Jump robust two time scale covariance estimation and realized volatility budgets

Kris Boudt and Jin Zhang
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Kris Boudt*    Jin Zhang†

Abstract

We estimate the daily integrated variance and covariance of stock returns using high-frequency data in the presence of jumps, market microstructure noise and non-synchronous trading. For this we propose jump robust two time scale (co)variance estimators and verify their reduced bias and mean square error in simulation studies. We use these estimators to construct the ex-post portfolio realized volatility (RV) budget, determining each portfolio component’s contribution to the RV of the portfolio return. These RV budgets provide insight into the risk concentration of a portfolio. Furthermore, the RV budgets can be directly used in a portfolio strategy, called the equal-risk-contribution allocation strategy. This yields both a higher average return and lower standard deviation out-of-sample than the equal-weight portfolio for the stocks in the Dow Jones Industrial Average over the period October 2007–May 2009.

KEYWORDS: High frequency data, Integrated (co)variance, Jumps, Market microstructure noise, Realized volatility budget.

JEL classification: C13, C15, G11

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1 Introduction

The availability of tick by tick transaction price data makes it possible to estimate precisely the daily integrated variance and covariance of stock prices. When sampling at ultra high frequencies, the variance estimators need to be robust to microstructure noise caused by bid-ask bounce and rounding, among other. For the estimation of the integrated covariance, the estimators also need to be able to handle the asynchronicity of the times at which transactions are recorded. Zhang et al. (2005) proposed the \textit{two time scale realized variance} (TSRV) estimator and later Zhang (2010) proposed the \textit{two time scale realized covariance} (TSCV) estimator in the bivariate setting that handled asynchronicity and microstructure noise in the absence of price jumps. In the presence of jumps (cojumps), the TSRV (TSCV) estimator is a biased estimator of the integrated variance (covariance). Jump robust covariance estimators have recently been proposed by Barndorff-Nielsen and Shephard (2004a); Boudt et al. (2010) and Mancini and Gobbi (2009), but none of them can cope with the microstructure noise typically present in ultra high frequency data. We fill this gap by introducing a jump robust version of the TSRV and TSCV estimator. An extensive simulation study confirms the robustness of this estimator to market microstructure noise, non-synchronous trading and (co)jumps in the high frequency stock price series.

Previous applications of high frequency data based estimation of the covariance matrix include the evaluation of the forecasting precision of multivariate GARCH models (Laurent et al., 2010), covariance matrix forecasting (Barndorff-Nielsen et al., 2010) and Markowitz optimal portfolio allocation (Fleming et al., 2003; De Pooter et al., 2008; Liu, 2009). In portfolio risk management, risk budgets are a popular tool to decompose total portfolio risk into the risk contribution of each position. In
practice they are usually based on rolling estimates of the daily sample covariance matrix (Qian, 2006). We propose to use realized volatility risk budgets for topdown attribution of total realized portfolio volatility. We illustrate this technique on an equal-weight portfolio invested in financial assets during the credit crisis. We then consider a portfolio strategy that is directly based on a realized volatility budget. This so-called equal risk contribution (ERC) portfolio, defined as the portfolio equalizing the realized volatility contributions, is shown to outperform the equal-weight portfolio invested in the Dow Jones Industrial Average components over the period October 2007-May 2009.

The outline of the paper is as follows. Section 2 states the model settings. Sections 3 and 4 propose a jump robust estimator for both the univariate and bivariate case that are robust to asynchronicity and microstructure noise. Section 5 introduces the realized volatility budget for topdown attribution of total realized portfolio volatility into the continuous and jump volatility caused by each of the portfolio components. Section 6 concludes the paper.

2 Model

We are interested in how to estimate the integrated quantity of the elements in the variance-covariance matrix of two assets $X$ and $Y$ over a fixed time horizon $[0, T]$, when the prices of the stocks are contaminated by market microstructure noise and observed at non-synchronized trading times. Consider the log-price processes of two assets $\{X_t\}$ and $\{Y_t\}$, that are a combination of a latent log-price process $\{\tilde{X}_t\}$ and
\{\tilde{Y}_t\}$ and noise $\{\varepsilon_t^X\}$ and $\{\varepsilon_t^Y\}$:

\[ \begin{align*}
X_t &= \tilde{X}_t + \varepsilon_t^X \quad (2.1) \\
Y_t &= \tilde{Y}_t + \varepsilon_t^Y. \quad (2.2)
\end{align*} \]

We assume $\tilde{X} \perp \varepsilon_X$, $\tilde{Y} \perp \varepsilon_Y$ (the symbol $\perp$ is used to denote stochastic independence), and $\varepsilon_x \overset{i.i.d.}{\sim} N(0, \sigma_{\varepsilon_X}^2)$, $\varepsilon_y \overset{i.i.d.}{\sim} N(0, \sigma_{\varepsilon_Y}^2)$.

The latent processes are supposed to be a Brownian semimartingale process with finite activity (BSMFAJ) jumps. This means that it can be decomposed into a drift, stochastic volatility and jump component:

\[ \begin{bmatrix}
    d\tilde{X}_t \\
    d\tilde{Y}_t
\end{bmatrix} = \mu_t dt + \Omega_t dW_t + K_t dq_t, \quad (2.3) \]

where $\mu_t$ is a $2 \times 1$ vector of locally bounded drifts, $W_t$ is a vector of 2 independent Brownian motions and $\Omega_t$ is the $2 \times 2$ càdlàg process such that

\[ \Sigma_t = \begin{pmatrix}
    (\sigma_t^X)^2 & \sigma_t^{XY} \\
    \sigma_t^{XY} & (\sigma_t^Y)^2
\end{pmatrix} = \Omega_t \Omega_t' \quad (2.4) \]

is the spot covariance matrix process. The vectorial counting process $q_t$ governs the occurrence of jumps. The jump process is supposed to be independent of the diffusion process and to have finite activity, which means that the change in the counting process over any interval of time is finite with probability 1. Finally, $K_t$ is the $2 \times 2$ process controlling the magnitude and transmission of jumps such that $K_t dq_t$ is the contribution of the jump process to the change in the log-price series at time $t$. 

4
We are interested in the estimation of the integrated variances

\[ \int_0^T (\sigma^X_t)^2 dt \quad \text{and} \quad \int_0^T (\sigma^Y_t)^2 dt \]

as well as the integrated covariance

\[ \int_0^T \sigma_{XY}^2 dt. \]

For this we develop a jump robust version of the two time scale estimator of Zhang et al. (2005) and Zhang (2010).

3 Integrated variance

3.1 TSRV Revisited

The estimator we propose is a modification of the two time scale realized volatility (TSRV) estimator of Zhang et al. (2005). Let \( n \) be the total number of returns within \([0, T]\). The price process is observed at time points \( 0 < t_1 < t_2 < \ldots < t_{n+1} \leq T \).

The standard realized variance calculated on the whole data set is

\[ [X, X]_T^{(all)} = \sum_{i=1}^{n} (X_{t_{i+1}} - X_{t_i})^2 \]

The TSRV paper is based on partitioning the whole sample into \( K \) subsamples, with \( K \) an integer. Denote the average \( K \) subsampled realized variance as:

\[ [X, X]_T^{(avg)} = \frac{1}{K} \sum_{i=1}^{n-K+1} (X_{t_{i+K}} - X_{t_i})^2 . \]
The TSRV is defined as the difference between the averaged realized variance computed over \( K \) step apart subsampled observations and the adjusted realized variance computed using all observations:

\[
TSRV = (1 - \frac{n}{n})^{-1}([X, X]_T^{(\text{avg})} - \frac{n}{n} [X, X]_T^{(\text{all})}),
\]

(3.1)

where \( n = (n - K + 1)/K \). Taking the difference between \([X, X]_T^{(\text{avg})}\) and \(\frac{n}{n} [X, X]_T^{(\text{all})}\) cancels the effect of the microstructure noise. The factor \((1 - n/n)^{-1}\) is a coefficient to adjust for finite sample bias.

Zhang et al. (2005) show that in the absence of jumps, TSRV is a consistent estimator for the daily integrated variance when prices are contaminated by microstructure noise. In the presence of jumps, TSRV estimates the integrated variance plus the sum of squared intraday jumps.

### 3.2 Jump-Robust TSRV

We propose a jump robust version of the TSRV estimator called the \textit{robust} TSRV (RTSRV). It removes from the estimator the returns that exceed a high threshold of their distribution under the assumption of no jumps.\(^1\) Under the Brownian semimartingale model we have that for data that are \( K \) steps apart,

\[
X_{t_{i+K}} - X_{t_i} = \hat{X}_{t_{i+K}} - \hat{X}_{t_i} + \varepsilon^X_{t_i} - \varepsilon^X_{t_i} = \int_{t_i}^{t_{i+K}} \mu_s ds + \int_{t_i}^{t_{i+K}} \sigma_s dW_s + \varepsilon^X_{t_{i+K}} - \varepsilon^X_{t_i}.
\]

\(^1\)The truncation technique to create a jump-robust alternative to an existing estimator is not new and, in settings without microstructure noise, it has already been successfully implemented in Boudt et al. (2010), Corsi et al. (2009) and Mancini and Renò (2009).
If we assume the drift term to be zero, it follows that

\[
\frac{X_{t+K} - X_t}{\sqrt{\int_{t_i}^{t_{i+K}} \sigma^2_s ds + 2\sigma^2_i \epsilon X}} \sim N(0, 1).
\]  

(3.2)

This leads us to define the following indicator function

\[
I^K_N(i; \eta) = \begin{cases} 
1 & \text{if } \frac{(X_{t_{i+K}} - X_{t_i})^2}{(f_{t_{i+K}} \sigma^2_{ds} + 2\sigma^2_i \epsilon X)} \leq \eta \\
0 & \text{otherwise.}
\end{cases}
\]  

(3.3)

Applying the truncation to the returns in the two components of the original TSRV, gives us:

\[
\{X, X\}^{(\text{avg})}_T = c_\eta \sum_{i=1}^{n-K+1} (X_{t+i} - X_t)^2 I^K_N(i; \eta),
\]  

(3.4)

\[
\{X, X\}^{(\text{all})}_T = c_\eta \sum_{i=1}^{n} (X_{t+i} - X_t)^2 I^1_N(i; \eta),
\]  

(3.5)

where the factor \(c_\eta = 1/F_{\chi^2_N}^3(\eta)\) is a constant to adjust for the bias due to the thresholding and \(F_{\chi^2_N}^3\) is the chi-square distribution with \(N\) degrees of freedom.\(^2\)

In the simulation and empirical application, we set \(\eta = 9\), which corresponds to truncating off returns that are larger than 3 standard deviations away from the mean of the normal distribution. To improve the performance of the estimator in finite samples, we also need to adjust for the percentage of data we truncated in

\[^2\text{More formally, } c_\eta \text{ is defined as the inverse of the expectation of a chi-square random variable with 1 degree of freedom, } w \in [0, \eta]. \text{ Since } w \text{ has probability density function } f(w) = \frac{1}{\sqrt{2\Gamma(\frac{1}{2})}} w^{\frac{3}{2}} e^{-w/2}, \text{ we have}
\]

\[
\int_0^\eta w f(w) dw = \int_0^\eta \frac{1}{\sqrt{2\Gamma(\frac{1}{2})}} w^{\frac{3}{2}} e^{-w/2} dw = \int_0^\eta \frac{1}{2^{\frac{3}{2}} \Gamma(\frac{3}{2})} w^{\frac{3}{2} - 1} e^{-w/2} dw = F_{\chi^2_3}(\eta).
\]
order to make the estimator more stable:

\[
\{X, X\}_{T}^{(avg)^{*}} = \frac{c^{*}_{\eta}}{K} \sum_{i=1}^{n-K+1} (X_{t+K} - X_{t})^{2} I_{X}^{K}(i; \eta) \\
\{X, X\}_{T}^{(all)^{*}} = \frac{c^{*}_{\eta}}{n} \sum_{i=1}^{n} (X_{t+1} - X_{t})^{2} I_{X}^{1}(i; \eta),
\]

with \(c^{*}_{\eta} = F_{\chi^{2}}^{1}(\eta)/F_{\chi^{2}}^{3}(\eta)\). For \(\eta = 9\), \(c^{*}_{\eta} \approx 1.027\).

The proposed RTSRV is defined as:

\[
\text{RTSRV} = (1 - \frac{n}{n})^{-1}(\{X, X\}_{T}^{(avg)^{*}} - \frac{n}{n} \{X, X\}_{T}^{(all)^{*}}).
\]

In order to calculate the indicator function, we need to estimate the variance of noise and \(\int_{t_{i}}^{t_{i+K}} \sigma^{2} ds\). In the absence of jumps, Zhang et al. (2005) show that

\[
\hat{\sigma}_{\varepsilon_{X}}^{2} = \frac{1}{2n} ([X, X]_{T}^{(all)} - \text{TSRV}) \overset{D}{\rightarrow} \sigma_{\varepsilon_{X}}^{2}.
\]

Since \([X, X]_{T}^{(all)}\) and TSRV are equally affected by jumps, this estimator remains consistent under the model with jumps. For the estimation of \(\int_{t_{i}}^{t_{i+K}} \sigma_{s}^{2} ds\), when \(K\) is relatively small, (e.g. \(K \leq 300\) when \(n = 23400\)), we approximate it by

\[
\int_{t_{i}}^{t_{i+K}} \sigma_{t}^{2} dt \approx \frac{t_{i+K} - t_{i}}{T} \int_{0}^{T} \sigma_{t}^{2} dt.
\]

For \(\int_{0}^{T} \sigma_{s}^{2} ds\), we take an iterative approach. We first estimate the daily integrated variance using the modulated bipower variation (MBPV) of Podolskij and Vetter (2009) to compute the jump robust RTSRV. And then we iterate with the new RTSRV estimate to compute the threshold until no large returns are further truncated.
3.3 Simulation Design

We now assess the finite sample relative bias and root mean squared error of the estimators through a simulation study. The simulation is based on a time interval of one day $t \in [0,1], (T = 1)$. The log-price series are simulated from the jump diffusion model in Huang and Tauchen (2006):

\[
X_t = \tilde{X}_t + \varepsilon_t^X
\]

\[
d\tilde{X}_t = \mu X_t dt + \exp(\beta_0 + \beta_1 v_t^X) dW_t^X + dJ_t^X
\]

\[
dv_t^X = \alpha_v v_t^X dt + dB_t^X
\]

\[
Corr(dW_t^X, dB_t^X) = \gamma,
\]

where $\gamma$ is the leverage correlation, $v_t^X$ is a stochastic volatility process, $J_t^X$ is a compound Poisson process with jump intensity $\lambda$ and the size of the jump follows a i.i.d. normal distribution $N(0, \sigma_J^2)$. The initial value of $v_t^X$ for each day is drawn from a normal distribution $N(0, (\alpha_v^2)^{-1})$. We assume the noise $\varepsilon_t^X \sim i.i.d. N(0, \sigma_{\varepsilon}^2)$. The parameters of our choice for the simulation are

\[(\mu_X, \beta_0, \beta_1, \alpha_v, \gamma, \lambda) = (0.03, -5/16, 1/8, -1/40, -0.3, 2).\]

Typically, the ratio between the noise variance and the integrated variance ranges between 0.001 and 0.01 for high frequency data. Based on this, we consider both the small and large noise cases $\sigma_{\varepsilon}^2 = 0.001$ and $\sigma_{\varepsilon}^2 = 0.01$ for the simulation. For the jump size, we consider the case of small ($\sigma_J = 0.25$) and large ($\sigma_J = 2.5$) jumps. The diffusion part of the model is simulated with an increment of 1 second per tick using the Euler Scheme, and we simulated realizations of $T = 1000$ days.

In the simulation study we compare the proposed RTSRV estimator with the
original TSRV (both implemented with $K = 300$), the BPV of Barndorff-Nielsen and Shephard (2004b) and the medRV of Andersen et al. (2009).

$$BPV = \frac{\pi}{2} \frac{M}{M-1} \sum_{i=2}^{M} |r_i||r_{i-1}| \quad (3.11)$$

$$medRV = \frac{\pi}{6 - 4\sqrt{3} + \pi} \left( \frac{M}{M-2} \right) \sum_{i=2}^{M-1} \text{median}(|r_{i-1}|, |r_i|, |r_{i+1}|)^2, \quad (3.12)$$

with $M$ the number of intraday returns sampled at the given frequency. These estimators are robust to jumps, but not to microstructure noise. Like the RTSRV, the MBPV of Podolskij and Vetter (2009) should be robust to noise and jumps and therefore serves as an ideal benchmark. The following implementation of the MBPV is recommended by the authors:

$$MBPV = \frac{c_1 c_2 MBPV^*-\nu_2 \hat{\omega}^2}{\nu_1}, \quad (3.13)$$

with

$$\hat{\omega}^2 = \frac{1}{2n} \sum_{i=1}^{n} (X_{t_{i+1}} - X_{t_i})^2, MBPV^* = \frac{\pi}{2} \sum_{m=1}^{M-1} \left| X^{(K)}_m \right| \left| X^{(K)}_{m+1} \right|, \quad (3.14)$$

and

$$X^{(K)}_m = \frac{1}{n - K + 1} \sum_{i=[(m-1)n]/M}^{(mn)/M-K} (X_{t_{i+K+1}} - X_{t_{i+1}}). \quad (3.15)$$

The parameters $K = c_1 n^{\frac{1}{2}}, M = n/(c_2 K) = n^{\frac{1}{2}}/(c_1 c_2), c_2 = 2, c_1 = 0.25$ for $\omega^2 = 0.01$ and $c_1 = 0.125$ for $\omega^2 = 0.001$, as suggested by the authors. For each of these estimators, we compute the relative bias and RMSE. Denote by $IV_i$ the integrated variance of day $i$ and let $\hat{IV}_i$ be the estimate for it. The relative bias and
root mean squared error are computed as follows:

$$\text{Relative Bias} = \frac{1}{S} \sum_{i=1}^{S} \frac{\hat{IV}_i - IV_i}{IV_i} \quad \text{and RMSE} = \sqrt{\frac{1}{S} \sum_{i=1}^{S} (\hat{IV}_i - IV_i)^2}. \quad (3.16)$$

### 3.4 Simulation Results

The results are reported in Table 1. We first focus on the no jumps small noise case. We see that TSRV has the smallest relative bias among all estimators. The RTSRV using the MBPV to detect the price jumps has a relative bias of -0.015. To explain the source of this bias, we report in the columns labeled with $E[\hat{I}_X^{1}]$ and $E[\hat{I}_X^{K}]$, the percentage of returns for 1 and $K$ step apart prices that were not detected as jumps. Under the model without jumps, the expected proportion of correctly classified return is 99.73% for our threshold value $\eta = 9$. We see that the RTSRV using the MBPV overdetects jumps, especially for the $K$-step apart returns. This leads to an underestimation of $\{X, X\}_{T}^{(\text{avg})^r}$, which explains the downward bias in the 1-iteration version of RTSRV. However, if we iterate by using the obtained RTSRV for the implementation of the truncation function, we see that after 20-iterations, we obtain that the average estimate of the truncation function is very close to the theoretical value. The advantage of the iteration algorithm is to allow for the flexibility to start the initial estimation value of \(\int_0^T \sigma_s^2 ds\) at a very wide range. We can use almost any jump robust estimator available to do the first step rough estimation of integrated variance. The final RTSRV has a negligible relative bias of -0.015. The MBPV that uses pre-averaging to create robustness to microstructure noise, has a relative bias of -0.173, but it has the smallest RMSE among all estimators. The relative bias of the BPV and medRV is around 1.45. Similar conclusions can be drawn for the no jumps large noise case.
### Table 1: Comparison of relative bias and RMSE of integrated variance estimators.

<table>
<thead>
<tr>
<th>Estimators</th>
<th>No jumps</th>
<th>Small noise</th>
<th>Large noise</th>
<th>Relative Bias</th>
<th>RMSE</th>
<th>$E[I]^i_x$</th>
<th>$E[I]^i_y$</th>
<th>Relative Bias</th>
<th>RMSE</th>
<th>$E[I]^i_x$</th>
<th>$E[I]^i_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPV</td>
<td>1.451</td>
<td>0.907</td>
<td>na</td>
<td>na</td>
<td>14.903</td>
<td>1.898</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>medRV</td>
<td>1.448</td>
<td>0.941</td>
<td>na</td>
<td>na</td>
<td>15.065</td>
<td>1.960</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>MBPV</td>
<td>-0.173</td>
<td>0.344</td>
<td>na</td>
<td>na</td>
<td>-0.228</td>
<td>0.506</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>TSRV</td>
<td>-0.013</td>
<td>0.580</td>
<td>na</td>
<td>na</td>
<td>-0.007</td>
<td>0.580</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>RTSRV (1 iter)</td>
<td>-0.039</td>
<td>0.563</td>
<td>99.73%</td>
<td>99.62%</td>
<td>-0.032</td>
<td>0.572</td>
<td>99.73%</td>
<td>99.71%</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>RTSRV (20 iter)</td>
<td>-0.015</td>
<td>0.621</td>
<td>99.73%</td>
<td>99.72%</td>
<td>-0.004</td>
<td>0.616</td>
<td>99.73%</td>
<td>99.73%</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>Small jumps ($\sigma_J = 0.25$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BPV</td>
<td>1.826</td>
<td>0.921</td>
<td>na</td>
<td>na</td>
<td>15.617</td>
<td>1.981</td>
<td>na</td>
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<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>medRV</td>
<td>1.605</td>
<td>0.941</td>
<td>na</td>
<td>na</td>
<td>15.543</td>
<td>2.024</td>
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<td>na</td>
</tr>
<tr>
<td>MBPV</td>
<td>-0.034</td>
<td>0.338</td>
<td>na</td>
<td>na</td>
<td>0.039</td>
<td>0.510</td>
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<tr>
<td>TSRV</td>
<td>1.042</td>
<td>0.604</td>
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<td>na</td>
<td>1.115</td>
<td>0.619</td>
<td>na</td>
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<td>na</td>
</tr>
<tr>
<td>RTSRV (1 iter)</td>
<td>0.029</td>
<td>0.565</td>
<td>99.72%</td>
<td>98.88%</td>
<td>0.379</td>
<td>0.578</td>
<td>99.73%</td>
<td>99.50%</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>RTSRV (20 iter)</td>
<td>0.047</td>
<td>0.622</td>
<td>99.73%</td>
<td>98.97%</td>
<td>0.446</td>
<td>0.627</td>
<td>99.73%</td>
<td>99.54%</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>Large jumps ($\sigma_J = 2.5$)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>BPV</td>
<td>7.948</td>
<td>3.407</td>
<td>na</td>
<td>na</td>
<td>28.064</td>
<td>4.895</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>medRV</td>
<td>4.631</td>
<td>2.914</td>
<td>na</td>
<td>na</td>
<td>19.124</td>
<td>3.689</td>
<td>na</td>
<td>na</td>
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<td>na</td>
<td>na</td>
</tr>
<tr>
<td>MBPV</td>
<td>1.001</td>
<td>0.828</td>
<td>na</td>
<td>na</td>
<td>3.193</td>
<td>1.641</td>
<td>na</td>
<td>na</td>
<td>na</td>
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<td>na</td>
</tr>
<tr>
<td>TSRV</td>
<td>104.813</td>
<td>19.431</td>
<td>na</td>
<td>na</td>
<td>104.785</td>
<td>19.433</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>RTSRV (1 iter)</td>
<td>0.027</td>
<td>0.578</td>
<td>99.73%</td>
<td>97.63%</td>
<td>0.298</td>
<td>0.608</td>
<td>99.72%</td>
<td>97.83%</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>RTSRV (20 iter)</td>
<td>-0.003</td>
<td>0.636</td>
<td>99.73%</td>
<td>97.53%</td>
<td>0.111</td>
<td>0.633</td>
<td>99.72%</td>
<td>97.70%</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
</tbody>
</table>

When there is on average 2 jumps a day in the prices, for small jumps, MBPV has the smallest relative bias and RMSE among all estimators. However, in the large noise small jump case, jump size $\sigma_J = 0.25$ and the noise standard deviation $\sigma_\epsilon = 0.1$, we can’t really distinguish between jump and noise.

When we look at the large jump cases, we can see the RTSRV estimator outperforms the BPV, medRV and MBPV in both relative bias and RMSE. The MBPV shows a positive bias which is known to be the property of bipower type of estimators in finite sample cases.

### 4 Integrated covariance

#### 4.1 TSCV Revisited

The estimator of the integrated covariance between stocks $X$ and $Y$ needs to account for the asynchronicity in trading times of the two assets on the one hand, and the
possible dependence between the microstructure noise $\varepsilon_t^X$ and $\varepsilon_t^Y$ on the other hand.

Let $\Gamma$ and $\Theta$ be the set of transaction times of $X$ and $Y$, respectively. We use $N_t^X$ and $N_t^Y$ to denote the counting processes governing the number of observations traded in assets $X$ and $Y$ up to time $t$. It follows that the trades in $X$ and $Y$ occur at times $\Gamma = \{\tau_1, \tau_2, ..., \tau_{N_t^X}\}$ and $\Theta = \{\theta_1, \theta_2, ..., \theta_{N_t^Y}\}$, respectively. Write $N = N_t^X + N_t^Y - 2$. The estimation of the integrated covariance first requires to synchronize the data. One of the synchronization schemes is the refresh time method proposed by Harris et al. (1995). It picks the so-called refresh times at which all assets have traded at least once since the last refresh time point. More formally, the first refresh time corresponds to the first time at which both stocks have traded, i.e. $\phi_1 = \max(\tau_1, \theta_1)$. The subsequent refresh time is defined as the first time when both stocks have again traded, i.e. $\phi_{j+1} = \max(\tau_{N_{\phi_j}^X+1}, \theta_{N_{\phi_j}^Y+1})$. The complete refresh time sample grid is $\Phi = \{\phi_1, \phi_2, ..., \phi_{M_N+1}\}$, where $M_N$ is the total number of paired returns. The sampling points of asset $X$ and $Y$ are defined to be $t_i = \max\{\tau \in \Gamma : \tau \leq \phi_i\}$ and $s_i = \max\{\theta \in \Theta : \theta \leq \phi_i\}$.

Zhang (2010) studies in detail the previous tick estimator for integrated covariation based on refresh time grid, defined as

$$[X, Y]_T = \sum_{i=1}^{M_N} (X_{t_{i+1}} - X_{t_i})(Y_{s_{i+1}} - Y_{s_i}).$$

If the matched log-price changes $(X_{t_{i+1}} - X_{t_i})$ and $(Y_{s_{i+1}} - Y_{s_i})$ are asynchronous, this previous tick covariance estimator is typically biased. This bias is known as the Epps effect (Epps (1979)). Zhang (2010) derives an analytical expression for this bias. Clearly, the size of the bias is more pronounced for less liquid assets.

As a bias-free alternative, Zhang (2010) provided a two time scale realized co-
variance (TSCV) estimator which simultaneously cancels the Epps effect and the effect of microstructure noise. Similarly as in the univariate case, it is defined as

\[
\text{TSCV} = c_N \left( [X, Y]_T^{(K)} - \frac{\bar{\pi}_K}{\bar{\pi}_J} [X, Y]_T^{(J)} \right),
\]

(4.1)

where \([X, Y]_T^{(K)}\) is the average lag \(K\) realized covariance:

\[
[X, Y]_T^{(K)} = \frac{1}{K} \sum_{i=1}^{M_N-K+1} (X_{t_i+K} - X_{t_i}) (Y_{s_i+K} - Y_{s_i}),
\]

and where \(\bar{\pi}_K = (M_N - K + 1)/K\), \(\bar{\pi}_J = (M_N - J + 1)/J\), \(c_N = 1 + o_p(M_N^{-1/6})\). We choose \(c_N = M_N/((K - J)\bar{\pi}_K)\).

### 4.2 Jump-Robust TSCV

The proposed jump robust two time scale realized covariance (RTSCV) is defined as

\[
\text{RTSCV} = c_N \left( \{X, Y\}_T^{(K)} - \frac{\bar{\pi}_K}{\bar{\pi}_J} \{X, Y\}_T^{(J)} \right),
\]

(4.2)

where

\[
\{X, Y\}_T^{(K)} = \frac{1}{K} \sum_{i=1}^{M_N-K+1} c_i (X_{t_i+K} - X_{t_i}) (Y_{s_i+K} - Y_{s_i}) \frac{I^K_X(i; \eta)I^K_Y(i; \eta)}{M_N-K+1 \sum_{i=1}^{M_N-K+1} I^K_X(i; \eta)I^K_Y(i; \eta)},
\]

(4.3)

with \(I^K_X(i; \eta)\) the indicator function as defined in (3.3), and similarly for \(I^K_Y(i; \eta)\).

Let \(f_{\rho^*}(u, v)\) be the density function of bivariate normal random variables \(u\) and \(v\) with correlation coefficient \(\rho^*\). The correction factor to remove the bias due to
truncation is defined as follows:

$$c_i = \rho_i^* \frac{\int_{-\sqrt{\eta}}^{\sqrt{\eta}} \int_{-\sqrt{\eta}}^{\sqrt{\eta}} f_\rho^*(u, v) dudv}{\int_{-\sqrt{\eta}}^{\sqrt{\eta}} \int_{-\sqrt{\eta}}^{\sqrt{\eta}} uv f_\rho^*(u, v) dudv},$$

(4.4)

with $\rho_i^* = E[u_{t_i}, v_{s_i}],$

$$u_{t_i} = \frac{X_{t_{i+K}} - X_{t_i}}{\left( \int_{t_i}^{t_{i+K}} (\sigma_X)^2 ds + 2\sigma_X^2 \right)^{\frac{1}{2}}} \quad \text{and} \quad v_{s_i} = \frac{Y_{s_{i+K}} - Y_{s_i}}{\left( \int_{s_i}^{s_{i+K}} (\sigma_Y)^2 ds + 2\sigma_Y^2 \right)^{\frac{1}{2}}}.$$  

The correction factor is plotted in Figure 1 for $\eta = 9$. It takes values between 1.027 ($|\rho^*| = 1$) and 1.056 ($\rho^* \approx 0$). Since $c_i$ varies between a small range for $\eta = 9$, the gain in accuracy of computing $c_i$ on a first-step estimate of $\rho_i$ is negligible. In the remainder of the paper, we set $c_i$ to $(1.027 + 1.056)/2 = 1.042$ when $\eta = 9$.  

Figure 1: Correction factor for robust TSCV with truncation at $\eta = 9$, as a function of the correlation between the series.
4.3 Simulation Design

The setup we use is similar to the one used in Barndorff-Nielsen et al. (2010) and can be seen as the bivariate version of the model in (3.10):

\[ X_t = \tilde{X}_t + \varepsilon_t^X, \quad Y_t = \tilde{Y}_t + \varepsilon_t^Y \]

\[ d\tilde{X}_t = \mu_X dt + \gamma_X \sigma_t^X dB_t^X + \sqrt{1-\gamma_X^2} \sigma_t^X dW_t + dJ_t^X \]

\[ d\tilde{Y}_t = \mu_Y dt + \gamma_Y \sigma_t^Y dB_t^Y + \sqrt{1-\gamma_Y^2} \sigma_t^Y dW_t + dJ_t^Y \]

\[ \sigma_t^X = \exp(\beta_0 + \beta_1 v_t^X), \quad dv_t^X = \alpha v_t^X dt + dB_t^X \]

\[ \sigma_t^Y = \exp(\beta_0 + \beta_1 v_t^Y), \quad dv_t^Y = \alpha v_t^Y dt + dB_t^Y, \quad (4.5) \]

with \( B_X \perp B_Y, B_X \perp W \) and \( B_Y \perp W \). The continuous part of the price processes are simulated using the Euler Scheme with an increment of 1 second per tick. There are 23400 intervals within \([0, 1]\). We simulated \( S = 1000 \) days. The parameters \((\mu_X, \mu_Y, \beta_0, \beta_1, \alpha, \gamma_X, \gamma_Y)\) are set to \((0, 0, -5/16, 1/8, -1/40, -0.3, -0.3)\) The initial value of \( v_t^X \) and \( v_t^Y \) for each day is drawn from a normal distribution \( N(0, (-2\alpha)^{-1}) \).

The spot correlation between the continuous part of the log-price changes is \( \rho = \sqrt{(1-\gamma_X^2)(1-\gamma_Y^2)} = 0.91 \) for our simulation. The noise processes are simulated as:

\[ \varepsilon_t^X \overset{i.i.d.}{\sim} N(0, \sigma_{\varepsilon_X}^2), \quad \varepsilon_t^Y \overset{i.i.d.}{\sim} N(0, \sigma_{\varepsilon_Y}^2), \quad \varepsilon_X \text{ and } \varepsilon_Y \text{ are independent, with } (\sigma_{\varepsilon_X}^2, \sigma_{\varepsilon_Y}^2) = (0.001, 0.001). \]

For the Poisson jump processes, we simulated both independent jumps and co-jumps in each price series such that on average, there is 1 independent jump and 1 cojump in \( X \) and \( Y \) per day. The magnitude of the jumps are drawn from i.i.d. normal distribution. The standard deviation for the size of independent jumps and cojumps on \( X \) an \( Y \) are \((\sigma_{\text{indJ}}^X, \sigma_{\text{indJ}}^Y, \sigma_{\text{coJ}}^X, \sigma_{\text{coJ}}^Y) = (0.25, 0.50, 1.25, 1.50)\), respectively. Independent Poisson sampling schemes are used to generate the actual transaction times of asset \( X \) and \( Y \) such that there is on average one observation of
and $Y$ every 1 and 2 seconds, respectively.

We examine how well the RTSCV estimator performs compared to the original TSCV in Zhang (2010) as well as the realized bipower covariation (BPCV) proposed by Barndorff-Nielsen and Shephard (2004a). The BPCV is implemented with the refresh time scheme to pick out the paired observations and using $K$ step apart previous tick prices for the computation of returns $r^X_i$ and $r^Y_i$:

$$
BPCV = \frac{\pi}{8} \left( \sum_{i=2}^{M} \left| r^X_i + r^Y_i \right| \left| r^X_{i-1} + r^Y_{i-1} \right| - \left| r^X_i - r^Y_i \right| \left| r^X_{i-1} - r^Y_{i-1} \right| \right),
$$

(4.6)

where $M$ is the total number of returns. For the implementation of TSCV and RTSCV, the choice of parameters $J$ and $K$ needs to satisfy $1 \leq J \leq K$. We tried different values of $J$ ranging from 1 to 20, and found it doesn’t change the result much. In line with the original TSCV paper, we choose $J = 3$ for the illustration purpose of the tables. We compare the relative bias and RMSE for the three estimators when $K$ equals to 30, 60, 90, 120, 150 respectively. We also give the result for the BPCV when sampled at the highest previous tick frequency. We compute the relative bias and RMSE analogously as in (3.16).

4.4 Simulation Results

The left panel of Table 2 reports the results for the case of no jumps. We see that TSCV is the most efficient estimator among the three estimators at all sampling frequencies. RTSCV has slightly smaller RMSE than BPCV, but has much smaller relative bias. For both TSCV and RTSCV, we see that the relative bias is positive and increases as $K$ becomes very small, this is due to the microstructure noise. The RMSE decreases as $K$ becomes smaller as we gain efficiency by using more data for
Table 2: Comparison of relative bias and RMSE of integrated covariance estimators

<table>
<thead>
<tr>
<th>K</th>
<th>BPCV</th>
<th>TSCV</th>
<th>RTSCV</th>
<th>BPCV</th>
<th>TSCV</th>
<th>RTSCV</th>
<th>BPCV</th>
<th>TSCV</th>
<th>RTSCV</th>
<th>Relative Bias</th>
<th>RMSE</th>
</tr>
</thead>
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<tr>
<td><strong>No jumps</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>-0.061</td>
<td>-0.003</td>
<td>0.002</td>
<td>0.560</td>
<td>0.384</td>
<td>0.376</td>
<td>1.532</td>
<td>16.042</td>
<td>-0.006</td>
<td>0.963</td>
<td>3.427</td>
</tr>
<tr>
<td>270</td>
<td>-0.052</td>
<td>-0.002</td>
<td>0.003</td>
<td>0.514</td>
<td>0.372</td>
<td>0.369</td>
<td>1.477</td>
<td>16.070</td>
<td>-0.004</td>
<td>0.873</td>
<td>3.440</td>
</tr>
<tr>
<td>240</td>
<td>-0.058</td>
<td>-0.001</td>
<td>0.003</td>
<td>0.559</td>
<td>0.357</td>
<td>0.352</td>
<td>1.583</td>
<td>16.105</td>
<td>-0.003</td>
<td>1.034</td>
<td>3.459</td>
</tr>
<tr>
<td>210</td>
<td>-0.043</td>
<td>-8e-4</td>
<td>0.003</td>
<td>0.483</td>
<td>0.345</td>
<td>0.333</td>
<td>1.436</td>
<td>16.113</td>
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<td>0.950</td>
<td>3.463</td>
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<tr>
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<td>3e-5</td>
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<td>0.326</td>
<td>0.313</td>
<td>1.318</td>
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<td>0.852</td>
<td>3.469</td>
</tr>
<tr>
<td>150</td>
<td>-0.026</td>
<td>-9e-5</td>
<td>0.003</td>
<td>0.403</td>
<td>0.305</td>
<td>0.296</td>
<td>1.179</td>
<td>16.148</td>
<td>0.002</td>
<td>0.681</td>
<td>3.473</td>
</tr>
<tr>
<td>120</td>
<td>-0.048</td>
<td>-0.001</td>
<td>0.002</td>
<td>0.411</td>
<td>0.279</td>
<td>0.269</td>
<td>1.215</td>
<td>16.131</td>
<td>0.001</td>
<td>0.799</td>
<td>3.477</td>
</tr>
<tr>
<td>90</td>
<td>-0.053</td>
<td>0.001</td>
<td>0.003</td>
<td>0.312</td>
<td>0.244</td>
<td>0.242</td>
<td>1.003</td>
<td>16.085</td>
<td>0.004</td>
<td>0.539</td>
<td>3.466</td>
</tr>
<tr>
<td>60</td>
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<td>0.003</td>
<td>0.004</td>
<td>0.265</td>
<td>0.212</td>
<td>0.211</td>
<td>1.064</td>
<td>16.027</td>
<td>0.006</td>
<td>0.521</td>
<td>3.445</td>
</tr>
<tr>
<td>30</td>
<td>-0.095</td>
<td>0.005</td>
<td>0.005</td>
<td>0.200</td>
<td>0.146</td>
<td>0.155</td>
<td>0.720</td>
<td>15.883</td>
<td>0.007</td>
<td>0.321</td>
<td>3.391</td>
</tr>
<tr>
<td>1</td>
<td>-0.043</td>
<td>na</td>
<td>na</td>
<td>0.422</td>
<td>na</td>
<td>na</td>
<td>0.649</td>
<td>na</td>
<td>na</td>
<td>0.433</td>
<td>na</td>
</tr>
</tbody>
</table>

the calculation of the average lag $K$ realized covariance.

The right panel of Table 2 reports the results for the case when there are independent jumps and cojumps in the price series. TSCV has the largest relative bias and RMSE, which is not surprising since the estimator is not robust to cojumps. The relative bias of RTSCV is between 2 and 3 orders of magnitude smaller than the relative bias of BPCV. The RMSE of RTSCV is also smaller than the RMSE of BPCV. This suggests that in terms of bias and RMSE, of all three estimators considered, the RTSCV is the best estimator for all sampling frequencies.

## 5 Realized volatility budgets

For financial applications involving the covariance matrix, the RTSRV and RTSCV estimates of the integrated variance and covariance need to be combined into a positive semidefinite estimate of the integrated covariance matrix. This section first describes the construction of this estimate, called the RTSCov. Next realized volatility budgets are defined and their usefulness in portfolio risk management and portfolio allocation is illustrated.
5.1 RTSCov

Suppose we want to estimate the integrated covariance matrix of a $p$-dimensional return series:

$$\int_0^T \Sigma_t dt,$$  \hspace{1cm} (5.1)

with $\Sigma_t$ the $p \times p$ spot covariance matrix, as defined in (2.4). Let $C$ ($RC$) be the $p \times p$ matrices whose diagonal elements are given by the TSRV (RTSRV) estimates of the integrated variance and the off-diagonal elements are set to the corresponding TSCV (RTSCV) estimates of the integrated covariance. Under the BSMFAJ model of Section 2, Zhang et al. (2005) observe that the TSRV estimator on the diagonal of $C$ is a consistent estimator for the integrated variance plus the sum of squared jumps of the corresponding components, while it follows from Zhang (2010) that the TSCV estimator in the off-diagonal elements of $C$ is a consistent estimator for the integrated covariance plus the sum of cojumps. By construction, the elements in the RC matrix only estimate the integrated variance and covariance.

The matrices $C$ and $RC$ are symmetric but not necessarily positive semidefinite, however. We will use the eigenvalue method to transform $C$ and $RC$ into a positive semidefinite covariance matrix (see e.g. Rousseeuw and Molenberghs 1993). Let $\Gamma$ be the orthogonal matrix consisting of the $p$ eigenvectors of $C$. Denote $\lambda_1^+, \ldots, \lambda_p^+$ its $p$ eigenvalues, whereby the negative eigenvalues have been replaced by zeroes. Under this approach, the positive semi-definite projection of $C$ is $C^+ = \Gamma'\text{diag}(\lambda_1^+, \ldots, \lambda_p^+)\Gamma$. The projection of $RC$ on $RC^+$ is analogous. Asymptotically, $C$ and $RC$ are always positive semidefinite. In the remainder of the paper we call $C^+$ the two time scale covariance matrix (TSCov) and $RC^+$ the Robust TSCov (RTSCov).
5.2 Realized volatility budgets

Risk budgets are a popular tool to decompose total portfolio risk into the risk contribution of each position. See e.g. the book of Pearson (2002) for a general review. We are the first to consider the use of high-frequency data to construct risk budgets. Let $S_d$ be a $p \times p$ positive semidefinite estimate of the daily covariance matrix on day $d$ based on high frequency data. We will consider both ex post estimates of $S_d$, such as the TSCov or RTSCov, and ex ante estimates. In the latter case, $S_d$ is a conditional covariance matrix estimated using daily returns, the TSCov or RTSCov. In a portfolio setting, De Pooter et al. (2008) and Fleming et al. (2003) recommend the following update scheme for a high frequency based conditional covariance estimate:

$$S_d = \exp(-\alpha)S_{d-1} + \alpha \exp(-\alpha)(V_{d-1} + \eta_{d-1}\eta_{d-1}'),$$

(5.2)

where $\alpha$ is the decay parameter, $V_{d-1}$ is an ex post estimate of the integrated covariance matrix on day $d - 1$ and $\eta_{d-1}$ is the close-to-open return between day $d - 2$ and day $d - 1$. To assess the performance gains of using high-frequency data, we compare also with the $S_d$ forecasts where $(V_{d-1} + \eta_{d-1}\eta_{d-1}')$ is replaced by the lagged daily return. Note that the higher $\alpha$ is, the higher is the weight of the lagged daily return, TSCov and RTSCov in the covariance forecast. In the application, we use a burn-in sample of 50 days and consider $\alpha = 0.05, 0.1$ and 0.2.

For a portfolio with $p \times 1$ vector of weights $w$, the contribution of asset $i$ to the portfolio’s (ex post or ex ante) Realized Volatility $\sqrt{w' S_d w}$ on day $d$ is defined as

$$C_{(i)d}(w) = w(i) \frac{(S_d w)(i)}{\sqrt{w' S_d w}}.$$

(5.3)
The portfolio RV budget is $C_{(i)d}(w), \ldots, C_{(p)d}(w)$, where we use the subscript $(i) = (1), \ldots, (p)$ to denote the $p$ components. Since the portfolio realized volatility is a one-homogeneous function of the portfolio weights and $C_{(i)d}(w) = w_{(i)} \partial \sqrt{w'Sdw}/\partial w_{(i)}$, it follows from Euler’s theorem that all realized volatility contributions sum up to the total portfolio realized volatility:

$$\sum_{i=1}^{p} C_{(i)d} = \sqrt{w'Sdw}.$$  

For portfolio allocation purposes, it will be interesting to consider the RV contribution in percent of total portfolio RV:

$$\%C_{(i)d}(w) = \frac{C_{(i)d}(w)}{\sqrt{w'Sdw}}.$$  

The ex post RV budget is useful for retrospective analysis of the portfolio risk allocation. The ex ante RV budget serves as an input for portfolio allocation. In particular, we follow Maillard et al. (2010) in constructing an equal-risk-contribution (ERC) portfolio. Its portfolio weights on day $d + 1$ are those that minimize the sum of squared deviances of the percentage RV contributions on day $d$ with respect to $1/p$, subject to a no short sales and full investment constraint:

$$w^{\text{ERC}}_{d+1} = \arg\min_w \sum_{i=1}^{p} (\%C_{(i)d}(w) - 1/p)^2, \text{ s.t. } \min_{1 \leq i \leq p} w_{(i)} \geq 0 \text{ and } \sum_{i=1}^{p} w_{(i)} = 1. \ (5.4)$$

Maillard et al. (2010) show that minimizing this type of risk concentration leads to portfolios that strike a good balance between a minimum risk and maximum diversification objective.
5.3 Illustration

This subsection compares first the out-of-sample performance of the ERC allocation strategy, with the equal-weight portfolio and the long only minimum variance portfolio, with an upper 20% bound on the portfolio weights. Our data is from the New York Stock Exchange Trades and Quotes database and covers the period October 1, 2007 to May 29, 2009. The investment universe covers 27 of the 30 Dow Jones Industrial Average constituents at the beginning of 2008.\(^3\) The data are first cleaned following the step by step procedure described in Barndorff-Nielsen et al. (2009) and implemented in Cornelissen and Boudt (2010).

Summary statistics on the out-of-sample returns are reported in Table 3. Since our data correspond to a bear market regime, we expect the minimum variance strategy to have the best return performance. This is confirmed by the data, the minimum variance strategy having both the highest average return and lowest standard deviation. The ERC strategy performs significantly better than the equal-weight portfolio.

In terms of turnover and portfolio concentration, the minimum variance investment strategy performs the worst however.\(^4\) We measure the portfolio concentration by the average Gini coefficient of the portfolio weights.\(^5\) It takes values between 0

---

\(^3\)Tickers of the stocks in the sample are: AA, AXP, BA, C, CAT, DD, DIS, GE, HD, HON, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MO, MRK, MSFT, PFE, PG, UTX, VZ, WMT, XOM. Because of too many missing observations AIG, GM and T were removed from our sample because of missing data.

\(^4\)The portfolio turnover is computed here as the annualized average absolute percentage of wealth traded at the end of day \(d = 1, \ldots, D - 1\):

\[
\text{Turnover} = 252 \cdot \frac{1}{D-1} \sum_{d=1}^{D-1} \sum_{i=1}^{p} |w(i)_{d+1} - w(i)_{d+}|,
\]

where \(w(i)_{d+1}\) is the optimal weight of asset \(i\) on day \(d + 1\) and \(w(i)_{d+}\) is the weight of that asset before rebalancing to that optimal weight.

\(^5\)The Gini index is a measure of dispersion using the Lorenz curve. Let \(z\) be a random variable on \([0, 1]\) with distribution function \(F\). The Gini index is calculated as \(1 - 2 \int_0^1 L(z)dz\), where \(L(z) = \frac{\int_0^z udF(u)}{\int_0^1 udF(u)}\).

<table>
<thead>
<tr>
<th>Equal-Weight</th>
<th>α</th>
<th>Mean</th>
<th>StdDev</th>
<th>Sharpe L/S</th>
<th>Turnover</th>
<th>Gini</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.298</td>
<td>0.393</td>
<td>na</td>
<td>0.016</td>
<td>0</td>
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</tr>
<tr>
<td>ERC, daily</td>
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<td>-0.283</td>
<td>0.340</td>
<td>0.057</td>
<td>0.013</td>
<td>0.234</td>
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<tr>
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<td>0.063</td>
<td>0.013</td>
<td>0.262</td>
</tr>
<tr>
<td></td>
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<td>0.335</td>
<td>0.093</td>
<td>0.013</td>
<td>0.300</td>
</tr>
<tr>
<td>ERC, TSCov</td>
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<td>0.349</td>
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Mean is the annualized average daily portfolio return. St. dev. represents the annualized standard deviation of the daily portfolio returns. Sharpe L/S represent the Sharpe ratio of the strategy, that goes short in the equal-weight portfolio and long in the strategy under consideration. Gini is the Gini coefficient of the portfolio weights. The decay parameter α governs the persistence in the conditional covariance forecast. The higher α, the higher the weight of the daily return, TSCov and RTSCov in the covariance forecast used to construct the equal-risk-contribution (ERC) and minimum variance (Min Var) portfolios.

(equal-weight portfolio) and 1 (portfolio concentrated on one asset). We see that the minimum variance portfolio is heavily concentrated on a few assets, compared to the ERC portfolio. The ERC portfolio thus strikes an attractive balance between a high risk adjusted return performance and a well diversified portfolio.

The equal-weight portfolio can be seen as a passive allocation rule to pursue portfolio diversification. In contrast, the ERC portfolio is rebalanced on a daily basis to ensure ex ante portfolio diversification based on the forecasted covariance matrix. We compute the Sharpe ratio of the portfolio that is long on the ERC strategy and short on the equal-weight portfolio to measure the gains of pursuing
an active diversification strategy. When using the TSCov/RTSCov, the Sharpe ratio is between 0.7 and 0.9. Since both the equal-weight and ERC portfolios have a low portfolio turnover, it seems profitable to pursue such a strategy.

The choice of ex post covariance measure has a second-order impact on the portfolio performance. Like Fleming et al. (2003) and Fan et al. (2010), we find that there is a clear advantage of using high-frequency data based covariance estimates. We see that for the minimum variance strategy, the TSCov/RTSCov based portfolio have always a lower out-of-sample portfolio standard deviation. The ERC portfolio aiming at risk diversification has a significantly lower Gini coefficient when using the TSCov/RTSCov. For this data set, there is no clear impact of the choice between the TSCov and RTSCov on the portfolio performance.

The above application showed the usefulness of ex ante RV budgets in portfolio allocation. On a smaller portfolio and time window, we now illustrate the application of ex post RV budgets in portfolio risk management. In Figure 2 we zoom in on a financial portfolio invested in American International Group (AIG), Citibank (C) and JPMorgan Chase (JPM) in the turbulent month September 2008. Important events are the Lehman Brothers’ bankruptcy and the bailout of AIG by the Federal Reserve on September 15 and 16, respectively. We study the TSCov and RTSCov RV budget of the equal-weight and ERC portfolios.\footnote{The ERC portfolio is implemented with the one-step ahead covariance forecasts in (5.2), $\alpha = 0.2$ and $V$ set to the RTSCov.}

Let us first focus on the portfolio RV, which is the dashed line in Figure 2. Consistent with the results in Table 3, we find that for most days, the out-of-sample RV of the ERC portfolio is lower than the RV of the equal-weight portfolio. The realized volatility of the equal-weight portfolio peaks on September 16 at a value of 31.7% using the TSCov and 27.7% using the RTSCov. On that day, the ERC portfolio is 11.3%, 45.4% and 43.3% invested in AIG, JPM and C. Because of the
Figure 2: TSCov and RTSCov realized volatility budgets of equal-weight and RTSCov equal-risk-contribution portfolio in September 2008, together with the weight allocation of the equal-risk-contribution portfolio.

lower investment in AIG, its TSCov (RTSCov) RV is only 16.2% (13.2%) for that day. The difference between the TSCov and RTSCov portfolio RV corresponds to the jump contribution to portfolio RV. The relative jump contribution to portfolio RV is the largest on Sep 19. For the equal-weight (ERC) strategy, the portfolio TSCov-RV is 12.0% (8.2%) while the RTSCov-RV is 7.4% (5.1%). As can be seen in
Figure 3: Intraday price series of AIG, C and JPM on September 19, 2008.

Figure 3 there were huge price swings in all three stocks that day. E.g. between EST 9:34:08 and 9:36:24 the AIG stock price dropped from USD 4 to 3.51, while between 10:16:04 and 10:19:18 it rose from USD 2.55 to 3.13. These big price movements in short periods of times are jumps compared to the average volatility of that day.

The shaded areas in the first four plots of Figure 2 indicate the contribution of each portfolio component to total portfolio RV. These should be compared with the portfolio weights, which are 1/3 for the equal-weight portfolio and plotted in the
bottom figure for the ERC portfolio. For the equal-weight portfolio, we see that in the days around the bailout, the AIG investment dominated the portfolio RV. Interestingly, AIG dominates the portfolio RV of the ERC portfolio on September 16 and the preceding days, but no longer in the days following the bailout. We see that from the beginning of September up to September 17, the ERC portfolio has in fact gradually reduced the AIG weight from 36.2% on Sep 2 to 8.6% on Sep 17.

6 Conclusion

Jump robust covariance estimators have recently been proposed by Barndorff-Nielsen and Shephard (2004a); Boudt et al. (2010) and Mancini and Gobbi (2009), but none of them can cope with the microstructure noise typically present in ultra high frequency data. We propose a jump robust version of the two time scale variance and covariance estimator of Zhang et al. (2005) and Zhang (2010) based on thresholding. The proposed covariance estimator is robust to microstructure noise, asynchronicity and price jumps. We verified the reduced bias and mean square error of the proposed estimator in a simulation study.

As a second contribution, we use these estimators to construct the ex-post portfolio realized volatility (RV) budget, determining each portfolio component’s contribution to the total portfolio RV. These RV budgets provide insight into the risk concentration of portfolio. Furthermore, the RV budgets can be directly used in a portfolio strategy, called the equal-risk-contribution allocation strategy. This portfolio is shown to outperform the equal-weight portfolio invested in the Dow Jones Industrial Average components over the period October 2007-May 2009.
References


