Conditional Quasi-Monte Carlo Sampling for Option Pricing under the LT Method

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Joint work with Nico Achtsis and Ronald Cools

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Outline

1. Pricing problems
2. Quasi-Monte Carlo methods
3. Conditional sampling and the linear transform
4. Conclusion
Derivative pricing

Aim: Calculate “fair price” of a derivative contract.

- They *derive* their value from underlying products.
- E.g.: Payoff formulated in terms of stock price paths.

→ Here: Options on multiple stocks with barrier conditions.
  Stock paths appear discretely sampled in payoff.
From expectation to high-dimensional integral

Assume: Stochastic model for stock paths.
→ For now: Black & Scholes log-normal prices
   (params: volatilities $\sigma_i$, correlations $\rho_{i,j}$ and risk-free rate $r$).

Given payoff function $g$ on $m$ stocks and $n$ time points, calculate expectation:

$$e^{-rT} \mathbb{E}[g(S)] = e^{-rT} \int_{\mathbb{R}^{n \times m}} g(S) \, dF(S)$$

by approximation

$$\frac{1}{N} \sum_{k=1}^{N} g(S^{(k)}), \quad \text{with } S^{(k)} \sim F.$$ 

→ Monte Carlo or quasi-Monte Carlo.
Path sampling

$$e^{-rT} E[g(S)] = e^{-rT} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g(S^{(k)})$$
Path sampling

\[ e^{-rT} \mathbb{E}[g(S)] = e^{-rT} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g(S^{(k)}) \]
Pricing problems

Quasi-Monte Carlo methods

Conditional sampling and the linear transform

Conclusion

Derivative pricing

Path sampling

\[ e^{-rT} \mathbb{E}[g(S)] = e^{-rT} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g(S^{(k)}) \]

And many many more…

(MC only has convergence \(O(N^{-0.5})\).)

(QMC does better… even higher order like \(O(N^{-3})\).)

(If you get lucky…)
This talk: Payoffs with barrier conditions

E.g.: Up-&-out condition (knock-out):

\[ g(S) = \max(f(S), 0) \mathbb{1}\left\{ \max_{1 \leq j \leq n} S_1(t_j) < B \right\} . \]

Barrier zeros out otherwise positive payoffs.
This talk: Payoffs with barrier conditions

E.g.: Up-&-out condition (knock-out):

$$g(S) = \max(f(S), 0) \cdot \mathbb{1}\left\{ \max_{1 \leq j \leq n} S_1(t_j) < B \right\}.$$
Quasi-Monte Carlo point sets (low discrepancy)

- QMC point sets are well distributed
  (middle: 64 Sobol’ points and right: 64 lattice points).
- In contrast MC shows large gaps and clusters
  (left: 64 mt19937 points).
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Improving convergence with Quasi-Monte Carlo

Single asset Asian option: \( g(S) = \max(\frac{1}{64} \sum_{j=1}^{64} S(t_j) - K, 0) \).

\( \Rightarrow O(N^{-0.9}) \)

Using Sobol’ or lattice sequence with PCA path construction.
Improving convergence showcase

Improving convergence with Quasi-Monte Carlo

Lookback option: \( g(S) = \max(\max_{1 \leq j \leq 5} S(t_j) - K, 0) \).

\[ \Rightarrow O(N^{-3}) \text{ (higher order convergence!)} \]

Using lattice sequence with periodization.
Pricing problems

Improving convergence showcase

Improving convergence with Quasi-Monte Carlo

Digital Asian: \( g(S) = 1 \left\{ \frac{1}{256} \sum_{j=1}^{256} S(t_j) > K \right\} \).

\( \Rightarrow O(N^{-0.85}) \)

Using the GELT adaptive QMC algorithm with Sobol’.

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Improving convergence with Quasi-Monte Carlo

Up-&-out Asian basket: \( g(S) = \max(\frac{1}{520} \sum_{i=1}^{4} \sum_{j=1}^{130} S_i(t_j) - K, 0) \ 1\{\max_{1 \leq j \leq 130} S_1(t_j) < B\} \).

\( \Rightarrow \) Up to \( O(N^{-0.7}) \)

Using the methods from this talk.
A crucial ingredient: path construction

- Path construction does not matter for Monte Carlo.
- But it does for quasi-Monte Carlo!

Given an $mn \times mn$ covariance matrix $\Sigma = CC^T$ sample the Brownian motion

$$w = CQz, \quad \forall z_k \sim N(0, 1),$$

where $Q$ is an orthogonal matrix.

By choosing $A$:
- Brownian bridge.
- PCA.
## Path constructions

### The LT method from Imai & Tan (2006)

For a payoff of the form $g(S) = \max(f(S), 0)$ the matrix $Q$ can be determined column-by-column by an optimization process:

\[
\text{maximize variance contribution of } f \text{ due to } k\text{th dimension} \\
\quad Q_{:,k} \in \mathbb{R}^{mn} \\
\text{subject to } \|Q_{:,k}\| = 1, \\
\quad \langle Q_{:,j}, Q_{:,k} \rangle = 0 \text{ for } j = 1, \ldots, k - 1.
\]
Low-discrepancy sequences software

You can find Matlab and C++ code

- `latticeseq_b2.m` or `lattice_sequence.hpp`: lattice sequence;
- `digitalseq_b2g.m`: digital sequence (e.g. Sobol');

on my home page:


Use like random number generator:

```matlab
>> latticeseq_b2 init0
>> Pfirst1024 = latticeseq_b2(200, 1024);
>> Pnext1024 = latticeseq_b2(200, 1024);
```
Conditional sampling

E.g.: Up-\&-out condition (knock-out):

\[ g(S) = \max(f(S), 0) \mathbb{1}\left\{ \max_{1 \leq j \leq n} S(t_j) < B \right\} . \]

⇒ Only generate paths which do not cross \( B \).

Here: need to be careful. Want to use LT!

Idea: Generate path by LT, then modify to not cross barrier.
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Conditional sampling

How to satisfy an up-&-out barrier in a nutshell

Under Black & Scholes we have, and need, for all $t_j$

$$S_1(t_j) = S_1(0) e^{(r - \sigma_i^2/2)t_j + \sigma_i W_i(t_j)} < B.$$

Which is equivalent to, for all $j = 1, \ldots, m$,

$$\sum_{k=1}^{mn} a_{j,k} \Phi^{-1}(u_k) < \log(B/S_1(0)) - (r - \sigma_i^2/2)t_j.$$

It happens we can force all $a_{j,1} > 0$ thus it suffices to force

$$u_1 < \Phi\left(\min_{1 \leq j \leq m} \frac{\log(B/S_1(0)) - (r - \sigma_i^2/2)t_j - \sum_{k=2}^{mn} a_{j,k} \Phi^{-1}(u_k)}{a_{j,1}}\right).$$
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Conditional sampling

Knock-out, knock-in, double barriers, …

Simplified algorithm for draw in QMC+LT+CS:

1. Draw uniform sample $u$ from QMC generator.
2. Calculate bounds on $u_1$ to satisfy barriers.
3. Calculate likelihood and payoff on rescaled sample.

This also works for knock-in options!

(They are more difficult in the Glasserman & Staum algorithm which also needs an explicit expression for the payoff without barrier.)
Numerical results

As an extension we throw in a root finding method for more drama (QMC+LT+CS+RF).

Up-&-out Asian basket: \( g(S) = \max\left( \frac{1}{520} \sum_{i=1}^{4} \sum_{j=1}^{130} S_i(t_j) - K, 0 \right) \mathbb{1}\{\max_{1 \leq j \leq 130} S_1(t_j) < B\} \).

<table>
<thead>
<tr>
<th>((P, \sigma_1, B, K))</th>
<th>QMC+LT+CS+RF</th>
<th>QMC+LT+CS</th>
</tr>
</thead>
<tbody>
<tr>
<td>((P_1, 0.25, 125, 100))</td>
<td>2039%</td>
<td>958%</td>
</tr>
<tr>
<td>((P_1, 0.25, 110, 100))</td>
<td>960%</td>
<td>446%</td>
</tr>
<tr>
<td>((P_1, 0.25, 105, 100))</td>
<td>638%</td>
<td>263%</td>
</tr>
<tr>
<td>((P_1, 0.25, 110, 90))</td>
<td>910%</td>
<td>737%</td>
</tr>
<tr>
<td>((P_1, 0.25, 105, 90))</td>
<td>757%</td>
<td>576%</td>
</tr>
<tr>
<td>((P_1, 0.25, 125, 110))</td>
<td>1923%</td>
<td>489%</td>
</tr>
<tr>
<td>((P_1, 0.55, 125, 100))</td>
<td>939%</td>
<td>437%</td>
</tr>
<tr>
<td>((P_1, 0.55, 125, 110))</td>
<td>1035%</td>
<td>234%</td>
</tr>
</tbody>
</table>

Reported numbers are the standard error of the MC+CS method divided by those of the QMC+LT+CS and QMC+LT+CS+RF methods for \( N = 163840 \) samples (using 10 shifts).
Conclusion

- Quasi-Monte Carlo for pricing problems!
- Improved rate of convergence! And constants!
- Current work in progress: Heston and Meixner.
- Software at my website:
- Recent results/preprints:
  - QMC+LT+CS(+RF), this talk, together with Nico Achtsis, currently submitted.