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Strategic stability of equilibria:
the missing paragraph

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Abstract

This paper introduces two set valued Nash equilibrium refinements that are a natural generalization of the concept of stable set of equilibria introduced in Kohlberg and Mertens (1986) and satisfy all the properties defined in Mertens (1989). It also establishes a connection between Nash equilibrium refinements and stochastic games as a tool to define a stable set of equilibria.

Keywords: game theory, equilibrium refinements, strategic stability, stochastic games.

JEL classification: A23, C72.

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1 Introduction

This paper is intended as the missing paragraph in the seminal work of Kohlberg and Mertens (1986) that introduced the concept of strategic stability without offering a definition of stable sets of equilibria satisfying all the desiderata listed by the authors. The final formulation of stability suggests that the properties of admissibility and backwards induction might be mutually exclusive. However, a natural generalization of stable sets of equilibria, in our formulation F - stable sets, satisfies all the desirable properties proposed in Kohlberg and Mertens (1986).

The backwards induction requirement is easily satisfied by widening the set of perturbed games allowing every pure strategy to be replaced with a set of its perturbations. Admissibility is easily verified, while the player splitting property introduced in Mertens (1989) is violated. This is not surprising since every agent of a splitted player will choose his perturbed strategy independently just to maximize his individual payoff. Conversely a single player would correlate his agents mistakes in order to maximize his overall payoff. Then the tension between these two requirements seems the real knot to be untied. This is confirmed also by the discussion in Hillas (1990)¹.

This conflict is solved by the definition of G - stable sets where the collection of games exploited to identify the set of equilibria includes perturbations both of the original game and of a class of new games each obtained from the initial game by introducing additional strategies that have no impact on the set of Nash equilibria.

Finally an innovative approach is proposed revealing a link between Nash equilibrium refinements and stochastic games. Even in this setting the resulting sets of equilibria satisfy all the properties as proposed in Mertens (1989).

2 F - stable equilibria

For the sake of convenience first recall the definition of hyperstable, fully stable and stable sets of equilibria and highlight the reasons why they are discarded.

Definition 1. *S is a hyperstable set of equilibria of a game Γ if it is minimal with respect to the following property:*

S is a closed set of Nash equilibria of Γ such that, for any equivalent game,

¹For a complete discussion of both empirical and theoretical contributions see ? and ?.

and for any perturbation of the normal form of that game, there is a Nash equilibrium close to S .

Definition 2. S is a fully stable set of equilibria of a game Γ if it is minimal with respect to the following property:

S is a closed set of Nash equilibria of the game Γ satisfying: for any $\epsilon > 0$ there exists some $\delta > 0$ such that, whenever each player's strategy set is restricted to some compact convex polyhedron in the interior of the simplex at an (Hausdorff) distance less than δ from the simplex, then the resulting game has an equilibrium point ϵ -close to S .

While both hyperstable and fully stable sets of equilibria of a normal form game always contain a proper² (hence perfect and sequential in every extensive form game with that normal form) equilibrium, they might fail to satisfy admissibility since each player's strategic choice is allowed to be affected by perturbations of his own strategies. A natural way to avoid this is to perturb each pure strategy s_i of each player i in the same amount towards the same completely mixed strategy. This led to the definition of stable set of equilibria:

Definition 3. S is a stable set of equilibria of a game Γ if it is minimal with respect to the following property:

S is a closed set of Nash equilibria of the game Γ satisfying: for any $\epsilon > 0$ there exists some $\delta_0 > 0$ such that for any completely mixed strategy vector $\sigma_1 \dots \sigma_n$ (n players) and for any $\delta_1 \dots \delta_n$ ($0 < \delta_i < \delta_0$), the perturbed game where every strategy $s_i \in S_i$ of player i is replaced by $(1 - \delta_i) s_i + \delta_i \sigma_i$ has an equilibrium ϵ -close to S .

However stable sets might not satisfy the backwards induction requirement. Thus a slight variation of the definition of stable set of equilibria is proposed in order to get a new set that, at least, always includes a sequential equilibrium of the game and satisfies admissibility.

3 The model

Let $\Gamma = \{I, \{\Sigma_i\}_{i \in I}, \{u_i\}_{i \in I}\}$ be a finite n player game, where I is the finite set of players indexed by i , Σ_i is player i 's compact, convex strategy-polyhedron (in Euclidean space) being S_i his pure strategy set and u_i his multi linear payoff function defined on $\Sigma = \prod_{i \in I} \Sigma_i$.

²See Myerson (1978).

Define the set P_ϵ of perturbations η as $P_\epsilon = \{\epsilon \cdot \bar{\sigma} \mid 0 < \epsilon < 1, \bar{\sigma} \in \Sigma^k \setminus \partial \Sigma^k, k \in \mathbb{Z}\}$ where Σ^k is the Cartesian power of Σ with k integer number.

Let $\eta_i = \epsilon \cdot \bar{\sigma}_i$ be the k -dimensional vector that represents the restriction to player i 's strategies in $\Sigma_i^k \setminus \partial \Sigma_i^k$ of the perturbation $\eta = \epsilon \cdot \bar{\sigma}$. For $\eta \in P_\epsilon$ let $\tau_\eta(s_i) = (\bar{1} - \bar{\epsilon}) s_i + \eta_i$ be the k -dimensional vector of strategies replacing strategy s_i in player i 's strategy set where $\bar{1}$ is the k -dimensional vector with each entry equal to one and $\bar{\epsilon}$ the k -dimensional vector with each entry equal to ϵ . Let $\Gamma(\eta)$ be the game with compact convex strategic polyhedron $\bar{\Sigma} \subset \Sigma$ obtained from Γ by replacing each pure strategy s_i of each player i by the vector $\tau_\eta(s_i)$. The definition of stable set of equilibria is then modified accordingly to define an F - stable set:

Definition 4. *S is an F - stable set of equilibria of a game Γ if it is a set of equilibria, minimal with respect to the following property F:*

Property (F). *S is a closed, set of Nash equilibria of Γ satisfying: for any $\delta > 0$ there exists some $\epsilon_0 > 0$ such that any perturbed game $\Gamma(\eta)$ with $\eta \in P_\epsilon$ and $\epsilon_0 > \epsilon > 0$ has an equilibrium δ -close to S .*

Note that the proposed definition is halfway the definitions of fully stable and stable set of equilibria. In particular, the new collection of strategy polyhedra includes all those allowed by the definition of stable equilibria as special cases in which $k = 1$, and is a proper subset of the ones defining a fully stable set.

Proposition 1 (Existence). *Every normal form game Γ has an F - stable equilibrium.*

Existence comes easily from existence of a fully stable set of equilibria for any normal form game Γ as proved in Kohlberg and Mertens (1986) since, as already pointed out, the definition of F - stable equilibria is less restrictive than the definition of fully stable equilibria.

Proposition 2 (Invariance). *Every F - stable set is also an F - stable set of any equivalent game (i.e. having the same reduced normal form).*

From a geometrical point of view, the definition considers just polyhedra within the strategic simplex of each player i that are the convex hull of any collection of polyhedra allowed by the definition of stable set of equilibria.

This easily implies that an F - stable set of equilibria of any game Γ depends only on its reduced normal form since if a new strategy \hat{s}_i , linear combination of pure strategies in S_i , was explicitly introduced as an additional pure strategy in S_i it would be perturbed as any other pure strategy, and each strategy in $\tau_\eta(\hat{s}_i)$ would be represented by a point on a side of some polyhedron generating the convex hull.

Proposition 3 (Admissibility). *Given any equilibrium in an F -stable set, every equilibrium strategy for every player i is undominated.*

It is well known that a stable set of equilibria satisfies admissibility. Despite a larger set of perturbations is now allowed, any F -stable set still satisfies this property since the set $\bar{\sigma}$ is in the interior of Σ^k and identical for all strategies in S_i for each player i .

Given these last two properties, one can verify immediately the (i, α) -ordinality of F -stable sets, using Theorem 2 in Mertens (2003).

Proposition 4 (Connectedness). *Every F -stable set is contained in a single connected component of the set of Nash equilibria.*

Since every fully stable set includes an F -stable set, every normal form game has an F -stable set which is contained in a single connected component of the set of Nash equilibria.

Proposition 5 (Backwards induction). *An F -stable set of any finite game Γ always includes a proper equilibrium of Γ .*

PROOF. Given Γ construct a perturbed game $\tilde{\Gamma}$ as follows: first for each player $i \in I$ define the set $E_i = \{e_j^i \in S_i^n \text{ with } j = 1, \dots, n\}$ of all orderings e_j^i of his pure strategies, where S_i^n is the Cartesian power of S_i and $n = |S_i|$; second, for every player i , construct, from each ordering e_j^i , a totally mixed strategy $\sigma(e_j^i)$ such that, when $\sigma(e_j^i)$ is chosen, the first strategy in the ordering e_j^i is played with probability $(1 - \epsilon) / (1 - \epsilon^n)$, the next one with probability $\epsilon(1 - \epsilon) / (1 - \epsilon^n)$, the next with probability $\epsilon^2(1 - \epsilon) / (1 - \epsilon^n)$ and so on. Thus for each player i it has been defined a set \tilde{E}_i of $n!$ totally mixed strategies. Finally define $\tilde{\Gamma}$ as the game in which each strategy s_i of each player i is replaced by the following set of perturbed strategies:

$$\{(1 - \epsilon) s_i + \epsilon \sigma(e_j^i)\}_{\sigma(e_j^i) \in \tilde{E}_i} \quad (1)$$

Therefore each player when choosing a pure strategy in the new game $\tilde{\Gamma}$ actually chooses with probability $(1 - \epsilon)$ a strategy in his strategy set S_i in the original game and, with probability ϵ a lottery over a given ordering of his pure strategies in S_i . Therefore the set of equilibria of this new game is identical to the set of equilibria of a game in which players choose simultaneously an ordering on their pure strategies, then, for each player i , nature picks his preferred choice with probability $(1 - \epsilon) + \epsilon(1 - \epsilon) / (1 - \epsilon^n)$, the next one with probability $\epsilon^2(1 - \epsilon) / (1 - \epsilon^n)$, the next one with probability $\epsilon^3(1 - \epsilon) / (1 - \epsilon^n)$ and so on. Pick an equilibrium point of the new game in

the neighbourhood of our set of equilibria. It is an ϵ -proper equilibrium of the initial game³.

Proposition 6 (Iterated dominance and forward induction). (A) An F -stable set of a game Γ contains the F -stable set of any game obtained from Γ by deleting a dominated strategy and (B) an F -stable set of a game Γ contains the F -stable set of any game obtained from Γ by deleting a strategy that is an inferior response in all the equilibria of the set (Forward induction).

PROOF. Given a perturbation $\bar{\Gamma}(\eta)$ with $\eta \in P_\epsilon$ of the game $\bar{\Gamma}$ without the eliminated strategy s_i , construct a close-by perturbation in two steps: first introduce the eliminated strategy in the strategy set S_i of player i and perturb it like any other strategy of the game. Then construct the perturbed game $\Gamma(\eta, z)$ by slightly perturbing any of player i 's strategies towards s_i by z . The game $\Gamma(\eta, z)$ is a perturbation of the initial game. Obviously in no equilibrium the eliminated strategy will be played and taking the limit for $z \rightarrow 0$ of these equilibria will give an equilibrium of $\bar{\Gamma}(\eta)$ close to the F -stable set.

Proposition 7 (Small worlds and Decomposition). An F -stable set of any finite game Γ satisfies small worlds and decomposition axioms.

This property is self evident⁴.

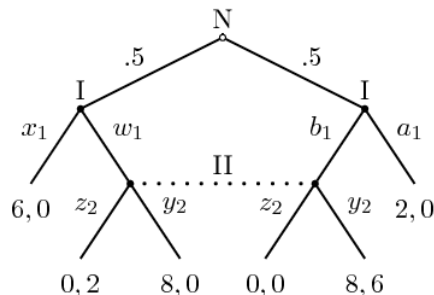
Proposition 8 (Player splitting). Given a partition of the information set of some player, such that no play intersects two different partition elements, consider the new game obtained by letting a different agent of this player manage each of these partition elements, and receive the same payoff as this player for those plays that intersect his own information sets—he receives an arbitrary payoff on the other plays. This new game, where this player is replaced by these agents, has the same stable sets as the old game.

This property is not satisfied in our model since there are some perturbations of the initial game that cannot be replicated once some player is splitted. The game represented in Figure 1 clarifies the point.

³Note that the new game $\tilde{\Gamma}$ can be seen as a game in which each strategy of each player i is a pair given by an ordering of his pure strategies and a lottery over this ordering. Each ordering is replicated $|S_i|$ times and two identical orderings still differ due to the associated lottery *i.e.* the probabilities nature will use to pick each pure strategy in the ordering. However, for any pair and any two pure strategies (s_i, \bar{s}_i) in S_i with s_i ranked before \bar{s}_i , nature will always pick s_i with probability $\sigma(s_i)$ and \bar{s}_i with probability $\sigma(\bar{s}_i)$ such that $\epsilon\sigma(s_i) \geq \sigma(\bar{s}_i)$.

⁴For a complete discussion of this property see Mertens (1992).

Figure 1: Game A - Extensive form



Source: "The analysis of conflict" Roger B. Myerson

Consider only two perturbations $(\hat{\sigma}_1, \tilde{\sigma}_1)$ of player 1's strategies in Figure 2 and redefine his strategy set as:

$$\dot{S}_1 = \{((1 - \epsilon) s + \epsilon \hat{\sigma}_1) \cup ((1 - \epsilon) s + \epsilon \tilde{\sigma}_1)\}_{s \in S_1}$$

Figure 2: Game A - Normal form

	$a_1 w_1$	$a_1 x_1$	$b_1 w_1$	$b_1 x_1$
y_2	5, 0	4, 0	8, 3	7, 3
z_2	1, 1	4, 0	0, 1	3, 0

Source: "The analysis of conflict" Roger B. Myerson

Note that this set of perturbations cannot be replicated when player 1 is replaced by his two agents since if agents' strategies in $S_{11} = (x_1, y_1)$ and $S_{12} = (z_1, w_1)$ were perturbed in order to replicate the marginal probabilities induced by $\hat{\sigma}_1$ and $\tilde{\sigma}_1$ the combination of these perturbations would lead to four distinct perturbations of each pure strategy of player 1. This is because agents choose independently their perturbed strategies while player 1, in making his strategic choice, induces an obvious correlation between the perturbations of his agents' strategies.

Therefore, while the set of perturbed games has to be enlarged to satisfy the property of backward induction, when this happens in a natural way by replacing every pure strategy with a set of its perturbations, the player splitting property is violated.

This approach and results, while reached independently, are analogue to the ones proposed in an unpublished paper by Reny⁵.

⁵Prof. P. Reny private communication.

3.1 G - stable equilibria

A second way to widen stable sets to include at least one proper equilibrium, is implicitly offered by Proposition 6 (Iterated dominance and Forward induction). Note first that the proof of Proposition 6 proposed in Kohlberg and Mertens (1986) can be easily extended to strategies that are either weakly dominated or never part of a Nash equilibrium. Thus, given the initial game Γ , create for each player i a new strategy set \bar{S}_i by adding to S_i a finite non-negative number of new strategies that are either (weakly) dominated or never played with strictly positive probability in any Nash equilibrium of the resulting game. Given $\{\bar{S}_i\}_{i \in I}$ with $S_i \subseteq \bar{S}_i$ for every $i \in I$ set, for each player $j \in I$ and each new strategy profile $\bar{s}_{-j} \in \bar{S}_{-j} \setminus S_{-j}$, a strategy $\bar{\sigma}_{-j} \in \Sigma_{-j}$ such that:

$$u_j(\bar{s}_i, s_j, \bar{s}_{-ij}) = u_j(\bar{\sigma}_i, s_j, \bar{\sigma}_{-ij}) \quad \text{for } \forall s_j \in S_j \quad \text{and} \quad \forall j \in I \quad (2)$$

If $S_j \subset \bar{S}_j$ define for every new strategy $\bar{s}_j \in \bar{S}_j \setminus S_j$ the corresponding payoffs:

$$u_j(s_i, \bar{s}_j, s_{-ij}) = \alpha u_j(\sigma_j, s_{-j}) + k \quad \text{with } \alpha, k \in \mathbb{R} \quad \text{and} \quad \sigma_j \in \Delta S_j \quad (3)$$

Condition (2) excludes that a (weakly) dominated strategy $s_j \in S_j$ for player j could become undominated given the introduction of a new strategy for any of his opponents. The generalization of Proposition 6 in Kohlberg and Mertens (1986) ensures that a stable set of any new game $\bar{\Gamma}$ always includes a stable set of the original game Γ . Equipped with this enlarged set of games, a new definition of stable sets is proposed:

Definition 5. *S is a G - stable set of equilibria of a game Γ if it is minimal with respect to the following property G:*

Property (G). *S is a closed set of Nash equilibria of the game Γ satisfying: for any game $\bar{\Gamma}$ obtained from Γ by adding to each S_i a finite non-negative number of new strategies that are either (weakly) dominated or never played with strictly positive probability in any Nash equilibrium of $\bar{\Gamma}$, and for any $\epsilon > 0$ there exists some $\delta_0 > 0$ such that for any completely mixed strategy vector $\sigma_1 \dots \sigma_n$ (n players) and for any $\delta_1 \dots \delta_n$ ($0 < \delta_i < \delta_0$), the perturbed game where every strategy $s_i \in \bar{S}_i$ of player i is replaced by $(1 - \delta_i) s_i + \delta_i \sigma_i$ has an equilibrium ϵ -close to S .*

Proposition 9 (Existence). *Every normal form game Γ has a G - stable set of equilibria.*

Existence of G - stable sets comes easily from existence of stable sets of equilibria for any normal form game as proved in Kohlberg and Mertens (1986) since, by construction, the set of Nash equilibria is identical for any considered game.

Proposition 10 (Invariance). *Every G -stable set is also a G -stable set of any equivalent game (i.e. having the same reduced normal form).*

The proof of invariance for G -stable sets is almost identical to the one proposed for stable sets and F -stable sets. Note that a randomly redundant strategy \dot{s}_i for player i would not change the set of new games since any new strategy profile $\bar{s}_{-j} \in \bar{S}_{-j} \setminus S_{-j}$, is payoff equivalent for player j to any (mixed) strategy $\bar{\sigma}_{-j}$ within Σ_{-j} that is unaffected by the introduction of \dot{s}_i .

Proposition 11 (Admissibility). *Given any equilibrium in a G -stable set, every equilibrium strategy for every player i is undominated.*

For this property to be verified, condition (2) is crucial since, as already pointed out, it ensures that a (weakly) dominated strategy for player j could not be made undominated by the introduction of a new strategy for some player i . Once excluded this possibility the property comes easily following the proof outlined for stable and F -stable equilibria.

Proposition 12 (Backwards induction). *A G -stable set of any finite game Γ always includes an equilibrium payoff equivalent to a sequential equilibrium of Γ .*

Suppose not, then consider any extensive form representation Γ^ϵ of the initial game Γ and let σ^ϵ be an ϵ -approximation of an equilibrium σ in the G -stable set, x^ϵ the (n -tuple of) behavioural strategies equivalent to σ^ϵ and μ^ϵ the vector of conditional probabilities that they imply on the information sets. Extract a subsequence along which all these objects converge. Since σ is not a sequential equilibrium, there exists a subgame $\tilde{\Gamma}^\epsilon$ out of the equilibrium path and a player i not entering the subgame and whose strategy x_i^ϵ is not optimal within $\tilde{\Gamma}^\epsilon$. Let $\tilde{\Gamma} = \left\{ \tilde{I}, \left\{ \tilde{S}_i \right\}_{i \in \tilde{I}}, \left\{ u_i \right\}_{i \in \tilde{I}} \right\}$ be the normal form subgame, as defined in Mailath and Swinkels (1993), corresponding to $\tilde{\Gamma}^\epsilon$ and \tilde{W}_i the set of all possible orderings ω_i of player i 's strategies in \tilde{S}_i ; each strategy in a selected ordering is played with probability $\frac{(1-\epsilon)\epsilon^{k-1}}{1-\epsilon^K}$ being k its position in the ordering and $K = |\tilde{S}_i|$. Given an ordering w_i , each strategy \bar{s}_i is played with strictly positive probability in σ_i , introduce a new (behavioural) strategy s_{ω_i, \bar{s}_i} for player i in $\tilde{\Gamma}^\epsilon$ such that:

$$u_i(s_{\omega_i, \bar{s}_i}, s_{-i}) = u_i(\bar{s}_i, s_{-i}) + \epsilon(1 + \epsilon u_i(\omega_i, s_{-i})) \quad (4)$$

For any other player $j \neq i$, each new strategy s_{ω_i, \bar{s}_i} is payoff equivalent to the mixed strategy obtained from \bar{s}_i by allowing the corresponding ordering ω_i to be played in the subgame $\tilde{\Gamma}$ with strictly positive probability ϵ . In the

new game, given σ_{-i} , player i will enter the subgame $\tilde{\Gamma}$ and will always play a best reply given his opponents' strategies.

Since, by assumption, in every sequential equilibrium strategy σ_i is not played, for an appropriate choice of $\epsilon > 0$ all the new strategies will be never be part of any equilibrium of the new game.

Note that, given the proposed setting, it cannot be proved that a G - stable set always contains a proper equilibrium.

To clarify the point consider the two player game represented in Figure 3: The unique sequential and proper equilibrium $([T]; \frac{1}{2}[L] + \frac{1}{2}[R])$ cannot

Figure 3: Game B

	L	R
T	2, 2	2, 2
M	1, 0	0, 1
B	0, 1	1, 0

be included in a G - stable set by introducing strategies that are not played with strictly positive probability in equilibrium.

Figure 4: Game C_1

	s_1^1	s_1^2
s_2^1	3, 1	0, 0
s_2^2	0, 0	3, 1

Figure 5: Game $C_{1.1}$ and Game $C_{1.2}$

	s_1^1	s_1^2		s_1^1	s_1^2
s_2^1	3, 1	0, 0	s_2^1	3, 1	0, 0
s_2^2	0, 0	3, 1	s_2^2	0, 0	3, 1
s_2^3	4, 0	0, 1	s_2^3	0, 1	4, 0

The drawback of G - stable sets is their dimension since they might be not even included in a connected component of equilibria: given game C_1 in Figure 4, consider the games $C_{1.1}$ and $C_{1.2}$ in Figure 5 each obtained by

including in S_2 a new strategy s_2^3 that is never part of an equilibrium and is not an inferior response in one stable set of the initial game. Then both equilibria $[s_1^1, s_2^1]$ and $[s_1^2, s_2^2]$ should be included in a G - stable set of C_1 . Therefore the definition of G - stable equilibria has to be reformulated in order to include just games with strategy sets $\{\bar{S}_i\}_{i \in I}$ where $S_i \subseteq \bar{S}_i$ and each strategy \bar{s}_i in $\bar{S}_i \setminus S_i$ is either dominated or an affine combination of pure strategies in S_i with $\bar{s}_i = (1 \pm \epsilon) s_i \pm \epsilon \sigma_i$ where $s_i \in S_i$, $\sigma_i \in \Sigma_i \setminus \partial \Sigma_i$ and $\epsilon \in \mathbb{R}$ arbitrarily small not necessarily different from zero.

Finally, redefine condition (2) and (3) for every player $j \in I$ as:

$$u_j(\bar{s}_i, s_j, \bar{s}_{-ij}) = u_j(\bar{\sigma}_i, s_j, \bar{\sigma}_{-ij}) \quad \text{for } \forall s_j \in \bar{S}_j \quad \text{and} \quad \forall \bar{s}_{-j} \in \bar{S}_{-j} \setminus S_{-j} \quad (5)$$

where $\bar{\sigma}_{-j}$ belongs to ΔS_{-j} and $\lim_{\epsilon \rightarrow 0} \bar{\sigma}_i = \lim_{\epsilon \rightarrow 0} \bar{s}_i$ for every player $i \neq j$.

Given this setting both the definition of G - stable equilibria and all the outlined poofs can be easily replicated. Note however that the introduction of strategies that can be part of a Nash equilibrium is now allowed. This allows the proof of backwards induction property to be extended to ensure that a G - stable set always includes a sequential equilibrium.

Finally, reconsider the player splitting property: it is easily verified in the setting of stable sets of equilibria as defined in Kohlberg and Mertens (1986) as proved in Mertens (1989). Therefore it has to be evaluated just the effect of the introduction of additional strategies. If player i had a unique agent k the proof would be immediate. Thus consider $k \geq 2$: in the agent normal form, add a new strategy $\tilde{s}_{i,k} = (1 \pm \epsilon) s_{i,k} \pm \epsilon \sigma_{i,k}$ for each agent k of player i with $k = 1, 2, \dots, n$. Assume that any strategy profile $\{\bar{s}_{i,k}\}_{k=1}^n \neq \{\tilde{s}_{i,k}\}_{k=1}^n$ including at least one new strategy is payoff equivalent to $\lim_{\epsilon \rightarrow 0} \{\bar{s}_{i,k}\}_{k=1}^n$ for any player $j \neq i$ while each strategy profile $(s_{-ij}, \{\tilde{s}_{i,k}\}_{k=1}^n)$ is payoff equivalent to some behavioural strategy $\tilde{\sigma}_{-j} \in \Sigma_{-j}$ depending on s_{-ij} with $\lim_{\epsilon \rightarrow 0} (\tilde{\sigma}_{-ij}, \{\tilde{\sigma}_{i,k}\}_{k=1}^n) = \left\{ s_{-ij}, \lim_{\epsilon \rightarrow 0} \{\tilde{s}_{i,k}\}_{k=1}^n \right\}$. Note that when some agent k' of player i chooses a new strategy $\tilde{s}_{i,k}$ the payoff to any other agent k' is irrelevant given that his strategic choices depend only on the strategies of players $j \neq i$.

Therefore even if in the agent normal form game the introduction of a new strategy for each agent k of player i leads to more than a unique new strategy for player i , the resulting game is strategically equivalent to a game obtained from Γ by including just one new strategy for player i .

Conversely if any new strategy were added to the strategy set of an agent the resulting new strategy profiles could be easily added to the strategy set of player i .

4 Stability and stochastic games

Given the initial game $\Gamma = \{I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}\}$, consider a collection P_ϵ of perturbed games Γ_ϵ derived from Γ by replacing each strategy $s_i \in S_i$ of each player i by $(1 - \epsilon_i) s_i + \epsilon_i \sigma_i$ with $\epsilon_i > 0$ for every i and $\sigma_i \in \Sigma_i \setminus \partial \Sigma_i$ completely mixed strategy. Let $\tilde{\Gamma}$ be an infinite stochastic⁶ game such that, at every stage, the game stops with probability $(1 - \delta)$ with $\delta > 0$ arbitrarily small. At the first stage Γ is played and, at every following stage $k > 1$, a perturbed game Γ_ϵ^k in P_ϵ occurs. The completely mixed strategy profile $\{\hat{\sigma}_i^k\}_{i \in I}$ characterizing the perturbed game in P_ϵ played at stage k represents the state of nature. At every stage the completely mixed strategy σ_i^k for every player i depends just on the history h_i^k defined as $h_i^k = \{v_{\sigma_{-i}^1}, v_{\sigma_{-i}^2}, \dots, v_{\sigma_{-i}^{k-1}}\}$ where σ_{-i}^h represents the strategy profile played at stage h by player i 's opponents and $v_{\sigma_{-i}^h} = BR_i(\sigma_{-i}^h)$ ⁷.

Thus, any equilibrium of the stochastic game $\tilde{\Gamma}$ is characterized, at every stage k , by a Nash equilibrium $\tilde{\sigma}^k$ of the corresponding game Γ_ϵ^k since each player always chooses a best response to his opponents' strategies: even if a not best response strategy might induce more favourable future states of nature, every future game has a vanishing probability not higher than δ .

Within the outlined class, consider just stochastic games with at least one quasi-absorbing state of nature and whose states of nature $\{\hat{\sigma}_i^k\}_{i \in I}$ for every player i at each stage k depend uniquely on $\{v_{\sigma_{-i}^{k-1}}\}_{i \in I}$ according to a set of bijective functions $F_i : \Delta S_i \rightarrow \Sigma_i \setminus \partial \Sigma_i$. At stage k a state of nature $\{\hat{\sigma}_i^k\}_{i \in I}$ is quasi-absorbing if it could characterize every following stage in an equilibrium of the stochastic game.

Given the infinite sequence of pairs $\{\Gamma_\epsilon^k, \tilde{\sigma}^k\}_{k=1}^\infty$ for each stochastic game, redefine the stable sets of equilibria as follows:

Definition 6. *S is an S -stable set of equilibria of a game Γ if it is a set of equilibria, minimal with respect to the following property S :*

Property (S). *S is a closed set of Nash equilibria of Γ satisfying: for every $\lambda > 0$ there exists some collection $\{\epsilon_i\}_{i \in I}$ such that for any stochastic game there is a convergent sequence $\{\Gamma_\epsilon^k, \tilde{\sigma}^k\}_{k=1}^\infty$ with limit point λ -close to $\{\Gamma, S\}$.*

Proposition 13 (Existence). *Every normal form game Γ has an S -stable equilibrium.*

⁶See Neyman and Sorin (2003).

⁷Every perturbed game allowed by the definition of stable sets in Kohlberg and Mertens (1986) is just a special case of the stochastic game $\tilde{\Gamma}$ in which the state of nature is unique and independent of the history.

Existence of an S-stable set of equilibria for any normal form game Γ is implied by the existence of a hyperstable set of equilibria as proved by Kohlberg and Mertens (1986): the limit of every convergent sequence of equilibria $\{\tilde{\sigma}^k\}_{k=1}^{\infty}$ is just an equilibrium of a perturbed game. Besides, by construction every stochastic game has a quasi absorbing state hence admits a convergent sequence.

The properties of admissibility, invariance, iterated dominance and forward induction come immediately.

First, the limit of every convergent sequence is an equilibrium and a completely mixed strategy profile; hence admissibility is easily satisfied. Second the introduction of a randomly redundant strategy is immaterial since at every stage $k > 1$ all strategies are identically perturbed. Moreover this variation does not modify the sets of best replies that determine the history h_i^k up to any stage.

Proposition 14 (Backwards induction). *A S-stable set of any finite game Γ always includes a proper, hence sequential, equilibrium of Γ .*

Consider the initial game Γ and define for every player i the set \hat{S}_i of all possible orderings over his pure strategies in S_i . Given an ordering each strategy is played with probability $\frac{(1-\epsilon)\epsilon^{n-1}}{1-\epsilon^N}$ being n its position in the ordering and $N = |S_i|$. Therefore each ordering corresponds to a unique completely mixed strategy. Finally assume that, at every stage $k > 1$, the perturbed strategy $\hat{\sigma}_i^k$ for every player i is a (mixed) ordering of his strategies that corresponds to a best response to his opponents' strategies at stage $k - 1$. Then any convergent sequence of equilibria $\{\tilde{\sigma}^k\}_{k=1}^{\infty}$ of the resulting stochastic game $\tilde{\Gamma}$ has an ϵ -proper equilibrium as its limit point. Conversely any sequence $\{\tilde{\sigma}^k\}_{k=1}^{\infty}$ with $\tilde{\sigma}^k$ identically equal, for every $k > 1$, to a given ϵ -proper equilibrium $\tilde{\sigma}^\epsilon$ of the initial game Γ converges and $\{BR_i(\tilde{\sigma}_{-i}^\epsilon)\}_{i \in I}$ characterize a quasi absorbing state of nature.

Proposition 15 (Player splitting). *Given a partition of the information set of some player, such that no play intersects two different partition elements, consider the new game obtained by letting a different agent of this player manage each of these partition elements, and receive the same payoff as this player for those plays that intersect his own information sets-he receives an arbitrary payoff on the other plays. This new game, where this player is replaced by these agents, has the same stable sets as the old game.*

Given the initial game Γ consider the new game ${}^a\Gamma$ where player i is replaced by the set of his agents indexed by k with $k = 1, 2, \dots, n$. At every stage $k > 1$ the state of nature characterizing Γ_ϵ^k can be easily replicated

in ${}^a\Gamma_\epsilon^k$: the main difference between the two games is that a single player will tremble in a completely correlated way while, once splitted, his agents tremble independently. However since only one partition of the game is going to occur, only the marginal probabilities matter. This implies that the histories themselves of both games can be overlapped: since every agent manage just one element of the realized partition of the information set of player i , his best response will depend just on σ_{-i}^k ; not on the other agents' strategies.

Finally, the small worlds and decomposition axioms come easily provided that the history is defined as a sequence of best replies. If the strategies of a set of players do not affect a player's payoffs a fortiori they won't affect his best replies.

5 Conclusions

The paper introduces two major contributions: first the class of perturbed games is widen by extending the concept of perturbations to affine combinations of pure strategies. Loosely speaking an equivalence relation is introduced so that two perturbed games are equivalent if and only if their limit game is identical. Second a link with stochastic games seem to emerge quite naturally. This approach can be regarded as a dynamic version of the definition of stable sets as proposed in Kohlberg and Mertens (1986). It remains unclear if the introduction of new properties might lead to prefer this second approach with respect to G - stable equilibria or M - stable equilibria proposed by Mertens (1989). A first insight could be offered by determining the geometric relation between the different sets following the contribution by Govindan (1995).

Appendix A: Gul example

The three player game proposed by Gul where Player 1 starts by either taking an outside option $[s_1^1]$ which yields payoffs $(2, 0, 0)$ or moving into a simultaneous move subgame represented by Figure 6 where each of the three players has two choices.

It is well known that this game admits a unique sequential equilibrium $\sigma^* = \{\frac{1}{2} [s_1^2] + \frac{1}{2} [s_1^3]; \frac{1}{2} [s_2^1] + \frac{1}{2} [s_2^2]; \frac{1}{2} [s_3^1] + \frac{1}{2} [s_3^2]\}$; however, there exists a stable set of equilibria $\{[s_1^1]; [s_2^1]; [s_3^1]\} \cup \{[s_1^1]; [s_2^2]; [s_3^2]\}$ that doesn't contain it.

The sequential equilibrium is preferred by player 1 to any scenario within the stable set. Despite player 1 moves first there is no chance for him to

Figure 6: Simultaneous move sub-game

	s_3^1	s_3^2	
s_2^1	0, 3, 3	5, 1, 1	
s_2^2	5, 1, 1	1, 0, 0	
			s_1^2

	s_3^1	s_3^2	
s_2^1	1, 0, 0	5, 1, 1	
s_2^2	5, 1, 1	0, 3, 3	
			s_1^3

induce σ^* : even if the equilibrium strategy $\{\frac{1}{2}[s_1^2] + \frac{1}{2}[s_1^3]\}$ were played within the subgame reached with vanishing probability ϵ , both player 2 and player 3 would prefer to play, respectively, strategy s_2^1 and strategy s_3^1 . Since, given the definition of stable sets, player 1 has no chance to change the perturbation of his pure strategies, he will confirm the choice of $(1 - \epsilon)[s_1^1] + \epsilon(\frac{1}{2}[s_1^2] + \frac{1}{2}[s_1^3])$.

Consider the introduction of a dominated strategy s_1^4 for player 1. Equation (6) implies that strategy s_1^4 is strictly dominated by the outside option. Second, equations from (7) to (10) define the new payoffs given the dominated strategy s_1^4 . As an example, equation (7) implies that the strategy profile (s_1^4, s_3^1) corresponds for player 2 to the strategy profile (s_2^1, s_3^1) in the original game.

Figure 7: Dominated strategy for Player 1

	s_3^1	s_3^2	
s_2^1	2 - ϵ , 2, 0	2 - ϵ , 0, 0	
s_2^2	2 - ϵ , 0, 0	2 - ϵ , 0, 9	
			s_1^4

$$u_1(s_1^4, s_{-1}) = u_i(s_1^1, s_{-1}) - \epsilon \quad \text{for } \forall s_{-1} \in S_{-1} \quad \epsilon > 0 \quad (6)$$

$$(s_1^4, s_3^1) = (s_2^1, s_3^1) \quad (7)$$

$$(s_1^4, s_3^2) = (s_3^1, s_3^2) \quad (8)$$

$$(s_1^4, s_2^1) = (s_2^1, s_2^1) \quad (9)$$

$$(s_1^4, s_2^2) = (s_3^1, s_2^2) \quad (10)$$

Therefore given s_1^4 , if player 3 played s_3^1 then player 2 would play $s_2^1 = BR_2(s_2^1, s_3^1)$ with associated payoff 3; however, if player 2 played s_2^1 then

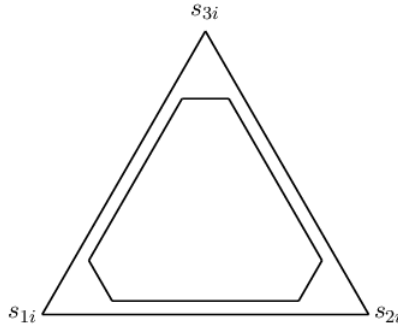
player 3 would play s_2^3 with associated payoff 1 that is what he would get in the original subgame if, once given the strategy profile (s_3^1, s_2^1) player 1 could freely choose between s_1^2 and s_1^3 . In other terms the dominated strategy mimics what would happen in the original game if player 1's outside strategy s_1^1 were replaced by an n-tuple of differently perturbed strategies as in the approach characterizing F - stable equilibria.

Appendix B

The proof of the property of backwards induction of F - stable equilibria is here graphically illustrated for a specific class of games.

Let $\Gamma = \{I, \{\Sigma_i\}_{i \in I}, \{u_i\}_{i \in I}\}$ be a finite n player game, where I is the set of players indexed by i , Σ_i player i 's compact, convex strategy-polyhedron (in Euclidean space) being S_i his pure strategy set where $S_i = \{s_{1i}, s_{2i}, s_{3i}\}$ for every player i .

Figure 8: Proper equilibria perturbation



Consider the perturbed game $\tilde{\Gamma}$ where the strategy simplex Σ_i of each player i is replaced by the polyhedron $\tilde{\Sigma}_i$ within Σ_i represented in Figure 8 and defined as follows: given a player i 's pure strategy, *e.g.* s_{1i} , replace it with two new strategies each represented by a distinct lottery. Each lottery picks that strategy s_{1i} with probability $\frac{1-\epsilon}{1-\epsilon^n}$. Strategy s_{2i} is chosen with probability $\epsilon \frac{1-\epsilon}{1-\epsilon^n}$ in the first lottery (resp. $\epsilon^2 \frac{1-\epsilon}{1-\epsilon^n}$ in the second lottery) and strategy s_{3i} with probability $\epsilon^2 \frac{1-\epsilon}{1-\epsilon^n}$ (resp. $\epsilon \frac{1-\epsilon}{1-\epsilon^n}$ in the second lottery). It is well known that any equilibrium of $\tilde{\Gamma}$ is an ϵ -proper equilibria.

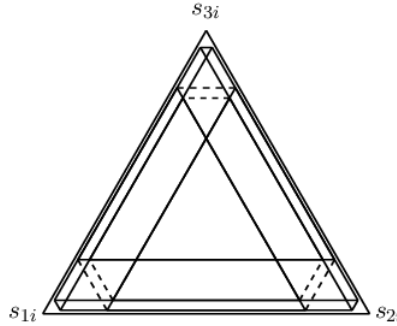
On the other hand, in the proof of the property of backwards induction, each strategy is assumed to be replaced by the six lotteries represented by the scheme in figure 9 again for strategy s_{1i} . Each player chooses a strategy and an ordering over his pure strategies. Nature picks player's favourite strategy

Figure 9: New strategy set

Lottery probabilities	Strategy orderings
$1 - \delta$	$s_{1i} \ s_{1i} \ s_{1i} \ s_{1i} \ s_{1i} \ s_{1i}$
$\delta \left(\frac{1-\delta}{1-\delta^3} \right)$	$s_{1i} \ s_{1i} \ s_{2i} \ s_{2i} \ s_{3i} \ s_{3i}$
$\delta^2 \left(\frac{1-\delta}{1-\delta^3} \right)$	$s_{2i} \ s_{3i} \ s_{1i} \ s_{3i} \ s_{1i} \ s_{2i}$
$\delta^3 \left(\frac{1-\delta}{1-\delta^3} \right)$	$s_{3i} \ s_{2i} \ s_{3i} \ s_{1i} \ s_{2i} \ s_{1i}$

s_{1i} with probability $1 - \delta$, and, with probability δ , a nested lottery defined as follows: given the chosen ordering the first ranked strategy is chosen with probability $\frac{1-\delta}{1-\delta^3}$ the second ranked strategy with probability $\delta \frac{1-\delta}{1-\delta^3}$ and the last strategy with probability $\delta^2 \frac{1-\delta}{1-\delta^3}$.

Figure 10: F-stable perturbation



Player i 's new strategy polyhedron within Σ_i is represented in Figure 8 and, for an appropriate choice of δ , is identical to the one represented in Figure 10. Hence the result.

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