Jumps in Intensity Models

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Abstract

This work presents intensity-based credit risk models where the default intensity of the point process is modeled by an Ornstein-Uhlenbeck type process completely driven by jumps. Under this model we compute the default probability over time by linking it to the characteristic function of the integrated intensity process. In case of the Gamma and Inverse Gaussian Ornstein-Uhlenbeck processes this leads to a closed form expression for the default probability and to a straightforward estimate of credit default swaps prices. The model is calibrated to a series of real-market term structures. Results are compared with the well known cases of Poisson and CIR dynamics. Possible extensions of the model to the multivariate setting are finally discussed.

Key words: Ornstein-Uhlenbeck process, Lévy process, survival probability, intensity-based model, credit default swap.

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1 Introduction

Modern finance has put much effort in developing new models for credit risk. This is related both to the growth in the credit derivatives volumes traded on the market and to the possibility given to financial institutions to assess in-house their credit exposure (advanced implementation option of Basel II).

Credit risk models are usually classified into two categories: structural models and intensity-based models. Structural models link an event of default to the asset value of the firm. Usually, the dynamics of the asset value is given and an event of default is defined in terms of boundary conditions on this process. The first structural models date back to Merton (1974) and Black and Cox (1976) but a lot of modifications/extensions can be found in the literature (e.g. Leland (1994), Longstaff and Schwartz (1995), Madan et al. (1998), Cariboni and Schoutens (2005)). A drawback of the structural approach is that the asset value is treated as a primary asset of the economy while it is usually dependent on other state variables (Madan and Unal, 1998).

Intensity-based models, known also as hazard rate or reduced-form models, focus directly on the modeling of the default probability. The intensity-based approach defines the time of default as the first jump-time of a counting process \( M_t \). As a consequence, a central role is played by the jumps intensity rate of \( M_t \). A standard example of counting process is the (homogeneous) Poisson process with constant default intensity \( \lambda > 0 \). The corresponding default model was developed by Jarrow and Turnbull (1995). Some generalizations allow the default intensities to be time-dependent, leading to the so-called inhomogeneous Poisson process, or allow for stochastic default intensities. In this latter case, the corresponding counting process is called a Cox-process. Madan and Unal (1998) considered the case where the default intensity \( \lambda_t \) is adapted to a Brownian filtration. Duffie and Singleton (1999) developed a basic affine model, which allows for jumps in the hazard dynamics.

In contrast to the structural approach, under the reduced-form models default is an inaccessible stopping time. Jarrow and Protter (2004) compare structural and intensity models and highlight that the key distinction between them is the information set assumed to be known. Structural models suppose that the available information is that observed by the firm’s managers; intensity models assume the modeler’s information observed on the market.

From a fundamental point of view it is important to have jumps in the intensity regime or in the firm’s value price process because changes in the creditworthiness (default intensity or firm’s value) are often shock driven: sudden events in reality cause important changes on the view on the company’s probability of default. Standard examples of such dramatic changes in the regime are discovery of fraud (e.g. Parmalat), a reviewing of company’s results, default of a competitor, a terroristic attack, etc.

This work introduces a new default model where the intensity of default is assumed to follow an Ornstein-Uhlenbeck (OU) process. Under this assumption, we show that the survival probability of the obligor can be expressed in terms of the characteristic function of the integrated OU process. We concentrate on
two special cases, the Gamma-OU and Inverse Gaussian-OU (IG-OU) processes. The names refer to the stationary law of the intensity process used. In these cases a closed-form expression for the characteristic function of the integrated process is available, leading to a straightforward estimate of the survival probability. The Gamma case can be rephrased as a special case of the basic affine model introduced by Duffie and Singleton (1999).

The model abilities are tested through a calibration exercise on Credit Default Swaps (CDS) term structures observed on the market. To this aim we consider the 125 CDS constituting the Itraxx Europe Index. For each asset, weekly term structure data are available for a total of 58 market observations (i.e. covering a time period of bit more than 1 year). This allows not only to investigate in depth the calibration capabilities of the OU-models, but also to check the stability of the processes’ parameters over time. For comparison purposes, we also calibrate intensity models based on the Poisson, inhomogeneous Poisson, and CIR dynamics.

Finally, we discuss possible extensions of the model to the multivariate setting. Following Joshi and Stacey (2005), we introduce dependency among assets by time-changing the intensity dynamics by a common subordinator. We focus on the special case of the Gamma-process and derive the expression for the corresponding survival probabilities. A review of other models for correlated defaults can be found in Elizalde (2005). Under the conditional independent approach, the defaults dependence is introduced via a set of common state variables. An extension is proposed by Duffie and Gärleanu (2001), which allow common shocks in the intensity dynamics. A different approach, known as default contagion, is based on an increase of credit risk caused by a default event (Davis and Lo, 2001; Jarrow and Yu, 2001). Finally, the copula approach model the defaults dependence through the use of copulas.

The paper is organized as follows. In the next section, we introduce the basic background on Lévy processes and OU processes driven by Lévy processes, concentrating on the Gamma and Inverse Gaussian OU processes. Section 3 presents our intensity based OU default model. We then introduce CDS and link CDS spreads to the integrated OU process. The last part of Section 3 presents the results of the calibration exercises. The penultimate section deals with the multivariate setting and the last section concludes.

2 Lévy Processes and Ornstein-Uhlenbeck Processes

2.1 Lévy Processes

Suppose $\phi(z)$ is the characteristic function of a distribution. If for every positive integer $n$, $\phi(z)$ is also the $n$th power of a characteristic function, we say that the distribution is infinitely divisible. One can define for any infinitely divisible distribution a stochastic process, $X = \{X_t, t \geq 0\}$, called Lévy process, which starts at zero, has independent and stationary increments and such that the
distribution of an increment over \([s, s + t]\), \(s, t \geq 0\), i.e. \(X_{t+s} - X_s\), has \((\phi(z))^t\) as characteristic function.

The function \(\psi(z) = \log \phi(z)\) is called the characteristic exponent and it satisfies the following Lévy-Khintchine formula (Bertoin, 1996):

\[
\psi(z) = i\gamma z - \frac{\varsigma^2}{2} z^2 + \int_{-\infty}^{+\infty} \left( \exp(izx) - 1 - izx1_{\{|x|<1\}} \right) \nu(dx),
\]

where \(\gamma \in \mathbb{R}, \varsigma^2 \geq 0\) and \(\nu\) is a measure on \(\mathbb{R}\setminus\{0\}\) with \(\int_{-\infty}^{+\infty} (1 \wedge x^2) \nu(dx) < \infty\).

From the Lévy-Khintchine formula, one sees that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part, and a pure jump part. We say that our infinitely divisible distribution has a triplet of Lévy characteristics \([\gamma, \varsigma^2, \nu(dx)]\). The measure \(\nu(dx)\) is called the Lévy measure of \(X\) and it dictates how the jumps occur. Jumps of sizes in the set \(A\) occur according to a Poisson process with parameter \(\int_A \nu(dx)\). If \(\varsigma^2 = 0\) and \(\int_{-1}^{1} |x| \nu(dx) < \infty\) it follows from standard Lévy process theory (Bertoin, 1996; Sato, 2000), that the process is of finite variation. For more details about the applications of Lévy processes in finance we refer to Schoutens (2003).

### 2.2 OU Processes

In this section we give a brief overview of OU processes (driven by Lévy processes) which were introduced by Barndorff-Nielsen and Shephard (2001a, 2001b, 2003) to describe volatility in finance. Further references on OU processes are Wolfe (1982), Sato and Yamazato (1982), Jyrek and Vervaat (1983), Sato et al. (1994).

An OU process \(y = \{y_t, t \geq 0\}\) is described by the following stochastic differential equation:

\[
dy_t = -\theta y_t dt + dz_t, \quad y_0 > 0,
\]

where \(\theta\) is the arbitrary positive rate parameter and \(z_t\) is a subordinator, i.e. a Lévy process with no Brownian component, nonnegative drift and only positive increments. \(z\) is often called Background Driving Lévy Process (BDLP).

As \(z\) is an increasing process and \(y_0 > 0\), it is clear that the process \(y\) is strictly positive. Moreover, it is bounded from below by the deterministic function \(y_0 \exp(-\theta t)\).

The process \(y = \{y_t, t \geq 0\}\) is strictly stationary on the positive half-line, i.e. there exists a law \(D\), called the stationary law or the marginal law, such that \(y_t\) will follow for every \(t\) the law \(D\), if initial \(y_0\) is chosen according to \(D\). The process \(y\) moves up entirely by jumps and then tails off exponentially. The fact that we have the parameter \(\theta\) in \(z_{\theta t}\) has to do with the separation of the stationary law from this decay parameter. In Barndorff-Nielsen and Shephard (2001a) some stochastic properties of \(y\) are studied. Barndorff-Nielsen and Shephard established the notation that if \(y\) is an OU process with marginal law \(D\), then we say that \(y\) is a \(D\)-OU process.

\[\text{1}\] Also OU processes based on a general Lévy process, not necessarily a subordinator, can be defined. However for our analysis we will only need the special case considered above.
In essence, given a one-dimensional distribution \( D \) (not necessarily restricted to the positive half line) there exists a (stationary) OU process whose marginal law is \( D \) (i.e. a \( D \)-OU process) if and only if \( D \) is self-decomposable (for definition see Sato, 2000). We have by standard results (Barndorff-Nielsen and Shephard, 2001a) that

\[
y_t = \exp(-\vartheta t)y_0 + \int_0^t \exp(-\vartheta(t-s))dz_s,
\]

In the case of a \( D \)-OU process, let us denote by \( k_D(u) \) the cumulant function of the self-decomposable law \( D \) and by \( k_z(u) \) the cumulant function of the BDLP at time \( t = 1 \), i.e. \( k_z(u) = \log E[\exp(-uz_1)] \), then both are related through the formula (see for example Barndorff-Nielsen, 2001):

\[
k_z(u) = u \frac{dk_D(u)}{du}.
\]

An important related process will be the integral of \( y_t \). Barndorff-Nielsen and Sheppard called this the integrated OU process (intOU); we will denote this process with \( Y = \{Y_t, t \geq 0\} \):

\[
Y_t = \int_0^t y_s ds.
\]

A major feature of the intOU process \( Y \) is

\[
Y_t = \vartheta^{-1} (z_{\vartheta t} - y_t + y_0) = \vartheta^{-1} (1 - \exp(-\vartheta t) y_0 + \vartheta^{-1} \int_0^t (1 - \exp(-\vartheta(t-s))) dz_s).
\]

One can show (see [2]) that given \( y_0 \),

\[
\log E[\exp(iuY_t)|y_0] = \vartheta \int_0^t k(u\vartheta^{-1}(1 - \exp(-\vartheta(t-s)))) ds + iuy_0 \vartheta^{-1}(1 - \exp(-\vartheta t)),
\]

where \( k(u) = k_z(u) = \log E[\exp(-uz_1)] \) is the cumulant function of \( z_1 \).

### 2.2.1 The Gamma-Ornstein-Uhlenbeck Process

The Gamma(\( a, b \))-OU process has as BDLP a compound Poisson process:

\[
z_t = \sum_{n=1}^{N_t} x_n
\]
where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $a$ (i.e. $E[N_t] = at$) and $\{x_n, n = 1, 2, \ldots, N_t\}$ is a sequence of independent identically distributed Exp($b$) variables, i.e. exponentially distributed with mean $1/b$. It turns out that the stationary law is given by a Gamma($a, b$) distribution with density function

$$f_{\text{Gamma}}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \quad x > 0,$$

which immediately explains the name. The Gamma-OU process has a finite number of jumps in every compact time interval.

If $y_t$ is a Gamma-OU process, the characteristic function of the intOU process $Y_t = \int_0^t y_s ds$ is given by:

$$\phi_{\text{Gamma-OU}}(u, t; \vartheta, a, b, y_0) = E[\exp(iuY_t) | y_0] = \exp\left(\frac{iu y_0}{\vartheta} (1 - e^{-\vartheta t}) + \frac{\vartheta a}{iu - \vartheta b} \left(b \log \left(\frac{b}{b - iu \vartheta^{-1}(1 - e^{-\vartheta t})}\right) - iut\right)\right). \quad (4)$$

A Gamma($a, b$)-OU process $\{y_t, t \geq 0\}$ can be simulated at time points $\{t = n\Delta t, n = 0, 1, 2, \ldots\}$ through its BDLP. First a Poisson processes $\{N_t, t \geq 0\}$ with intensity parameter $a\vartheta$ has to be simulated at the same time points. Then we have to set:

$$y_{n\Delta t} = \exp(-\vartheta \Delta t)y_{(n-1)\Delta t} + \sum_{n=N(n-1)\Delta t+1}^{N_n\Delta t} x_n \exp(-u_n \vartheta \Delta t), \quad (5)$$

where $\{x_n, n = 0, 1, \ldots\}$ are (independent) Exp($b$) random numbers and $\{u_n, n = 0, 1, \ldots\}$ are (independent) standard uniform random numbers. Note that the exponential term and the uniform random numbers $u_n$ in the sum allows the jumps to happen somewhere in between two time steps. Figure 1 (top graph) shows a path of a Gamma-OU process with parameters $\vartheta = 4, a = 4, b = 18$ and $y_0 = 0.08$.

### 2.2.2 The Inverse Gaussian-Ornstein-Uhlenbeck Process

We start with recalling the definition of the Inverse Gaussian ($\text{IG}(a, b)$) law, by stating its density function:

$$f_{\text{IG}}(x; a, b) = \frac{a}{\sqrt{2\pi}} \exp(ab)x^{-3/2} \exp(-(a^2 x^{-1} + b^2 x)/2), \quad x > 0.$$  

This IG($a, b$) belongs to the class of the self-decomposable distributions and hence an IG-OU process exists. In the case of the IG($a, b$)-OU process the BDLP is a sum of two independent Lévy processes $z = \{z_t = z_t^{(1)} + z_t^{(2)}, t \geq 0\}$. $z^{(1)}$ is an IG-Lévy process with parameters $a/2$ and $b$, while $z^{(2)}$ is of the form:

$$z_t^{(2)} = b^{-1} \sum_{n=1}^{N_t} y_n^2,$$
where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter $ab/2$, i.e. $E[N_t] = abt/2$. $\{v_n, n = 1, 2, \ldots\}$ is a sequence of independent and identically distributed random variables: each $v_n$ follows a $\text{Normal}(0,1)$ law independent from the Poisson process $N$. Since the BDLP (via $z^{(1)}$) jumps infinitely often in any finite (time) interval, also the IG-OU process jumps infinitely often in every interval. The cumulant of the BDLP (at time 1) is given by

$$k(u) = -uab^{-1}(1+2ub^{-2})^{-1/2}.$$ 

In the IG-OU case the characteristic function of the intOU process $Y_t = \int_0^t y_s ds$ can also be given explicitly. The following expression was independently derived in Nicolato and Venardos (2002) and Tompkins and Hubalek (2000):

$$\phi_{\text{IG-OU}}(u,t; \vartheta, a, b, y_0) = E[\exp(iuY_t)|y_0] = \exp(iuy_0\vartheta(1-\exp(-\vartheta t)) + \frac{2aiu}{b\vartheta}A(u,t)), \quad (6)$$

where

$$A(u,t) = \frac{1}{\kappa} - \sqrt{\frac{1}{1 + \kappa(1-\exp(-\vartheta t))}}$$

$$\kappa = -2b^{-2}ia/\vartheta.$$ 

The IG-OU process can be simulated by first simulating its BDLP and then applying the Euler’s scheme to the defining stochastic differential equation (2). Fast simulation of the BDLP is achieved by recalling that the BDLP is the sum of two independent Lévy processes.

Figure 1 (bottom plot) shows a path of an IG-OU process with parameters $\vartheta = 2$, $a = 1.5$, $b = 12$ and $y_0 = 0.08$.

### 3 The Intensity OU-Model

Intensity models assume an event of default to occur at the first jump of a counting process $M = \{M_t, t \geq 0\}$. The intensity rate $\lambda = \{\lambda_t, t \geq 0\}$, known also as hazard rate, represents the instantaneous default probability. If we indicate with $\tau$ the default time, the intensity of default is defined as:

$$\lambda_t = \lim_{h \to 0} \frac{P[\tau \in (t, t+h)|\tau > t]}{h}.$$ 

(8)

Roughly speaking, this means that for a small time interval $\Delta t > 0$:

$$P[\tau \leq t + \Delta t | \tau > t] \approx \lambda_t \Delta t.$$
As a consequence, the dynamics of the default intensity governs the credit quality of the corresponding asset. If $\lambda_t$ is a stochastic process, $M_t$ is called Cox process.

In our model we assume that the default intensity follows a Gamma-OU or an IG-OU process, as described in the previous section. Hence, the intensity process is modeled by the stochastic differential equation:

$$d\lambda_t = -\vartheta \lambda_t dt + d\tilde{z}_t, \lambda_0 > 0.$$  \hspace{1cm} \text{(9)}

As before, $\vartheta$ is the arbitrary positive rate parameter and $z_t$ is the BDLP, which we assume to be a subordinator in order to force the intensity process to be a positive process. Note that the Gamma-OU case can be rephrased as a special case of the basic affine model introduced by Duffie and Singleton (1999):

$$d\lambda_t = \vartheta (\kappa - \lambda_t) dt + \sigma \sqrt{\lambda_t} dW_t + \Delta J_t,$$

where $W_t$ is a standard Brownian motion, and $\Delta J_t$ denotes any jump of a pure jump process $J_t$ occurring at time $t$. $J_t$ has jump sizes independent and exponentially distributed with mean $\mu$ and arrival rate $l \vartheta$. Moreover, $J_t$ is independent of $W_t$. The Gamma-OU process can be reformulated under this notation by setting $\sigma = 0, \kappa = 0, l = a$, and $\mu = 1/b$.

Under our assumptions, the time of default $\tau$ is defined as the first jump of $M_t$:

$$\tau = \inf\{t \in \mathbb{R}^+ | M_t > 0\}.$$

The implied survival probability from 0 to $t$, $P(t)$, is given by:

$$P(t) = P[M_t = 0] = P[\tau > t] = E\left[\exp\left(-\int_0^t \lambda_s ds\right)\right]$$  \hspace{1cm} \text{(10)}

where $Y_t = \int_0^t \lambda_s ds$ is the intOU process.

For the Gamma-OU and the IG-OU dynamics, using Equations (4) and (6), we obtain the following closed-form solutions for the survival probabilities:

$$P_{\text{Gamma-OU}}(t) = E\left[\exp\left(-\int_0^t \lambda_s ds\right)\right] = \varphi_{\text{Gamma-OU}}(i, t; \vartheta, a, b, \lambda_0) = \exp\left[-\frac{y_0}{A} (1 - e^{-\vartheta t}) - \frac{\vartheta a}{1 + \vartheta b} \left(b \log\left(\frac{b}{b + \vartheta^{-1}(1 - e^{-\vartheta t})}\right) + t\right)\right],$$

$$P_{\text{IG-OU}}(t) = E\left[\exp\left(-\int_0^t \lambda_s ds\right)\right] = \varphi_{\text{IG-OU}}(i, t; \vartheta, a, b, \lambda_0) = \exp\left[-\frac{y_0}{A} (1 - \exp(-\lambda t)) - \frac{2a}{bA} A(i, t)\right],$$

8
where the function $A$ is given by Equation (8).

### 3.1 Calibration of the model on CDS term structures

A CDS is a derivative that provides the buyer an insurance against the default of a company (the reference entity) on its debt. The buyer of this protection makes (continuous) predetermined payments to the seller. The payments continue until the maturity date of the contract or until default occurs, whichever is earlier. If default occurs, the buyer delivers a bond on the underlying defaulting asset in exchange for its face value. The market for CDS is well established and trading is increasing in related products like forwards and options on these CDS. For pricing techniques for these forwards and options we refer to Hull and White (2003). The price of a CDS of maturity $T$ is given by the difference of the discounted spread and the loss payments:

\[
CDS = (1 - R) \left( - \int_0^T \exp(-r_s s) dP(s) \right) - c \int_0^T \exp(-r_s s) P(s) ds,
\]

where $R$ is the recovery rate, $\{r_t, t \geq 0\}$ is the term-structure of default-free discount rate and $P(t)$ indicates the survival probability up to time $t$. The par spread $c^*$ that makes this price equals to zero is:

\[
c^* = \frac{(1 - R) \left( - \int_0^T \exp(-r_s s) dP(s) \right)}{\int_0^T \exp(-r_s s) P(s) ds} \tag{11}
\]

Under the Gamma-OU and IG-OU dynamics, $c^*$ can be estimated using the equations derived at the end of the previous section for the survival probability.

We calibrate our intensity OU models to the CDS term structures of the Itraxx Europe Index. Each term structure includes prices of CDS for five different times to maturity (respectively $T_1 = 1y$, $T_2 = 3y$, $T_3 = 5y$, $T_4 = 7y$, and $T_5 = 10y$ years). We set for convenience $T_0 = 0$. For each component, we consider the complete weekly time series from the 5th of January 2005 to the 8th of February 2006. This allows not only to check the calibration capabilities of the OU-models, but also to investigate the stability of their parameters over time. In the calibrations the discounting term-structure $r = r_t, t \geq 0$ is taken from the market on the corresponding day. The recovery rates for all the Itraxx Europe Index assets is fixed at $R = 0.4$.

In the calibrations we use the Nelder-Mead simplex (direct search) method to minimize the root mean square error (rmse) given by:

\[
rmse = \sqrt{\frac{\sum_{\text{CDS prices}} (\text{Market CDS price} - \text{Model CDS price})^2}{\text{number of CDS prices}}} \tag{12}
\]

The cpu time required to calibrate our OU-model to all the 125 CDS term structures together for a given point in time (i.e. for a given week) is around 1 minute.
For comparison purposes, the capabilities of the OU model are tested by calibrating on the same term structures the following models:

1. the homogeneous Poisson (HP) model (Jarrow and Turnbull, 1995), where the default intensity is constant;

2. the inhomogeneous Poisson (IHP) model with piecewise constant default intensity

\[ \lambda_t = K_j, \quad T_{j-1} \leq t < T_j, \quad j = 1, 2, \ldots, 5; \quad (13) \]

3. the Cox-Ingersoll-Ross (CIR) model (Cox et al., 1985), where the default intensity is stochastic.\(^2\)

To compare the quality of the fits, we compute for each model the average absolute error as a percentage of the mean price \( \text{ape} \) and the average relative percentage error \( \text{arpe} \):

\[
\text{ape} = \frac{1}{\text{mean CDS price}} \sum_{\text{CDS}} \frac{|\text{Market CDS price} - \text{Model CDS price}|}{\text{number of CDS prices}}
\]

\[
\text{arpe} = \frac{1}{\text{number of CDS prices}} \sum_{\text{CDS}} \frac{|\text{Market CDS price} - \text{Model CDS price}|}{\text{Market CDS price}}.
\]

To discuss the outcomes, we first concentrate on two companies, Zurich Insurance and Continental, and present the calibrations results matching market data as of 21st July 2005. Similar results are obtained for all the other Itraxx Europe index components for each week. Figures 2 plots the calibrated term structures (left plots) and the default probabilities as a function of time (right plots) for the five models. In Tables 1 and 2 we report on the calibration exercises for the two companies. In the former, market CDS prices are compared with the prices obtained using the models. For each calibration the value of \( \text{ape} \) and \( \text{arpe} \) are also given. In the latter, the optimal parameters are presented.

Results highlight the complete failure of the HP model to match market data. Concerning the IHP case, the model can match perfectly the market

\(^{2}\)Under the CIR model the default intensity is described by the following equation:

\[
d\lambda_t = \kappa(\eta - \lambda_t)dt + \sqrt{\lambda_t}dW_t
\]

where \( W = \{W_t, t \geq 0\} \) is a standard Brownian motion, \( \eta \) is the long-run rate of time-change, \( \kappa \) is the rate of mean reversion, and \( \theta \) governs the volatility of time change. A closed-form expression for the characteristic function of the integrated process is given by (Cox et al., 1985):

\[
\phi_{CIR}(u, t; \kappa, \eta, \theta, \lambda_0) = \frac{\exp(\kappa^2 \eta t / \theta^2) \exp(2\lambda_0 i u / (\kappa + \gamma \coth(\gamma t / 2)))}{(\coth(\gamma t / 2) + \kappa \sinh(\gamma t / 2) / \gamma)^{2\kappa \eta / \theta^2}}.
\]
quotes; however the behavior of the term structure between two subsequent
time horizons is clearly unreliable, due to the piecewise constant assumption.
The CIR, Gamma-OU and IG-OU models can all be nicely calibrated to market
data. In the following we will concentrate only on these three models and present
further analysis.

For these models, we consider two different approaches for the calibrations:
in the first the starting values for the models' parameters are fixed at each week;
we will refer to this approach as **Fixed** approach. In the second approach, the
values obtained for a given week are used as initial guess for the next week; we
will refer to this approach as **Dynamic**. The **Fixed** approach guarantees that
the new calibration do not persist around a minimum which might be local. On
the other hand, initializing the values at a fixed point in the parameters' space
increases the probability for the parameters to be unstable over time, which is
inconvenient in case of an hedging strategy.

The distributions of the $ape$ and $arpe$ for the three models are plotted in
Figures 3 and 4. The histograms refer to the entire set of 7250 calibrations
obtained considering all the Itraxx components at each of the 58 time points.
The first row refers to the CIR process, the second to the Gamma-OU, and the
third to the IG-OU; left plots show results for the **Fixed** approach while right
plots concern with the **Dynamic** approach. Table 3 presents the average values
of $ape$ and $arpe$ for each model. Results show that the three models can be all
nicely calibrated to market observations. Model prices can better fit real data
under the **Fixed** approach, i.e. if the initial values for the parameters are fixed
at each time step. This is more evident for the Gamma-OU and CIR dynamics.

We further investigate our intensity models by analyzing the stability of
the optimal parameters over time. For each model and each parameter we
obtain a weekly time series of optimal values. In Figures 5 and 6 we present
the behavior of the these time series for **Zurich Insurance** and **Continental**. In
these figures dotted and solid lines refer to the **Fixed** and **Dynamic** approach
respectively. The plots clearly show that the stability of the parameters is much
enhanced under the **Dynamic** method. Among the models, the IG-OU is the
most unstable.

To investigate the overall behavior of the parameters Figures 7, 8 and 9 show
the behavior of some given percentiles of the distribution of the parameters
over time. In each plot the values of the 95%, 75%, 50%, 25% and 5% quantiles
of the distribution of a given parameter over the 125 companies of the index
are presented. The average value of the parameter is also shown (red line).
Left plots concern with the **Fixed** approach; right plots illustrate results of the
**Dynamic** method. The graphs confirm that the **Fixed** approach produces much
more stable parameters and that the IG-OU parametrization is the least steady.
Moreover, the average value of the parameters is often higher than the median
value, highlighting the presence of big but isolated jumps in the parameters
values. This is more enhanced in the cases of the parameters $\vartheta$ for the CIR
model, $\vartheta$ and $b$ for the Gamma-OU dynamics, and even more for the $a$ and $b$
parameters under the IG-OU patterns. In these cases the average parameters’
values are even higher than the values of the 75% (and for IG-OU 95%) quantiles.
To complete the analysis of the parameters behavior, we present in Figure 10 the lag $-1$ autocorrelation distributions for each parameter and each model. We recall that, given a discrete time series of data $X = \{X_t, t = 1, 2, \ldots, N\}$, the lag-$k$ autocorrelation is given by:

$$
\rho_k = \frac{E[(X_t - \overline{X})(X_{t+k} - \overline{X})]}{\sqrt{E[(X_t - \overline{X})^2] E[(X_{t+k} - \overline{X})^2]}}.
$$

(14)

Autocorrelation is thus a correlation coefficient. However, instead of correlation between two different variables, the correlation is between two values of the same variable at times $t_i, t_{i+k}$. Since we use autocorrelation to investigate whether there is randomness in the optimal values of the parameters, we first consider only the lag $-1$ autocorrelation (i.e. $k = 1$).

In the plots, red bars refer to the Fixed approach; black histograms plot the Dynamic method results. The Dynamic approach results in distributions much shifted to the right, i.e. with higher dependency between two subsequent values of the parameters. If we concentrate on the Fixed approach, which best fits market data, a comparison of the three models highlight that the Gamma-OU dynamics gives higher autocorrelation values. In Figure 11 we finally show the values of the autocorrelation for higher lags for each model and each parameter.

4 Moving Towards a Multivariate Setting

Based on the idea of Joshi and Stacey (2005) and Luciano and Schoutens (2005) one can introduce dependency by time-changing. We consider $n$ assets described by $n$ independent individual intensity models $\lambda^{(i)} = \{\lambda_{t}^{(i)}, t \geq 0, i = 1, \ldots, n\}$. The default of each asset is defined by the first jump-time of a Cox process $M^{(i)} = \{M_{t}^{(i)}, t \geq 0\}$. We assume that the corresponding default intensities are described by the OU model introduced in Section 3:

$$
d\lambda_{t}^{(i)} = -\vartheta^{(i)} \lambda_{t}^{(i)} dt + dz_{t}^{(i)}, \lambda_{0}^{(i)} > 0, i = 1, \ldots, n.
$$

(15)

As before, $\vartheta^{(i)}$ is the arbitrary positive rate parameter for the $i$th firm and $z_{t}^{(i)}$ is the $i$th firm’s BDLP, which we assume to be a subordinator.

We introduce dependency by time-changing the individual Cox processes $M^{(i)}$ by a common subordinator. A tractable choice for this subordinator is the below described Gamma process. Other choices, like the $IG$ subordinator, are also possible.

So, let $G = \{G_t, t \geq 0\}$ be a Gamma process, i.e. a process which starts at zero, has stationary independent increments; which follows a gamma distribution over the time interval $[s, s + t]$. More precisely, $G$ is a Lévy process (a subordinator), where the defining distribution of $G_1$ is a Gamma($\alpha, \beta$) distribution with density function

$$
f_{\text{Gamma}}(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-x\beta), x \geq 0
$$
and characteristic function given by

\[ \phi_{\text{Gamma}}(x; \alpha, \beta) = (1 - iu/\beta)^{-\alpha}. \]

Standard Lévy process theory (see for example Bertoin, 1996; Sato, 2000; Schoutens, 2003) teaches us that increments over intervals of length \( s \) then are Gamma(\( \alpha s, \beta \)) distributed. For obvious, normalization reasons, we will work in terms of the parameters imply that \( \alpha = \beta \).

The time to default \( \tau^{(i)} \) of the \( i \)th firm is again defined as:

\[ \tau^{(i)} = \inf\{t \in R^+ | M^{(i)}_{G_t} > 0\} \]

while the implied survival probability from 0 to \( t \), \( P_t^{(i)} \) of this firm, is given by

\[
P(t) = P[M^{(i)}_{G_t} = 0] = P[\tau^{(i)} > G_t] = E \left[ \exp \left( -\int_{G_t}^{G_t} \lambda^{(i)}_s ds \right) \right] \]

\[
= E \left[ \exp \left( -Y^{(i)}_{G_t} \right) \right] \tag{16}
\]

where \( Y^{(i)}_t = \int_0^t \lambda^{(i)}_s ds \) is the integrated OU process of the default intensity of asset \( i \).

In the case of a OU process, we have the following expression for the survival probabilities:

\[
P^{(i)}(t) = E \left[ \exp(-Y^{(i)}_{G_t}) \right] \]

\[
= \int_0^\infty \phi_{\text{OU}}(i, s) \frac{\alpha t}{\Gamma(\alpha t)} s^{\alpha t - 1} \exp(-s\alpha) ds,
\]

where \( \phi_{\text{OU}} \) is the characteristic function of the intOU process. For our special cases of the Gamma-OU and IG-OU dynamics \( \phi_{\text{OU}} \) is given by Equations (4) and (6).

5 Conclusions

This work has presented an intensity-based Lévy model to price credit derivatives. An event of default is assumed to occur at the first jump-time of a Cox process. The dynamics of the default intensity is described by a Gamma or Inverse Gaussian Ornstein-Uhlenbeck process. We have shown that under these hypothesis the survival probability can be expressed in closed-form using the characteristic function of the integrated Ornstein-Uhlenbeck process. This allows to estimate straightforwardly the par spread of a credit default swap.

The capabilities of the model have been tested through a comparative calibration exercise on the 125 credit default swaps constituting the Itraxx Europe
Index. The calibration of the model is quite fast: in one minute a standard pc station can calibrate the model to the complete set of the 125 CDS composing the Index. The Ornstein-Uhlenbeck model has been compared with the homogeneous and inhomogeneous Poisson models and with the Cox-Ingersoll-Ross dynamics. Results have shown that while homogeneous and inhomogeneous Poisson models fail in replicating real market structures, the CIR, Gamma-OU and IG-OU models can be nicely calibrated to market data.

We have further analyzed these three models by introducing two different approaches for the calibration. In one case (Fixed approach) the starting values for the models’ parameters are fixed at each week; in the second (Dynamic approach) the values obtained for a given week are used as initial guesses for the next week. Under both the approaches we have analyzed the performance of the three models and the stability of their parameters. Results show that the IG-OU models slightly outperforms the other two in terms of calibration capabilities. However, this models lacks of parameters stability. As opposite, the Gamma-OU model nicely matches market features and its parameters are also quite stable.

The choice between the Fixed or Dynamic approach shall be driven by the objective of the analysis. If the model is used within an hedging strategy, stable parameters are preferable and the Dynamic approach provides the modeler with much appropriate results. On the other hand, when the aim of the calibration is a spot replication of market data, the Fixed approach provides the user with fits which are much closer to market observations.

Finally, we have briefly discussed a possible extension of the model to multivariate setting. We have assumed that each asset is characterized by default intensity following a Ornstein-Uhlenbeck process. Dependence among default is achieved by time changing the corresponding Cox processes with a common subordinator. The formulation is detailed for the case of the Gamma subordinator.
References


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Table 1: Examples of calibrated CDS term structures versus market data (in bp). The corresponding *ape* and *arpe* are also reported.
Table 2: Parameters for the examples presented in Table 1.

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<th>Dynamic</th>
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<td>$\lambda = 0.0055$</td>
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<td>$\lambda = \text{piecewise constant}$</td>
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<tr>
<td>Continental</td>
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Table 3: Mean $ape$ and $arpe$ for distributions in Figure 3 and 4.

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Figure 1: Gamma-OU (top plot) and an Inverse Gaussian-OU (bottom plot) simulated sample paths.
Figure 2: Term structures (left plots) and default probabilities (right plots) for the five models.
Figure 3: Distributions of the ape for the CIR (first row), Gamma-OU (second row) and IG-OU models. Left plots show results for the Fixed approach; right plots refer to the Dynamic approach.
Figure 4: Distributions of the \( arpe \) for the CIR (first row), Gamma-OU (second row) and IG-OU models. Left plots show results for the \textit{Fixed} approach; right plots refer to the \textit{Dynamic} approach.
Figure 5: Behavior of the optimal parameters as a function of time for Zurich Insurance: CIR (left column plots), Gamma-OU (central column plots), and IG-OU (right column plots).
Figure 6: Behavior of the optimal parameters as a function of time for Continental: CIR (left column plots), Gamma-OU (central column plots), and IG-OU (right column plots).
Figure 7: Overall behavior of the optimal parameters as a function of time for the CIR model.
Figure 8: Overall behavior of the optimal parameters as a function of time for the Gamma-OU model.
Figure 9: Overall behavior of the optimal parameters as a function of time for the IG-OU model.
Figure 10: Lag 1 autocorrelation distributions for CIR (left column), Gamma-OU (central column) and IG-OU (right column).
Figure 11: Autocorrelation behavior for each model and each parameter as a function of the first five time-lags.