zopounidis (2004) or Spronk, Steuer, and Zopounidis (2005)). Discrete evaluation methods are used to assist decision-making on diverse financial problems ranging from bankruptcy and credit risk assessment over investment appraisal to portfolio selection and management, among others.

\(^3\)Aboul-Enein, Dionne, and Papageorgiou (2011), Anson (2006) and Elkaim and Papageorgiou (2006) all develop a four moment PGP model as well. Hafner and Wallmeier (2008) also normalize the deviation variables, but their portfolio model has only implicitly four dimensions: they maximize skewness and minimize kurtosis, but maximize the Sharpe ratio to combine the first two moments.
dislikes for even moments) on the other hand (see Briec and Kerstens (2010) for a general duality result). One of the main practical advantages of the shortage function approach is its capability of providing geometrical representations of portfolio frontiers under a wide variety of restrictions on portfolio weights. This has been studied from a methodological angle for MVS portfolio frontiers in Kerstens, Mounir, and Van de Woestyne (2011). Other empirical portfolio work based on this approach includes Jurczenko, Maillet, and Merlin (2006), Jurczenko and Yanou (2010), and Lozano and Gutiérrez (2008), among others. Compared to PGP, it is an approach that makes some claim to generality in that it bridges the gap between primal and dual (i.e., utility-based) approaches (see Briec and Kerstens (2010) in particular).

An immediate question is how these two primal approaches can be related to one another. In particular, this contribution basically attempts to answer the following pertinent questions: (i) Are both PGP and shortage function approaches situated on the same MVS frontier? (ii) What are the mathematical relations between the PGP and shortage function approaches? (iii) Can PGP be used for reconstructing portfolio frontiers, and if so how could one proceed? (iv) Can the PGP approach be easily extended beyond its initial portfolio setting, as claimed by Lai (1991)? In answering these questions, we make ample use of appropriate graphical illustrations based on the shortage function and we develop new two-dimensional geometrical portfolio representations in MVS based upon both the shortage function and the PGP approaches.

The remainder of the paper is organized as follows. Section 2 presents the basic portfolio setting. In section 3, we explain the PGP technique introduced by Lai (1991) as well as the shortage function approach based on the work initiated by Briec, Kerstens, and Lesourd (2004) and Briec, Kerstens, and Jokung (2007). Section 4 develops all geometrical representations and their interpretations in a series of subsections. First, PGP results are interpreted within the context of a MVS reconstruction based on the shortage function following Kerstens, Mounir, and Van de Woestyne (2011). This also leads to a new, two-dimensional reconstruction formulation based on the shortage function. Then, it is shown that PGP results are MVS efficient. Thereafter, we offer a substantial result showing that a single Mean-Skewness (MS) section is sufficient to reconstruct the MVS frontier under a risk-free asset and shorting. This is a kind of generalized two fund separation result. Next, we explore in great detail the ways of developing geometrical representations using the PGP approach. Finally, we discuss some geometrical representations of portfolio models PGP currently fails to generate, while the shortage function approach manages to deliver. To remedy these problems, we propose a new, revised PGP formulation allowing for two-dimensional reconstructions in more general portfolio settings. These resulting reconstructions are similar to the new two-dimensional reconstruction formulations based on the shortage function introduced earlier. Section 5 concludes

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4 For an expected utility optimizer, the more basic equivalence between multimoment preferences and a polynomial utility function of corresponding degree is proven in, e.g., Mülle and Machina (1987).

5 We are unaware of any other systematic contribution on higher order moment reconstruction. Obviously, geometric reconstructions play a major role in the practice of portfolio management. For instance, Anagnostopoulos and Mamanis (2010) extend the MV space with one dimension representing the optimal number of assets.
2 Basic portfolio framework

We start by describing the basic portfolio setting and introduce some notation. Consider the problem of selecting a portfolio from \( n \) financial products (addressed as assets hereafter although other risky products such as funds might equally be considered) and a risk-free asset.

A portfolio \((w, w_{RF}) = (w_1, \ldots, w_n, w_{RF}) \in \mathbb{R}^{n+1}\) is a vector of proportions in each of these \( n \) financial assets and the risk-free asset with

\[
\sum_{i=1}^{n} w_i + w_{RF} = 1.
\]

If a risk-free asset is absent or cannot be selected, then \( w_{RF} = 0 \). In case short selling is not allowed, all proportions \( w_i \) for \( i \in \{1, \ldots, n\} \) and \( w_{RF} \) must be positive. The set of admissible portfolios is given by

\[
\mathcal{F} = \left\{(w, w_{RF}) \in \mathbb{R}^{n+1} ; \sum_{i=1}^{n} w_i + w_{RF} = 1 \right\}.
\]

The assets from which the investor makes a choice are characterized by their returns \( R_i \) for \( i \in \{1, \ldots, n\} \). From these, the expected returns \( E[R_i] \), for \( i \in \{1, \ldots, n\} \), can be derived, as also the covariance matrix \( \Omega \) with

\[
\Omega_{ij} = \text{Cov}[R_i, R_j] = E[(R_i - E[R_i])(E_j - E[R_j])],
\]

for \( i, j \in \{1, \ldots, n\} \) and the coskewness tensor of rank three \( \Lambda \) with

\[
\Lambda_{ijk} = E[(R_i - E[R_i])(R_j - E[R_j])(R_k - E[R_k])],
\]

for \( i, j, k \in \{1, \ldots, n\} \). Obviously, the risk-free asset has expected return \( R_{RF} \) and zero covariances and coskewnesses with itself and the other assets.

The return of portfolio \((w, w_{RF})\) is defined by \( R(w, w_{RF}) = \sum_{i=1}^{n} w_i R_i + w_{RF} R_{RF} \). The expected return of this portfolio \((w, w_{RF})\), its variance and its skewness are straightforwardly computed as follows:

\[
E[R(w, w_{RF})] = \sum_{i=1}^{n} w_i E[R_i] + w_{RF} R_{RF},
\]

\[
\text{Var}[R(w, w_{RF})] = E[(R(w, w_{RF}) - E[R(w, w_{RF})])^2] = \sum_{i,j=1}^{n} w_i w_j \Omega_{ij},
\]
Sk[R(w, w_{RF})] = E[(R(w, w_{RF}) - E[R(w, w_{RF})])^3] = \sum_{i,j,k=1}^{n} w_i w_j w_k \Lambda_{ijk}. \quad (4)

Note that the skewness actually refers to the third central moment. This is the case for the remainder of this text.

Let \( \Phi : \mathcal{S} \to \mathbb{R}^3 \) be the function defined by

\[
\Phi(w, w_{RF}) = (\Phi_M(w, w_{RF}), \Phi_V(w, w_{RF}), \Phi_S(w, w_{RF}))
= (E[R(w, w_{RF})], \text{Var}[R(w, w_{RF})], \text{Sk}[R(w, w_{RF})]).
\]

It provides the expected return, variance and skewness of a given portfolio \((w, w_{RF})\). The functions \( \Phi_M, \Phi_V \) and \( \Phi_S \) represent the coordinate functions of \( \Phi \).

In the remainder, an arbitrary element \( \alpha = (\alpha_M, \alpha_V, \alpha_S) \) of \( \mathbb{R}^3 \) is called a MVS point. Thus, a MVS point can be the image by \( \Phi \) of a portfolio, or any arbitrary point in this three-dimensional space. The MVS-image of \( \mathcal{S} \) is obtained by

\[
\Phi(\mathcal{S}) = \{ \Phi(w, w_{RF}); (w, w_{RF}) \in \mathcal{S} \}.
\]

This set can be extended by defining a MVS disposal representation set

\[
\mathcal{DR} = \Phi(\mathcal{S}) + K, \quad (5)
\]
with \( K = \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \) an octant of \( \mathbb{R}^3 \).

There are two important subsets of this MVS disposal representation set \( (\mathcal{DR}) \). The weakly efficient frontier is simply the boundary of \( \mathcal{DR} \). The part of this weakly efficient frontier that can actually be reached by a real portfolio determines the strongly efficient frontier. These subsets correspond to the notions of weakly and strongly undominated solutions respectively.

Instead of working with expected return, variance and skewness, it is sometimes convenient to switch to the normalized moments determined by expected return, normalized variance (i.e., square root of variance or standard deviation) and normalized skewness (i.e., cubic root of skewness). Using the letter ‘n’ in the notation when referring to normalized moments (e.g., nVar refers to normalized variance), we also introduce the normalized function \( n\Phi : \mathcal{S} \to \mathbb{R}^3 \) mapping an arbitrary portfolio into normalized MVS space:

\[
n\Phi(w, w_{RF}) = (E[R(w, w_{RF})], \text{nVar}[R(w, w_{RF})], \text{nSk}[R(w, w_{RF})])
= (E[R(w, w_{RF})], \sqrt{\text{Var}[R(w, w_{RF})]}, \sqrt[3]{\text{Sk}[R(w, w_{RF})]}).
\]
In the next section, we turn to the description of two basic methodologies for determining optimal MVS portfolios: the PGP approach, and the shortage function method.

3 Methodology: Polynomial goal programming and shortage function

3.1 Polynomial goal programming

In this section, we explain the multi-objective approach for selecting MVS optimal portfolios proposed by Lai (1991). His main idea in developing a feasible MVS model assuming shorting and a risk-free asset is to search for a portfolio maximizing both expected excess return $Z_1$ and skewness $Z_3$, for a given level of variance.

One way to rationalize this approach is as follows. It follows directly from (1) and (2) that

$$Z_1 = \sum_{i=1}^{n} w_i (E[R_i] - R_{rf}) = \sum_{i=1}^{n} w_i E[R_i] + (w_{rf} - 1)R_{rf} = E[R(w, w_{rf})] - R_{rf}. \quad (6)$$

Under the assumption of short selling, the expected excess return $Z_1$ can become arbitrary large making the problem of maximizing $Z_1$ unbounded. Therefore, some restriction is needed. Lai (1991) proposes to focus on unit variance portfolios solely. Under the current assumptions, this makes sense since a non-unit variance portfolio can always be rescaled (as can be easily seen from (1) and (3)). This guarantees that the problem of maximizing both expected excess return $Z_1$ and skewness $Z_3$ is feasible.

Ideally, one searches for a unit variance portfolio maximizing both $Z_1$ and $Z_3$. However, it is very unlikely, if not impossible, to achieve both goals simultaneously. Therefore, the following PGP-program is suggested in Lai (1991) and has been adopted in the ensuing literature:

Definition 3.1. For given parameter values $\alpha, \beta \in \mathbb{R}_+$, the PGP-model is defined by

$$\text{PGP}(\alpha, \beta) = \min_{(w, w_{rf}) \in \Omega} \{ d_1^\alpha + d_3^\beta ; d_1 = Z_1^* - Z_1, d_3 = Z_3^* - Z_3, \text{Var}[R(w, w_{rf})] = 1 \},$$

with

$$Z_1^* = \max_{(w, w_{rf}) \in \Omega} \{ Z_1 ; \text{Var}[R(w, w_{rf})] = 1 \} \quad (7)$$

and

$$Z_3^* = \max_{(w, w_{rf}) \in \Omega} \{ Z_3 ; \text{Var}[R(w, w_{rf})] = 1 \}. \quad (8)$$

Thus, the target expected excess return $Z_1^*$ and the target portfolio skewness $Z_3^*$ are first determined in two separate portfolio optimization programs. (i) The program determined by (7)
maximizes expected excess return subject to a unit portfolio variance constraint. (ii) The program determined by (8) maximizes portfolio skewness subject to the same unit portfolio variance constraint. The PGP($\alpha, \beta$) program simultaneously minimizes the deviations between expected excess return and its target, and between portfolio skewness and its target, subject to the unit portfolio variance constraint. The powers $\alpha$ and $\beta$ in Definition 3.1 related to the deviation variables $d_1$ and $d_3$ are determined according to the investor’s preferences towards expected excess return and skewness. A larger value of $\alpha$ reflects a higher importance of maximizing expected excess return, while a larger value of $\beta$ corresponds with a higher interest in maximizing portfolio skewness.

In the remainder, we focus on the initial Lai (1991) formulation and ignore any variations on this basic framework. For instance, Davies, Kat, and Lu (2009) normalize the deviation variables and have a slightly different objective function.\(^6\)

### 3.2 Shortage function

Based on Briec, Kerstens, and Lesourd (2004), Briec, Kerstens, and Jokung (2007) and Kerstens, Mounir, and Van de Woestyne (2011), we now adapt the shortage function to the basic assumptions of having a risk-free asset and being able to sell short. Moreover, we include additional adapted versions suitable for generating in subsection 4.5 hereafter the geometrical representation of the MS frontier among others. We start with the shortage function in MVS space introduced in the following definition:

**Definition 3.2.** Let $g = (g_M, g_V, g_S) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and $g \neq 0$. The shortage function $S_g$ in the direction of vector $g$ is the function $S_g : \mathbb{R}^3 \rightarrow \mathbb{R}_+ \cup \{-\infty, +\infty\}$, with

$$S_g(y) = \sup_{\delta \in \mathbb{R}_+} \{\delta : y + \delta g \in \mathcal{DR}\}.$$

Starting from a given MVS point, the shortage function seeks for simultaneous improvements of expected return and skewness and reduction of the variance in the direction of vector $g$. The choice of this direction vector depends on the investor’s preferences. To give some examples, if an investor mainly wants to minimize variance without influencing the expected return and skewness level, then a direction vector $(0, -1, 0)$ would be appropriate. If variance should be decreased and skewness be increased without altering the given expected return level, then one could opt for the direction vector $(0, -1, 1)$. If improvements need to be realized on all three levels, then one could use the direction vector $g = (|y_M|, -|y_V|, |y_S|)$ for a given MVS point $y = (y_M, y_V, y_S)$. This choice, labeled the position dependent projection direction, has the additional advantage that the shortage function value has a proportional interpretation. Proportionality is convenient for practitioners when using the shortage function as an efficiency measure. These are merely some of the choices for

\(^6\)See also Canela and Colloza (2007), among others.
the direction vector one could come up with. The shortage function is an efficiency gauge, whereby a zero value indicates efficiency.

The computation of the shortage function value specified in Definition 3.2 for a MVS point \(y = (y_M, y_V, y_S)\) in the direction of the vector \(g = (g_M, g_V, g_S)\) can be obtained by solving the following cubic non-linear programming model:

\[
S_g(y) = \max_{(w,w_{RF}) \in \mathcal{A}} \{ \delta ; E[R(w, w_{RF})] \geq y_M + \delta g_M, \quad \Var[R(w, w_{RF})] \leq y_V + \delta g_V, \quad Sk[R(w, w_{RF})] \geq y_S + \delta g_S \}. 
\] (9)

When solving model (9), one not only obtains the efficiency value \(S_g(y)\) but additionally, also the left-hand and right-hand sides of the constraints in the optimal value provide useful information. Indeed, when denoting the optimal value by \(\delta^*\) and the optimal portfolio by \((w^*, w_{RF}^*)\), then the MVS point

\[
(E[R(w^*, w_{RF}^*)], Var[R(w^*, w_{RF}^*)], Sk[R(w^*, w_{RF}^*)])
\]

derived from the left-hand sided of the constraints, is always positioned on the strongly efficient frontier (see Kerstens, Mounir, and Van de Woestyne (2011)). The MVS point

\[
(y_M + \delta^* g_M, y_V + \delta^* g_V, y_S + \delta^* g_S)
\]
deducted from the right-hand sides of the constraints, is always situated on the weakly efficient frontier. Thus, \(S_g(y) = 0\) if and only if the MVS point \(y\) is part of the weakly efficient frontier.

One can force possible slacks between the left-hand and right-hand sides of the constraints in (9) to be zero by replacing the inequalities with equalities. Consequently, one obtains

\[
S^g_{\geq}(y) = \max_{(w,w_{RF}) \in \mathcal{A}} \{ \delta ; E[R(w, w_{RF})] = y_M + \delta g_M, \quad \Var[R(w, w_{RF})] = y_V + \delta g_V, \quad Sk[R(w, w_{RF})] = y_S + \delta g_S \}. 
\] (10)

If an optimal solution is found, then the corresponding MVS point obtained from the left-hand sides of the equality constraints is positioned on the boundary of \(\Phi(\mathcal{A})\).

One of the variables can even be dropped to obtain two-dimensional shortage functions relative to a more basic two-dimensional portfolio model. For instance, if the skewness is omitted in (9), then the MV shortage function is obtained:

\[
S_g^{-S}(y) = \max_{(w,w_{RF}) \in \mathcal{A}} \{ \delta ; E[R(w, w_{RF})] \geq y_M + \delta g_M, \Var[R(w, w_{RF})] \leq y_V + \delta g_V \}. 
\] (11)
3.3 Concluding comments

Within the same portfolio setting (i.e., a risk free rate and shorting), both methodologies determine MVS optimal portfolios using a slightly different approach. The PGP method computes an MVS optimal portfolio with unit variance and some compromise between excess return and skewness. Evaluating some arbitrary portfolio, the shortage function computes an efficiency value in a given direction and determines the corresponding optimal portfolio on the MVS frontier. If this arbitrary portfolio happens to be situated on the MVS frontier, then the shortage function value equals zero to indicate its efficiency.

It has been noted in Briec, Kerstens, and Jokung (2007) that the PGP approach has the weakness that the powers on the deviation variables have no obvious link with investor preferences. By contrast, the shortage function has been firmly linked to the MV and MVS utility functions in Briec, Kerstens, and Lesourd (2004) and Briec, Kerstens, and Jokung (2007) respectively.

4 Comparing polynomial goal programming and shortage function: New results and their illustration

To illustrate the PGP-framework set out in subsection 3.1, we use the example provided by Lai (1991) for which all statistical information is publicly available. In particular, the risk-free return, expected returns, covariance matrix and coskewness tensor are all published in the Lai (1991) article. Indeed, we simply have taken the risk-free asset of 0.0058 and the expected returns from Table 1 on page 298 in Lai (1991). The covariance matrix and coskewness tensor are available from the table in the appendix on page 302 in Lai (1991).

We now start our systematic comparison of both the PGP and shortage function approaches. This leads to several theoretical results hitherto unavailable in the literature. Each of these new findings is illustrated by either geometric representations in MVS space or the Lai (1991) data.

4.1 PGP results within the context of a MVS reconstruction based on the shortage function

Since it is possible to obtain points on the weakly and the strongly efficient frontier and on the boundary of $\Phi(\mathcal{S})$ by applying the shortage functions $S_g$ and $S_g^\tau$ introduced in (9) and (10) respectively, we can come up with techniques for generating portfolio frontiers. These are explained in detail in Kerstens, Mounir, and Van de Woestyne (2011). To summarize, it turns out that projecting some planar grid in fixed directions parallel to the coordinate axes obtains the best results. In fact, combining a series of such two-dimensional grids offers better results than starting off from a three-dimensional grid.
This technique of combining three planar grids and projecting the grid points in fixed directions parallel to the coordinate axes has been used for generating Figure 1. The shortage functions (9) and (10) have been used, both with direction vector \( g = (0,0,1) \) starting from a single planar grid situated in the MV plane. Figure 1a is in MVS space, while Figure 1b is in the space of mean return, normalized variance (i.e., standard deviation) and normalized skewness (i.e., cubic root of skewness). Part of the MVS frontier has been visualized in Figure 1 as a point cloud. This MVS frontier consists of the upper part of the boundary of \( \Phi(3) \) when looking in the skewness direction. This is the vertical direction in the figure. The horizontal plane consists of the MV plane.

The MV frontier visible in Figure 1 has been computed by means of the MV shortage function defined in (11). This MV frontier is obtained by minimizing variance given a fixed expected return. It turns out to be the lower boundary of the MVS frontier. Note that the skewness of all these MV optimal portfolios has been used for visualizing the MV frontier. Consequently, the MV frontier observed is actually embedded into MVS space. If one projects this embedded MV frontier into the MV plane, then a traditional two-dimensional MV curve is obtained.

Furthermore, observe in Figure 1 the vertical unit variance plane intersecting the MVS frontier along a planar curve. This curve is visualized separately in Figure 2. It consists of the two parts labeled ‘Boundary of \( \Phi(3) \)’ and ‘Strongly efficient frontier’. For simplicity, we refer to this curve as the mean skewness (MS) frontier. The scaling of the axes differs somewhat from the scaling used in Figure 1. However, it is mathematically the same curve. Notice the presence of a curve labeled ‘Weakly efficient frontier’ that continues into the strongly efficient frontier.

We briefly clarify the meaning of all these different frontier notions in Figure 2. As introduced earlier, the MVS disposal representation set \( \mathcal{DR} \) is an extension of \( \Phi(3) \) by adding the octant \( K \) (see (5)). By intersecting the weakly and strongly efficient frontiers with the vertical unit variance plane, corresponding frontier curves in MS space arise. From a finance viewpoint, only the strongly efficient part of the weakly efficient frontier is of importance since this part can be attained by real portfolios. The other part is only of importance in the mathematics underlying the shortage function and in comparison with non-parametric efficiency techniques (see Kerstens, Mounir, and Van de Woestyne (2011) for further details).

This MS section in Figure 2 can also be generated by the shortage function (10) with in addition the value of the portfolio variance being fixed at unity \( (V_0 = 1) \). Inspired by this, we introduce the notion of variance fixed shortage function.

**Definition 4.1.** Let \( g = (g_M, g_V, g_S) \in \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \) and \( g \neq 0 \). The variance fixed shortage function \( \mathcal{S}_g^{V=V_0} \) in the direction of vector \( g \) and fixed at variance level \( V = V_0 \) is the function
This new and special formulation turns out to be crucial in establishing a link with the PGP approach. Notice that the portfolio variance is now fixed \((V_0)\) instead of being linked to some portfolio under evaluation (as in models (9) and (10)). Also note that the variance component \(g_v\) of the direction vector \(g\) has no influence on the variance fixed shortage function value and, therefore, can be put to zero. Consequently, the direction vector specializes to \((g_M, 0, g_S)\). In particular, this variance fixed shortage function allows to reconstruct the MS frontier from a line grid along the fixed variance dimension and covering the range of the return dimension by projection into the remaining skewness dimension solely (i.e., the direction vector equals \((0, 0, 1)\)).

The PGP points computed with the powers indicated in the headings of Tables 2 and 3 on pages 299-300 in Lai (1991) are superimposed on the same MS section. The six PGP points are defined in the legend to Figure 2 where the exact powers are specified. Data on these same six PGP points in Figure 2 can be found in Table 1. Table 1 contains six parts separated by horizontal lines. The first part contains the optimal values of the PGP objective function. Then, we have the optimal values \(d_1^\ast\) and \(d_3^\ast\) of the deviation variables \(d_1\) and \(d_3\) respectively. The third part reports the optimal portfolio weights \((w^\ast, w_{Rf}^\ast)\). Part four shows the MVS coordinates of the optimal PGP point (obviously, its variance equals one). Finally, the last two parts list the values obtained from applying the MVS \((S_g)\) and MV \((S_g^{−S})\) shortage functions on the optimal PGP points in part four.

Currently, we have three main comments on Table 1. First, the optimal value of the PGP objective function has no straightforward interpretation in terms of investor preferences. For \(\alpha = 0\) and \(\beta = 1\), for instance, the optimal deviation values \(d_1^\ast = 0.027\) and \(d_3^\ast = 0\) indicate a positive deviation from the target expected excess return and no deviation from the target portfolio skewness, respectively. This exclusive preference for skewness yields an objective function value of unity. Exactly the opposite preferences (i.e., \(\alpha = 1\) and \(\beta = 0\)) yield also an objective function value of unity, even though now there is no deviation from the target expected excess return and a larger positive deviation from the target portfolio skewness (i.e., \(d_1^\ast = 0\) and \(d_3^\ast = 0.309\)). Thus, different deviation variables and opposite powers yield an identical objective function value of unity. Second, observe that the variance of all optimal solutions equals unity. Third, we did not succeed in duplicating the original results in Lai (1991) exactly (compare with his Tables 2 and 3 on pages 299-300).
In conclusion of these empirical results, PGP points seem to be located on the MS section with unit variance which is part of a MVS portfolio set reconstructed using the shortage function. Furthermore, this two-dimensional MS section can also be completely reconstructed using the variance fixed shortage function.

4.2 PGP results are MVS efficient

In this subsection, we want to firmly establish the efficiency status of these PGP optimal points. We observed in Figure 1 that all corresponding MVS points are located at the intersection of the MVS frontier with the unit variance vertical plane. In Table 1, the PGP optimal unit variance portfolios have been computed for distinct values of the parameters $\alpha$ and $\beta$. To verify the efficiency status of these PGP points and making use of the efficiency gauge interpretation of the shortage function, we compute the shortage function values for all these PGP optimal points. We do this for both the MVS and the MV shortage function defined by (11) and (9), respectively. In both cases, we use the position dependent direction vector. The optimal values for the MVS and MV shortage functions are reported in the last two parts of Table 1. Since all shortage function values equal zero, we conclude that all PGP optimal portfolios are MVS efficient. When using the MVS shortage function, these PGP points are projected onto themselves and therefore obtain an efficiency indicator equal to zero. This illustrates empirically that all PGP points are actually located on the MVS frontier. This confirms the observations made with respect to Figure 1.

Apart from this empirical observation, establishing whether or not PGP yields MVS efficient portfolios is important for several theoretical reasons. First, in the literature some doubts on the efficiency status of the PGP approach are present. For instance, Jurczenko, Maillet, and Merlin (2006) conjecture in a MVS-kurtosis context that “minimising deviations from the first four moments simultaneously only guarantees a solution close to the mean-variance-skewness-kurtosis efficient frontier.” (page 52). By contrast, Chunhachinda, Dandapani, Hamid, and Prakash (1997) claim that the “existence of an optimal solution” is a key feature of the PGP approach (page146). Second, in the case of the shortage function the MVS efficiency has been well established. In general, Briec and Kerstens (2010) prove that the shortage function guarantees a global optimal solution for a large class of convex problems. Furthermore, it yields a set of weakly efficient portfolio solutions.
that contains at least one strongly efficient solution. In particular, Briec, Kerstens, and Jokung (2007) establish sufficient conditions to guarantee MVS efficiency. These results clearly make the shortage function approach stand out compared to some of the other primal approaches listed in the introduction where efficiency claims seem only rarely established.

In fact, the above empirical observation with respect to the MVS efficiency of the PGP approach can now be formally proven in the following proposition contradicting the Jurczenko, Maillet, and Merlin (2006) conjecture.

**Proposition 4.1.** All PGP optimal portfolios are MVS efficient.

*Proof.* Denote the expected return of the optimal portfolio obtained from problem (7) by $P_M$ and the skewness of the portfolio solving problem (8) by $P_S$. Then, because of (6) and using the coordinate functions of $\Phi$,

$$d_1 = Z_1^* - Z_1 = P_M - \Phi_M(w, w_{RF}) \quad \text{and} \quad d_3 = Z_3^* - Z_3 = P_S - \Phi_S(w, w_{RF}). \quad (13)$$

Furthermore, denote the MVS point of the PGP optimal portfolio by $(y_{P\,GP}^M, y_{P\,GP}^V, y_{P\,GP}^S)$. Notice that the variance component equals unity by construction.

Consequently, the optimal values of $d_1$ and $d_3$ for the PGP optimal portfolio can be written as

$$d_1^{P\,GP} = P_M - y_{M}^{P\,GP} \quad \text{and} \quad d_3^{P\,GP} = P_S - y_{S}^{P\,GP}. \quad (14)$$

We now project the PGP optimal portfolio by means of the shortage function in the direction of vector $g = (g_M, g_V, g_S) \in \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+$ with not both $g_M$ and $g_S$ equal to zero. Assume the PGP optimal portfolio is not MVS efficient. Then, the corresponding shortage function value $\delta^* > 0$. From (9), it follows that the coordinates of the corresponding optimal MVS point are given by

$$y_{M}^{S} = y_{M}^{P\,GP} + \delta^* g_M, \quad y_{V}^{S} = y_{V}^{P\,GP} + \delta^* g_V \quad \text{and} \quad y_{S}^{S} = y_{S}^{P\,GP} + \delta^* g_S. \quad (15)$$

Using (13) for the optimal MVS point, we obtain

$$d_1^{S} = P_M - y_{M}^{S} \quad \text{and} \quad d_3^{S} = P_S - y_{S}^{S}. \quad (16)$$

Combining (14), (15) and (16) leads to

$$(d_1^{S})^\alpha + (d_3^{S})^\beta = (P_M - y_{M}^{P\,GP} - \delta^* g_M)^\alpha + (P_S - y_{S}^{P\,GP} - \delta^* g_S)^\beta \quad (17)$$

$$= (d_1^{P\,GP} - \delta^* g_M)^\alpha + (d_3^{P\,GP} - \delta^* g_S)^\beta \quad (18)$$

From Definition 3.1, we know that $PGP(\alpha, \beta) = (d_1^{P\,GP})^\alpha + (d_3^{P\,GP})^\beta$ must be a minimum.

---

9Hence, if a unique solution exists, then it is strongly efficient.
Consequently,

\[(d_1^{\text{PGP}} - \delta^* g_M)^\alpha + (d_3^{\text{PGP}} - \delta^* g_S)^\beta \geq (d_1^{\text{PGP}})^\alpha + (d_3^{\text{PGP}})^\beta.\]  

(19)

However, since \(\delta^* > 0, g_M \geq 0, g_S \geq 0\) and not both \(g_M\) and \(g_S\) are equal to zero, it follows

\[(d_1^{\text{PGP}} - \delta^* g_M)^\alpha + (d_3^{\text{PGP}} - \delta^* g_S)^\beta < (d_1^{\text{PGP}})^\alpha + (d_3^{\text{PGP}})^\beta,\]  

(20)

which contradicts (19). Consequently, the PGP optimal portfolio must be MVS efficient.

Turning now to the last part of Table 1 computed using the MV shortage function (11), notice first that this shortage function value is zero (meaning MV efficient) only for the PGP point obtained with \(\alpha = 1\) and \(\beta = 0\). This should not come as a surprise, since this choice for the power parameter values does not attach any importance to the skewness in the PGP optimization. Consequently, the optimal portfolio with the unit variance that minimizes the deviation with respect to the expected excess return must be on the MV frontier. The latter can also be visually verified from Figure 1 as the intersection point of the unit variance MS frontier and the MV frontier. All other combinations of \(\alpha\) and \(\beta\) lead to MV inefficient points, which can also be concluded from the non-zero shortage function values in the last section of Table 1.

Summing up, PGP points are situated on the unit variance MS section of the MVS frontier and are MVS efficient. This theoretical results is also illustrated using the shortage functions (9) and (10). Only one PGP point is also MV efficient according to the shortage function (11).

4.3 A MS section is sufficient to reconstruct the MVS frontier under shorting and a risk-free asset

As well-known in the traditional MV portfolio models, the combined assumptions of the availability of a risk-free asset and shorting lead to a linear relation between return and normalized variance. This is related to the fact that return and normalized risk can be rescaled at will.

Based on casual inspection of Figure 1b, one could conjecture that also a linear relationship prevails in the normalized MVS world. Indeed, for the given unit normalized variance, any combination of return and normalized skewness along the normalized MS section spans a line with the risk-free point containing frontier points. The rationale behind this phenomenon is similar to the one mentioned above.

While some intuitions underlying this result must be around in the literature (e.g., Hafner and Wallmeier (2008) on page 161), we are unaware of any precise statement in the multi-moment portfolio literature similar to ours. We formalize this intuition in the following proposition.

**Proposition 4.2.** Assume the presence of a risk-free asset and the possibility of short selling this
same risk-free asset. Then, the following statements hold true:

(a) An arbitrary risky portfolio (i.e., with non-zero variance) can be transformed to a unit variance portfolio such that for both portfolios, excess return, normalized variance and normalized skewness are proportional;

(b) Conversely, a unit variance portfolio can be transformed to a portfolio with arbitrary strictly positive variance such that for both portfolios, excess return, normalized variance and normalized skewness are proportional;

(c) The normalized MVS frontier takes the shape of a cone with vertex the risk-free asset;

(d) To generate the normalized MVS frontier, it suffices to generate a planar section of this frontier not going through the risk-free point and to construct the cone over this intersecting curve with vertex the risk-free asset.

Proof. (a) Let \((w, w_{RF})\) be an arbitrary risky portfolio and denote its variance by \(\sigma^2(>0)\). From (3), it follows that \(\sigma^2 = \sum_{i,j=1}^{n} w_i w_j \Omega_{ij}\). Consider now the new portfolio \((\bar{w}, \bar{w}_{RF})\) with \(\bar{w}_i = \frac{w_i}{\sigma}\), for \(i \in \{1, \ldots, n\}\). Because of (1), the proportion of the risk-free asset in this new portfolio must be \(\bar{w}_{RF} = 1 - \frac{1}{\sigma}(1 - w_{RF})\). Depending on the values of \(w_{RF}\) and \(\sigma\), \(\bar{w}_{RF}\) may well become negative. The latter explains the necessity to allow for shorting the risk-free asset. Following (6), the expected excess return of the new portfolio equals

\[
Z_1 = \sum_{i=1}^{n} \bar{w}_i (E[R_i] - R_{RF}) = \frac{1}{\sigma} Z_1.
\]  

Using (3) on the new portfolio, we obtain a variance of

\[
\text{Var}[R(\bar{w}, \bar{w}_{RF})] = \sum_{i,j=1}^{n} \bar{w}_i \bar{w}_j \Omega_{ij} = \frac{1}{\sigma^2} \text{Var}[R(w, w_{RF})] = 1.
\]  

Analogously, the skewness of the new portfolio is derived from (4) and leads to

\[
\text{Sk}[R(\bar{w}, \bar{w}_{RF})] = \sum_{i,j,k=1}^{n} \bar{w}_i \bar{w}_j \bar{w}_k \Lambda_{ijk} = \frac{1}{\sigma^3} \text{Sk}[R(w, w_{RF})].
\]

Rewriting (21), (22) and (23) with respect to normalized coordinates leads to

\[
\frac{Z_1}{Z_1} = \frac{n \text{Var}[R(\bar{w}, \bar{w}_{RF})]}{n \text{Var}[R(w, w_{RF})]} = \frac{n \text{Sk}[R(\bar{w}, \bar{w}_{RF})]}{n \text{Sk}[R(w, w_{RF})]} = \frac{1}{\sigma}.
\]

The proportionality observed in (24) combined with the unit variance in (22) proves (a).

(b) A similar argument to (a) leads to the proof for (b). Therefore, it is omitted.
(c) This result follows directly from the proportionality relation (24).

(d) This general result holds true for every cone and in particular for the cone in (c).

We add two comments to this proposition. Firstly, the presence of a risk-free asset in Proposition 4.2 is essential. Indeed, without a risk-free asset, the vertex point of the cone cannot be identified. Secondly, the short selling assumption on the risk-free asset is only required to guarantee the existence of the new portfolio \((\bar{w}, \bar{w}_{RF})\), since \(\bar{w}_{RF}\) might be negative depending on the given data. However, if \(\bar{w}_{RF}\) is positive in a particular case where short selling is excluded, then part (a) of Proposition 4.2 still holds true. In terms of visualization, we can then observe that the normalized MVS frontier contains a partial cone. In Figure 11b below, this phenomenon is clearly noticeable near the return axis. Notice that short selling of all assets is not necessary (as some authors in the PGP literature claim: see, e.g., Chunhachinda, Dandapani, Hamid, and Prakash (1997) on page 147).

The key advantage of Proposition 4.2 (part (d)) is that it provides a new method for geometrically reconstructing the normalized MVS frontier from a two-dimensional normalized MS section obtained at unit variance level. Moreover, because of the straightforward relations between normalized and non-normalized coordinates, a normalized MVS frontier can be easily transformed into a non-normalized MVS frontier.

We illustrate the usage of this new reconstruction technique in Figure 3. This figure visualizes the same frontier as the one present in Figure 1, except that it is reconstructed from the two-dimensional MS section in Figure 2. Any line starting from the risk-free asset and going through any point of the two-dimensional MS section spans a part of the MVS frontier. Thus, the traditional PGP model functions well in a portfolio setting with shorting and the availability of a risk-free asset, because it is in principle capable to generate a MS section providing the basis for a MVS portfolio reconstruction. Note that, within this portfolio setting, any arbitrarily fixed variance level \((V_0)\) instead of the unit level could have been selected to generate some MS section. This MS section would have allowed to reconstruct exactly the same MVS surface. In the PGP literature or elsewhere, we are unaware of any published results on these issues.

Apart from the generalization of the traditional two fund separation results (see, e.g., Luenberger (1998), Ch. 6) to the MVS portfolio model, it goes without saying that the possibility of generating the complete MVS frontier from a simple MS section leads to substantial gains in computer time. All figures in this contribution have been generated using Maple version 14. Just to provide some idea: while it takes more than 5000 seconds to reconstruct Figure 1, one only needs less than 100 seconds for Figure 3 on a Dell Latitude D610 with 4 Gb RAM.

In brief, the unit variance MS section inferred from the PGP points or from the variance fixed
shortage function is sufficient to reconstruct the MVS portfolio surface in a portfolio setting with shorting and a risk-free asset. This reconstruction strategy saves computer time. This result seems to have gone unnoticed in the literature so far.

4.4 Geometrical interpretation of PGP and its representation

As observed in subsections 4.1 and 4.2, varying the values of the parameters $\alpha$ and $\beta$ in Definition 3.1 leads to different points on the MS frontier that are MVS efficient. Returning to Figures 1 and 2, the MVS points corresponding with the PGP-optimal portfolios have been marked and labeled. It turns out that these MVS points are situated on the MS frontier. Therefore, these points are also part of the MVS frontier. By altering the values of the parameters $\alpha$ and $\beta$, different regions on the MS frontier can be reached.

Zghal, Audet, and Savard (2007) remark that the PGP problem in fact involves some relation between an exterior point constituted by the optimal solutions to the target expected excess return $Z_1^*$ and the target portfolio skewness $Z_3^*$ on the one hand, and the PGP optimal points on the other hand. This can be illustrated by glancing at Figure 2. Apart from the six PGP optimal points one can notice the ideal point in the upper right corner that is exterior to the MS section of the MVS frontier. This observation leads to the following natural questions: (i) What is the exact geometrical relation between this exterior ideal point and the PGP optimal points? This leads to a geometric interpretation of the PGP optimization framework. (ii) Can the PGP model be used for the geometric reconstruction of portfolio frontiers?

Starting with the first question, we need to understand more clearly how PGP-optimization determined by Definition 3.1 geometrically selects points on the MS frontier. Put differently, we need to grasp the relation between the choice of the parameters $\alpha$ and $\beta$ and the resulting position on the MS frontier. From Definition 3.1, it follows that

$$\text{PGP}(\alpha, \beta) = \min_{(w, w_{RF}) \in \mathcal{S}} \left\{ |Z_1 - Z_1^*|^{\alpha} + |Z_3 - Z_3^*|^{\beta}; \text{Var}[R(w, w_{RF})] = 1 \right\}. \quad (25)$$

As in the proof of Proposition 4.1, denote the expected return of the optimal portfolio obtained from the problem (7) by $P_M$ and the skewness of the portfolio solving the problem (8) by $P_S$. Then, because of (6) and using the coordinate functions of $\Phi$, $Z_1 - Z_1^* = \Phi_M(w, w_{RF}) - P_M$ and $Z_3 - Z_3^* = \Phi_S(w, w_{RF}) - P_S$. Consequently,

$$\text{PGP}(\alpha, \beta) = \min_{(w, w_{RF}) \in \mathcal{S}} \left\{ \bar{r}; \bar{r} = |\Phi_M(w, w_{RF}) - P_M|^{\alpha} + |\Phi_S(w, w_{RF}) - P_S|^{\beta} \text{ and } \Phi_V(w, w_{RF}) = 1 \right\}. \quad (25)$$
Thus, when solving (25), one needs to look for unit variance portfolios \((w, w_{Rf})\) minimizing
\[
\bar{r} = |\Phi_M(w, w_{Rf}) - P_M|^\alpha + |\Phi_S(w, w_{Rf}) - P_S)|^\beta.
\]
Since the point \((\Phi_M(w, w_{Rf}), \Phi_S(w, w_{Rf}))\) is contained in the unit variance MS plane, it is natural to consider the geometrical object in this MS plane described by the implicit equation
\[
|M - P_M|^\alpha + |S - P_S|^\beta = \bar{r},
\]
(26)
for some fixed \(\bar{r} \in \mathbb{R}_+\).

Consider now the example of parameter values \(\alpha = 2\) and \(\beta = 2\). Then, (26) can be simplified to \((M - P_M)^2 + (S - P_S)^2 = \bar{r}\), which represents the circle with radius \(\sqrt{\bar{r}}\) and center \((P_M, P_S)\).

Since we would rather not have the square root in the previous special case, we propose the transformation \(\bar{r} = r^{1/(\alpha + \beta)}\) for the general case. Consequently, (25) can now be rewritten as
\[
\text{PGP}(\alpha, \beta) = \left\{ r \in \mathbb{R}_+ : r^{1/(\alpha + \beta)} = |\Phi_M(w, w_{Rf}) - P_M|^\alpha + |\Phi_S(w, w_{Rf}) - P_S)|^\beta \right\},
\]
and (26) becomes
\[
|M - P_M|^\alpha + |S - P_S|^\beta = r^{1/(\alpha + \beta)}.
\]
(28)

It is easily verified that the geometrical object described by (28) is the circle in the unit variance MS-plane with radius \(r\) and center \((P_M, P_S)\) in the case \(\alpha = 2\) and \(\beta = 2\). Therefore, we propose the following definition:

**Definition 4.2.** The **PGP-circle** in the unit variance MS plane with radius \(r\) and center \((P_M, P_S)\) for parameter values \(\alpha\) and \(\beta\) is the geometrical object described by the implicit equation
\[
|M - P_M|^\alpha + |S - P_S|^\beta = r^{1/(\alpha + \beta)}.
\]

Figures 4, 5 and 6 provide several examples of PGP-circles, each with different radii and different parameter values. Clearly, the notion of a ‘circle’ should be relaxed. Only for the special case of \(\alpha = 2\) and \(\beta = 2\), we obtain regular Euclidean circles (as indicated earlier). In Figure 5 (left-hand side), we see the visualization for this special case. Observe that increasing the radius by adding a fixed constant (0.001 in the figure) results in equally spaced circles. Thus, in this case, \(r\) is considered as the distance from the center to any point on the PGP-circle.

[Figures 4, 5 and 6 about here]
However, for other combinations of $\alpha$ and $\beta$, this interpretation need no longer be correct. E.g., consider Figure 5 (right-hand side) where $\alpha = 2$ and $\beta = 5$. Here, adding a fixed constant of 0.001 no longer results in equally spaced PGP-circles. Actually, without going into the details, a proper distance interpretation is only possible in those cases where $\alpha = \beta$. Still, in the general case, a smaller $r$ results in a smaller PGP-circle. Consequently, (27) determines in the unit variance MS plane the minimal possible radius $r$ of a PGP-circle around $(P_M, P_S)$ that remains in contact with $\Phi(\mathcal{I})$. Clearly, at the minimum the PGP-circle is tangent to the boundary of $\Phi(3)$.

**Proposition 4.3.** The MVS image of the PGP optimal portfolio is locally a tangency point of the PGP-circle in the unit variance MS plane with radius $\text{PGP}(\alpha, \beta) = \frac{1}{\alpha + \beta}$ and center $(P_M, P_S)$ for parameter values $\alpha$ and $\beta$, and the unit variance MS section of the MVS frontier.

*Proof.* From (27) and (28), it follows directly that the MVS image of the PGP optimal portfolio is located on the PGP-circle in the unit variance MS plane with radius $\text{PGP}(\alpha, \beta) = \frac{1}{\alpha + \beta}$ and center $(P_M, P_S)$ for parameter values $\alpha$ and $\beta$. Additionally, we know from Proposition 4.1 that the PGP optimal portfolio is MVS efficient. Therefore, its MVS image is located on the unit variance MS section of the MVS frontier. Since the latter is a boundary, the result follows suit.

This proposition is illustrated in Figures 7, 8 and 9 for the same parameter settings as in Figures 4, 5 and 6, respectively. Note that the proportions of the PGP-circles in Figures 7 to 9 differ from those in Figures 4 to 6. This is simply a consequence of the scaling of the axes that has been applied in Figures 7, 8 and 9. E.g., in Figure 8 (left-hand side), the PGP-circle has become an ellipse because of this scaling.

[Figure 7, 8 and 9 about here]

Observe that in all these figures the point $(P_M, P_S)$ derived from the two-dimensional problems (7) and (8) can be considered an ultimate, but unreachable goal, since it is situated outside the MS section of the portfolio frontier. Therefore, this point is labeled ‘Ideal portfolio with maximal return and maximal skewness’. Around this point, the radius minimizing PGP-circle is drawn. This PGP-circle is tangent to the boundary of $\Phi(3)$ in the point visualized by a small square (□) in all figures. This tangent point is actually the MS point of the portfolio minimizing the radius.

Turning to the second question, we now highlight four potential difficulties. First, understanding the geometry behind PGP, we now return to the idea of varying the values of $\alpha$ and $\beta$ for computing sufficient points on the MS frontier to become visible. We know that this variation somehow leads to differently shaped PGP-circles, each tangent to the boundary of $\Phi(3)$. Compared to the shortage function approach where an initial point is projected according to a direction vector, the link between the parameter values for $\alpha$ and $\beta$ and the PGP-circle on the one hand and the resulting
boundary point is more involved. This directly leads to a first difficulty of controlling the position of this optimal PGP point.

A second difficulty is related to the presence of two degrees of freedom for generating the MS frontier which is a curve (i.e., a one-dimensional object). In general, increasing the value of $\alpha$ increases the importance of maximizing expected return, shifting the optimal PGP point down along the MS frontier. Similarly, increasing the value of $\beta$ increases the weight of maximizing skewness, resulting in an upward shift along the MS frontier. However, by increasing both $\alpha$ and $\beta$, effects from the first parameter partially outweight those from the second parameter. Thus, identical or neighboring PGP-optimal points can be found with quite different parameter settings.

Thirdly, the parameters $\alpha$ and $\beta$ can vary over an infinitely large interval, i.e., the set $\mathbb{R}_+$. Therefore, it can turn out that some extreme regions are hard to reach. This implies that one should make a correct selection in the combination of parameter values. This problem could be remedied by a transformation mapping the infinitely large parameter domain to a finite interval, picking appropriate values in this finite domain and then mapping these back into the original set. Quite a few transformations performing these operations could be considered, e.g., the function $f(x) = \frac{x}{1+x}$ with inverse $f^{-1}(y) = \frac{y}{1+y}$. This function $f$ maps the infinitely large interval $[0, +\infty)$ into the finite length interval $[0, 1)$.

Fourthly, it is clear from the geometrical interpretation of PGP-optimization that only the strongly efficient part of the MS frontier can be generated. Although this is the interesting part from a finance viewpoint, it might be preferably also being able to generate the whole MS frontier.

Two final remarks can be formulated. First, Zghal, Audet, and Savard (2007) reconstruct the MS section by developing a variation on the PGP approach. Second, once PGP can be used for reconstructing an MS section of the MVS portfolio frontier, then one can also conceive of MV and VS sections of the same MVS portfolio frontier. We return to this point below.

In conclusion, the PGP approach reconstructs the MS section starting from a ‘Ideal portfolio with maximal return and maximal skewness’ situated outside the portfolio frontier. By contrast, the shortage function starts off from a grid situated in the interior of the portfolio frontier. Subject to the four remarks listed, the PGP approach is capable to reconstruct the same MS section of the portfolio frontier.

4.5 Extending PGP to alternative portfolio models

In this subsection, we turn to the question whether the PGP approach in its current formulation is capable to handle other portfolio models. Lai (1991) claims that the assumption of short selling

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10Returning to the original information underlying the Lai (1991) data, Zghal, Audet, and Savard (2007) demonstrate that the original Lai (1991) results are situated somewhat below the frontier, thereby illustrating the technical progress in optimisation algorithms (see also footnote 8).
is non-essential (see footnote 6 on page 303). PGP results without shorting have been reported in Chang, Dupoyet, and Prakash (2008a) and in Prakash, Chang, and Pactwa (2003), among others. Thus, the PGP approach could equally well impose nonnegativity on all portfolio weights, excluding short selling of assets.

However, this claim is incorrect. We illustrate these shortcomings of the current PGP approach by considering two alternative portfolio models. Following Briec and Kerstens (2010), we illustrate how the shortage function can serve equally well for generating MV and MVS frontiers in the absence of a risk-free asset and when shorting is not allowed. In particular, we include Figure 10 without shorting and without risk-free asset, and Figure 11 with a risk-free asset but without shorting. For these figures, the same initial assets are chosen as those for Figure 1. The geometrical representations each time show the weakly (gray) and strongly (blue) efficient MVS frontier. Again, the MV frontiers can be observed as the lower boundary of the MVS frontiers.

Looking in detail at Figure 10 without risk-free asset and without shorting, it is now critical to observe that the maximal value for the optimal portfolio variances along the variance axis (part (a)) or the standard deviation axis (part (b)) is way below the unit level. Thus, the PGP model based on a unit variance constraint is infeasible in the absence of a risk-free asset and shorting and is unable to generate a MS section.

Similarly, looking in detail at Figure 11 with a risk-free asset but no shorting, one again observes that the maximal value for the optimal portfolio variances along the variance axis (part (a)) or the standard deviation axis (part (b)) is far below unity. Again, the PGP model based on a unit variance constraint would be infeasible in the presence of a risk-free asset without shorting and no MS section can be reconstructed. Our computations on the Lai (1991) data confirm this infeasibility.

In conclusion, if the basic portfolio setting does not include a risk-free asset or short selling is not allowed, then rescaling to unit variance need no longer be a feasible option to reconstruct a MS section of the MVS frontier. Obviously, as the empirical PGP results without shorting reported in Chang, Dupoyet, and Prakash (2008a) and Prakash, Chang, and Pactwa (2003) illustrate, the PGP approach may or may not work depending on the underlying return data. Thus, the claim of Lai (1991) is unfounded in general, as demonstrated by our counter-example in, e.g., Figure 11.

But, in these cases it may suffice to use other values for the variance constraint to make the optimization process as proposed in Lai (1991) a feasible option. This implies generalizing the definition of the PGP-model described by Definition 3.1 to this new one:
Definition 4.3. For given parameter values \( \alpha, \beta \in \mathbb{R}_+ \) and for some variance level \( V_0 \), the generalized PGP-model is defined by

\[
P_{\alpha, \beta}^{V_0}(w, w_{Rf}) = \min_{(w, w_{Rf}) \in \mathcal{I}} \left\{ d_1^* + d_3^* ; d_1 = Z_1^*(V_0) - Z_1, d_3 = Z_3^*(V_0) - Z_3, \text{Var}[R(w, w_{Rf})] = V_0 \right\},
\]

with

\[
Z_1^*(V_0) = \max_{(w, w_{Rf}) \in \mathcal{I}} \{ Z_1 ; \text{Var}[R(w, w_{Rf})] = V_0 \}
\]

and

\[
Z_3^*(V_0) = \max_{(w, w_{Rf}) \in \mathcal{I}} \{ Z_3 ; \text{Var}[R(w, w_{Rf})] = V_0 \}.
\]

While it may be a priori difficult to know which constraining values should be imposed to make the PGP model feasible in such contexts, it is straightforward to come up with a workable empirical strategy. One simple solution is to fix a variance level that is situated within the range of variance levels observed in the underlying return data. This guarantees that the PGP approach is feasible.

Proposition 4.4. The generalized PGP-model is feasible if \( V_0 \) is situated between the minimal possible and maximal possible variance levels observed in the underlying return data.

Proof. Under the given conditions, the set of portfolios over which to optimize the models (29) and (30) is nonempty. Therefore, solutions can be obtained. Consequently, also the set of portfolios over which to minimize the main model from Definition 4.3 is nonempty. Since zero is a lower boundary, a minimum must exist.

Obviously, once one can generate arbitrary two-dimensional MS sections using the PGP approach, it is rather straightforward to come up with alternative two-dimensional sections based on slightly rewritten PGP problems. For instance, as mentioned earlier, two-dimensional MV and VS sections are possible for a fixed return level situated within the observed range of return levels and a fixed skewness level situated within the observed range of skewness levels, respectively.

Apart from a portfolio setting with shorting and a risk-free asset, a more fundamental problem related to all two-dimensional sections is that some arbitrary two-dimensional section from an otherwise unknown three-dimensional MVS frontier model may be of limited value for portfolio management, despite the obvious time gain in calculation. Three-dimensional MVS frontiers convey much more information for investors. However, it is not trivial to come up with three-dimensional MVS frontier reconstructions using a PGP approach. This observation calls for a generalization of the current PGP approach allowing for more flexibility in the assumptions regarding the portfolio weights and capable of making three-dimensional MVS frontier reconstructions. This remains, to
the best of our knowledge, an open question. Perhaps the framework developed in Leung, Daouk, and Chen (2001) can be a starting base.

Thus, the claim in Lai (1991) that the PGP approach can also function with a risk-free asset and no shorting is incorrect. Generalizations of this PGP approach allowing for more flexibility regarding portfolio weights are possible by restricting the variance constraint within the empirical range.

5 Conclusions

In this contribution, we have made a first attempt to bridge the gap between seemingly two different approaches to determine MVS optimal portfolios. We are now in a position to summarize the four main theoretical results.

First, it has been demonstrated that PGP optimal portfolios are MVS efficient, a result that seems new in the literature. This shows up in the empirical illustration by the shortage function being nil for these PGP optimal portfolios. Second, we demonstrate that a single MS section is sufficient to reconstruct the MVS frontier in the presence of a risk-free asset and its shorting. This new theoretical result offers a strong justification for the basic Lai (1991) model in this portfolio setting. Third, we develop a geometrical interpretation for the PGP model that was hitherto unknown. In particular, the MVS image of a PGP optimal portfolio is locally a tangency point of a PGP-circle and the unit variance MS section of the MVS frontier. Fourth, it has first been empirically demonstrated that the claims in Lai (1991) about extending his approach to the case of no shorting are incorrect. However, we have developed a new, generalized PGP model whereby the unit variance goal is replaced by a value conditioned by the empirical range of variance levels observed in the underlying return distributions.

These results related to the PGP approach have been interpreted within the context of a three-dimensional reconstruction based on the shortage function (following Kerstens, Mounir, and Van de Woestyne (2011)) to show that the basic PGP model reconstructs a unit variance MS section of a MVS portfolio frontier. This has also led to a new variance fixed shortage function capable to generate any two-dimensional MS section. Furthermore, the above geometrical interpretation of PGP has allowed to outline a new way to generate a MS section of a MVS portfolio frontier, as an alternative to the new variance fixed shortage function. It also opens up perspectives for a better understanding of the role of the powers $\alpha$ and $\beta$ in PGP modeling.

Wrapping up, this first attempt to bridge the gap between two seemingly unrelated approaches to multi-moment portfolio modeling has found quite some common ground. Obviously, there is still a large variety of alternative approaches around for which family resemblances remain to be identified. However, this could be a fruitful avenue for future methodological research. From a more practical point of view, we think the main contribution lies in the new ways of reconstructing two-
dimensional sections of MVS portfolio sets. The possibility of these two new ways to reconstruct MS sections of the MVS portfolio frontier leads to substantial gains in computer time in a portfolio setting with a risk-free asset and shorting. But, on top of that it is obvious to extend both the new variance fixed shortage function and the new generalized PGP model to define any two-dimensional section of the MVS portfolio frontier.

The main open challenges we see to further develop the PGP approach is first to come up with a procedure for three-dimensional MVS frontier reconstruction in general portfolio settings. An even more ambitious goal is to try to develop a link between the PGP powers $\alpha$ and $\beta$ and investor preferences, keeping in mind that the duality between shortage function and MVS utility function has been firmly established (Briec, Kerstens, and Jokung (2007)).

References


Figure 1: Geometrical representation of the MVS frontier (a) and normalized MVS frontier (b), the intersection with the unit variance plane and the position of some PGP optimal portfolios.

Figure 2: Unit variance section of the MVS frontier and position of some PGP optimal portfolios.
Figure 3: Geometrical representation of the MVS frontier (a) and normalized MVS frontier (b), generated by scaling the optimal unit variance frontier points.

Figure 4: Visualization of PGP-circles for different radii and values of $\alpha$ and $\beta$.

Figure 5: Visualization of PGP-circles for different radii and values of $\alpha$ and $\beta$. 
Figure 6: Visualization of PGP-circles for different radii and values of $\alpha$ and $\beta$

Figure 7: Visualization of the PGP optimization process for different values of $\alpha$ and $\beta$

Figure 8: Visualization of the PGP optimization process for different values of $\alpha$ and $\beta$

Figure 9: Visualization of the PGP optimization process for different values of $\alpha$ and $\beta$
Figure 10: Geometrical representation of the weakly (gray) and strongly (blue) efficient MVS frontier (a) and normalized efficient MVS frontier (b) without shorting and without risk-free asset, with the corresponding MV frontier visible as lower boundary.

Figure 11: Geometrical representation of the weakly (gray) and strongly (blue) efficient MVS frontier (a) and normalized efficient MVS frontier (b) without shorting and with risk-free asset, with the corresponding MV frontier visible as lower boundary.
<table>
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<th>$\alpha = 1, \beta = 2$</th>
<th>$\alpha = 2, \beta = 1$</th>
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<td>1.000000</td>
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Table 1: The results from optimizing with PGP for distinct combinations of $\alpha$ and $\beta$ and the projection of this optimal portfolios using the MVS and MV shortage functions.