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We consider nonlinear Choquard equation $-\Delta u + Vu = (I_\alpha * |u|^{(\alpha/N + 1)})|u|^{(\alpha/N - 1)} u$ where $N \geq 3$, $V \in L^\infty(\mathbb{R}^N)$ is an external potential and $I_\alpha(x)$ is the Riesz potential of order $\alpha \in (0, N)$. The power in the nonlocal part of the equation is critical with respect to the Hardy–Littlewood–Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if $\liminf_{r \to \infty} (1-V(x))|x|^2 > N^2(N-2)/(4(N+1))$ then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis–Lieb lemma.

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GROUNDSTATES OF NONLINEAR CHOQUARD EQUATIONS: HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENT

VITALY MOROZ AND JEAN VAN SCHAFTINGEN

Abstract. We consider nonlinear Choquard equation

$$-\Delta u + Vu = (I_\alpha * |u|^{p+1})|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $V \in L^\infty(\mathbb{R}^N)$ is an external potential and $I_\alpha(x)$ is the Riesz potential of order $\alpha \in (0, N)$. The power $\frac{N}{2} + 1$ in the nonlocal part of the equation is critical with respect to the Hardy-Littlewood-Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if

$$\liminf_{|x| \to \infty} (1 - V(x))|x|^2 > \frac{N(N-2)}{4(N+1)},$$

then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis-Lieb lemma.

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1. Introduction and results

We consider a nonlinear Choquard type equation

$$(P) \quad -\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \in \mathbb{N}$, $\alpha \in (0, N)$, $p > 1$, $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$ defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^{\alpha} \pi^{N/2} \Gamma(\frac{\alpha}{2})}|x|^{N-\alpha},$$

and $V \in L^\infty(\mathbb{R}^N)$ is an external potential.

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For $N = 3$, $\alpha = 2$ and $p = 2$ equation $\mathcal{P}$ is the Choquard-Pekar equation which goes back to the 1954’s work by S. I. Pekar on quantum theory of a Polaron at rest [6] Section 2.1; [20] and to 1976’s model of P. Choquard of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [8]. In the 1990’s the same equation reemerged as a model of self-gravitating matter [7,19] and is known in that context as the Schrödinger-Newton equation.

Mathematically, the existence and qualitative properties of solutions of Choquard equation $\mathcal{P}$ have been studied for a few decades by variational methods, see [8; 11; 12 Chapter III: 14] for earlier and [2–5,13;16–18] for recent work on the problem and further references therein.

The following sharp characterisation of the existence and nonexistence of nontrivial solutions of $\mathcal{P}$ in the case of constant potential $V$ can be found in [16].

**Theorem 1** (Ground states of $\mathcal{P}$ with constant potential [16 theorems 1 and 2]). Assume that $V \equiv 1$. Then $\mathcal{P}$ has a nontrivial solution $u \in H^1(\mathbb{R}^N) \cap L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ with $\nabla u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^{\frac{2N}{N-\alpha}}_{\text{loc}}(\mathbb{R}^N)$ if and only if $p \in \left(\frac{\alpha}{N} + 1, \frac{N + \alpha}{N - 2 \alpha}\right]$.

If $p \in \left(\frac{\alpha}{N} + 1, \frac{N + \alpha}{N - 2 \alpha}\right]$ then $H^1(\mathbb{R}^N) \subset L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ by the Sobolev inequality, and moreover, every $H^1$-solution of $\mathcal{P}$ belongs to $W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for any $p \geq 1$ by a regularity result in [17] proposition 3.1. This implies that the Choquard equation $\mathcal{P}$ with a positive constant potential has no $H^1$-solutions at the end-points of the above existence interval.

In this note we are interested in the existence and nonexistence of solutions to $\mathcal{P}$ with nonconstant potential $V$ at the lower critical exponent $p = \frac{\alpha}{N} + 1$, that is, we consider the problem

$$(\mathcal{P}_*) \quad -\Delta u + Vu = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^\frac{\alpha}{N}-1 \quad \text{in } \mathbb{R}^N.$$  

The exponent $\frac{\alpha}{N} + 1$ is critical with respect to the Hardy-Littlewood-Sobolev inequality, which we recall here in a form of minimization problem

$$c_\infty = \inf \left\{ \int_{\mathbb{R}^N} |u|^2 \mid u \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1 \right\} > 0.$$  

**Theorem 2** (Optimal Hardy-Littlewood-Sobolev inequality [9 theorem 3.1; 10 theorem 4.3]). The infimum $c_\infty$ is achieved if and only if

$$u(x) = C \left( \frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{N/2},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $\lambda \in (0, \infty)$ are parameters.

The form of minimizers in theorem 2 suggests that a loss of compactness in $\mathcal{P}_*$ may occur by translations and dilations.

In order to characterise the existence of nontrivial solutions for the lower critical Choquard equation $\mathcal{P}_*$ we define the critical level

$$c_* = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \mid u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1 \right\}.$$
It can be checked directly that if \( u \in H^1(\mathbb{R}^N) \) achieves the infimum \( c_* \), then a multiple of the minimizer \( u \) is a weak solution of Choquard equation \((P)\).

Using a Brezis-Lieb type lemma for Riesz potentials \([16, \text{lemma 2.4}]\) and a concentration compactness argument (lemma 10), we establish our main abstract result.

**Theorem 3** (Existence of a minimizer). Assume that \( V \in L^\infty(\mathbb{R}^N) \) and
\[
\liminf_{|x| \to \infty} V(x) \geq 1.
\]
If \( c_* < c_\infty \) then the infimum \( c_* \) is achieved and every minimizing sequence for \( c_* \) up to a subsequence converges strongly in \( H^1(\mathbb{R}^N) \).

The inequality for the existence of minimizers is sharp, as shown by the following lemma for constant potentials.

**Lemma 4.** If \( V \equiv 1 \), then \( c_* = c_\infty \).

Since problem \((P)\) with \( V \equiv 1 \) has no \( H^1 \)-solutions, this shows that the strict inequality \( c_* < c_\infty \) is indeed essential for the existence of a minimizer for \( c_* \).

In fact, the strict inequality \( c_* < c_\infty \) is necessary at least for the strong convergence of all minimizing sequences.

**Proposition 5.** Let \( V \in L^\infty(\mathbb{R}^N) \). If
\[
\limsup_{|x| \to \infty} V(x) \leq 1,
\]
then
\[
c_* \leq c_\infty.
\]
In addition, if
\[
c_* = c_\infty,
\]
then there exists a minimizing sequence for \( c_* \) which converges weakly to \( 0 \) in \( H^1(\mathbb{R}^N) \).

Using Hardy-Littlewood-Sobolev minimizers \((1.1)\) as a family of test functions for \( c_* \), we establish a sufficient condition for the strict inequality \( c_* < c_\infty \).

**Theorem 6.** Let \( V \in L^\infty(\mathbb{R}^N) \). If
\[
\liminf_{|x| \to \infty} (1 - V(x)) |x|^2 > \frac{N^2(N-2)+4(N+1)}{4(N+1)},
\]
then \( c_* < c_\infty \) and hence the infimum \( c_* \) is achieved.

In particular, if \( N = 1, 2 \) then condition \((1.3)\) reduces to
\[
\limsup_{|x| \to \infty} (1 - V(x)) |x|^2 > 0,
\]
that is, the potential \( 1 - V \) should not decay to zero at infinity faster then the inverse square of \( |x| \).

Employing a version of Pohožaev identity for Choquard equation \((P)\) (see proposition \((11)\) below), we show that a certain control on the potential \( V \) is indeed necessary for the strict inequality \( c_* < c_\infty \).
Proposition 7. Let $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If
\[
\sup_{x \in \mathbb{R}^N} \left\{ \int_{\mathbb{R}^N} \frac{1}{2} (\nabla V(x) | x) \varphi(x)^2 \, dx \mid \varphi \in C^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |\nabla \varphi|^2 \leq 1 \right\} < 1,
\]
then Choquard equation (P*) does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_\text{loc}(\mathbb{R}^N)$.

In particular, combining (1.4) with Hardy’s inequality on $\mathbb{R}^N$, we obtain a simple nonexistence criterion.

Proposition 8. Let $N \geq 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If for every $x \in \mathbb{R}^N$,
\[
\sup_{x \in \mathbb{R}^N} |x|^2 (\nabla V(x) | x) < \frac{(N - 2)^2}{2},
\]
then Choquard equation (P*) does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_\text{loc}(\mathbb{R}^N)$.

For example, for $N \geq 3$ and $\mu > 0$, we consider a model equation
\[
- \Delta u + \left( 1 - \frac{\mu}{1 + |x|^2} \right) u = (I_\alpha * |u|^\frac{N+1}{N-1} |u|^\frac{N-1}{N-1} u \quad \text{in } \mathbb{R}^N.
\]
Then proposition 8 implies that (1.6) has no nontrivial solutions for $\mu < \frac{(N-2)^2}{4}$, while for $\mu > \frac{N^2(N-2)}{4(N+1)}$, assumption (1.3) is satisfied and hence (P*) admits a ground state. We note that
\[
\frac{(N-2)^2}{4} = 1 - \frac{N - 2}{N^2},
\]
so that the two bounds are asymptotically sharp when $N \to \infty$. We leave as an open question whether (1.6) admits a ground state for $\mu \in \left[ \frac{(N-2)^2}{4}, \frac{N^2(N-2)}{4(N+1)} \right]$.

We emphasise that unlike the asymptotic sufficient existence condition (1.3), nonexistence condition (1.5) is a global condition on the whole of $\mathbb{R}^N$. For example, a direct computation shows that for $a = 0$ and every $\lambda > 0$, a multiple of the Hardy-Littlewood-Sobolev minimizer (1.1) solves the equation
\[
- \Delta u + \left( 1 + \frac{N(2|x|^2 - N\lambda^2)}{(|x|^2 + \lambda^2)^2} \right) u = (I_\alpha * |u|^\frac{N+1}{N-1} |u|^\frac{N-1}{N-1} u \quad \text{in } \mathbb{R}^N.
\]
Here (1.3) fails on an annulus centered at the origin, while $V(x) > 1$ and $(\nabla V(x) | x) < 0$ for all $|x|$ sufficiently large. Moreover,
\[
\lim_{|x| \to \infty} (1 - V(x)) |x|^2 = -2N < 0 \leq \frac{N^2(N-2)}{4(N+1)}.
\]
Note that the constructed solution $u_\lambda$ satisfies
\[
\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = 0.
\]
In particular, we are unable to conclude that $c_\star < c_\infty$. We do not know whether $u_\lambda$ is a ground state of (1.17). However, if $u_\lambda$ was not a ground state, then we would have $c_\star < c_\infty$ and (1.17) would then have a ground state by theorem 2.
2. Existence of minimizers under strict inequality: proof of theorem \( \Box \)

In order to prove theorem \( \Box \) we will use a special case of the classical Brezis-Lieb lemma \([1]\) for Riesz potentials.

**Lemma 9** (Brezis-Lieb lemma for the Riesz potential \([16, \text{lemma 2.4}]\)). Let \( N \in \mathbb{N} \), \( \alpha \in (0, N) \), and \((u_n)_{n \in \mathbb{N}}\) be a bounded sequence in \( L^2(\mathbb{R}^N) \). If \( u_n \to u \) almost everywhere on \( \mathbb{R}^N \) as \( n \to \infty \), then

\[
\int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{\alpha}{\alpha + 1}) |u|^\frac{\alpha}{\alpha + 1} = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{\alpha}{\alpha + 1}) |u_n|^\frac{\alpha}{\alpha + 1}
- \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^\frac{\alpha}{\alpha + 1}) |u_n - u|^\frac{\alpha}{\alpha + 1}.
\]

Our second result is a concentration type lemma.

**Lemma 10.** Assume that \( V \in L^\infty(\mathbb{R}^N) \) and \( \liminf_{|x| \to \infty} V(x) \geq 1 \). If the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in \( L^2(\mathbb{R}^N) \) and converges in \( L^2_{\text{loc}}(\mathbb{R}^N) \) to \( u \) as \( n \to \infty \), then

\[
\int_{\mathbb{R}^N} V|u|^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} V|u_n|^2 - \int_{\mathbb{R}^N} |u_n - u|^2.
\]

**Proof.** Since the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in \( L^2(\mathbb{R}^N) \) and converges in measure to \( u \), we deduce by the Brezis-Lieb lemma \([1]\) (see also \([10, \text{theorem 1.9}]\)) that

\[
\int_{\mathbb{R}^N} V|u|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} V|u_n|^2 - \int_{\mathbb{R}^N} V|u_n - u|^2.
\]

Now, we observe that for every \( R > 0 \) and every \( n \in \mathbb{N} \),

\[
\int_{B_R} (1 - V)|u_n - u|^2 \leq \int_{B_R} (1 - V)|u_n - u|^2 + (1 - \inf_{\mathbb{R}^N \setminus B_R} V) \int_{\mathbb{R}^N} |u_n - u|^2.
\]

By the local \( L^2_{\text{loc}}(\mathbb{R}^N) \) convergence, we note that

\[
\lim_{n \to \infty} \int_{B_R} (1 - V)|u_n - u|^2 = 0.
\]

Since \( \lim_{R \to \infty} (1 - \inf_{\mathbb{R}^N \setminus B_R} V)_+ = 0 \) and \((u_n - u)_{n \in \mathbb{N}}\) is bounded in \( L^2(\mathbb{R}^N) \), we conclude that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} (1 - V)|u_n - u|^2 \leq 0;
\]

the conclusion follows. \( \square \)

**Proof of theorem \( \Box \)** Let \((u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^N)\) be a minimizing sequence for \( c_* \), that is

\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{\alpha}{\alpha + 1}) |u_n|^\frac{\alpha}{\alpha + 1} = 1
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |
abla u_n|^2 + V|u_n|^2 \to c_*.
\]

In view of our assumption \([12]\) we observe that the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in \( H^1(\mathbb{R}^N) \). So, there exists \( u \in H^1(\mathbb{R}^N) \) such that, up to a subsequence, the sequence \((u_n)_{n \in \mathbb{N}}\) converges to \( u \) weakly in \( H^1(\mathbb{R}^N) \) and, by the classical Rellich-Kondrachov
compactness theorem, strongly in $L^2_{loc}(\mathbb{R}^N)$. By the lower semi-continuity of the norm under weak convergence,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 = c_*.$$ 

and by Fatou’s lemma

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+1}{N}})|u_n|^{\frac{N}{N+1}} \leq 1.$$ 

In order to conclude, it suffices to prove that equality is achieved in the latter inequality.

We observe that by lemma 9,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{N+1}{N}})|u_n - u|^{\frac{N}{N+1}} = 1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+1}{N}})|u|^{\frac{N}{N+1}}$$

while by lemma 10 and by the lower-semicontinuity of the norm under weak convergence,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \liminf_{n \to \infty} \int_{\mathbb{R}^N} V|u_n|^2 - |u_n - u|^2$$

$$= c_* - \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n - u|^2.$$ 

(2.1)

By definition of $c_\infty$, we have

$$\int_{\mathbb{R}^N} |u_n - u|^2 \geq c_\infty \left( \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{N+1}{N}})|u_n - u|^{\frac{N}{N+1}} \right)^{\frac{N}{N+1}}.$$ 

Therefore, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq c_* - c_\infty \left( 1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+1}{N}})|u|^{\frac{N}{N+1}} \right)^{\frac{N}{N+1}}.$$ 

In view of the definition of $c_*$ this implies that

$$c_* \geq c_\infty \left( 1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+1}{N}})|u|^{\frac{N}{N+1}} \right)^{\frac{N}{N+1}} + c_* \left( \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+1}{N}})|u|^{\frac{N}{N+1}} \right)^{\frac{N}{N+1}}.$$ 

Since by assumption $c_* < c_\infty$, we conclude that

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+1}{N}})|u|^{\frac{N}{N+1}} = 1,$$

and hence, by definition of $c_*$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 = c_*,$$

that is the infimum $c_*$ is achieved at $u$. Moreover, from (2.1) we conclude that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Since $V \in L^\infty(\mathbb{R}^N)$, this implies that $Vu_n \to Vu$ in $L^2(\mathbb{R}^N)$. Using (2.1) again, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2.$$ 

Since $(u_n)_{n \in \mathbb{N}}$ converges to $u$ weakly in $H^1(\mathbb{R}^N)$, this implies that $(u_n)_{n \in \mathbb{N}}$ also converges to $u$ strongly in $H^1(\mathbb{R}^N)$. □
3. Optimality of the strict inequality

In this section we prove lemma 4 and proposition 5.

Proof of lemma 4. Let us denote by $\tilde{c}_\infty$ the infimum on the right-hand side. By density of the space $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ and by continuity in $L^2$ of the integral functionals involved in the definition of $c_\infty$, it is clear that $\tilde{c}_\infty \geq c_\infty$. We choose now $u \in H^1(\mathbb{R}^N)$ and define for every $\lambda > 0$ the function $u_\lambda \in H^1(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$ by

$$u_\lambda(x) = \lambda^{N/2} u(\lambda x).$$

We compute for every $\lambda > 0$ that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^{\frac{N+1}{\alpha}})|u_\lambda|^{\frac{N+1}{\alpha}} = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+1}{\alpha}})|u|^{\frac{N+1}{\alpha}}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2.$$

Hence,

$$\inf_{\lambda > 0} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \int_{\mathbb{R}^N} |u|^2,$$

and we conclude that $\tilde{c}_\infty \leq c_\infty$. □

Proof of proposition 5. For $\lambda > 0$, let

$$u_\lambda(x) = C \left( \frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{N}{2}} = \lambda^{-\frac{N}{2}} u_1 \left( \frac{x}{\lambda} \right)$$

be a family of minimizers for $c_\infty$ given in (1.1). We observe that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^{\frac{N+1}{\alpha}})|u_\lambda|^{\frac{N+1}{\alpha}} = 1,$$

whereas by a change of variables,

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + \int_{\mathbb{R}^N} V \left( \frac{y}{\lambda} \right) \frac{C^2}{1 + |y|^2} \, dy.$$

By Lebesgue’s dominated convergence theorem

$$\limsup_{\lambda \to 0} \int_{\mathbb{R}^N} V \left( \frac{y}{\lambda} \right) \frac{C^2}{1 + |y|^2} \, dy \leq \int_{\mathbb{R}^N} \frac{C^2}{1 + |y|^2} \, dy = c_\infty,$$

so we conclude that $c_* \leq c_\infty$. If, in addition, $c_* = c_\infty$ then for any $\lambda_0 \to 0$, $(u_{\lambda_0})_{n \in \mathbb{N}}$ is a minimizing sequence for $c_*$, and the conclusion follows. □

4. Sufficient conditions for the strict inequality: proof of theorem 6

For $a \in \mathbb{R}^N$ and $\lambda > 0$, let

$$u_\lambda(x) = C \left( \frac{\lambda}{\lambda^2 + |x - a|^2} \right)^{N/2}$$

be a family of minimizers for $c_\infty$ as in (1.1). Then

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = c_\infty + \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \int_{\mathbb{R}^N} (V - 1) |u_\lambda|^2.$$
Denote
\[ I_V(a, \lambda) := \lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \lambda^2 \int_{\mathbb{R}^N} (V - 1)|u_\lambda|^2 < 0. \]

To obtain a sufficient conditions for \( c_* < c_\infty \) it is enough to show that for some \( a \in \mathbb{R}^N \),
\[ (4.1) \inf_{\lambda > 0} I_V(a, \lambda) < 0, \]

**Proof of theorem 6.** If \( N \leq 2 \), then by (1.3) there exists \( \mu > 0 \) such that
\[ \liminf_{|x| \to \infty} (1 - V(x))|x|^2 \geq \mu. \]

Therefore
\[ \lim_{\lambda \to \infty} \lambda^2 \int_{\mathbb{R}^N} (1 - V)|u_\lambda|^2 = \lim_{\lambda \to \infty} \int_{\mathbb{R}^N} \frac{\lambda^2(1 - V(\lambda x))}{(1 + |x|^2)N} \, dx \geq \int_{\mathbb{R}^N} \frac{\mu}{|x|^2(1 + |x|^2)^N} \, dx = \infty. \]

Since for every \( \lambda > 0 \),
\[ \lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \int_{\mathbb{R}^N} |\nabla u_1|^2 < \infty, \]
the condition (4.1) is satisfied.

If \( N \geq 3 \), we observe that for every \( \lambda > 0 \),
\[ \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \frac{N^2(N - 2)}{4(N + 1)} \int_{\mathbb{R}^N} \frac{|u_\lambda(x)|^2}{|x|^2} \, dx. \]

This follows from the fact that
\[ \int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^{N+2}} \, dx = \frac{N - 2}{4(N + 1)} \int_{\mathbb{R}^N} \frac{1}{|x|^2(1 + |x|^2)^N} \, dx, \]
which can be proved by two successive integrations by parts. Then, after a transformation \( x = \lambda y + a \),
\[ I_V(a, \lambda) = \int_{\mathbb{R}^N} \left( \frac{N^2(N - 2)}{4(N + 1)} \frac{1}{|y|^2} - \lambda^2(1 - V(a + \lambda y)) \right) \frac{C^2}{(1 + |y|^2)^N} \, dy, \]
and in view of (1.3), sufficient condition is (4.1) is satisfied for \( a = 0 \), so we conclude that \( c_* < c_\infty \). \( \square \)

Note that if the function \( \lambda \mapsto \lambda^2(1 - V(a + \lambda y)) \) is nondecreasing for every \( y \in \mathbb{R}^N \),
then \( \lambda \mapsto I_V(a, \lambda) \) is nonincreasing. Therefore \( I_V(a, \lambda) \) admits negative values if and only if it has a negative limit as \( \lambda \to \infty \). The latter is ensured in theorem 6 via asymptotic condition (1.3). This explains that if the function \( \lambda \mapsto \lambda^2(1 - V(a + \lambda y)) \) is nondecreasing, like for instance, in the special case
\[ V(x) = 1 - \frac{\mu}{1 + |x|^2}, \]
then integral sufficient condition (4.1) is in fact equivalent to the asymptotic sufficient condition (1.3).
5. Pohožaev Identity and Necessary Conditions for the Existence

We establish a Pohožaev type identity, which extends the identities (5.1) obtained previously for constant potentials $V$ [4, lemma 2.1; 15, proposition 3.1; 17, theorem 3].

**Proposition 11.** Let $N \geq 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u \in W^{1,2}(\mathbb{R}^N)$. If

$$\sup_{x \in \mathbb{R}^N} |(\nabla V(x))x| < \infty,$$

and $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfies Choquard equation $\mathcal{P}_u$ then

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x|)u(x)^2 \, dx.$$

**Proof.** We fix a cut-off function $\phi \in C^1_c(\mathbb{R}^N)$ such that $\phi = 1$ on $B_1$ and we test for $\lambda \in (0, \infty)$ the equation against the function $v_\lambda \in W^{1,2}(\mathbb{R}^N)$ defined for every $x \in \mathbb{R}^N$ by

$$v_\lambda(x) = \phi(\lambda x)(\eta u(x)|x|)$$

to obtain the identity

$$\int_{\mathbb{R}^N} (\nabla u|\nabla v_\lambda) + \int_{\mathbb{R}^N} Vuv_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{2N}{N-2}})|u|^{\frac{2N}{N-2} - 2} v_\lambda.$$

We compute for every $\lambda > 0$, by definition of $v_\lambda$, the chain rule and by the Gauss integral formula,

$$\int_{\mathbb{R}^N} Vuv_\lambda = \int_{\mathbb{R}^N} V(x)u(x)\phi(\lambda x)(|x|\nabla u(x)) \, dx$$

$$= \int_{\mathbb{R}^N} V(x)\phi(\lambda x)(|x|\nabla (\frac{|u|^2}{2})) \, dx$$

$$= -\int_{\mathbb{R}^N} ((NV(x) + (\nabla V(x)|x|)\phi(\lambda x) + V(x)\lambda x|\nabla \phi(\lambda x))|\frac{|u|^2}{2} \, dx.$$}

Since $\sup_{x \in \mathbb{R}^N}(\nabla V(x)|x| < \infty$, by Lebesgue’s dominated convergence theorem it holds

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} Vuv_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} |u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x|)|u|^2.$$}

By Lebesgue’s dominated convergence again, since $u \in W^{1,2}(\mathbb{R}^N)$, we have (see [16] proof of proposition 3.1) for the details)

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} (\nabla u|\nabla v_\lambda) = -\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2.$$}

and

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{2N}{N-2}})|u|^{\frac{2N}{N-2} - 2} v_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{2N}{N-2}})|u|^{\frac{2N}{N-2} - 2}.$$}

We have thus proved the Pohožaev type identity

\begin{equation}
\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x|)|u(x)|^2 \, dx
= \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{2N}{N-2}})|u|^{\frac{2N}{N-2} - 2}.
\end{equation}
If we test the equation against \( u \), we obtain the identity
\[
\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V|u|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u|^\alpha + 1)|u|^\alpha + 1;
\]
the combination of those two identities yields the conclusion. \( \square \)

**Proof of propositions 7 and 8.** Proposition 7 is a direct consequence of proposition 11, while proposition 8 follows from proposition 11 and the classical optimal Hardy inequality on \( \mathbb{R}^N \),
\[
\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2
\]
which is valid for all \( u \in H^1(\mathbb{R}^N) \) (see for example [21, theorem 6.4.10 and exercise 6.8]). \( \square \)

**References**


