Poisson boundary of the discrete quantum group $\widehat{A_u}(F)$

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Abstract

We identify the Poisson boundary of the dual of the universal compact quantum group $A_u(F)$ with a measurable field of ITPFI factors.

1 Introduction and statement of main result

Poisson boundaries of discrete quantum groups were introduced by Izumi [5] in his study of infinite tensor product actions of $SU_q(2)$. Izumi was able to identify the Poisson boundary of the dual of $SU_q(2)$ with the quantum homogeneous space $L^\infty(SU_q(2)/S^1)$, called the Podleś sphere. The generalization to $SU_q(n)$ was established by Izumi, Neshveyev and Tuset [6], yielding $L^\infty(SU_q(n)/S^{n-1})$ as the Poisson boundary. A more systematic approach was given by Tomatsu [10] who proved the following very general result: if $G$ is a compact quantum group with commutative fusion rules and amenable dual $\hat{G}$, the Poisson boundary of $\hat{G}$ can be identified with the quantum homogeneous space $L^\infty(G/K)$, where $K$ is the maximal closed quantum subgroup of Kac type inside $G$. Tomatsu’s result provides the Poisson boundary for the duals of all $q$-deformations of classical compact groups.

All examples discussed in the previous paragraph concern amenable discrete quantum groups. In [12], we identified the Poisson boundary for the (non-amenable) dual of the compact quantum group $A_o(F)$ with a higher dimensional Podleś sphere. Although the dual of $A_o(F)$ is non-amenable, the representation category of $A_o(F)$ is monoidally equivalent with the representation category of $SU_q(2)$ for the appropriate value of $q$. The second author and De Rijdt provided in [4] a general result explaining the behavior of the Poisson boundary under the passage to monoidally equivalent quantum groups. In particular, a combination of the results of [4] and [5] give a more conceptual approach to our identification in [12].

The quantum random walks studied on a discrete quantum group $\hat{G}$ have a semi-classical counterpart, being a Markov chain on the (countable) set $\text{Irred}(\hat{G})$ of irreducible representations of $G$ (modulo unitary equivalence). All the examples above, share the feature that the semi-classical random walk on $\text{Irred}(\hat{G})$ has trivial Poisson boundary.

In this paper, we identify the Poisson boundary for the dual of $G = A_u(F)$. In that case, $\text{Irred}(\hat{G})$ can be identified with the Cayley tree of the monoid $\mathbb{N} * \mathbb{N}$ and, by results of [9], has a non-trivial Poisson boundary: the end compactification of the tree with the appropriate harmonic measure. Before discussing in more detail our main result, we introduce some terminology and notations.
For a more complete introduction to Poisson boundaries of discrete quantum groups, we refer to [15, Chapter 4].

Compact quantum groups were originally introduced by Woronowicz in [17] and their definition finally took the following form.

Definition 1.1 (Woronowicz [18, Definition 1.1]). A compact quantum group \( G \) is a pair consisting of a unital C*-algebra \( C(G) \) and a unital \(*\)-homomorphism \( \Delta : C(G) \to C(G) \otimes C(G) \), called comultiplication, satisfying the following two conditions.

- **Co-associativity**: \((\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta\).
- \(\text{span} \Delta(C(G))(1 \otimes C(G))\) and \(\text{span} \Delta(C(G))(C(G) \otimes 1)\) are dense in \(C(G) \otimes C(G)\).

In the above definition, the symbol \( \otimes \) denotes the minimal (i.e. spatial) tensor product of C*-algebras.

Let \( G \) be a compact quantum group. By [18, Theorem 1.3], there is a unique state \( h \) on \( C(G) \) satisfying \((\text{id} \otimes h)\Delta(a) = h(a)1 = (h \otimes \text{id})\Delta(a)\) for all \( a \in C(G) \). We call \( h \) the Haar state of \( G \).

A unitary representation of \( G \) on the finite dimensional Hilbert space \( H \) is a unitary operator \( U \in \mathcal{L}(H) \otimes C(G) \) satisfying \((\text{id} \otimes \Delta)(U) = U_{12}U_{13} \). Given unitary representations \( U_1, U_2 \) on \( H_1, H_2 \), we put

\[
\text{Mor}(U_2, U_1) := \{ S \in \mathcal{L}(H_1, H_2) \mid (S \otimes 1)U_1 = U_2(S \otimes 1) \}.
\]

Let \( U \) be a unitary representation of \( G \) on the finite dimensional Hilbert space \( H \). The elements \((\xi^* \otimes 1)U(\eta \otimes 1) \in C(G) \) are called the coefficients of \( U \). The linear span of all coefficients of all finite dimensional unitary representations of \( G \) forms a dense \(*\)-subalgebra of \( C(G) \) (see [18, Theorem 1.2]). We call \( U \) irreducible if \( \text{Mor}(U, U) = \mathbb{C}1 \). We call \( U_1 \) and \( U_2 \) unitarily equivalent if \( \text{Mor}(U_2, U_1) \) contains a unitary operator.

Let \( U \) be an irreducible unitary representation of \( G \) on the finite dimensional Hilbert space \( H \). By [18, Proposition 5.2], there exists an anti-linear invertible map \( j : H \to \overline{H} \) such that the operator \( U^c \in \mathcal{L}(\overline{H}) \otimes C(G) \) defined by the formula \((j(\xi)^* \otimes 1)U^c(j(\eta) \otimes 1) = (\eta^* \otimes 1)U^*(\xi \otimes 1)\) is unitary. One calls \( U^c \) the contragredient of \( U \). Since \( U \) is irreducible, the map \( j \) is uniquely determined up to multiplication by a non-zero scalar. We normalize in such a way that \( Q := j^*j \) satisfies \( \text{Tr}(Q) = \text{Tr}(Q^{-1}) \). Then, \( j \) is determined up to multiplication by \( \lambda \in S^1 \) and \( Q \) is uniquely determined. We call \( \text{Tr}(Q) \) the quantum dimension of \( U \) and denote it by \( \text{dim}_q(U) \). Note that \( \text{dim}_q(U) \geq \dim(H) \) with equality holding iff \( Q = 1 \).

The tensor product \( U \otimes V \) of two unitary representations is defined as \( U_{13}V_{23} \).

Given a compact quantum group \( G \), we denote by \( \text{Irred}(G) \) the set of irreducible unitary representations of \( G \) modulo unitary conjugacy. For every \( x \in \text{Irred}(G) \), we choose a representative \( U^x \) on the Hilbert space \( H_x \). We denote by \( Q_x \in \mathcal{L}(H_x) \) the associated positive invertible operator and define the state \( \psi_x \) on \( \mathcal{L}(H_x) \) by the formula

\[
\psi_x(A) := \frac{\text{Tr}(Q_x A)}{\text{Tr}(Q_x)}.
\]

The dual, discrete quantum group \( \hat{G} \) is defined as the \( \ell^\infty \)-direct sum of matrix algebras

\[
\ell^\infty(\hat{G}) := \prod_{x \in \text{Irred}(G)} \mathcal{L}(H_x).
\]
We denote by $p_x, x \in \text{Irred}(G)$, the minimal central projections in $\ell^\infty(\hat{G})$. Denote by $\epsilon \in \text{Irred}(G)$ the trivial representation and by $\hat{\epsilon} : \ell^\infty(\hat{G}) \to \mathbb{C}$ the co-unit given by $ap_x = \hat{\epsilon}(a)p_x$. Whenever $x, y, z \in I$, we use the short-hand notation $\text{Mor}(x \otimes y, z) := \text{Mor}(U^x \otimes U^y, U^z)$ and we write $z \subset x \otimes y$ if $\text{Mor}(x \otimes y, z) \neq \{0\}$.

The von Neumann algebra $\ell^\infty(\hat{G})$ carries a comultiplication $\hat{\Delta} : \ell^\infty(\hat{G}) \to \ell^\infty(\hat{G}) \otimes \ell^\infty(\hat{G})$, uniquely characterized by the formula

$$\hat{\Delta}(a)(p_x \otimes p_y)S = Sap_z \quad \text{for all} \quad x, y, z \in \text{Irred}(G) \quad \text{and} \quad S \in \text{Mor}(x \otimes y, z).$$

Denote by $L^\infty(G)$ the weak closure of $C(G)$ in the GNS representation of the Haar state $h$. One defines the unitary $V \in \ell^\infty(\hat{G}) \otimes L^\infty(G)$ by the formula

$$V := \bigoplus_{x \in \text{Irred}(G)} U^x.$$

The unitary $V$ implements the duality between $G$ and $\hat{G}$, in the sense that it satisfies

$$(\hat{\Delta} \otimes \text{id})(V) = V_{13}V_{23} \quad \text{and} \quad (\text{id} \otimes \Delta)(V) = V_{12}V_{13}.$$

Discrete quantum groups can also be defined intrinsically, see [13]. Whenever $\omega \in \ell^\infty(\hat{G})_*$ is a normal state, we consider the Markov operator $P_\omega : \ell^\infty(\hat{G}) \to \ell^\infty(\hat{G}) : P_\omega(a) = (\text{id} \otimes \omega)\hat{\Delta}(a)$.

By [7, Proposition 2.1], the Markov operator $P_\omega$ leaves globally invariant the center $Z(\ell^\infty(\hat{G}))$ of $\ell^\infty(\hat{G})$ if and only if $\omega = \psi_\mu := \sum_{x \in \text{Irred}(G)} \mu(x)\psi_x$ where $\mu$ is a probability measure on $\text{Irred}(G)$.

We only consider states $\omega$ of the form $\psi_\mu$ and denote by $P_\mu$ the corresponding Markov operator. Note that we can define a convolution product on the probability measures on $\text{Irred}(G)$ by the formula

$$P_{\mu \ast \eta} = P_\mu \circ P_\eta.$$

Considering the restriction of $P_\mu$ to $\ell^\infty(\text{Irred}(\hat{G})) = \mathcal{Z}(\ell^\infty(\hat{G}))$, every probability measure $\mu$ on $\text{Irred}(G)$ defines a Markov chain on the countable set $\text{Irred}(G)$ with $n$-step transition probabilities given by

$$p_zp_n(x, y) = p_zP_n^\mu(p_y).$$

Note that the 1-step transition probabilities are given by

$$p_1(x, y) = \sum_{z \in x \otimes y} \frac{\mu(z)\dim_y(y)}{\dim_y(x)\dim_y(z)}.$$

The probability measure $\mu$ is called generating if for every $x, y \in \text{Irred}(G)$, there exists an $n \in \mathbb{N} \setminus \{0\}$ such that $p_n(x, y) > 0$. 

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\textbf{Definition 1.2.} Let $\mathbb{G}$ be a compact quantum group and $\mu$ a generating probability measure on $\text{Irred}(\mathbb{G})$. The Poisson boundary of $\hat{\mathbb{G}}$ with respect to $\mu$ is defined as the space of $P_{\mu}$-harmonic elements in $\ell^\infty(\hat{\mathbb{G}})$.

$$H^\infty(\hat{\mathbb{G}}, \mu) := \{ a \in \ell^\infty(\hat{\mathbb{G}}) \mid P_{\mu}(a) = a \}.$$  

The weakly closed vector subspace $H^\infty(\hat{\mathbb{G}}, \mu)$ of $\ell^\infty(\hat{\mathbb{G}})$ is turned into a von Neumann algebra using the product (cf. [5, Theorem 3.6])

$$a \cdot b := \lim_{n \to \infty} P^n_{\mu}(ab)$$

and where the sequence at the right hand side is strongly\(^*\) convergent.

- The restriction of $\hat{\alpha}$ to $H^\infty(\hat{\mathbb{G}}, \mu)$ is a faithful normal state on $H^\infty(\hat{\mathbb{G}}, \mu)$.
- The restriction of $\hat{\Delta}$ to $H^\infty(\hat{\mathbb{G}}, \mu)$ defines a left action

$$\alpha_G : H^\infty(\hat{\mathbb{G}}, \mu) \to \ell^\infty(\hat{\mathbb{G}}) \hat{\otimes} H^\infty(\hat{\mathbb{G}}, \mu) : a \mapsto \hat{\Delta}(a)$$

of $\hat{\mathbb{G}}$ on $H^\infty(\hat{\mathbb{G}}, \mu)$.
- The restriction of the adjoint action to $H^\infty(\hat{\mathbb{G}}, \mu)$ defines an action

$$\alpha_G : H^\infty(\hat{\mathbb{G}}, \mu) \to H^\infty(\hat{\mathbb{G}}, \mu) \hat{\otimes} L^\infty(\mathbb{G}) : a \mapsto \forall(a \otimes 1)V^*.$$  

We denote by $H^\infty_{\text{centr}}(\hat{\mathbb{G}}, \mu) := H^\infty(\hat{\mathbb{G}}, \mu) \cap Z(\ell^\infty(\hat{\mathbb{G}}))$ the space of bounded $P_{\mu}$-harmonic functions on $\text{Irred}(\mathbb{G})$. Defining the conditional expectation

$$E : \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\text{Irred}(\hat{\mathbb{G}})) : E(a)p_x = \psi_x(a)p_x,$$

we observe that $E$ also provides a faithful conditional expectation of $H^\infty(\hat{\mathbb{G}}, \mu)$ onto the von Neumann subalgebra $H^\infty_{\text{centr}}(\hat{\mathbb{G}}, \mu)$.

We now turn to the concrete family of compact quantum groups studied in this paper and introduced by Van Daele and Wang in [14]. Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $F \in \text{GL}(n, \mathbb{C})$. One defines the compact quantum group $\mathbb{G} = A_n(F)$ such that $C(\mathbb{G})$ is the universal unital $C^*$-algebra generated by the entries of an $n \times n$ matrix $U$ satisfying the relations

$$U \quad \text{and} \quad \overline{FUF}^{-1} \quad \text{are unitary, with} \quad (\overline{U})_{ij} = (U_{ij})^*$$

and such that $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$. By definition, $U$ is an $n$-dimensional unitary representation of $A_n(F)$, called the fundamental representation.

Fix $F \in \text{GL}(n, \mathbb{C})$ and put $\mathbb{G} = A_n(F)$. For reasons to become clear later, we assume that $F$ is not a scalar multiple of a unitary $2 \times 2$ matrix.

By [2, Théorème 1], the irreducible unitary representations of $\mathbb{G}$ can be labeled by the elements of the free monoid $I := \mathbb{N} * \mathbb{N}$ generated by $\alpha$ and $\beta$. We represent the elements of $I$ as words in $\alpha$ and $\beta$. The empty word is denoted by $\epsilon$ and corresponds to the trivial representation of $\mathbb{G}$, while $\alpha$ corresponds to the fundamental representation and $\beta$ to the contragredient of $\alpha$. We denote by $x \mapsto \pi$ the unique antimultiplicative and involutive map on $I$ satisfying $\pi = \beta$. This involution corresponds to the contragredient on the level of representations. The fusion rules of $\mathbb{G}$ are given by

$$x \otimes y \cong \bigoplus_{z \in \mathbb{I}, x = x_0z, y = y_0} x_0 y_0.$$
So, if the last letter of $x$ equals the first letter of $y$, the tensor product $x \otimes y$ is irreducible and given by $xy$. We denote this as $xy = x \otimes y$.

Denote by $\partial I$ the compact space of infinite words in $\alpha$ and $\beta$. For $x \in \partial I$, the expression

$$x = x_1 \otimes x_2 \otimes \cdots$$

means that the infinite word $x$ is the concatenation of the finite words $x_1x_2\cdots$ and that the last letter of $x_n$ equals the first letter of $x_{n+1}$ for all $n \in \mathbb{N}$. All elements $x$ of $\partial I$ can be decomposed as in (2), except the countable number of elements of the form $x = y\alpha\beta\alpha\beta\cdots$ for some $y \in I$.

Below, we will only deal with non-atomic measures on $\partial I$, so that almost every point of $\partial I$ has a decomposition as in (2). We denote by $\partial_0 I$ the subset of $\partial I$ consisting of the infinite words that have a decomposition of the form (2).

The following is the main result of the paper.

**Theorem 1.3.** Let $F \in \text{GL}(n, \mathbb{C})$ such that $F$ is not a scalar multiple of a unitary $2 \times 2$ matrix. Write $\mathbb{G} = A_n(F)$ and suppose that $\mu$ is a finitely supported, generating probability measure on $I = \text{Irred}(\mathbb{G})$. Denote by $\partial I$ the compact space of infinite words in the letters $\alpha, \beta$. There exists

- a non-atomic probability measure $\nu_\epsilon$ on $\partial I$,
- a measurable field $M$ of ITPFI factors over $(\partial I, \nu_\epsilon)$ with fibers

$$(M_x, \omega_x) = \bigotimes_{k=1}^\infty (\mathcal{L}(H_{x_k}), \psi_{x_k})$$

whenever $x \in \partial I$ is of the form $x = x_1x_2x_3\cdots = x_1 \otimes x_2 \otimes x_3 \otimes \cdots$,

- an action $\hat{\beta}_\mathbb{G}$ of $\hat{\mathbb{G}}$ on $M$ concretely given by (3) below,

such that, with $\omega_\infty = \int \oplus \omega_x \, d\nu_\epsilon(x)$, the Poisson integral formula

$$\Theta_\mu : M \to \mathcal{H}^\infty(\hat{\mathbb{G}}, \mu) : \Theta_\mu(a) = (\text{id} \otimes \omega_\infty)\hat{\beta}_\mathbb{G}(a)$$

defines a $\ast$-isomorphism of $M$ onto $\mathcal{H}^\infty(\hat{\mathbb{G}}, \mu)$, intertwining the action $\hat{\beta}_\mathbb{G}$ on $M$ with the action $\alpha_\mathbb{G}$ on $\mathcal{H}^\infty(\hat{\mathbb{G}}, \mu)$.

Moreover, defining the action $\beta_\mathbb{G}$ of $\mathbb{G}$ on $M_x$ as the infinite tensor product of the inner actions $a \mapsto U^{x_k}(a \otimes 1)(U^{x_k})^*$, we obtain the action $\beta_\mathbb{G}$ of $\mathbb{G}$ on $M$. The $\ast$-isomorphism $\Theta_\mu$ intertwines $\beta_\mathbb{G}$ with $\alpha_\mathbb{G}$.

The comultiplication $\hat{\Delta} : \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}}) \otimes \ell^\infty(\hat{\mathbb{G}})$ can be uniquely cut down into completely positive maps $\hat{\Delta}_{x \otimes y, z} : \mathcal{L}(H_z) \to \mathcal{L}(H_x) \otimes \mathcal{L}(H_y)$ in such a way that

$$\hat{\Delta}(a)(p_x \otimes p_y) = \sum_{z \subseteq x \otimes y} \hat{\Delta}_{x \otimes y, z}(ap_z)$$

for all $a \in \ell^\infty(\hat{\mathbb{G}})$.

We denote by $|x|$ the length of a word $x \in I$. 

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If now \(x, y \in I, z \in \partial I\) with \(yz = y \otimes z\) and \(|y| > |x|\), we define for all \(s \subset x \otimes y\),
\[
\hat{\Delta}_{x \otimes y, sz} : M_{sz} \to \mathcal{L}(H_x) \otimes M_{yz}
\]
by composing \(\hat{\Delta}_{x \otimes y, s} \otimes \text{id}\) with the identifications \(M_{sz} \cong \mathcal{L}(H_s) \otimes M_z\) and \(M_{yz} \cong \mathcal{L}(H_y) \otimes M_z\). The action \(\beta_{\hat{G}} : M \to \ell^\infty(\hat{G}) \otimes M\) of \(\hat{G}\) on \(M\) is now given by
\[
\beta_{\hat{G}}(a)_{x, yz} = \sum_{s \subset x \otimes y} \hat{\Delta}_{x \otimes y, sz}(a_{sz})
\]
whenever \(a \in M, x, y \in I, z \in \partial I, |y| > |x|\) and \(yz = y \otimes z\). Note that we identified \(\ell^\infty(\hat{G}) \otimes M\) with a measurable field over \(I \times \partial I\) with fiber in \((x, z)\) given by \(\mathcal{L}(H_x) \otimes M_z\).

**Further notations and terminology**

Fix \(F \in \text{GL}(n, \mathbb{C})\) and put \(\hat{G} = A_u(F)\). We identify Irred(\(\hat{G}\)) with \(I := \mathbb{N} \ast \mathbb{N}\). We assume that \(F\) is not a multiple of a unitary \(2 \times 2\) matrix. Equivalently, \(\dim_q(\alpha) > 2\). The first reason to do so, is that under this assumption, the random walk defined by any non-trivial probability measure \(\mu\) on \(I\) (i.e. \(\mu(\epsilon) < 1\)), is automatically *transient*, which means that
\[
\sum_{n=1}^\infty p_n(x, y) < \infty
\]
for all \(x, y \in I\). This statement can be proven in the same was as [7, Theorem 2.6]. For the convenience of the reader, we give the argument. Denote by \(\dim_{\text{min}}(y)\) the dimension of the carrier Hilbert space of \(y\), when \(y\) is viewed as an irreducible representation of \(A_u(I_2)\). Since \(F\) is not a multiple of a unitary \(2 \times 2\) matrix, it follows that \(\dim_q(y) > \dim_{\text{min}}(y)\) for all \(y \in I \setminus \{\epsilon\}\). Denote by \(\text{mult}(z; y_1 \otimes \cdots \otimes y_n)\) the multiplicity of the irreducible representation \(z \in I\) in the tensor product of the irreducible representations \(y_1, \ldots, y_n\). Since the fusion rules of \(A_u(F)\) and \(A_u(I_2)\) are identical, it follows that
\[
\text{mult}(z; y_1 \otimes \cdots \otimes y_n) \leq \dim_{\text{min}}(y_1) \cdots \dim_{\text{min}}(y_n).
\]
One then computes for all \(x, y \in I, n \in \mathbb{N}\),
\[
p_n(x, y) = \sum_{z \subset \mathcal{T} \otimes y} \mu^x(z) \frac{\dim_q(y)}{\dim_q(x) \dim_q(z)}
\]
\[
= \frac{\dim_q(y)}{\dim_q(x)} \sum_{z \subset \mathcal{T} \otimes y} \sum_{y_1, \ldots, y_n \in I} \text{mult}(z; y_1 \otimes \cdots \otimes y_n) \frac{\mu(y_1) \cdots \mu(y_n)}{\dim_q(y_1) \cdots \dim_q(y_n)}
\]
\[
\leq \frac{\dim_q(y)}{\dim_q(x)} \dim(\mathcal{T} \otimes y) \rho^n
\]
where \(\rho = \sum_{y \in I} \mu(y) \frac{\dim_{\text{min}}(y)}{\dim_q(y)}\). Since \(\mu\) is non-trivial and \(F\) is not a multiple of a \(2 \times 2\) unitary matrix, we have \(0 < \rho < 1\). Transience of the random walk follows immediately.

An element \(x \in I\) is said to be *indecomposable* if \(x = y \otimes z\) implies \(y = \epsilon\) or \(z = \epsilon\). Equivalently, \(x\) is an alternating product of the letters \(\alpha\) and \(\beta\).

For every \(x \in I\), we denote by \(\dim_q(x)\) the *quantum dimension* of the irreducible representation labeled by \(x\). Since \(\dim_q(\alpha) > 2\), take \(0 < q < 1\) such that \(\dim_q(\alpha) = \dim_q(\beta) = q + 1/q\). An
important part of the proof of Theorem 1.3 is based on the technical estimates provided by Lemma 6.1 and they require \( q < 1 \), i.e. \( \dim_q(\alpha) > 2 \).

Denote the \( q \)-numbers
\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.
\]
Writing \( x = x_1 \otimes \cdots \otimes x_n \) where the words \( x_1, \ldots, x_n \) are indecomposable, we have
\[
\dim_q(x) = [ |x_1| + 1]_q \cdots [ |x_n| + 1]_q. \tag{4}
\]
For later use, note that it follows that
\[
\dim_q(xy) \geq q^{-|y|} \dim_q(x) \tag{5}
\]
for all \( x, y \in I \).

Whenever \( x \in I \cup \partial I \), we denote by \( [x]_n \) the word consisting of the first \( n \) letters of \( x \) and by \( [x]^n \) the word that arises by removing the first \( n \) letters from \( x \). So, by definition, \( x = [x]_n[x]^n \).

## 2 Poisson boundary of the classical random walk on \( \text{Irred}(\mathbb{G}) \)

Given a probability measure \( \mu \) on \( I := \text{Irred}(\mathbb{G}) \), the Markov operator \( P_\mu : \ell^\infty(\hat{G}) \rightarrow \ell^\infty(\hat{G}) \) preserves the center \( \mathcal{Z}(\ell^\infty(\hat{G})) = \ell^\infty(I) \) and hence, defines an ordinary random walk on the countable set \( I \) with \( n \)-step transition probabilities
\[
p_x p_n(x, y) = p_x P_\mu^n(p_y). \tag{6}
\]
As shown above, this random walk is transient whenever \( \mu(\epsilon) < 1 \). Denote by \( H^\infty_{\text{cent}}(\hat{G}, \mu) \) the commutative von Neumann algebra of bounded \( P_\mu \)-harmonic functions in \( \ell^\infty(I) \), with product given by \( a \cdot b = \lim_n P_\mu^n(ab) \) and the sequence being strongly \( * \)-convergent. Write \( p(x, y) = p_1(x, y) \).

The set \( I \) becomes in a natural way a tree, the Cayley tree of the semi-group \( \mathbb{N} \ast \mathbb{N} \). Let \( \mu \) be a generating probability measure on \( I \) with finite support.

**Lemma 2.1.** There exists a \( \delta > 0 \) such that \( p(x, y) > 0 \) implies that \( p(x, y) \geq \delta \).

**Proof.** Take \( L, \delta_0 > 0 \) such that for all \( z \in \text{supp} \mu \), we have \( |z| \leq L \) and \( \mu(z) \geq \delta_0 \). By (1), if \( p(x, y) > 0 \), we get \( z \) with \( |z| \leq L \), \( y \in x \otimes z \) and
\[
p(x, y) \geq \delta_0 \frac{\dim_q(y)}{\dim_q(x) \dim_q(z)}.
\]
Write \( x = x_0 r, z = rz_1 \) and \( y = x_0 z_1 \). Put \( \eta = q + 1/q \). Then,
\[
p(x, y) \geq \delta_0 \frac{\dim_q(x_0)}{\dim_q(x_0) \eta^{|r|} \eta^{|z_1|}} \geq \delta_0 \eta^{-2L}.
\]
So, we can put \( \delta = \delta_0 \eta^{-2L} \).

The following properties of the random walk on \( I \) can be checked easily.
• Uniform irreducibility: there exists an integer $M$ such that, for any pair $x, y \in I$ of neighboring edges of the tree, there exists an integer $k \leq M$, such that $p_k(x, y) > 0$.

• Bounded step-length: there exists an integer $N$ such that $p(x, y) > 0$ implies that $d(x, y) \leq N$ where $d(x, y)$ equals the length of the unique geodesic path from $x$ to $y$.

Combining these remarks with Lemma 2.1, we can apply [9, Theorem 2] and identify the Poisson boundary of the random walk on $I$, with the boundary $\partial I$ of infinite words in $\alpha, \beta$, equipped with a probability measure in the following way.

**Theorem 2.2** (Picardello and Woess, [9, Theorem 2]). Let $\mu$ be a finitely supported generating measure on $I = \text{Irred}(A_u(F))$, where $F$ is not a scalar multiple of a $2 \times 2$ unitary matrix. Consider the associated random walk on $I$ with transition probabilities given by (6) and the compactification $I \cup \partial I$ of $I$.

- The random walk converges almost surely to a point in $\partial I$.

- Denote, for every $x \in I$, by $\nu_x$ the hitting probability measure on $\partial I$, where $\nu_x(U)$ is defined as the probability that the random walk starting in $x$ converges to a point in $U$. Then, the formula

$$\Upsilon(F)(x) = \int_{\partial I} F(z)d\nu_x(z)$$

defines a $\ast$-isomorphism $\Upsilon : L^\infty(\partial I, \nu) \to H^\infty_{\text{centr}}(\hat{G}, \mu)$.

In fact, Theorem 2 in [9], identifies $\partial I$ with the Martin compactification of the given random walk on $I$. It is a general fact (see [16, Theorem 24.10]), that a transient random walk converges almost surely to a point of the minimal Martin boundary and that the hitting probability measures provide a realization of the Poisson boundary through the Poisson integral formula (7), see [16, Theorem 24.12].

Since a continuous function on the compact space $I \cup \partial I$ is entirely determined by its values on $I$, we can and will view $C(I \cup \partial I)$ as a $C^\ast$-subalgebra of $\ell^\infty(I)$.

The rest of this section is devoted to the proof of the non-atomicity of the harmonic measures $\nu_x$.

**Lemma 2.3.** For all $x, y \in I$ and $z \in \partial I$, the sequence

$$\left(\frac{\dim_q(x[z]n)}{\dim_q(y[z]n)}\right)_n$$

converges. By a slight abuse of notation, we denote the limit by $\dim_q\left(\frac{xz}{yw}\right)$. The following properties hold.

1. For all $x, y \in I$, the map $\partial I \to \mathbb{R}_+: z \mapsto \dim_q\left(\frac{xz}{yw}\right)$ is continuous.

2. For all $x, y \in I$ and $w \in \partial I$, the sequence of continuous functions

$$\partial I \to \mathbb{R}_+: z \mapsto \dim_q\left(\frac{x[w]n.z}{y[w]n.z}\right)$$

converges uniformly on $\partial I$ to the constant function $\dim_q\left(\frac{x[w]}{y[w]}\right)$.
Proof. Fix \( x, y \in I \). Whenever \( z \in \partial I \) and \( n \in \mathbb{N} \), denote

\[
f_n(z) = \frac{\dim_q(x[z]_n)}{\dim_q(y[z]_n)}.
\]

If \( z \notin \{ \alpha \beta \alpha \cdots , \beta \alpha \beta \cdots \} \), write \( z = z_1 \otimes z_2 \) for some \( z_1 \in I \), \( z_1 \neq \epsilon \) and some \( z_2 \in \partial I \). Denote by \( U \) the neighborhood of \( z \) consisting of words of the form \( z_1 \otimes z' \). For all \( s \in U \) and all \( n \geq |z_1| \), we have

\[
f_n(s) = \frac{\dim_q(xz_1)}{\dim_q(yz_1)}.
\]

Hence, for all \( s \in U \), the sequence \( n \mapsto f_n(s) \) is eventually constant and converges to a limit that is constant on \( U \).

Also for \( z \in \{ \alpha \beta \alpha \cdots , \beta \alpha \beta \cdots \} \), the sequence \( f_n(z) \) is convergent. Take \( z = \alpha \beta \alpha \cdots \). Write \( x = x_0 \otimes x_1 \) where \( x_1 \) is the longest possible (and maybe empty) indecomposable word ending with \( \beta \). Write \( y = y_0 \otimes y_1 \) similarly. It follows that

\[
f_n(z) = \dim_q(x_0) \frac{|n + |x_1| + 1|_q}{|n + |y_1| + 1|_q} \dim_q(y_0) \rightarrow q^{\beta_1 \cdots |\beta_1|_q}.
\]

The convergence of \( f_n(z) \) for \( z = \beta \alpha \beta \cdots \) is proven analogously.

Write \( f(z) = \lim_n f_n(z) \). We have seen above that every \( z \in \partial I \), \( z \notin \{ \alpha \beta \alpha \cdots , \beta \alpha \beta \cdots \} \), has a neighborhood on which \( f \) is constant. We now prove that \( f \) is also continuous in \( z = \alpha \beta \alpha \cdots \) and in \( z = \beta \alpha \beta \cdots \). In both cases, define for every \( n \in \mathbb{N} \), the neighborhood \( U_n \) of \( z \) consisting of all \( s \in \partial I \) with \( |s|_n = |z|_n \). For every \( s \in U_n \), \( s \neq z \), there exists \( m \geq n \) such that \( f(s) = f_m(z) \). The continuity of \( f \) in \( z \) follows and we have proven statement 1.

It remains to prove statement 2. If \( w \) is decomposable, i.e. \( w = w_0 \otimes w_1 \) with \( |w_0| \geq 1 \), then for all \( n > |w_0| \), we have

\[
\dim_q\left( x[w]_n \frac{z}{y[w]_n z} \right) = \frac{\dim_q(xw_0)}{\dim_q(yw_0)}
\]

and hence statement 2 follows. If \( w \) is indecomposable, let us assume that \( w = \alpha \beta \alpha \cdots \); the case \( w = \beta \alpha \beta \cdots \) being analogous. Write \( x = x_0 \otimes x_1 \) and \( y = y_0 \otimes y_1 \), where \( x_1, y_1 \) are maximal, possibly empty, indecomposable words ending with the letter \( \beta \). If \( z \) is indecomposable, the expression \( \dim_q\left( x[w]_n \frac{z}{y[w]_n z} \right) \) is alternatingly equal to

\[
\dim_q(x_0) \frac{|x_1| + n + 1|_q}{|y_1| + n + 1|_q} \quad \text{and} \quad \dim_q(y_0) \frac{|y_1| - |x_1|}{q}.
\]

When \( z = z_0 \otimes z_1 \) where \( z_0 \) is an indecomposable word with length at least 1, the expression \( \dim_q\left( x[w]_n \frac{z}{y[w]_n z} \right) \) is alternatingly equal to

\[
\dim_q(x_0) \frac{|x_1| + n + 1|_q}{|y_1| + n + 1|_q} \quad \text{and} \quad \dim_q(x_0) \frac{|x_1| + n + |z_0| + 1}{|y_1| + n + |z_0| + 1|_q}.
\]

Since the four expressions appearing in (8) and (9) converge uniformly in \( z \), to

\[
\dim_q(x_0) \frac{|y_1| - |x_1|}{q} = \dim_q\left( x[w]_n \frac{z}{y[w]_n z} \right)
\]

when \( n \to \infty \), statement 2 is proven. \( \square \)
Whenever \( x, y \in I \cup \partial I \), define \((x|y):= \max\{n \mid [x]_n = [y]_n \}\).

**Lemma 2.4.** Let \( x, z \in I \) with \(|x| \leq |z|\). Denote by \( U_z \) the subset of \( \partial I \) consisting of infinite words that start with \( z \). For every \( 0 \leq k \leq (x|z) \), define the function \( f_k \in C(\partial I) \) with support \( \cup_{[x]_k[z]_k}^k \), given by

\[
f_k ([x]_k[z]_k y) = \frac{1}{\dim_q(x)} \dim_q \left( \frac{z}{[x]_k[z]_k y} \right).
\]

We then have

\[
\nu_x(U_z) = \sum_{k=0}^{(x|z)} \int_{\partial I} f_k(y) \, d\nu_x(y).
\]

Moreover, for all \( w \in \partial I \), we have

\[
\nu_x(\{w\}) = \frac{1}{\dim_q(x)} \sum_{k=0}^{(x|w)} \dim_q \left( \frac{[w]_k[w]_k^k}{[x]_k[z]_k^k} \right) \nu_x(\{[x]_k[w]_k^k\}).
\]

**Proof.** By Lemma 2.3, the functions \( f_k \) are well defined and belong to \( C(\partial I) \). By Theorem 2.2, our random walk converges almost surely to a point of \( \partial I \) and we denoted by \( \nu_x \) the hitting probability measure. So, \((\psi_x \otimes \psi_{\nu_x})\Delta \to \nu_x \) weakly\(^*\) in \( C(I \cup \partial I)^* \).

Recall that \( E : \ell^\infty(\mathbb{G}) \to \ell^\infty(I) \) denotes the conditional expectation defined by \( E(b)p_y = \psi_y(b)p_y \).

Whenever \(|z| \geq |x|\), we have

\[
E(\psi_x \otimes \text{id})\hat{\Delta}(p_z) = \sum_{k=0}^{(x|z)} \frac{\dim_q(z)}{\dim_q(x) \dim_q([x]_k[z]_k^k)} \nu_x(\{[x]_k[z]_k^k\}).
\]

Denote \( q_z = \sum_{s \in I} p_{zs} \) and observe that \( q_z \in C(I \cup \partial I) \). It follows that for all \(|z| \geq |x|\),

\[
E(\psi_x \otimes \text{id})\hat{\Delta}(q_z) = \sum_{k=0}^{(x|z)} F_k
\]

where \( F_k \in \ell^\infty(I) \) is defined by \( F_k(y) = 0 \) if \( y \) does not start with \([x]_k[z]_k^k\) and

\[
F_k([x]_k[z]_k^k y) = \frac{1}{\dim_q(x)} \frac{\dim_q(z y)}{\dim_q([x]_k[z]_k^k y)}.
\]

Note that \( F_k \in C(I \cup \partial I) \subseteq \ell^\infty(I) \) and that \( F_k \) is a continuous extension of \( f_k \). Hence, it follows that, for \(|z| \geq |x|\)

\[
\nu_x(U_z) = \sum_{k=0}^{(x|z)} \int_{\partial I} f_k(y) \, d\nu_x(y).
\]

Finally, let \( w \in \partial I \). Write \( w = w_0 w_1 \), where \(|w_0| \geq |x|\). Let \( n \in \mathbb{N} \). We apply the above formula to \( z = w_0[w_1]_n \). Since \( U_{w_0[w_1]_n} \) decreases to \( \{w\} \), we have

\[
\nu_x(U_{w_0[w_1]_n}) \to \nu_x(\{w\}).
\]

On the other hand, because \((x|w_0[w_1]_n) = (x|w_0)\), we have

\[
\nu_x(U_{w_0[w_1]_n}) = \sum_{k=0}^{(x|w_0)} g_k^n(y) \, d\nu_x(y),
\]

where 

where \( g^n_k \in C(\partial I) \) is supported on the words that start with \( [x]^k[w_0]^k[w_1]^n \) and is given by

\[
g^n_k([x]^k[w_0]^k[w_1]^n y) = \frac{1}{\dim_q(x)} \dim_q \left( \frac{w_0[w_1]^n}{[x]^k[w_0]^k[w_1]^n y} \right).
\]

By Lemma 2.3.2, when \( n \to \infty \), the right-hand side of this last expression converges uniformly in \( y \) to

\[
\frac{1}{\dim_q(x)} \dim_q \left( \frac{w_0[w_1]^n}{[x]^k[w_0]^k[w_1]^n y} \right) = \frac{1}{\dim_q(x)} \dim_q \left( \frac{[w]^k}{[x]^k [w]^k} \right).
\]

Since \( U_{[x]^k[w_0]^k[w_1]^n} \) decreases to \( \{ [x]^k [w]^k \} \) and since \( (x|w) = (x|w_0) \), the lemma is proven.

**Proposition 2.5.** The support of the harmonic measure \( \nu_e \) is the whole of \( \partial I \). The harmonic measure \( \nu_e \) has no atoms in words ending with \( \alpha \beta \alpha \beta \cdots \)

**Remark 2.6.** The same methods as in the proof of Proposition 2.5 given below, but involving more tedious computations, show in fact that \( \nu_e \) is non-atomic. To prove our main theorem, it is only crucial that \( \nu_e \) has no atoms in words ending with \( \alpha \beta \alpha \beta \cdots \). We believe that it should be possible to give a more conceptual proof of the non-atomicity of \( \nu_e \) and refer to [15, Proposition 8.3.10] for an ad hoc proof along the lines of the proof of Proposition 2.5.

**Proof of Proposition 2.5.** In order to prove that the support of \( \nu_e \) is the whole of \( \partial I \), it suffices to show that \( \nu_e(U_z) > 0 \) for all \( z \in I \). Since \( \nu_e \) and \( \nu_z \) are absolutely continuous, it suffices to show that \( \nu_z(U_z) > 0 \) for all \( z \in I \). By Lemma 2.4, we have

\[
\nu_z(U_z) \geq \frac{1}{\dim_q(x)} \int_{\partial I} \dim_q \left( \frac{\hat{z} y}{y} \right) d\nu_z(y).
\]

Since the integral of a strictly positive function, is strictly positive, it follows that \( \nu_z(U_z) > 0 \).

Because of Lemma 2.4 and the equality

\[
\nu_e = \sum_{x \in I} \mu^k(x) \nu_x
\]

for all \( k \geq 1 \), we observe that if \( w \) is an atom for \( \nu_e \), then all \( w' \) with the same tail as \( w \) are atoms for all \( \nu_x, x \in I \). So, we assume that \( w := \alpha \beta \alpha \beta \cdots \) is an atom for \( \nu_e \) and derive a contradiction.

Denote by \( \delta_w \) the function on \( \partial I \) that is equal to 1 in \( w \) and 0 elsewhere. Using the \(*\)-isomorphism in Theorem 2.2, it follows that the bounded function

\[
\xi \in \ell^\infty(\hat{\Sigma}) : \xi(x) := \nu_e(\{w\}) = \int_{\partial I} \delta_w \ d\nu_x
\]

is harmonic.

We will prove that \( \xi \) attains its maximum and apply the maximum principle for irreducible random walks (see e.g. [16, Theorem 1.15]) to deduce that \( \xi \) must be constant. This will lead to a contradiction.

Denote

\[
w^n_\alpha := \underbrace{\alpha \beta \alpha \cdots}_n \quad \text{and} \quad w^n_\beta := \underbrace{\beta \alpha \beta \cdots}_n.
\]
Note that all elements of $I$ are either of the form
\[ w_{2n+1}^\alpha x \quad \text{where} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \{ \epsilon \} \cup \alpha I \]
or of the form
\[ w_{2n}^\alpha x \quad \text{where} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \{ \epsilon \} \cup \beta I . \]

By Lemma 2.4 and formula (4), we get that for $n \in \mathbb{N}$ and $x \in \{ \epsilon \} \cup \alpha I$,
\[
\xi(w_{2n+1}^\alpha x) = \sum_{k=0}^{2n+1} \frac{1}{2(n+1)q} \dim_q(x)^2 \dim_q\left( \frac{w_k}{w_{2n+1-k}} \right) \nu_\epsilon(\bar{\alpha} \beta \beta \cdots )
\]
\[
= \sum_{k=0}^{2n+1} \frac{1}{2(n+1)q} \dim_q(x)^2 q^{2(n-k)+1} \nu_\epsilon(\bar{\alpha} \beta \beta \cdots ) = \frac{\nu_\epsilon(\bar{\alpha} \beta \beta \cdots )}{\dim_q(x)^2}.
\]

Since $\nu_\epsilon$ is a probability measure, it follows that $x \mapsto \xi(w_{2n+1}^\alpha x)$ is independent of $n$ and summable over the set $\{ \epsilon \} \cup \alpha I$. Analogously, it follows that $x \mapsto \xi(w_{2n}^\alpha x)$ is independent of $n$ and summable over the set $\{ \epsilon \} \cup \beta I$. As a result, $\xi$ attains its maximum on $I$. By the maximum principle, $\xi$ is constant. Since $\xi(\epsilon) \neq 0$, this constant is non-zero and we arrive at a contradiction with the summability of $x \mapsto \xi(w_{2n+1}^\alpha x)$ over the infinite set $\{ \epsilon \} \cup \alpha I$. \hfill \qed

## 3 Topological boundary and boundary action for the dual of $A_u(F)$

Before proving Theorem 1.3, we construct a compactification for $\hat{G}$, i.e. a unital C*-algebra $B$ lying between $c_0(\hat{G})$ and $\ell^\infty(\hat{G})$. This C*-algebra $B$ is a non-commutative version of $C(I \cup \partial I)$. The construction of $B$ follows word by word the analogous construction given in [11, Section 3] for $\mathbb{G} = A_u(F)$. So, we only indicate the necessary modifications.

For all $x, y \in I$ and $z \subset x \otimes y$, we choose an isometry $V(x \otimes y, z) \in \text{Mor}(x \otimes y, z)$. Since $z$ appears with multiplicity one in $x \otimes y$, the isometry $V(x \otimes y, z)$ is uniquely determined up to multiplication by a scalar $\lambda \in S^1$. Therefore, the following unital completely positive maps are uniquely defined (cf. [11, Definition 3.1]).

**Definition 3.1.** Let $x, y \in I$. We define unital completely positive maps
\[ \psi_{xy,x} : \mathcal{L}(H_x) \to \mathcal{L}(H_{xy}) : \psi_{xy,x}(A) = V(x \otimes y, xy)^*(A \otimes 1)V(x \otimes y, xy) . \]

**Theorem 3.2.** The maps $\psi_{xy,x}$ form an inductive system of completely positive maps. Defining
\[ B = \{ a \in \ell^\infty(\hat{G}) \mid \forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ such that } \| ap_{xy} - \psi_{xy,x}(ap_x) \| < \varepsilon \}
\]
\[ \text{for all } x, y \in I \text{ with } |x| \geq n \}, \]
we get that $B$ is a unital C*-subalgebra of $\ell^\infty(\hat{G})$ containing $c_0(\hat{G})$.

- The restriction of the comultiplication $\hat{\Delta}$ yields a left action $\beta_{\hat{G}}$ of $\hat{G}$ on $B$:
\[ \beta_\hat{G} : B \to M(c_0(\hat{G}) \otimes B) : a \mapsto \hat{\Delta}(a) . \]
• The restriction of the adjoint action of $\mathbb{G}$ on $\ell^\infty(\hat{\mathbb{G}})$ yields a right action of $\mathbb{G}$ on $\mathcal{B}$:

$$\beta_G : \mathcal{B} \to \mathcal{B} \otimes C(\mathbb{G}) : a \mapsto \mathcal{V}(a \otimes 1)\mathcal{V}^*.$$ 

Here, $\mathcal{V} \in \ell^\infty(\hat{\mathbb{G}}) \otimes L^\infty(\hat{\mathbb{G}})$ is defined as $\mathcal{V} = \sum_{x \in I} U^x$. The action $\beta_G$ is continuous in the sense that $\text{span} \beta_G(\mathcal{B})(1 \otimes C(\mathbb{G}))$ is dense in $\mathcal{B} \otimes C(\mathbb{G})$.

**Proof.** One can repeat word by word the proofs of [11, Propositions 3.4 and 3.6]. The crucial ingredients of these proofs are the approximate commutation formulae provided by [11, Lemmas A.1 and A.2] and they have to be replaced by the inequalities provided by Lemma 6.1. \hfill \Box

We denote $\mathcal{B}_\infty := \mathcal{B}/c_0(\hat{\mathbb{G}})$ and call it the topological boundary of $\hat{\mathbb{G}}$. Both actions $\beta_G$ and $\beta_\hat{\mathbb{G}}$ preserve the ideal $c_0(\hat{\mathbb{G}})$ and hence yield actions on $\mathcal{B}_\infty$ that we still denote by $\beta_G$ and $\beta_\hat{\mathbb{G}}$.

As before, we view $C(I \cup \partial I) \subset \ell^\infty(I)$ by restricting continuous functions on $I \cup \partial I$ to $I$. A bounded function on $I$ extends continuously to $I \cup \partial I$ if and only if for every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $|f(xy) - f(x)| < \varepsilon$ for all $x, y \in I$ with $|x| \geq n$. Hence, when viewing $C(I \cup \partial I)$ as a C*-subalgebra of $\ell^\infty(I)$, we get $C(I \cup \partial I) = \mathcal{B} \cap \mathcal{Z}(\ell^\infty(\mathbb{G})) = \mathcal{B} \cap \ell^\infty(I)$. Taking the quotient with $c_0(I)$, we view $C(\partial I) \subset \mathcal{B}_\infty$.

We partially order $I$ by writing $x \leq y$ if $y = xz$ for some $z \in I$. Define

$$\psi_{\infty,x} : \mathcal{L}(H_x) \to \mathcal{B} : \psi_{\infty,x}(A)p_y = \begin{cases} \psi_{y,x}(A) & \text{if } y \geq x \\ 0 & \text{else} \end{cases}.$$ 

We use the same notation for the composition of $\psi_{\infty,x}$ with the quotient map $\mathcal{B} \to \mathcal{B}_\infty$, yielding the map $\psi_{\infty,x} : \mathcal{L}(H_x) \to \mathcal{B}_\infty$.

Observe that the linear span of all $\psi_{\infty,x}(\mathcal{L}(H_x))$ is dense in $\mathcal{B}_\infty$. Indeed, whenever $a \in \mathcal{B}$ and $\varepsilon > 0$, we can take $n \in \mathbb{N}$ such that $\|ap_{xy} - \psi_{x,y}(ap_x)\| \leq \varepsilon$ whenever $|x| \geq n$. If $x_1, \ldots, x_m$ is an enumeration of all elements in $I$ of length $n$, it follows that

$$\left\| \pi(a) - \sum_{k=1}^m \psi_{\infty,x_k}(ap_{x_k}) \right\| \leq \varepsilon.$$ 

**Lemma 3.3.** The inclusion $C(\partial I) \subset \mathcal{B}_\infty$ defines a continuous field of unital C*-algebras. Denote, for every $x \in \partial I$, by $J_x$ the closed two-sided ideal of $\mathcal{B}_\infty$ generated by the functions in $C(\partial I)$ vanishing in $x$.

For every $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \partial_0 I$, there exists a unique surjective *-homomorphism

$$\pi_x : \mathcal{B}_\infty \to \bigotimes_{k=1}^n \mathcal{L}(H_{x_k})$$

satisfying $\text{Ker} \pi_x = J_x$ and $\pi_x(\psi_{\infty,x_1\cdots x_n}(A)) = A \otimes 1$ for all $A \in \bigotimes_{k=1}^n \mathcal{L}(H_{x_k}) = \mathcal{L}(H_{x_1\cdots x_n})$.

**Proof.** Given $x \in \partial I$, define the decreasing sequence of projections $e_n \in \mathcal{B}$ given by

$$e_n := \sum_{y \in I} p_{|x|_n y}.$$
Denote by \(\pi : \mathcal{B} \to \mathcal{B}_\infty\) the quotient map. It follows that
\[
\|\pi(a) + J_x\| = \lim_n \|ae_n\| \tag{10}
\]
for all \(a \in \mathcal{B}\).

To prove that \(C(\partial I) \subset \mathcal{B}_\infty\) is a continuous field, let \(y \in I\), \(A \in \mathcal{L}(H_y)\) and define \(a \in \mathcal{B}\) by \(a := \psi_{\infty,y}(A)\). Put \(f : \partial I \to \mathbb{R}_+ : f(x) = \|\pi(a) + J_x\|\). We have to prove that \(f\) is a continuous function. Define \(U \subset \partial I\) consisting of infinite words starting with \(y\). Then, \(U\) is open and closed and \(f\) is zero, in particular continuous, on the complement of \(U\). Assume that the last letter of \(y\) is \(\alpha\) (the other case, of course, being analogous). If \(x \in U\) and \(x \neq y\beta\alpha\beta\alpha\cdots\), write \(x = yz \otimes u\) for some \(z \in I\), \(u \in \partial I\). Define \(V\) as the neighborhood of \(x\) consisting of infinite words of the form \(yzu'\) where \(u' \in \partial I\) and \(yzu' = yz \otimes u'\). Then, \(f\) is constantly equal to \(\|\psi_{yz,y}(A)\|\) on \(V\). It remains to prove that \(f\) is continuous in \(x := y\beta\alpha\beta\alpha\cdots\). Let
\[
w_n = \underbrace{\beta\alpha\beta\cdots}_{n \text{ letters}}.
\]
Then, the sequence \(\|\psi_{yw_n,y}(A)\|\) is decreasing and converges to \(f(x)\). If \(U_n\) is the neighborhood of \(x\) consisting of words starting with \(yw_n\), it follows that
\[
f(x) \leq f(u) \leq \|\psi_{yw_n,y}(A)\|
\]
for all \(u \in U_n\). This proves the continuity of \(f\) in \(x\). So, \(C(\partial I) \subset \mathcal{B}_\infty\) is a continuous field of \(C^*\)-algebras.

Let now \(x = x_1 \otimes x_2 \otimes \cdots\) be an element of \(\partial_0 I\). Put \(y_n = x_1 \otimes \cdots \otimes x_n\) and
\[
f_n := \sum_{z \in I} p_{y_n z}.
\]
The map \(A \mapsto f_{n+1}\psi_{\infty,y_n}(A)\) defines a unital \(*\)-homomorphism from \(\mathcal{L}(H_{y_n})\) to \(f_{n+1}\mathcal{B}\). Since \(\pi(1 - f_{n+1}) \in J_x\), we obtain the unital \(*\)-homomorphism \(\theta_n : \mathcal{L}(H_{y_n}) \to \mathcal{B}_\infty/J_x\). The \(*\)-homomorphisms \(\theta_n\) are compatible and combine into the unital \(*\)-homomorphism
\[
\theta : \bigotimes_{k=1}^\infty \mathcal{L}(H_{x_k}) \to \mathcal{B}_\infty/J_x.
\]
By (10), \(\theta\) is isometric. Since the union of all \(\psi_{\infty,y_n}(\mathcal{L}(H_{y_n})) + J_x\), \(n \in \mathbb{N}\), is dense in \(\mathcal{B}_\infty\), it follows that \(\theta\) is surjective. The composition of the quotient map \(\mathcal{B}_\infty \to \mathcal{B}_\infty/J_x\) and the inverse of \(\theta\) provides the required \(*\)-homomorphism \(\pi_x\).

### 4 Proof of Theorem 1.3

We prove Theorem 1.3 by performing the following steps.

- Construct on the boundary \(\mathcal{B}_\infty\) of \(\hat{\mathcal{G}}\), a faithful KMS state \(\omega_\infty\), to be considered as the harmonic state and satisfying \((\psi_\mu \otimes \omega_\infty)_{\beta_\mathcal{G}} = \omega_\infty\). Extend \(\beta_\mathcal{G}\) to an action
  \[
  \beta_\mathcal{G} : (\mathcal{B}_\infty, \omega_\infty)'' \to \ell^\infty(\hat{\mathcal{G}}) \otimes (\mathcal{B}_\infty, \omega_\infty)'',
  \]
  and denote by \(\Theta_\mu := (\text{id} \otimes \omega_\infty)_{\beta_\mathcal{G}}\) the Poisson integral.
• Prove a quantum Dirichlet property: for all \( a \in \mathcal{B} \), we have \( \Theta_\mu(a) - a \in \mathfrak{c}_0(\hat{\mathcal{G}}) \). It will follow that \( \Theta_\mu \) is a normal and faithful \(*\)-homomorphism of \( (\mathcal{B}_\infty, \omega_\infty)'' \) onto a von Neumann subalgebra of \( \mathcal{H}_\infty(\hat{\mathcal{G}}, \mu) \).

• By Theorem 2.2, \( \Theta_\mu \) is a \(*\)-isomorphism of \( \mathcal{L}^\infty(\partial I, \nu_\epsilon) \subset (\mathcal{B}_\infty, \omega_\infty)'' \) onto \( \mathcal{H}_\text{centr}(\hat{\mathcal{G}}, \mu) \). Deduce that the image of \( \Theta_\mu \) is the whole of \( \mathcal{H}_\infty(\hat{\mathcal{G}}, \mu) \).

• Use Lemma 3.3 to identify \( (\mathcal{B}_\infty, \omega_\infty)'' \) with a field of ITPFI factors.

**Proposition 4.1.** The sequence \( \psi_{\mu^n} \) of states on \( \mathcal{B} \) converges weakly* to a KMS state \( \omega_\infty \) on \( \mathcal{B} \). The state \( \omega_\infty \) vanishes on \( \mathfrak{c}_0(\hat{\mathcal{G}}) \). We still denote by \( \omega_\infty \) the resulting KMS state on \( \mathcal{B}_\infty \). Then, \( \omega_\infty \) is faithful on \( \mathcal{B}_\infty \).

We have \((\psi_\mu \otimes \omega_\infty)\beta_{\hat{\mathcal{G}}} = \omega_\infty\), so that we can uniquely extend \( \beta_{\hat{\mathcal{G}}} \) to an action

\[
\beta_{\hat{\mathcal{G}}}: (\mathcal{B}_\infty, \omega_\infty)'' \rightarrow \ell^\infty(\hat{\mathcal{G}}) \otimes (\mathcal{B}_\infty, \omega_\infty)''
\]

that we still denote by \( \beta_{\hat{\mathcal{G}}} \).

The state \( \omega_\infty \) is invariant under the action \( \beta_{\hat{\mathcal{G}}} \) of \( \mathcal{G} \) on \( \mathcal{B}_\infty \). We extend \( \beta_{\hat{\mathcal{G}}} \) to an action on \((\mathcal{B}_\infty, \omega_\infty)''\) that we still denote by \( \beta_{\hat{\mathcal{G}}} \).

The normal, completely positive map

\[
\Theta_\mu: (\mathcal{B}_\infty, \omega_\infty)'' \rightarrow \mathcal{H}_\infty(\hat{\mathcal{G}}, \mu): \Theta_\mu = (\text{id} \otimes \omega_\infty)\beta_{\hat{\mathcal{G}}}
\]  

(11)

is called the Poisson integral. It satisfies the following properties (recall that \( \alpha_{\hat{\mathcal{G}}} \) and \( \alpha_\mathcal{G} \) were introduced in Definition 1.2).

- \( \hat{\epsilon} \circ \Theta_\mu = \omega_\infty \).
- \( (\Theta_\mu \otimes \text{id}) \circ \beta_{\hat{\mathcal{G}}} = \alpha_{\mathcal{G}} \circ \Theta_\mu \).
- \( (\text{id} \otimes \Theta_\mu) \circ \beta_{\hat{\mathcal{G}}} = \alpha_{\hat{\mathcal{G}}} \circ \Theta_\mu \).

For every \( x = x_1 \otimes x_2 \otimes \cdots \) in \( \partial_0 I \), denote by \( \omega_x \) the infinite tensor product state on \( \bigotimes_{k=1}^\infty \mathcal{L}(\mathcal{H}_{x_k}) \), of the states \( \psi_{x_k} \) on \( \mathcal{L}(\mathcal{H}_{x_k}) \). Using the notation \( \pi_x \) of Lemma 3.3, we have

\[
\omega_\infty(a) = \int_{\partial_0 I} \omega_x(\pi_x(a)) \, d\nu_\epsilon(x)
\]

(12)

for all \( a \in \mathcal{B}_\infty \).

**Proof.** Define the 1-parameter group of automorphisms \((\sigma_t)_{t \in \mathbb{R}}\) of \( \ell^\infty(\hat{\mathcal{G}}) \) given by

\[
\sigma_t(a)p_x = Q_x^{it}aQ_x^{-it}.
\]

Since \( \sigma_t(\psi_{\infty,x}(A)) = \psi_{\infty,x}(Q_x^{it}AQ_x^{-it}) \), it follows that \( \sigma_t \) is norm-continuous on the C*-algebra \( \mathcal{B} \). By Theorem 2.2, the sequence of probability measures \( \mu^{*n} \) on \( I \cup \partial I \) converges weakly* to \( \nu_\epsilon \). It follows that \( \psi_{\mu^n}(a) \rightarrow 0 \) whenever \( a \in \mathfrak{c}_0(\hat{\mathcal{G}}) \). Given \( x \in I \) and \( A \in \mathcal{L}(\mathcal{H}_x) \), put \( a := \psi_{\infty,x}(A) \).
Denote, as before, by $\mathcal{U}_x$ the set of infinite words starting with $x$ and by $\mathcal{U}_x^0$ its intersection with $\partial_0(I)$. We get, using Proposition 2.5,

$$
\psi_{n+}(a) = \sum_{y \in I} \mu^n(y) \psi_y(\psi_{\infty,x}(A)) = \sum_{y \in x} \mu^n(y) \psi_x(A) - \psi_x(A) \nu_x(\mathcal{U}_x^0) = \int_{\partial_0 I} \omega_y(\pi_y(a)) \, d\nu_x(y).
$$

So, the sequence $\psi_{n+}$ of states on $B$ converges weakly* to a state on $B$ that we denote by $\omega_\infty$ and that satisfies (12). Since all $\psi_{n+}$ satisfy the KMS condition w.r.t. $(\sigma_t)$, also $\omega_\infty$ satisfy a KMS state. If $a \in B_\infty$ and $\omega_\infty(a) = 0$, it follows from (12) that $\omega_\infty(\pi_x(a)) = 0$ for $\nu_x$-almost every $x \in \partial I$. Since $\omega_x$ is faithful, it follows that $\|\pi(a) + J_x\| = 0$ for $\nu_x$-almost every $x \in \partial I$. By Proposition 2.5, the support of $\nu_x$ is the whole of $\partial I$ and by Lemma 3.3, $x \mapsto \|\pi(a) + J_x\|$ is a continuous function. It follows that $\|\pi(a) + J_x\| = 0$ for all $x \in \partial I$ and hence, $a = 0$. So, $\omega_\infty$ is faithful.

Since $(\psi_x \otimes \psi_{n+}) \beta^G = \psi_{n+}$, it follows that $(\psi_x \otimes \omega_\infty) \beta^G = \omega_\infty$. So, $(\psi_x \otimes \omega_\infty) \beta^G = \omega_\infty$ for all $k \in \mathbb{N}$. Since $\mu$ is generating, there exists for every $x \in I$, a $C_x > 0$ such that $(\psi_x \otimes \omega_\infty) \beta^G \leq C_x \omega_\infty$. As a result, we can uniquely extend $\beta^G$ to a normal $*$-homomorphism

$$(B_\infty, \omega_\infty)^{\prime\prime} \to \ell^\infty(\hat{G}) \otimes (B_\infty, \omega_\infty)^{\prime\prime}.$$  

Since $\beta^G$ is an action, the same holds for the extension to the von Neumann algebra $(B_\infty, \omega_\infty)^{\prime\prime}$.

Because $(\psi_x \otimes \omega_\infty) \beta^G = \beta^G$ and because $\beta^G$ is an action, the Poisson integral defined by (11) takes values in $H^\infty(\hat{G}, \mu)$. It is straightforward to check that $\Theta_\mu$ intertwines $\beta^G$ with $\alpha_G$ and $\beta^G$ with $\alpha_{G^\ast}$.

**Theorem 4.2.** The compactification $\mathcal{B}$ of $\hat{G}$ satisfies the quantum Dirichlet property, meaning that, for all $a \in \mathcal{B}$,

$$\|(\Theta_\mu(a) - a)p_x\| \to 0$$

if $|x| \to \infty$.

In particular, the Poisson integral $\Theta_\mu$ is a normal and faithful $*$-homomorphism of $(\mathcal{B}_\infty, \omega_\infty)^{\prime\prime}$ onto a von Neumann subalgebra of $H^\infty(\hat{G}, \mu)$.

We will deduce Theorem 4.2 from the following lemma.

**Lemma 4.3.** For every $a \in \mathcal{B}$ we have that

$$\sup_{y \in I} \|(\id \otimes \psi_y) \hat{\Delta}(a)p_x - ap_x\| \to 0$$

when $|x| \to \infty$.

**Proof.** Fix $a \in \mathcal{B}$ with $\|a\| \leq 1$. Choose $\varepsilon > 0$. Take $n$ such that $\|ap_{x_0 x_1} - \psi_{x_0 x_1, x_0}(ap_{x_0})\| < \varepsilon$ for all $x_0, x_1 \in I$ with $|x_0| = n$.

Denote $d_{S^1}(V, W) = \inf\{\|V - \lambda W\| \mid \lambda \in S^1\}$. By formula (18) in the appendix, take $k$ such that

$$d_{S^1}(V(x_0 \otimes x_1 x_2, x_0 x_1 x_2) \otimes 1) V(x_0 x_1 x_2 \otimes \overline{a} u, x_0 x_1 u),$$

$$(1 \otimes V(x_1 x_2 \otimes \overline{a} u, x_1 u)) V(x_0 \otimes x_1 u, x_0 x_1 u) < \varepsilon$$

(14)
whenever \(|x_1| \geq k\).

Finally, take \(l\) such that \(q^{2l} < \varepsilon\). We prove that
\[
\|(\text{id} \otimes \psi_y)\tilde{\Delta}(a)p_x - ap_x\| < 5\varepsilon
\]
for all \(x, y \in I\) with \(|x| \geq n + k + l\).

Choose \(x, y \in I\) with \(|x| \geq n + k + l\) and write \(x = x_0x_1x_2\) with \(|x_0| = n, |x_1| = k\) and hence, \(|x_2| \geq l\). We get
\[
(\text{id} \otimes \psi_y)\tilde{\Delta}(a)p_x = \sum_{z \in x \otimes y} (\text{id} \otimes \psi_y)(V(x \otimes y, z)ap_z V(x \otimes y, z)^*)
\]
\[
= \sum_{z \in x \otimes y} \frac{\dim_q(z)}{\dim_q(x) \dim_q(y)} V(z \otimes y, x)^*(ap_z \otimes 1)V(z \otimes y, x)
\]
\[
= \sum_{z \in x \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} V(x_0x_1z \otimes y, x)^*(ap_{x_0x_1} \otimes 1)V(x_0x_1z \otimes y, x)
\]
\[+ \text{remaining terms.}
\]

In order to have remaining terms, \(y\) should be of the form \(y = x_2y_0\) and then, using (5) and the assumption \(\|a\| \leq 1\),
\[
\sum \|\text{remaining terms}\| = \sum_{z \in x_0x_1 \otimes y_0} \frac{\dim_q(z)}{\dim_q(x_0x_1x_2) \dim_q(x_2y_0)}
\]
\[\leq \sum_{z \in x_0x_1 \otimes y_0} q^{|x_2|} \frac{\dim_q(z)}{\dim_q(x_0x_1) \dim_q(y_0)} = q^{|x_2|} < \varepsilon.
\]

Combining this estimate with the fact that \(\|ap_{x_0x_1} - \psi_{x_0x_1,z_0}(ap_{x_0})\| < \varepsilon\), it follows that
\[
\|(\text{id} \otimes \psi_y)\tilde{\Delta}(a)p_x - ap_x\| \leq 2\varepsilon + \|ap_x - \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} V(x_0x_1z \otimes y, x)^*(\psi_{x_0x_1,z_0}(ap_{x_0}) \otimes 1)V(x_0x_1z \otimes y, x)\|.
\]

But now, (14) implies that
\[
\|(\text{id} \otimes \psi_y)\tilde{\Delta}(a)p_x - ap_x\| \leq 3\varepsilon + \|ap_x - \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} \psi_{x,x_0}(ap_{x_0})\|.
\]

Since \(\|\psi_{x,x_0}(ap_{x_0}) - ap_x\| < \varepsilon\) and \(\|a\| \leq 1\), we get
\[
\|(\text{id} \otimes \psi_y)\tilde{\Delta}(a)p_x\| \leq 4\varepsilon + \left(1 - \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)}\right).
\]

The second term on the right hand side is zero, unless \(y = x_2y_0\), in which case, it equals
\[
\sum_{z \in x_0x_1 \otimes y_0} \frac{\dim_q(z)}{\dim_q(x_0x_1x_2) \dim_q(x_2y_0)} \leq \sum_{z \in x_0x_1 \otimes y_0} q^{2|x_2|} \frac{\dim_q(z)}{\dim_q(x_0x_1) \dim_q(y_0)} \leq \varepsilon
\]
because of (5). Finally, (15) follows and the lemma is proven. \(\square\)
Proof of Theorem 4.2. Let \( a \in \mathcal{B} \). Given \( \varepsilon > 0 \), Lemma 4.3 provides \( k \) such that
\[
\| (\text{id} \otimes \psi_{\mu}^n) \Delta(a) p_x - ap_x \| \leq \varepsilon
\]
for all \( n \in \mathbb{N} \) and all \( x \) with \( |x| \geq k \). Since \( \psi_{\mu}^n \to \omega_\infty \) weakly*, it follows that
\[
\| (\Theta_\mu(a) - a) p_x \| \leq \varepsilon
\]
whenever \( |x| \geq k \). This proves (13).

It remains to prove the multiplicativity of \( \Theta_\mu \). We know that \( \Theta_\mu : \mathcal{B}_\infty \to \mathcal{H}_\infty(\hat{G}, \mu) \) is a unital, completely positive map. Since \( \hat{\epsilon} \circ \Theta_\mu = \omega_\infty, \Theta_\mu \) is faithful. Denote by \( \pi : \mathcal{H}_\infty(\hat{G}, \mu) \to \mathcal{E}(\hat{G})/c_0(\hat{G}) \) the quotient map, which is also a unital, completely positive map. By (13), we have \( \pi \circ \Theta_\mu = \text{id} \). So, for all \( a \in \mathcal{B}_\infty \), we find
\[
\pi(\Theta_\mu(a)^* \cdot \Theta_\mu(a)) \leq \pi(\Theta_\mu(a^*a)) = a^*a = \pi(\Theta_\mu(a))^* \pi(\Theta_\mu(a)) \leq \pi(\Theta_\mu(a)^* \cdot \Theta_\mu(a))
\]

We claim that \( \pi \) is faithful. If \( a \in \mathcal{H}_\infty(\hat{G}, \mu)^+ \cap c_0(\hat{G}) \), we have \( \hat{\epsilon}(a) = \psi_{\mu}^n(a) \) for all \( n \) and the transience of \( \mu \) combined with the assumption \( a \in c_0(\hat{G}) \) implies that \( \hat{\epsilon}(a) = 0 \) and hence, \( a = 0 \). So, we conclude that \( \Theta_\mu(a)^* \cdot \Theta_\mu(a) = \Theta_\mu(a^*a) \) for all \( a \in \mathcal{B}_\infty \). Hence, \( \Theta_\mu \) is multiplicative on \( \mathcal{B}_\infty \) and also on \( (\mathcal{B}_\infty, \omega_\infty) \) by normality.

\( \square \)

Remark 4.4. We now give a reinterpretation of Theorem 2.2. Denote by \( \Omega = \mathbb{P}^n \) the path space of the random walk with transition probabilities (6). Elements of \( \Omega \) are denoted by \( x \), \( y \), etc. For every \( x \in I \), one defines the probability measure \( \mathbb{P}_x \) on \( \Omega \) such that \( \mathbb{P}_x(\{x\} \times I \times I \times \cdots) = 1 \) and
\[
\mathbb{P}_x(\{(x, x_1, x_2, \cdots, x_n) \times I \times I \times \cdots\} = p(x, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n).
\]

Choose a probability measure \( \eta \) on \( I \) with \( \eta = \text{supp} \eta \). Write \( \mathbb{P} = \sum_{x \in I} \eta(x) \mathbb{P}_x \).

Define on \( \Omega \) the following equivalence relation: \( x \sim y \) iff there exist \( k, l \in \mathbb{N} \) such that \( x_{n+k} = y_{n+l} \) for all \( n \in \mathbb{N} \). Whenever \( F \in \mathcal{H}_\infty(\hat{G}, \mu) \), the martingale convergence theorem implies that the sequence of measurable functions \( \Omega \to \mathbb{C} : x \mapsto F(x_n) \) converges \( \mathbb{P} \)-almost everywhere to a \( \sim \)-invariant bounded measurable function on \( \Omega \), that we denote by \( \pi_\infty(F) \). Denote by \( \mathcal{L}_\infty(\Omega/\sim, \mathbb{P}) \) the von Neumann subalgebra of \( \sim \)-invariant functions in \( \mathcal{L}_\infty(\Omega, \mathbb{P}) \). As such, \( \pi_\infty : \mathcal{H}_\infty(\hat{G}, \mu) \to \mathcal{L}_\infty(\Omega, \mathbb{P}) \) is a \( * \)-isomorphism.

By Theorem 2.2, we can define the measurable function \( \text{bnd} : \Omega \to \partial I \) such that \( \text{bnd} \ x = \lim_n x_n \) for \( \mathbb{P} \)-almost every \( x \in \mathbb{P} \) and where the convergence is understood in the compact space \( I \cup \partial I \). Recall that we denote, for \( x \in I \), by \( \nu_x \) the hitting probability measure on \( \partial I \). So, \( \nu_x(A) = \mathbb{P}_x(\text{bnd}^{-1}(A)) \) for all measurable \( A \subset \partial I \) and all \( x \in I \).

Again by Theorem 2.2, \( \pi_\infty \circ \mathcal{Y} \) is a \( * \)-isomorphism of \( \mathcal{L}_\infty(\partial I, \nu_x) \) onto \( \mathcal{L}_\infty(\Omega/\sim, \mathbb{P}) \). We claim that for all \( F \in \mathcal{L}_\infty(\partial I, \nu_x) \), we have
\[
((\pi_\infty \circ \mathcal{Y})(F))(x) = F(\text{bnd} \ x) \quad \text{for } \mathbb{P} \text{-almost every } x \in \Omega.
\]

Let \( A \subset \partial I \) be measurable. Define \( F_n : \Omega \to \mathbb{R} : F_n(x) = \nu_{x_n}(A) \). Then, \( F_n \) converges almost everywhere with limit equal to \( (\pi_\infty \circ \mathcal{Y})(\chi_A) \). If the measurable function \( G : \Omega \to \mathbb{C} \) only depends on \( x_0, \ldots, x_k \), one checks that
\[
\int_{\Omega} F_n(x) G(x) d\mathbb{P}(x) = \int_{\text{bnd}^{-1}(A)} G(x) d\mathbb{P}(x) \quad \text{for all } n > k.
\]
From this, the claim follows.

Since the \$\ast\$-isomorphism \$\pi_\infty \circ \Upsilon\$ is given by \$\text{bnd}\$, it follows that for every \$\sim\$-invariant bounded measurable function \$F\$ on \$\Omega\$, there exists a bounded measurable function \$F_1\$ on \$\partial I\$ such that \$F(x) = F_1(\text{bnd } x)\$ for \$\mathbb{P}\$-almost every path \$x \in \Omega\$.

As before, we view \$C(\partial I)\$ as a \$C^\ast\$-subalgebra of \$\mathcal{B}_\infty\$. The restriction of the state \$\omega_\infty\$ to \$C(\partial I)\$ is, by definition, given by integration along \$\nu\$. So, we can and will view \$L^\infty(\partial I, \nu_\epsilon)\$ as a von Neumann subalgebra of \$(\mathcal{B}_\infty, \omega_\infty)''\$. But then, both \$\Upsilon\$ and \$\Theta_\mu\$ are normal \$\ast\$-homomorphisms from \$L^\infty(\partial I, \nu_\epsilon)\$ to \$H^\infty(\hat{G}, \mu)\$. We claim that, viewed in this way, \$\Upsilon = \Theta_\mu\$ on \$L^\infty(\partial I, \nu_\epsilon)\$. Since almost every path \$x\$ converges to \$\text{bnd } x\$, Theorem 4.2 implies that \$\pi_\partial I\$ is weakly dense in \$L^\infty(\partial I, \nu_\epsilon)\$ and since \$\pi_\infty \circ \Upsilon\$ and \$\pi_\infty \circ \Theta_\mu\$ are both normal, we conclude that \$\pi_\infty \circ \Upsilon = \pi_\infty \circ \Theta_\mu\$ and hence, \$\Upsilon = \Theta_\mu\$ on \$L^\infty(\partial I, \nu_\epsilon)\$.

We are now ready to prove the main Theorem 1.3.

**Proof of Theorem 1.3.** Because of Theorem 4.2 and Lemma 3.3, it remains to show that

\[
\Theta_\mu : (\mathcal{B}_\infty, \omega_\infty)'' \to H^\infty(\hat{G}, \mu)
\]

is surjective.

Whenever \$\gamma : N \to N \otimes L^\infty(G)\$ is an action of \$G\$ on the von Neumann algebra \$N\$, we denote, for \$x \in I\$, by \$N^x \subset N\$ the spectral subspace of the irreducible representation \$x\$. By definition, \$N^x\$ is the linear span of all \$S(H_x)\$, where \$S\$ ranges over the linear maps \$S : H_x \to N\$ satisfying \$\gamma(S(\xi)) = (S \otimes \text{id}) (U_x (\xi \otimes 1))\$. The linear span of all \$N^x, x \in I\$, is a weakly dense \$\ast\$-subalgebra of \$N\$, called the spectral subalgebra of \$N\$. For \$n \in \mathbb{N}\$, we denote by \$N^n\$ the linear span of all \$N^x, |x| \leq n\$.

Fixing \$x, y \in I\$, consider the adjoint action \$\gamma : \mathcal{L}(H_{xy}) \to \mathcal{L}(H_{xy}) \otimes C(G)\$ given by \$\gamma(A) = U_{xy}(A \otimes 1)U_{xy}^\ast\$ \$\forall A \in \mathcal{A}_\infty(F)\$. The fusion rules of \$G = A_\infty(F)\$ imply that \$\mathcal{L}(H_{xy})^{\otimes [2]} = \psi_{xy,x}(\mathcal{L}(H_x))\$.

For the rest of the proof, put \$M := (\mathcal{B}_\infty, \omega_\infty)''\$ \$\forall G\$ and the action \$\alpha_G\$ of \$G\$ on \$H^\infty(\hat{G}, \mu)\$. It suffices to prove that \$H^\infty(\hat{G}, \mu)^k \subset \Theta_\mu(M)\$ for all \$k \in \mathbb{N}\$.

Define, for all \$y \in I\$, the subset

\[
V_y := \{yz \mid z \in I \text{ and } yz = y \otimes z\}
\]

Define the projections

\[
q_y = \sum_{z \in V_y} p_z \in \mathcal{B}
\]

and consider \$q_y\$ also as an element of the von Neumann algebra \$M\$. Define \$W_y \subset \partial I\$ as the subset of infinite words of the form \$yu\$, where \$u \in \partial I\$ and \$yu = y \otimes u\$.

Fix \$y \in I\$. Let \$F \in C(W_y)\$ and \$A \in \mathcal{L}(H_y)\$. Let \$\tilde{F} \in C(I \cup \partial I)\$ be a continuous extension of \$F\$. Define \$b \in \ell^\infty(\hat{G})\$ by the formula \$bp_{yz} = \tilde{F}(yz) \psi_{yz,y}(A)\$ when \$yz = y \otimes z\$ and \$bp_r = 0\$ elsewhere. Note that \$b \in \mathcal{B}\$ and that the image \$\pi(b)\$ of \$b\$ in \$\mathcal{B}_\infty\$ actually belongs to \$Mq_y\$. We put \$\zeta(F \otimes A) := \pi(b)\$.

As such, we have defined, for every \$y \in I\$, the unital \$\ast\$-homomorphism

\[
\zeta : C(W_y) \otimes \mathcal{L}(H_y) \to Mq_y
\]

**Claim.** For all \$y \in I\$, there exists a linear map

\[
T_y : H^\infty(\hat{G}, \mu)^{\otimes [2]} \cdot \Theta_\mu(q_y) \subset H^\infty(\hat{G}, \mu) \to L^\infty(W_y) \otimes \mathcal{L}(H_y)
\]

satisfying the following conditions.
\begin{itemize}
  \item $T_\gamma$ is isometric for the 2-norm on $H^\infty(\widehat{\mathcal{G}}, \mu)$ given by the state $\widehat{\epsilon}$ and the 2-norm on $L^\infty(W_y) \otimes L(H_y)$ given by the state $\nu_\epsilon \otimes \psi_y$.
  \item $(T_\gamma \circ \Theta_\mu \circ \zeta)(F) = F$ for all $F \in C(W_y) \otimes L(H_y)$.
\end{itemize}

To prove this claim, we use the notations and results introduced in Remark 4.4. Fix $y \in I$. Consider $a \in H^\infty(\widehat{\mathcal{G}}, \mu)^2[y]$ . $\Theta_\mu(q_y)$. If $x \in \Omega$ is such that $\text{bnd}_x \in W_y$, then, for $n$ big enough, $x_n$ will be of the form $x_n = y \otimes z_n$. By definition of $\alpha_G$, we have that $ap_{x_n} \in L(H_{x_n})^2[y]$. So, we can take elements $a_{x,n} \in \mathcal{L}(H_y)$ such that $ap_{x_n} = \psi_{x_n,y}(a_{x,n})$. We prove that, for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$, the sequence $(a_{x,n})_n$ is convergent. We then define $T_y(a) \in L^\infty(W_y) \otimes L(H_y)$ such that $T_y(a)(\text{bnd}_x) = \lim_n a_{x,n}$ for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$.

Take $d \in \mathcal{L}(H_y)$. Then, for $\mathbb{P}$-almost every path $x$ such that $\text{bnd}_x \in W_y$ and $n$ big enough, we get that

$$\psi_y(da_{x,n}) = \psi_{x_n}(\psi_{x_n,y}(da_{x,n})) = \psi_{x_n}(\psi_{x_n,y}(d)\psi_{x_n,y}(a_{x,n})) = \psi_{x_n}(\psi_{x_n,y}(d)ap_{x_n})$$

In the second step, we used the multiplicativity of $\psi_{x_n,y} : \mathcal{L}(H_y) \to \mathcal{L}(H_{x_n})$ which follows because $x_n = y \otimes z_n$. Also note that $\|a_{x,n}\| \leq \|a\|$. From Theorem 4.2, it follows that

$$\|\Theta_\mu(\zeta(1 \otimes d))p_{x_n} - \psi_{x_n,y}(d)p_{x_n}\| \to 0$$

whenever $x_n$ converges to a point in $W_y$. This implies that

$$\|\psi_y(da_{x,n}) - \psi_{x_n}(\Theta_\mu(\zeta(1 \otimes d))ap_{x_n})\| \to 0$$

for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$.

From [6, Proposition 3.3], we know that for $\mathbb{P}$-almost every path $x$, $\|\psi_{x_n}(\Theta_\mu(\zeta(1 \otimes d))ap_{x_n})p_{x_n} - \mathcal{E}(\Theta_\mu(\zeta(1 \otimes d)) \cdot a)p_{x_n}\| \to 0$.

As before, $\mathcal{E}(b)p_x = \psi_z(b)p_x$. It follows that

$$\|\psi_y(da_{x,n})p_{x_n} - \mathcal{E}(\Theta_\mu(\zeta(1 \otimes d)) \cdot a)p_{x_n}\| \to 0$$

Note that $\mathcal{E}$ maps $H^\infty(\widehat{\mathcal{G}}, \mu)$ onto $H^\infty_{\text{centr}}(\widehat{\mathcal{G}}, \mu)$. Whenever $F \in H^\infty_{\text{centr}}(\widehat{\mathcal{G}}, \mu)$, the sequence $F(x_n)$ converges for $\mathbb{P}$-almost every path $x$. We conclude that for every $d \in \mathcal{L}(H_y)$, the sequence $\psi_y(da_{x,n})$ is convergent for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$. Since $\mathcal{L}(H_y)$ is finite dimensional, it follows that the sequence $(a_{x,n})_n$ in $\mathcal{L}(H_y)$ is convergent for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$.

By Remark 4.4, we get $T_y(a) \in L^\infty(W_y) \otimes L(H_y)$ such that $T_y(a)(\text{bnd}_x) = \lim_n a_{x,n}$ for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$. From the definition of $a_{x,n}$, we get that

$$\|\psi_{x_n,y}(T_y(a)(\text{bnd}_x)) - ap_{x_n}\| \to 0 \quad (16)$$

for $\mathbb{P}$-almost every path $x$ such that $\text{bnd}_x \in W_y$.

The map $T_y$ is isometric. Indeed, by the defining property (16) and again by [6, Proposition 3.3], we have, for $\mathbb{P}$-almost every path $x$ with $\text{bnd}_x \in W_y$,

$$\psi_y(T_y(a)(\text{bnd}_x)T_y(a)(\text{bnd}_x)) = \lim_{n \to \infty} \psi_{x_n}(a^*ap_{x_n}) = (\pi_\infty \circ \mathcal{E})(a^* \cdot a)(x).$$
Here, \( \pi_\infty \) denotes the \(*\)-isomorphism \( H^\infty_{\text{centr}}(\hat{\mathbb{G}}, \mu) \to L^\infty(\Omega/\alpha, \mathbb{P}) \) introduced in Remark 4.4. On the other hand, by Remark 4.4, \( (\pi_\infty \circ \Theta_\mu)(q_y)(x) = 0 \) for \( \mathbb{P} \)-almost every path \( x \) with \( \text{bnd} x \notin W_y \). Since
\[
\int_\Omega ((\pi_\infty \circ \mathcal{E})(b))(x) \, d\mathbb{P}_x(x) = \hat{c}(b)
\]
for all \( b \in H^\infty(\hat{\mathbb{G}}, \mu) \), it follows that \( T_y \) is an isometry in 2-norm.

We next prove that \( (T_y \circ \Theta_\mu \circ \zeta)(F) = F \) for all \( F \in C(W_y) \otimes \mathcal{L}(H_y) \). Let \( \hat{\alpha} \in C(I \cup \partial I) \subset \ell^\infty(I) \) and let \( a \) be the restriction of \( \hat{\alpha} \) to \( \partial I \). Take \( A \in \mathcal{L}(H_y) \). It suffices to take \( F = a \otimes A \). Theorem 4.2 implies that
\[
\left\| \hat{a}p_{x_n} \psi_{x_n,y}(A) - (\Theta_\mu \circ \zeta)(a \otimes A)p_{x_n} \right\| \to 0
\]
for \( \mathbb{P} \)-almost every path \( x \). On the other hand, for \( \mathbb{P} \)-almost every path \( x \) with \( \text{bnd} x \in W_y \), the scalar \( \hat{a}p_{x_n} \) converges to \( a(\text{bnd} \hat{x}) \). In combination with (16), it follows that \( (T_y \circ \Theta_\mu \circ \zeta)(a \otimes A) = a \otimes A \), concluding the proof of the claim.

Having proven the claim, we now show that for all \( y \in I \), \( H^\infty(\hat{\mathbb{G}}, \mu)^{2|y|} \cdot \Theta_\mu(q_y) \subset \Theta_\mu(M) \). Take \( a \in H^\infty(\hat{\mathbb{G}}, \mu)^{2|y|} \cdot \Theta_\mu(q_y) \). Let \( d_n \) be a bounded sequence in the \( C^* \)-algebra \( C(W_y) \otimes \mathcal{L}(H_y) \) converging to \( T_y(a) \) in 2-norm. Since \( T_y \circ \Theta_\mu \) is an isometry in 2-norm, it follows that \( \zeta(d_n) \) is a bounded sequence in \( M \) that converges in 2-norm. Denoting by \( c \in M \) the limit of \( \zeta(d_n) \), we conclude that \( T_y(\Theta_\mu(c)) = T_y(a) \) and hence, \( \Theta_\mu(c) = a \).

Fix \( k \in \mathbb{N} \). A fortiori, \( H^\infty(\hat{\mathbb{G}}, \mu)^k \cdot \Theta_\mu(q_y) \subset \Theta_\mu(M) \) for all \( y \in I \) with \( 2|y| \geq k \). By Proposition 2.5, the harmonic measure \( \nu_k \) has no atoms in infinite words ending with \( \alpha \beta \alpha \beta \cdots \). As a result, 1 is the smallest projection in \( M \) that dominates all \( q_y \), \( y \in I \), \( 2|y| \geq k \). So, \( H^\infty(\hat{\mathbb{G}}, \mu)^k \subset \Theta_\mu(M) \) for all \( k \in \mathbb{N} \). This finally implies that \( \Theta_\mu \) is surjective. \( \square \)

5 Solitude and the Akemann-Ostrand property

In Section 3, we followed the approach of [11] to construct the compactification \( \mathcal{B} \) of \( \hat{\mathbb{G}} \). In fact, more of the constructions and results of [11] carry over immediately to the case \( \mathbb{G} = A_u(F) \). We continue to assume that \( F \) is not a multiple of a \( 2 \times 2 \) unitary matrix.

Denote by \( L^2(\mathbb{G}) \) the GNS Hilbert space defined by the Haar state \( h \) on \( C(\mathbb{G}) \). Denote by \( \lambda : C(\mathbb{G}) \to \mathcal{L}(L^2(\mathbb{G})) \) the corresponding GNS representation and define \( C_{\text{red}}(\mathbb{G}) := \lambda(C(\mathbb{G})) \). We can view \( \lambda \) as the left-regular representation. We also have a right-regular representation \( \rho \) and the operators \( \lambda(a) \) and \( \rho(b) \) commute for all \( a, b \in C(\mathbb{G}) \) (see [11, Formulae (1.3)]).

Repeating the proofs of [11, Proposition 3.8 and Theorem 4.5], we arrive at the following result.

**Theorem 5.1.** The boundary action \( \beta_\mathbb{G} \) of \( \hat{\mathbb{G}} \) on \( \mathcal{B} \) defined in Theorem 3.2 is

- amenable in the sense of [11, Definition 4.1];
- small at infinity in the following sense: the comultiplication \( \hat{\Delta} \) restricts as well to a right action of \( \hat{\mathbb{G}} \) on \( \mathcal{B} \); this action leaves \( c_0(\hat{\mathbb{G}}) \) globally invariant and becomes the trivial action on the quotient \( \mathcal{B}_\infty \).

By construction, \( \mathcal{B} \) is a nuclear \( C^* \)-algebra and hence, as in [11, Corollary 4.7], we get that
• $G$ satisfies the Akemann-Ostrand property: the homomorphism

$$C_{\text{red}}(G) \otimes_{\text{alg}} C_{\text{red}}(G) \to C(L^2(G)) \otimes_{\text{min}} C(L^2(G)) : a \otimes b \mapsto \lambda(a)\rho(b) + \mathcal{K}(L^2(G))$$

is continuous for the minimal $C^*$-tensor product $\otimes_{\text{min}}$.

• $C_{\text{red}}(G)$ is an exact $C^*$-algebra.

As before, we denote by $L^\infty(G)$ the von Neumann algebra acting on $L^2(G)$ generated by $\lambda(C(G))$. From [2, Théorème 3], it follows that $L^\infty(G)$ is a factor, of type II$_1$ if $F$ is a multiple of an $n \times n$ unitary matrix and of type III in the other cases.

Applying [8, Theorem 6] (in fact, its slight generalization provided by [11, Theorem 2.5]), we get the following corollary of Theorem 5.1. Recall that a II$_1$ factor $M$ is called solid if for every diffuse von Neumann subalgebra $A \subset M$, the relative commutant $M \cap A'$ is injective. An arbitrary von Neumann subalgebra $M$ is called generalized solid if the same holds for every diffuse von Neumann subalgebra $A \subset M$ which is the image of a faithful normal conditional expectation.

**Corollary 5.2.** When $n \geq 3$ and $G = A_n(I_n)$, the II$_1$ factor $L^\infty(G)$ is solid. When $n \geq 2$, $F \in \text{GL}(n, \mathbb{C})$ is not a multiple of an $n \times n$ unitary matrix and $G = A_n(F)$, the type III factor $L^\infty(G)$ is generalized solid.

### 6 Appendix: approximate intertwining relations

We fix an invertible matrix $F$ and assume that $F$ is not a scalar multiple of a unitary $2 \times 2$ matrix. Define $G = A_n(F)$ and label the irreducible representations of $G$ by the monoid $\mathbb{N} \ast \mathbb{N}$, freely generated by $\alpha$ and $\beta$. The representation labeled by $\alpha$ is the fundamental representation of $G$ and $\beta$ is its contragredient. Define $0 < q < 1$ such that $\dim_q(\alpha) = \dim_q(\beta) = q + \frac{1}{q}$. Recall from Section 3 that whenever $z \subset x \otimes y$, we choose an isometry $V(x \otimes y, z) \in \text{Mor}(x \otimes y, z)$. Observe that $V(x \otimes y, z)$ is uniquely determined up to multiplication by a scalar $\lambda \in S^1$. We denote by $p^\otimes_{xy}$ the projection $V(x \otimes y, z)V(x \otimes y, z)^*.$

**Lemma 6.1.** There exists a constant $C > 0$ that only depends on $q$ such that

$$
\| (V(xr \otimes y, xy) \otimes 1_z)p^\otimes_{xy} - (1_x \otimes p^\otimes_{yz})(V(xr \otimes y, xy) \otimes 1_z) \| \leq Cq|y|,
\| (1_x \otimes V(yr \otimes z, yz))p^\otimes_{yz} - (p^\otimes_{yr} \otimes 1_{rz})(1_x \otimes V(yr \otimes z, yz)) \| \leq Cq|y|
$$

(17)

for all $x, y, z, r \in I$.

One way of proving Lemma 6.1 consists in repeating step by step the proof of [11, Lemma A.1]. But, as we explain now, Lemma 6.1 can also be deduced more directly from [11, Lemma A.1].

**Sketch of proof.** Whenever $y = y_1 \otimes y_2$ with $y_1 \neq \epsilon \neq y_2$, the expressions above are easily seen to be 0. Denote

$$v_n = \alpha \otimes \beta \otimes \alpha \otimes \cdots \text{ and } w_n = \beta \otimes \alpha \otimes \beta \otimes \cdots.$$
The remaining estimates that have to be proven, reduce to estimates of norms of operators in \( \text{Mor}(v_n, v_m) \) and \( \text{Mor}(w_n, w_m) \). Putting these spaces together in an infinite matrix, one defines the C*–algebras

\[
A := \left( \text{Mor}(v_n, v_m) \right)_{n,m} \quad \text{and} \quad B := \left( \text{Mor}(w_n, w_m) \right)_{n,m}
\]
generated by the subspaces \( \text{Mor}(v_n, v_m) \) and \( \text{Mor}(w_n, w_m) \), respectively. Choose unit vectors \( t \in \text{Mor}(\alpha \otimes \beta, \epsilon) \) and \( s \in \text{Mor}(\beta \otimes \alpha, \epsilon) \) such that \( (t^* \otimes 1)(1 \otimes s) = (q + 1/q)^{-1} \). By [2, Lemme 5], the C*–algebra \( A \) is generated by the elements \( 1 \otimes 2^k \otimes t \otimes 1 \otimes l \), \( 1 \otimes (2^k + 1) \otimes s \otimes 1 \otimes l \). A similar statement holds for \( B \).

Denote by \( U \) the fundamental representation of the quantum group \( SU_{-q}(2) \) and let \( t_0 \in \text{Mor}(U \otimes U, \epsilon) \) be a unit vector. The proofs of [3, Theorems 5.3 and 6.2] (which heavily rely on results in [1, 2]), imply the existence of ∗–isomorphisms \( \pi_A : (\text{Mor}(U^\otimes n, U^\otimes m))_{n,m} \to A \) and \( \pi_B : (\text{Mor}(U^\otimes n, U^\otimes m))_{n,m} \to B \) satisfying

\[
\pi_A(1 \otimes 2^k \otimes t_0 \otimes 1 \otimes l) = 1 \otimes 2^k \otimes t \otimes 1 \otimes l \quad \text{and} \quad \pi_A(1 \otimes (2^k+1) \otimes t_0 \otimes 1 \otimes l) = 1 \otimes (2^k+1) \otimes s \otimes 1 \otimes l
\]

and similarly for \( \pi_B \).

As a result, the estimates to be proven, follow directly from the corresponding estimates for \( SU_{-q}(2) \) proven in [11, Lemma A.1].

Using the notation

\[
d_{S^1}(V, W) = \inf\{\|V - \lambda W\| \mid \lambda \in S^1\},
\]

several approximate commutation relations can be deduced from Lemma 6.1. For instance, after a possible increase of the constant \( C \), (17) implies that

\[
d_{S^1}\left(\left(1_x \otimes V(yr \otimes \tau z, yz)\right)V(x \otimes yz, xyz), \left(V(x \otimes yr, xyr) \otimes 1_R\right)V(xyr \otimes \tau z, xyz)\right) \leq Cq|y|
\]

for all \( x, y, z, r \in I \). We again refer to [11, Lemma A.1] for a full list of approximate intertwining relations.

References


