A Characterization of Stochastically Stable Networks*

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Abstract

Jackson and Watts [*J. of Econ. Theory* 71 (2002), 44-74] have examined the dynamic formation and stochastic evolution of networks. We provide a refinement of pairwise stability, $p-$pairwise stability, which allows us to characterize the stochastically stable networks without requiring the "tree construction" and the computation of resistance that may be quite complex. When a $\frac{1}{2}-$pairwise stable network exists, it is unique and it coincides with the unique stochastically stable network. To solve the inexistence problem of $p-$pairwise stable networks, we define its set-valued extension with the notion of $p-$pairwise stable set. The $\frac{1}{2}-$pairwise stable set exists and is unique. Any stochastically stable networks is included in the $\frac{1}{2}-$pairwise stable set. Thus, any network outside the $\frac{1}{2}-$pairwise stable set must be considered as a non-robust network. We also show that the $\frac{1}{2}-$pairwise stable set can contain no pairwise stable network and we provide examples where a set of networks is more "stable" than a pairwise stable network.

Keywords: Network formation, Pairwise stability, Stochastic stability.
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1 Introduction

The organization of individual agents into networks and groups or coalitions has an important role in the determination of the outcome of many social and economic interactions.[1] There are many possible approaches to model network formation. One is simply to model it explicitly as a non-cooperative game (see e.g. Aumann and Myerson, 1988). A different approach is to analyze the networks that one might expect to emerge in the long run and to examine a sort of stability requirement that individuals not benefit from altering the structure of the network. This is the approach that was taken by Jackson and Wolinsky (1996) when defining pairwise stable networks. A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Another approach is to analyze the process of network formation in a dynamic framework.[2] Jackson and Watts (2002) have proposed a dynamic process in which individuals form and sever links based on the improvement that the resulting network offers them relative to the current network. This deterministic dynamic process may end at stable networks or in some cases may cycle. To explore whether some networks might be regarded as more reasonable than others, Jackson and Watts (2002) add to this deterministic process random perturbations and examine the distribution over networks as the level of random perturbations vanishes.

Exploiting the tree construction of Freidlin and Wentzel (1984), Jackson and Watts (2002) have shown that the outcome of their selection process (called stochastically stable networks) can be fully characterized in terms of resistances. However, these results are not always helpful in determining the outcome, because the required computation for resistances and the tree construction may be quite complex. To be more precise, this problem is known to be NP-complete in complexity theory.[3] Thus we do not have much knowledge on which network will arise in these processes in general. In order to extend the applicability of these results, more succinct criteria are needed to determine the outcome of this selection theory. One goal of the paper is to find a criterion for network selection that is free from the computation of resistances and the tree construction.[4]

[1] Jackson (2003, 2005) has provided a survey of models of network formation.

[2] Watts (2001) has extended the Jackson and Wolinsky model to a dynamic process but she has limited attention to the specific contest of the connections model and a particular deterministic dynamic.

[3] See Garey and Johson (1979, p.206). We know that for NP-complete problems, all *known* algorithms to solve the problem require time which is exponential in the problem size (for instance in the number of individuals considered).

[4] In noncooperative games Young (1993), Ellison (1993), Kandori, Mailath and Rob (1993) among others have applied the Freidlin and Wentzell (1984) techniques in order to provide evolutionary models that select among (strict) Nash equilibria. But these results are submitted to the same criticism than

We propose a new concept, $p-$pairwise stability, which is a refinement of the notion of pairwise stability. A network is said to be $p-$pairwise stable if when we add a set of links to this network (or sever a set of links), then if we allow players to successively create or delete links, they will come back to the initial network. The parameter $p \in [0, 1]$ indicates the "number" of links that can be modified: $p = 0$ means that all links may be modified, $p = 1$ means that no link may be added or severed. Thus, $1-$pairwise stability reverts to Jackson and Wolinsky (1996) pairwise stability concept. Also, a network is said to be $\frac{1}{2}-$pairwise stable if when we add a set of links to this network (or sever a set of links) such that the number of changes is less than half the total of possible changes, then if we allow players to successively create or delete links, they will come back to the initial network.

We show that when a $\frac{1}{2}-$pairwise stable network exists, it is unique. Moreover it is the only stochastically stable network in Jackson and Watts (2002) stochastic evolutionary process. But while our notion of a $\frac{1}{2}-$pairwise stable network leads to a unique selection when it exists, it does not always exist. Therefore, we define its set-valued extension with the notion of $\frac{1}{2}-$pairwise stable set of networks that is proved to exist and to coincide with the $\frac{1}{2}-$pairwise stable network when it exists. We also show that if a network is stochastically stable then it belongs to the $\frac{1}{2}-$pairwise stable set of networks. Thus, any network outside the $\frac{1}{2}-$pairwise stable set must be considered as a non-robust network. Interestingly, the $\frac{1}{2}-$pairwise stable set of networks can contain no pairwise stable network. We see this as a drawback of pairwise stability, and we provide examples where a set of networks is more "stable" than a pairwise stable network.

The paper is organized as follows. In Section 2 we define the notion of $p-$pairwise stable network and we study its properties. In Section 3 we propose a set-valued extension, the $p-$pairwise stable set of networks. In Section 4 we provide an evolutionary foundation to the $\frac{1}{2}-$pairwise stable set of networks. In Section 5 we conclude.

2 $p-$Pairwise Stable Networks

Let $N = \{1, ..., n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a

Jackson and Watts (2002) and so they are not always helpful in determining the selected action profiles. Then, some authors have looked for criteria for equilibrium (or non-equilibrium) selection that are free from the computation of resistances and the tree construction. For instance, Young (1993) has shown that in a two player, two action game, only the risk-dominant equilibrium (in the sense of Harsanyi and Selten (1988)) is stochastically stable. This result was generalized by Maruta (1997) and Durieu, Solal and Tercieux (2003) to two players finite games.

non-directed graph.[5] Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. If we are considering a pair of individuals i and j, then $\{i, j\} \in g$ indicates that i and j are linked under the network g. For simplicity, we write ij to represent the link $\{i, j\}$, and so $ij \in g$ indicates that i and j are linked under the network g. Let g^N be the set of all subsets of N of size 2. G^N denotes the set of all possible networks or graphs on N, with g^N being the complete network. The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network obtained by deleting link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g.

Different network configurations lead to different values of overall production or overall utility to players. These various possible valuations are represented via a value function. A *value function* is a function $v : G^N \to \mathbb{R}$. The set of all possible value functions is denoted $\mathcal{V}$. A value function only keeps track of how the total societal value varies across different networks. We also wish to keep track of how that value is allocated or distributed among the players forming a network. An allocation rule is a function $Y : G^N \times \mathcal{V} \to \mathbb{R}^N$ such that $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g. It is important to note that an allocation rule depends on both g and v. This allows an allocation rule to take full account of a player i's role in the network. This includes not only what the network configuration is, but also and how the value generated depends on the overall network structure.

In evaluating societal welfare, we may take various perspectives.[6] A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \subseteq G^N$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with strict inequality for some i. This definition of efficiency of a network takes Y as fixed, and hence can be thought of as applying to situations where no intervention is possible. A network $g \subseteq G^N$ is *strongly efficient* relative to v if $v(g) \geq v(g')$ for all $g' \subseteq G^N$. This is a strong notion of efficiency as it takes the perspective that value is fully transferable.

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that agents not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player

[5] Bala and Goyal (2000) have studied network formation in directed networks. See also Dutta and Jackson (2000).

[6] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. We also use the symbols $\vee$ and $\wedge$ which mean "or" and "and", respectively. Finally, $\#$ will refer to the notion of cardinality.

benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

Definition 1 *A network g is pairwise stable with respect to value function v and allocation rule Y if*

(i) for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and

(ii) for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$.

Let us say that g' is adjacent to g if $g' = g + ij$ or $g' = g - ij$ for some ij. A network g' defeats g if either $g' = g - ij$ and $Y_i(g', v) > Y_i(g, v)$, or if $g' = g + ij$ with $Y_i(g', v) \geq Y_i(g, v)$ and $Y_j(g', v) \geq Y_j(g, v)$ with at least one inequality holding strictly. Pairwise stability is equivalent to saying that a network is pairwise stable if it is not defeated by another (necessarily adjacent) network. The following example shows the main insight of the stability requirement we will introduce. In particular, the example shows that a network that is both pareto-dominant and pairwise stable can be "less stable" than another network.

Example 1. Consider a situation where four players can form links. The payoffs they obtained from the different network configurations are (see Figure 1): for a non-empty network g, $Y_i(g) = \#(g)$ if $i \in N(g)$ with $\#(g)$ being the number of links in g, $Y_i(g) = 0$ if $i \notin N(g)$, and $Y_i(g) = 10$ if g is the empty network. Both the empty network and the complete network are pairwise stable networks. The empty network is also the efficient network.

Suppose that at least two links are added to the empty network to form g'. Then, from g' all "undefeated" improving paths go to the complete network and none goes back to the empty network. An improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both agree to its addition, with at least one of the two strictly benefiting from the addition of the link. If a link is deleted, then it must be that at least one of the two players involved in the link strictly benefits from its deletion. By an "undefeated" improving path, we mean that the final network in the sequence of the improving path is not defeated. Suppose now that at most four links are deleted from the complete network to form g''. Then, from g'' all "undefeated" improving paths go back to the complete network. Thus, we say that the empty network (while being the efficient network) is "less stable" than the complete network, while both are pairwise stable.

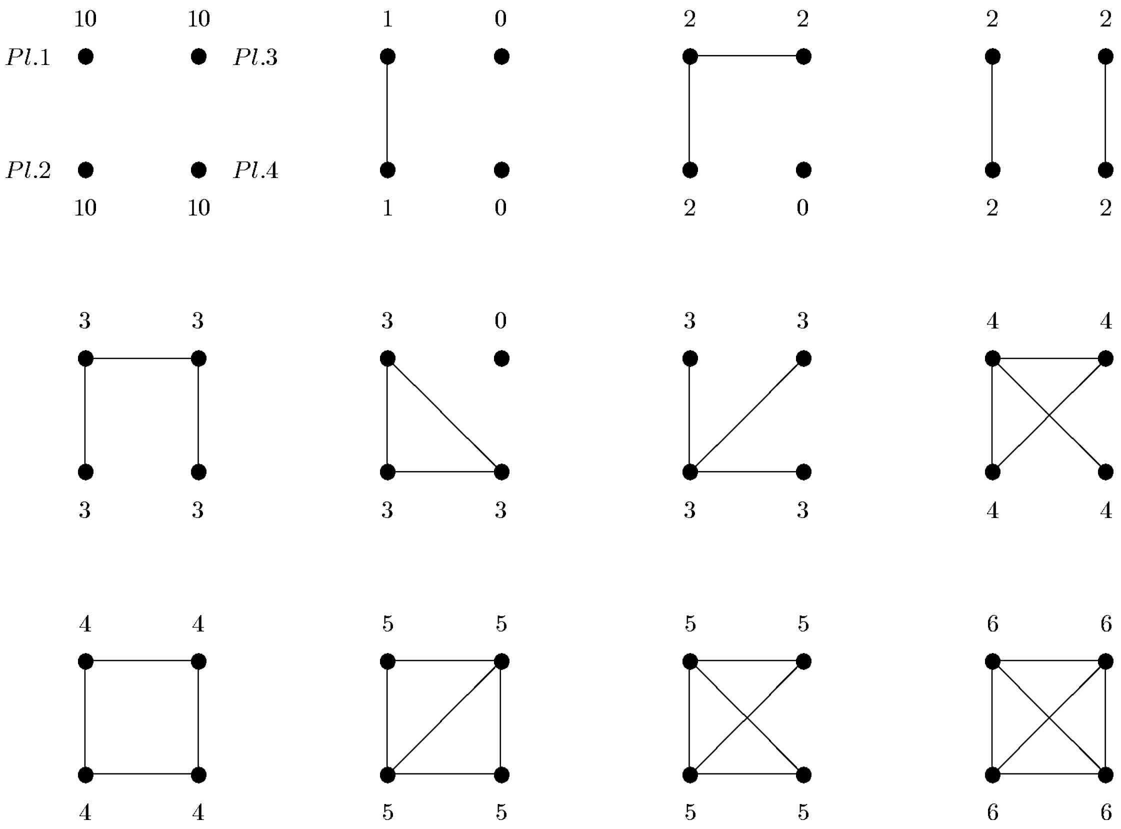

Figure 1: The empty and complete networks are pairwise stable (Example 1).

In order to formalize such refinement of pairwise stability, we first define a notion of distance between two networks. For $g, g' \subseteq g^N$ we denote by

$$d(g,g') \equiv \frac{\#\{ij \in g^N \mid (ij \in g \wedge ij \notin g') \vee (ij \notin g \wedge ij \in g')\}}{\#g^N}$$

the *distance* between g and g'. That is, $d(g,g')$ is the number of links that g does have while g' does not, plus the number of links that g does not have while g' does, the total being divided by the maximum number of links. Thus, $0 \leq d(g,g') \leq 1$. The formal definition of an improving path is due to Jackson and Watts (2002). An improving path from a network g to a network g' is a finite sequence of graphs $g_1, ..., g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, ..., K-1\}$ either:

(i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or

(ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$.

The length of an improving path is $K-1$, $K \geq 2$. If there exists an improving path from g' to g, then as Jackson and Watts (2002) we use the symbol $g' \rightarrow g$. For a given network, g, let $im(g) = \{g' \subseteq g^N \mid g' \rightarrow g\}$. This is the set of networks for which there is an improving path leading from g' to g. An improving path from g' to g is of maximum length if g_K is not defeated by any $g'' \subseteq g^N$. Remember that a network g' defeats g if either $g' = g - ij$ and $Y_i(g', v) > Y_i(g, v)$, or if $g' = g + ij$ with $Y_i(g', v) \geq Y_i(g, v)$ and $Y_j(g', v) \geq Y_j(g, v)$ with at least one inequality holding strictly. Thus, for any improving path from g' to g, if g is not defeated by any other g'', then the path is of maximum length. We say that an improving path $g_1, ..., g_K$ with $g_1 = g'$ and $g_K = g$ goes **directly** from g' to g if for all $k \in \{1, ..., K-1\}$, we have $d(g_{k+1}, g) \leq d(g_k, g)$.

For $g' \neq g$, we write $g' \longmapsto g$ if:

(i) all improving paths of maximum length from g' go **directly** to g,

(ii) there does not exist an infinite improving path from g'.[7]

Thus, $g' \longmapsto g$ means that g should be the endpoint of any improving path of maximum length if g' is its initial point, and that all improving paths of maximum length from g' should go to g without moving away from g. We write $g \longmapsto g$ if from g there is no improving path. For a given network g, let $IM(g) = \{g' \subseteq g^N \mid g' \longmapsto g\}$. In the sequel, we note $\phi(p)$ the largest number smaller or equal to p such that $\phi(p) \cdot \#g^N$ is an integer. This notation will be useful in defining our notion of $p-$pairwise stable networks.

Definition 2 *Let $p \in [0, 1]$. A network g is $p-$pairwise stable with respect to allocation rule Y and value function v if for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$, we have $g' \in IM(g)$.*

Any network g that is $p-$pairwise stable is $p'-$pairwise stable for $p' \geq p$. The notion of $p-$pairwise stability is a refinement of pairwise stability in the following sense. A network g is pairwise stable if and only if it is $1-$pairwise stable. Thus, any network g that is $p-$pairwise stable is pairwise stable.[8]

[7] An infinite improving path from g' is an infinite sequence of graphs $g_1, g_2, ...$ such that for any $k \in \{1, 2, 3, ...\}$ either (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$. Thus, no network is not defeated in an infinite improving path.

[8] In Appendix C we provide an alternative definition, $p-$stability, which is shown to be equivalent to $p-$pairwise stability if Y and v exhibit no indifference. A network is said to be $p-$stable if when we take two linked (not linked) players and we add and sever a set of links to the network, then both players still do not want to sever (add) the link between them. In other words, $p-$stability asks a pairwise stable link (no-link) to be robust against perturbations to the network.

Proposition 1 *Let $p \leq \frac{1}{2}$. A $p-$pairwise stable network is unique when it exists.*

Proof. We proceed by contradiction. Let us assume that g^1 and g^2 are two distinct $p-$pairwise stable networks where $p \leq \frac{1}{2}$. Then, they are $\frac{1}{2}-$pairwise stable. If $d(g^1, g^2) \leq 1 - \phi(\frac{1}{2})$, we have a straightforward contradiction. (Since we must have $g^1 \in IM(g^2)$, i.e. $g^1 \longmapsto g^2$ which is not possible since $g^1 (\neq g^2)$ is pairwise stable (or indifferently $1-$pairwise stable)).
Assume now that $d(g^1, g^2) > 1 - \phi(\frac{1}{2})$. Pick g^1 and delete some elements in $\{(ij \in g^1 \wedge ij \notin g^2)\}$ and add some elements in $\{(ij \notin g^1 \wedge ij \in g^2)\}$ so that the total number of changes is $(1 - \phi(\frac{1}{2}))\#g^N$. We obtain a network g' satisfying $d(g', g^1) = 1 - \phi(\frac{1}{2})$. By construction, this network g' satisfies $d(g', g^2) \leq \phi(\frac{1}{2}) \leq 1 - \phi(\frac{1}{2})$. Then, since g^1 and g^2 are $\frac{1}{2}-$pairwise stable, we have that $g' \in IM(g^1)$, i.e. $g' \longmapsto g^1$, and $g' \in IM(g^2)$, i.e. $g' \longmapsto g^2$, which is not possible since $g^2 \neq g^1$. ■

In Example 1, the empty network is pairwise stable and is the unique strongly stable network.[9] However, the complete network is the unique $\frac{1}{2}-$pairwise stable network. The reason is that from any network g' with $\#(g') \geq 3$ (or $d(g', g^N) \leq \frac{1}{2}$) any "undefeated" improving paths go directly to the complete network g^N, but none goes to the empty network.[10] The next two examples show that a $\frac{1}{2}-$pairwise stable network may fail to exist while a pairwise stable network exists. In the first example, none of the two pairwise stable networks is $\frac{1}{2}-$pairwise stable, because there exists a network at mid distance from which there are improving paths going to both pairwise stable networks. In the second example, the unique pairwise stable is not $\frac{1}{2}-$pairwise stable because improving paths are enclosed in a cycle.

Example 2. Consider a situation where four players can form links. The payoffs they obtained from the different network configurations are (see Figure 2): $Y_i(g) = [\#(g)]^2 - c \cdot \#\{j \in N \text{ such that } ij \in g\}$ if $i \in N(g)$, $Y_i(g) = 0$ if $i \notin N(g)$, (and so, $Y_i(g) = 0$ if g is the empty network). The parameter $c > 0$ is the individual cost of forming a link. For $c < 11$ the complete network is pairwise stable, for $c > 1$ the empty network is pairwise stable. For $c < 5$ our refinement will select the complete network which is the unique $\frac{1}{2}-$pairwise stable network. For $c > 7$ the empty network is the unique $\frac{1}{2}-$pairwise

[9] Jackson and van den Nouweland (2005) have introduced the notion of strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component.

[10] Note that in all examples of the paper, we will choose the number of players N so that $\#g^N = \frac{N(N-1)}{2}$ is even. This will allow us to have $\phi(\frac{1}{2}) = \frac{1}{2}$.

stable network. But, if $5 < c < 7$ then a $\frac{1}{2}$−pairwise stable network fails to exist. The reason is that at $g' = \{12, 13, 34\}$ players 2 and 4 have incentives to form the link 24 but at the same time players 1 or 3 has an incentive to sever the link he has with 2 or 4. So, from g' some improving paths go to the empty network, while others go to the complete network. It follows that no $\frac{1}{2}$−pairwise stable network exists.

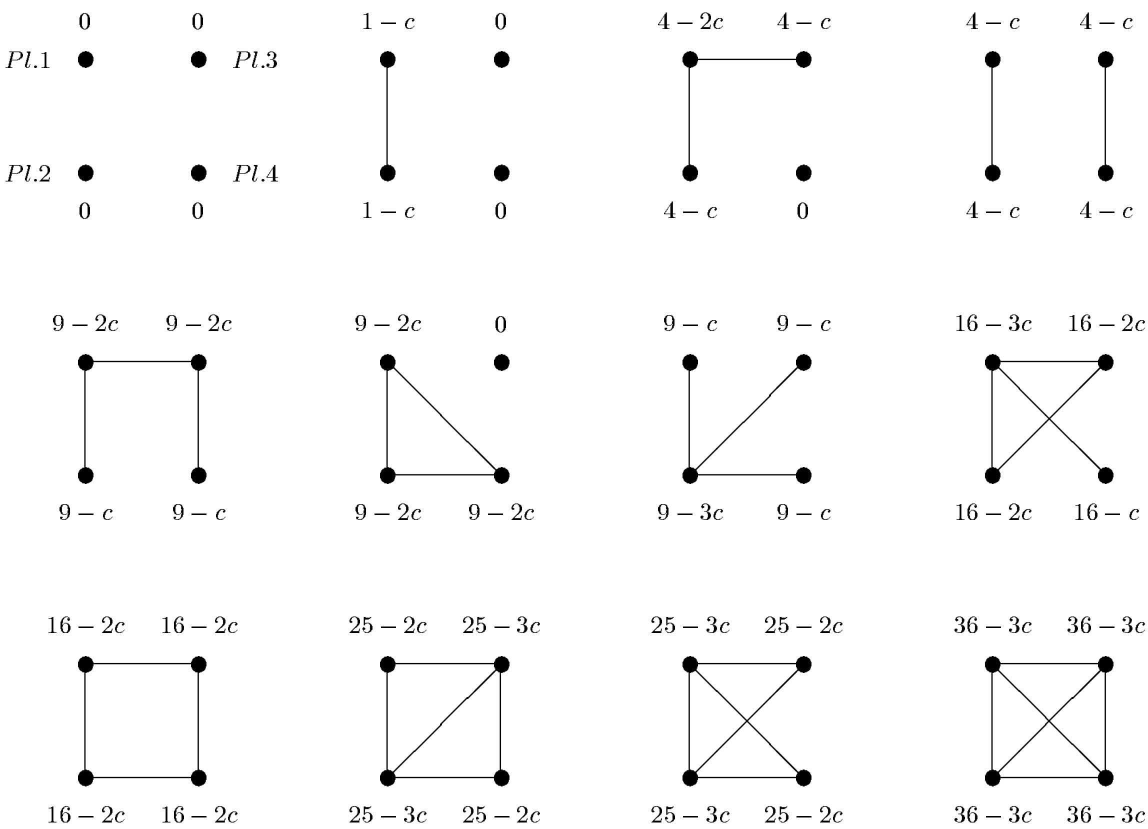

Figure 2: Non-existence of $\frac{1}{2}$−pairwise stable networks (Example 2).

Example 3. Suppose that five players can form links. In the complete network, $Y_i(g) = 8$ for all i. In any network g players $i \notin N(g)$ have a payoff $Y_i(g) = 0$. In networks g such that $\#(g) \in [3, 9]$, we have $Y_i(g) = 9 - \#(g)$ if $i \in N(g)$. In any g such that $\#(g) = 1$ or 2 and players 4 or 5 belong to $N(g)$ then $Y_i(g) = 0$ for all i. In any g such that $\#(g) = 2$ and players 4 and 5 do not belong to $N(g)$, we have that $Y_i(g) = 7$ for $i \in N(g)$. Finally, let $Y_1(\{12\}) = Y_3(\{13\}) = Y_2(\{23\}) = 6$, $Y_2(\{12\}) = Y_1(\{13\}) = Y_3(\{23\}) = 8$. Figure 3 presents some of these network configurations. In this example there is a unique pairwise stable network, the complete network. But, there does not exist a $\frac{1}{2}$−pairwise stable network. Indeed, from any g' such that $d(g', g^N) \geq \frac{1}{5}$, no improving path goes to g^N.

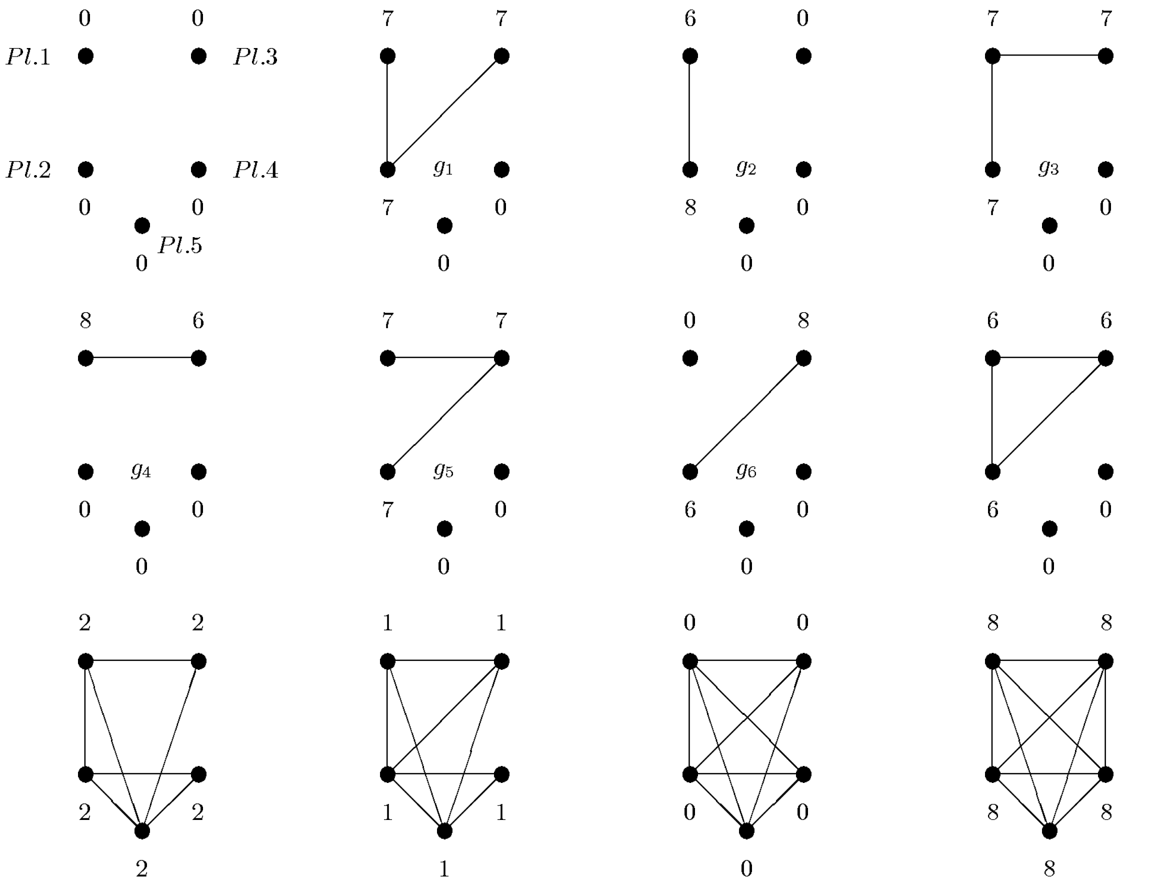

Figure 3: Another example of non-existence of $\frac{1}{2}-$pairwise stable networks (Example 3).

Thus, a $\frac{1}{2}-$pairwise stable network does not always exist. In the spirit of Tercieux (2005) we aim to solve the problem of non-existence of $\frac{1}{2}-$pairwise stable networks by providing a set-valued extension. Interestingly, such an approach will put into relief that a set of networks that are not pairwise stable can be more "stable" than a pairwise stable network.

3 $p-$Pairwise Stable Sets of Networks

Let us first restate the definition of an improving path. An improving path from a network g to a set of networks $G \subseteq G^N$ is a finite sequence of graphs $g_1, ..., g_K$ with $g_1 = g$ and $g_K \in G$ such that for any $k \in \{1, ..., K-1\}$ either:

(i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or

(ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$.

The length of an improving path is $K - 1$, $K \geq 2$. An improving path from g' to $G \subseteq G^N$ is of maximum length if g_K is not defeated by any $g'' \notin G$. For $g' \notin G$, we write $g' \longmapsto G$ if:

(i) all improving paths of maximum length from g' go **directly** to G,

(ii) for any infinite improving path from g', there exists K such that for all $k \geq K$, $g_k \in G$.

We write $g \longmapsto G$ if from $g \in G$ there is no improving path going to $g' \notin G$. For a given set of network G, let $IM(G) = \{g' \subseteq g^N \mid g' \longmapsto G\}$. Note that, in the following, for $G \subseteq G^N$, and $g' \subseteq g^N$ we will note $d(g', G) \leq 1 - \phi(p)$ if $d(g', g) \leq 1 - \phi(p)$ for some $g \in G$.

Definition 3 *Let $p \in [0, 1]$. A set of networks $G \subseteq G^N$ is $p-$pairwise stable with respect to allocation rule Y and value function v if*

(1) *for all $g' \subseteq g^N$ such that $d(g', G) \leq 1 - \phi(p)$, we have $g' \in IM(G)$,*

(2) *there does not exist $G' \subsetneq G$ such that G' satisfies **(1)**.*

Remark 1 *The set G^N (trivially) satisfies **(1)** in Definition 3 for any $p \in [0, 1]$.*

Proposition 2 *Let $p \in [0, 1]$. Two (distinct) $p-$pairwise stable set of networks must be disjoint.*

Proof. We proceed by contradiction. Assume that G and G' are two (distinct) $p-$pairwise stable sets of networks and $G \cap G' \neq \emptyset$. Then, for all $g' \subseteq g^N$ such that $d(g', G \cap G') \leq 1 - \phi(p)$, we have $g' \in IM(G)$. But since this assertion is also true for G', we have that for all $g' \subseteq g^N$ such that $d(g', G \cap G') \leq 1 - \phi(p)$, $g' \in IM(G \cap G')$. Thus $G \cap G'$ satisfies **(1)** in Definition 3, contradicting the fact that G (and G') are $p-$pairwise stable sets, i.e. the minimality is violated (point **(2)** in Definition 3 of $p-$pairwise stable sets). ■

As underlined earlier, the main drawback of our definition of $\frac{1}{2}-$pairwise stable networks is that existence may fail also when a pairwise stable network exists. We now show that our set-valued notion of $\frac{1}{2}-$pairwise stable set always exists. As will become clear (for instance through Example 3), when there does not exist any $\frac{1}{2}-$pairwise stable network, our notion allows to eliminate many possibilities. Moreover, it is possible that the $\frac{1}{2}-$pairwise stable set of networks does not contain any pairwise stable network (see Example 3). We claim that this last point is important and underlines an important drawback of pairwise stability. The selection result we will introduce in the next section will give a foundation to this informal argument since we will prove that any network outside the $\frac{1}{2}-$pairwise stable set is not robust in a precise sense.

Proposition 3 *Let $p \in [0,1]$. There always exists at least one $p-$pairwise stable set of networks.*

Proof. Let us proceed by contradiction. Let $p \in [0,1]$ and assume that there does not exist any set of networks $G \subseteq G^N$ that is $p-$pairwise stable. This means that for any $G^{\mathbf{0}} \subseteq G^N$ that satisfies **(1)** in Definition 3 (there always exist such a $G^{\mathbf{0}}$, see Remark 1), we can find a proper subset G^1 that satisfies **(1)**. But again for G^1, we can find a proper subset G^2 that satisfies **(1)**. Iterating the reasoning we can build an infinite (decreasing) sequence $\{G^k\}_{k\geq 0}$ of **distinct** elements of G^N satisfying **(1)**. But since $\#G^N < \infty$, this is not possible; so the proof is completed. ∎

Note first that if g is a $\frac{1}{2}-$pairwise stable network then $\{g\}$ is a $\frac{1}{2}-$pairwise stable set of networks. What our next result shows in particular is that $\{g\}$ is the only $\frac{1}{2}-$pairwise stable set of networks and thus the two notions coincide in that special case.

Proposition 4 *Let $p \leq \frac{1}{2}$. There always exists a unique $p-$pairwise stable set of networks.*

Proof. We proceed by contradiction. Assume that G^1 and G^2 are two distinct $p-$pairwise stable networks where $p \leq \frac{1}{2}$. Then, they satisfy **(1)** in Definition 3 for $p = \frac{1}{2}$.
If $d(G^1, G^2) \leq 1-\phi(\frac{1}{2})$. Then we have a straightforward contradiction. (Since from $g \in G^1$ we must have $g \in IM(G^1)$, i.e. $g \longmapsto G^1$ and $g \in IM(G^2)$, i.e. $g \longmapsto G^2$ which is not possible since $G^1 \cap G^2 = \emptyset$.
If $d(G^1, G^2) > 1 - \phi(\frac{1}{2})$, we take $g^1(\in G^1)$ and $g^2(\in G^2)$. Then, pick g^1 and delete some elements in $\{(ij \in g^1 \wedge ij \notin g^2)\}$ and add some elements in $\{(ij \notin g^1 \wedge ij \in g^2)\}$ so that the total number of changes is $(1 - \phi(\frac{1}{2}))\#g^N$. We obtain a network g' satisfying $d(g', G^1) = 1-\phi(\frac{1}{2})$. By construction, this network g' satisfies $d(g', G^2) \leq \phi(\frac{1}{2}) \leq 1-\phi(\frac{1}{2})$. Then, since G^1 and G^2 are $p-$pairwise stable for $p \leq \frac{1}{2}$ (i.e. they both satisfy **(1)** in Definition 3 for $p = \frac{1}{2}$), we have that $g' \in IM(G^1)$, i.e. $g' \longmapsto G^1$ and $g' \in IM(G^2)$, i.e.

$g' \longmapsto G^2$ which again is not possible since $G^1 \cap G^2 = \emptyset$. ■

In Example 2 we have that, for $5 < c < 7$, there does not exist a $\frac{1}{2}-$pairwise stable network, but the set formed by the complete and empty networks is the $\frac{1}{2}-$pairwise stable set of networks. In Example 3, the complete network is the unique pairwise stable network and there is no $\frac{1}{2}-$pairwise stable network. However, the $\frac{1}{2}-$pairwise stable set of networks is $G' = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ (see Figure 3), which does not include the complete network, because there is a cycle $g_1 \rightarrow g_2 \rightarrow g_3 \rightarrow g_4 \rightarrow g_5 \rightarrow g_6 \rightarrow g_1$ and all "undefeated" improving paths from any g' such that $d(g', G') \leq \frac{1}{2}$ go directly to G' and stay in G'. By an "undefeated" improving path, we mean that the final network in the sequence of the improving path is not defeated by a network that does not belong to G'.

Our set-valued notion generalizes many existing concepts of the literature. We can easily link this to two definitions, the first one is the well-known definition of pairwise stable networks of Jackson and Wolinsky (1996). The second one is the one of closed cycle provided by Jackson and Watts (2002). The following straightforward proposition is stated without proof.

Proposition 5 *$\{g\}$ is a $p-$pairwise stable set if and only if g is a $p-$pairwise stable network. And so, $\{g\}$ is a $1-$pairwise stable set if and only if it is a pairwise stable network.*

The following definition is due to Jackson and Watts (2002, p.273). A set of networks G, form a *cycle* if for any $g \in G$ and $g' \in G$, there exists an improving path connecting g to g'. A cycle G is a *closed cycle* if no network in G lies on an improving path leading to a network that is not in G.

Proposition 6 *G is a $1-$pairwise stable set if and only if it is a closed cycle.*

Proof. The proof can be found in Appendix A. ■

4 Evolutionary Selection

In this section, we show that our notion of $\frac{1}{2}-$pairwise stable networks (and $\frac{1}{2}-$pairwise stable set of networks) is relevant in the stochastic evolutionary process proposed by Jackson and Watts (2002).

4.1 The Process

Let us recall first the Jackson and Watts (2002)'s process. At a discrete set of times, $\{1, 2, 3, ...\}$ decisions to add or sever a link are made. At each date, a pair of players ij

is randomly identified with probability $p(ij) > 0$. The (potential) link between these two players is the only link that can be altered at that time. If the link is already in the network, then the decision is whether to sever it, and otherwise the decision is whether to add the link. The players involved act myopically, adding the link if it makes each at least as well off and one strictly better off, and severing the link if its deletion makes either player better off. After the action is taken, there is some small probability $\varepsilon > 0$ that a mutation (or tremble, or mistake) occurs and the link is deleted if it is present, and added if it is absent.[11]

The above process defines a (finite) Markov chain with states being the network in place at the end of a given period. Note that with mutations as part of the process, each state of the system is reachable with positive probability from every other state. The Markov chain is said to be irreducible and aperiodic, and thus has a unique corresponding stationary distribution (see Freidlin and Wentzel, 1984). As ε goes to zero, the stationary distribution converges to a unique limiting stationary distribution. A network that is in the support of the limiting (as ε goes to zero) stationary distribution of the above-described Markov process is said to be *stochastically stable.* Intuitively, a stochastically stable network is one that is observed infinitely many more times than others when the probability of mutations is infinitely small. Jackson and Watts (2002) provides a characterization of stochastically stable networks using the tree construction of Freidlin and Wentzell (1984). In the following, we prove that our concept can be used to avoid this complex construction.

4.2 Relationship between $p-$Pairwise Stability and Stochastic Stability

The following theorem shows that under the process we have just described, the only networks that will arise with a significant frequency in the long run (i.e., the stochastically stable one) are in the $\frac{1}{2}-$pairwise stable set.

Theorem 1 *Let G be the $\frac{1}{2}-$pairwise stable set of networks. The set of stochastically stable networks is included in G.*

Proof. See Appendix B. ■

Thus any network outside the $\frac{1}{2}-$pairwise stable set must be considered as a non-robust network. To be more precise, the stochastic process presented above can be thought of as

[11] Mutations may be due to exogenous unmodeled factors that are beyond player's control. Alternatively, players may simply make errors in calculating whether adding or severing a link is beneficial. Finally, we could think to players having a limited information. Thus they occasionally experiment to see if adding or severing a link will make them better off (endogenous mutations have been formalized in several papers, see for instance van Damme and Weibull (2002) or Maruta (2002)).

a check on the robustness of pairwise networks or cycles. Although a number of networks may be pairwise stable, they can differ in how resilient they are to the random mutations. For instance, it may be relatively hard to leave and easy to get back to some networks, our above theorem tells us that such networks are included in the $\frac{1}{2}-$pairwise stable set of networks. This result also tells us that any network that is not in the $\frac{1}{2}-$pairwise stable set is relatively easy to leave and hard to get back.

In order to understand these points, note that once the process has reached the $\frac{1}{2}-$pairwise stable set of networks G, it cannot leave it without further mutations. On the first hand, in order to get off that set, it is necessary that strictly more than $\frac{\#g^N}{2}$ mutations occur (notice that in order to give the intuition of our result, we skip some technical points in assuming that N is such that $\frac{\#g^N}{2}$ is an integer). If it is not the case, the process will come back to G with no further mutation. On the other hand, as it will become clear, if the process has reached a network that is outside G, it is sufficient that less than $\frac{\#g^N}{2}$ mutations occur to allow the process to reach a network that belong to G. In order to see why it is so, note that from a network g' that does not belong to G, with (less than) $\frac{\#g^N}{2}$ mutations, one can reach a network $\bar{g}$ such that $d(g,\bar{g}) \leq \frac{1}{2}$ where g belongs to G. Thus, by definition, the process will move to G without any further mutations. To see how we can build $\bar{g}$, we just have to add links to g' that belong to g and not to g' or to delete links that do not belong to g but belong to g'. By repeating this procedure less than $\frac{\#g^N}{2}$ times, we can reach such a $\bar{g}$. Thus there exist networks in G which are the easiest to reach from other networks, where - again - "easiest" is interpreted as requiring the fewest mutations. These networks are stochastically stable. The formal argument is given in the appendix.

Of course, we would like to have a full characterization of the set of stochastically stable networks. In order to do so, we provide several sufficient conditions that go in that sense. These results are corollaries of Theorem 1. The first one shows that if there exists a $\frac{1}{2}-$pairwise stable network then it must be the unique stochastically stable network. Note that this result can be seen as a parallel to the one of Young (1993) [Theorem 3, p.72] in noncooperative games.

Corollary 1 *Assume that a network g is the $\frac{1}{2}-$pairwise stable network. Then g is the unique stochastically stable network.*

The following two corollaries directly come from the fact that if g is stochastically stable then g is part of a $1-$pairwise stable set of networks. Furthermore, if $g \in G$ is stochastically stable and G is a $1-$pairwise stable set then all $g' \in G$ are stochastically stable (this follows from Lemma 2 in Jackson and Watts (2002) together with our Proposition 6 that establishes the equivalence between a $1-$pairwise stable set and a closed cycle).

Corollary 2 *Let G be the $\frac{1}{2}-$pairwise stable set of networks. If G is $1-$pairwise stable then G is the set of stochastically stable networks.*

Corollary 3 *Let G be the $\frac{1}{2}-$pairwise stable set of networks. If $G' \subseteq G$ is the unique $1-$pairwise stable set in G then G' is the set of stochastically stable networks.*

Example 4. Suppose that three players can form links (see Figure 4). In the complete network, $Y_i(g) = 3$ for all i. In any network g players $i \notin N(g)$ have a payoff $Y_i(g) = 0$. In any g such that $\#(g) = 2$, $Y_i(g) = 2$ if $i \in N(g)$. Finally, let $Y_1(\{12\}) = Y_3(\{13\}) = Y_2(\{23\}) = 1$, $Y_2(\{12\}) = Y_1(\{13\}) = Y_3(\{23\}) = 4$. In this example there is a unique pairwise stable network, the complete network. There does not exist a $\frac{1}{2}-$pairwise stable network, $\{g^N\}$ is the $1-$pairwise stable set, and all networks except the empty one belong to the $\frac{1}{2}-$pairwise stable set of networks.

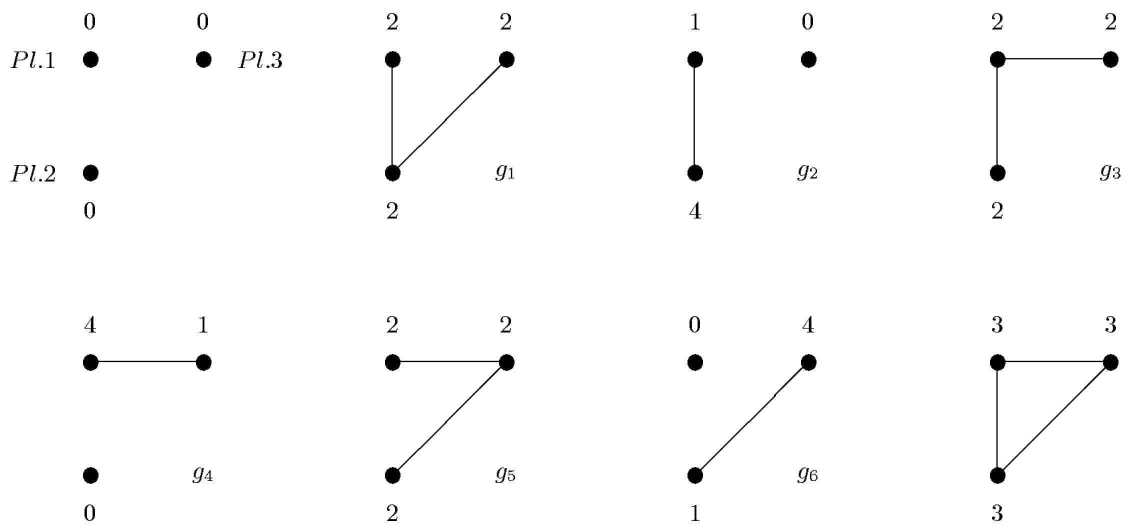

Figure 4: $\frac{1}{2}-$pairwise stable set and stochastically stable networks (Example 4).

In Example 4, the complete network is the unique pairwise stable network and there is no $\frac{1}{2}-$pairwise stable network because of the cycle $g_1 \to g_2 \to g_3 \to g_4 \to g_5 \to g_6 \to g_1$. The $\frac{1}{2}-$pairwise stable set of networks is $G' = \{g_1, g_2, g_3, g_4, g_5, g_6, g^N\}$ but this set is not $1-$pairwise stable. Indeed, $\{g^N\}$ is the unique $1-$ pairwise stable set and so by corollary 3 is the unique stochastically stable network.

The next example shows that our sufficient conditions are quite tight in the following sense: a $p-$pairwise stable network with $p = \frac{1}{2} + \varepsilon$ (ε small) may not be a stochastically stable network.

Example 5. Suppose that fifty players can form links. For $\#(g) \leq 611$, let $Y_i(g) = 611 - \#(g)$ if $i \in N(g)$ and $Y_i(g) = 0$ otherwise. For $\#(g) \geq 612$, let $Y_i(g) = \#(g) - 611$ if $i \in N(g)$ and $Y_i(g) = 0$ otherwise. The empty network is a $p-$pairwise stable network for $p \geq (615/1225) \simeq 0.502$, but the empty network is not stochastically stable. The unique stochastically stable network is the complete one, which is also the unique $\frac{1}{2}-$pairwise stable network.

5 Conclusion

In this paper, we have defined a refinement of pairwise stability: $p-$pairwise stability. When a $\frac{1}{2}-$pairwise stable network exists, we have shown that it is unique and that it coincides with the unique stochastically stable network. To solve the inexistence problem of $p-$pairwise stable networks, we have defined its set-valued extension with the notion of $p-$pairwise stable set. We have shown that $\frac{1}{2}-$pairwise stable set exists and is unique. In addition, any stochastically stable networks is included in the $\frac{1}{2}-$pairwise stable set.

Appendix

A Proof of Proposition 6

In this part we prove Proposition 6 that establishes the equivalence between our notion of $1-$pairwise stability and the notion of a closed cycle proposed by Jackson and Watts (2002). In order to do so, we first state and prove some useful lemmas. The following lemma is stated without proof.[12]

Lemma 1 *If G is such that for all $g \in G$, $g \in IM(G)$ (note that this is **(1)** in Definition 3 of a $1-$pairwise stable set) then there exists $C \subseteq G$ that is a closed cycle.*

Our next lemma provides a first step in establishing a link between $1-$pairwise stability and closed cycles.

Lemma 2 *If C is a closed cycle then there exists $G \subseteq C$ that is $1-$pairwise stable.*

Proof. Since C is a closed cycle, we know that for all $g \in C$, $g \in IM(C)$. Then C satisfies **(1)** of Definition 3 of a $1-$pairwise stable set. Now assume that there does not exist any $G \subseteq C$ that is $1-$pairwise stable. Then any $G \subseteq C$ has a proper subset that satisfies **(1)** in the definition of $1-$pairwise stable sets. Now, as in the proof of Proposition

[12] A complete proof would mimic the proof of Lemma 1 in Jackson and Watts (2002, p.273).

3, this implies that there exists an infinite decreasing sequence $\{G^k\}_{k\geq 0}$ where $G^0 = C$ and $G^{k+1} \subsetneq G^k$ for all $k \geq 0$. But since $\#G^N < \infty$, this is not possible; so the proof is completed. ■

Now we are ready to complete the proof of Proposition 6. We first prove the **"if" part**. Suppose that G is a closed cycle but G is not $1-$pairwise stable and show that this lead to a contradiction. This last point can be due to the violation of **(1)** or **(2)** in the definition of a $1-$pairwise stable set. Assume first that **(1)** is violated. Such a violation implies in particular that there exists $g \in G$ and $g' \notin G$ such that $g \to g'$. Which contradicts the definition of a closed cycle. Assume now that **(2)** is violated. This means that there exists $G' \subsetneq G$ that satisfies **(1)** in the definition of a $1-$pairwise stable set i.e., for all $g' \in G'$, $g' \in IM(G')$. But by Lemma 1, we know that there exists a closed cycle $C \subseteq G' \subsetneq G$. Then, we have the following: first, because G is a (closed) cycle, we have that for all $g, g' \in G$, $g \to g'$. But we also have, because C is a closed cycle, that for all $g \in C(\subsetneq G)$ and $g' \in G - C$, $g \to g'$ is wrong. Thus we obtain a contradiction.

We now prove the **"only if" part**. We know by Lemma 1 that since G is $1-$pairwise stable, there exists $C \subseteq G$ that is a closed cycle. We must prove that $C = G$. So let us proceed by contradiction and assume that $C \subsetneq G$. We know by Lemma 2 that there exists $G' \subseteq C \subsetneq G$ that is $1-$pairwise stable. This leads to a straightforward contradiction since it contradicts **(2)** (the minimality) in the $1-$pairwise stability of G. This completes the proof of Proposition 6.

B Proof of Theorem 1

In order to prove Theorem 1, we first introduce some useful definitions and notations.

B.1 Definitions

For a given network g, remember that $im(g) = \{g' \subseteq g^N \mid \text{there exists an improving path from } g' \text{ to } g\}$. A path $\mathbf{p} = \{g_1, ..., g_K\}$ is a sequence of adjacent networks. The resistance of a path $\mathbf{p} = \{g_1, ..., g_K\}$ from g' to g, denoted $r(\mathbf{p})$, is computed by $r(\mathbf{p}) = \sum_{i=1}^{K-1} I(g_i, g_{i+1})$, where

$$I(g_i, g_{i+1}) = \begin{cases} 0 & \text{if } g_i \in im(g_{i+1}) \\ 1 & \text{otherwise} \end{cases}.$$

Resistance keeps track of how many mutations must occur along a special path to follow that path from one network to another. A mutation is necessary to move from one network to an adjacent one whenever it is not in the relevant player's interests to sever or add the link that distinguishes the two adjacent networks.

Let $r(g', g) = \min\{r(\mathbf{p}) \mid \mathbf{p}$ is a path from g' to $g\}$ and set $r(g, g) = 0$. Note that $r(g', g) = 0$ iff $g' \in im(g)$ or $g' = g$. Thus (by proposition 6) if $g, g' \in G$ where G is $1-$pairwise stable, then $r(g', g) = 0$.

Given a network g, a $g-$tree is a directed graph which has as vertices all networks and has a unique directed path leading from each g' to g. Let $T(g)$ denote all the $g-$trees, and represent a $t \in T(g)$ as a collection of ordered pairs of networks, so that $g'g'' \in t$ if and only if there is a directed edge connecting g' to g'' in the $g-$tree t. The resistance of a network g is computed as $r(g) = \min_{t \in T(g)} \sum_{g'g'' \in t} r(g', g'')$.

In addition as noted in Jackson and Watts (2002), only closed cycles (a pairwise stable network is a closed cycle) matter in the dynamic process. Given two closed cycles C, C', let $r(C, C') = r(g, g')$ where $g \in C$ and $g' \in C'$ and set $r(C, C) = 0$. In the sequel, the set of closed cycles will be denoted Ξ.

Given a closed cycle C, a $C-$tree is a directed graph which has as its root C, and as other vertices closed cycles and has a unique directed path from each vertex to C. Denote the set of $C-$trees by $T(C)$, and represent a $t \in T(C)$ as a collection of ordered pairs of networks, so that $C'C'' \in t$ if and only if there is a directed edge connecting C' to C'' in the $C-$tree t. The resistance of a closed cycle C is computed as $r(C) = \min_{t \in T(C)} \sum_{C'C'' \in t} r(C', C'')$.

It is well-known (see Young (1993), Jackson and Watts (2002)) that a network g is a stochastically stable network if and only if g belongs to a closed cycle C such that $r(C) \leq r(C')$ for all $C' \in \Xi$. We will use this characterization in order to prove our main results.

B.2 The Proof

The proof is divided into two parts:

(1) We give a lower bound on the resistance of the transitions that begin at $g \in G$ and end at any $g' \notin G$ where $d(g', G) > 1 - \phi(\frac{1}{2})$. By definition of $p-$pairwise stability for $p \leq \frac{1}{2}$, $r(g, g') > (1 - \phi(\frac{1}{2})) \cdot \#g^N \geq \frac{\#g^N}{2}$.

We give now an upper bound on the resistance of paths that begin at any $g' \notin G$ and end in G. Pick $g' \notin G$. (Note that if $d(g', G) \leq 1 - \phi(\frac{1}{2})$ then, by definition of G, $g' \in IM(G)$ i.e. no mutation is necessary to go to G. Thus we will implicitly assume that $d(g', G) > 1 - \phi(\frac{1}{2})$.) Picking $g \in G$, we delete some elements in $\{(ij \in g' \wedge ij \notin g)\}$ and add some elements in $\{(ij \notin g' \wedge ij \in g)\}$ so that the total number of changes is $(1 - \phi(\frac{1}{2})) \cdot \#g^N$. We obtain a network $\bar{g}$ satisfying $d(\bar{g}, g') = 1 - \phi(\frac{1}{2})$. By construction, this network $\bar{g}$ satisfies $d(\bar{g}, g) \leq \phi(\frac{1}{2}) \leq 1 - \phi(\frac{1}{2})$ where $g \in G$. But G is a $p-$pairwise stable set of networks for $p \leq \frac{1}{2}$ and so $\bar{g} \longmapsto G$. Therefore with less than $(1 - \phi(\frac{1}{2})) \cdot \#g^N$ mutations, we will reach a network in closed cycle included in G (note that once the process has reached G, we cannot leave it without mutations). Therefore, $r(g', \tilde{C}) \leq (1 - \phi(\frac{1}{2})) \cdot \#g^N$ for some closed cycle $\tilde{C} \subset G$. Such a closed cycle will be denoted $C(g')$. Thus for every $g' \notin G$, $r(g', C(g')) \leq (1 - \phi(\frac{1}{2})) \cdot \#g^N$.

(2) Suppose by contradiction that $g' \notin G$ is stochastically stable. Let C' be the closed cycle so that $g' \in C'$. First note that it must be that $d(g', G) > 1 - \phi(\frac{1}{2})$. Denote by t' (one of) the $C'-$tree(s) ($t' \in T(C')$) that minimizes resistance. We know that there is a sequence $C_1, ..., C_n$ with $C_1 = C(g')(\subset G)$ and $C_n = C'$ such that: $C_l C_{l+1} \in t'$ for every $l = 1, ..., n-1$.

In addition, there exists two closed cycles $\tilde{C}$ and $\bar{C}$ such that $\tilde{C}\bar{C} \in t'$ and $\tilde{C} \cap G \neq \emptyset$ and $\bar{C} \cap G = \emptyset$. Delete this edge and add one from C' to C_1. We obtain a tree $t'' \in T(\tilde{C})$ where $\tilde{C} \cap G \neq \emptyset$. It is easy to show that indeed, $\tilde{C} \subset G$ (because once the process has reached G it cannot go out without mutations).

By construction, $r(\tilde{C}) = r(C') - r(\tilde{C}, \bar{C}) + r(C', C_1)$. But $r(\tilde{C}, \bar{C}) = r(\tilde{g}, \bar{g})$ where $\tilde{g} \in G$ and $\bar{g} \notin G$, and so as proved above, $r(\tilde{C}, \bar{C}) > (1 - \phi(\frac{1}{2})) \cdot \#g^N$. In addition, $r(C', C_1) = r(g', C(g'))$ and so again as proved above, $r(C', C_1) \leq (1 - \phi(\frac{1}{2})) \cdot \#g^N$. Hence, $r(\tilde{C}) < r(C')$. This contradicts the fact that g' minimizes stochastic potential. This completes the proof.

C An Alternative Definition to $p-$Pairwise Stability

Definition 4 *Let $p \in [0, 1]$. A network g is $p-$stable with respect to the value function v and to the allocation rule Y if*

1.) *for all $ij \in g$, for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$ and $ij \in g'$,*

$$Y_i(g', v) \geq Y_i(g' - ij, v) \text{ and } Y_j(g', v) \geq Y_j(g' - ij, v),$$

and,

2.) *for all $ij \notin g$, for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$ and $ij \notin g'$,*

$$[Y_i(g', v) < Y_i(g' + ij, v) \Longrightarrow Y_j(g', v) > Y_j(g' + ij, v)].$$

Any network g that is $p-$stable is $p'-$stable for $p' \geq p$. A network g is pairwise stable if and only if it is $1-$stable. Thus, any network g that is $p-$stable is pairwise stable. Two networks g and g' are adjacent if they differ by one link.

No indifference Y and v exhibit no indifference if for any g and g' that are adjacent either g defeats g' or g' defeats g.

Proposition 7 *Suppose that Y and v exhibit no indifference. Then, g is $p-$stable if and only if g is $p-$pairwise stable.*

Proof. We have that g is $1-$stable if and only if g is $1-$pairwise stable. Suppose now that g is $p-$stable with $(1-\phi(p))\cdot\#(g^N)=1$. We will show that g is $p-$pairwise stable. That is, we have to show that for all g' such that $0\leq d(g',g)\leq 1-\phi(p)$ (i.e. for all g' that have up to one link different of g), $g'\in IM(g)$. We have: (i) $g\in IM(g)$. (ii) Player i and player j do not want to delete the link ij such that $ij\in g$ and $ij\in g'$, and they do not want to add the link ij such that $ij\notin g$ and $ij\notin g'$. Denote by kl a link such that $kl\notin g$ but $kl\in g'$, or $kl\in g$ but $kl\notin g'$. Since Y and v exhibit no indifference, we have that: if $kl\in g$ then player k and player l have incentives to add this link to g'; if $kl\notin g$ then player k or player l have incentives to sever this link from g'. Thus, $g'\in IM(g)$.

Suppose now that g is $p-$stable with $(1-\phi(p))\cdot\#(g^N)=2$ and $p'-$pairwise stable with $(1-\phi(p'))\cdot\#(g^N)=1$. We will show that g is $p-$pairwise stable. That is, we have to show that for all g' such that $0\leq d(g',g)\leq 1-\phi(p)$ (i.e. for all g' that have up to two links different of g), $g'\in IM(g)$. We have: (i) $g\in IM(g)$. (ii) For all g'' such that $g''=g-ij$ or $g''=g+ij$, $g''\in IM(g)$. (iii) Take any g' such that g' has 2 links different of g. Player i and player j do not want to delete the link ij such that $ij\in g$ and $ij\in g'$, and they do not want to add the link ij such that $ij\notin g$ and $ij\notin g'$. Denote by kl (or mn) a link such that $kl\notin g$ but $kl\in g'$, or $kl\in g$ but $kl\notin g'$. Since Y and v exhibit no indifference, we have that: if $kl\in g$ then player k and player l have incentives to add this link to g'; if $kl\notin g$ then player k or player l have incentives to sever this link from g'. Indeed, if $g'=g-kl+mn$ or $g'=g-kl-mn$ (i.e. $g'=g''-kl$), players k and l have incentives to create the link kl because none of them had incentives to delete it at $g''=g+mn$ or $g''=g-mn$; if $g'=g+kl+mn$ or $g'=g+kl-mn$ ($g'=g''+kl$), players k and l have incentives to sever the link kl because none of them wanted to create it at $g''=g+mn$ or $g''=g-mn$. Thus, $g'\in IM(g'')$. Since $g''\in IM(g)$ and g' and g'' are adjacent, we have $g'\in IM(g)$.

Suppose now that g is $p-$stable with $(1-\phi(p))\cdot\#(g^N)=L$ and $p'-$pairwise stable with $(1-\phi(p'))\cdot\#(g^N)=L-1$. We will show that g is $p-$pairwise stable. That is, we have to show that for all g' such that $0\leq d(g',g)\leq 1-\phi(p)$ (i.e. for all g' that have up to L links different of g), $g'\in IM(g)$. We have: (i) $g\in IM(g)$. (ii) For all g'' such that $0\leq d(g'',g)\leq 1-\phi(p')$ (i.e. for all g'' that have up to $L-1$ links different of g), $g''\in IM(g)$. (iii) Take any g' such that g' has L links different of g. Player i and player j do not want to delete the link ij such that $ij\in g$ and $ij\in g'$, and they do not want to add the link ij such that $ij\notin g$ and $ij\notin g'$. Denote by kl a link such that $kl\notin g$ but $kl\in g'$, or $kl\in g$ but $kl\notin g'$. Since Y and v exhibit no indifference, we have that: if $kl\in g$ then player k and player l have incentives to add this link to g' (none of them had incentives to delete it at $g'' = g' + kl$); if $kl \notin g$ then player k or player l have incentives to sever this link from g' (none of them wanted to create it at $g'' = g' - kl$). Thus, $g' \in IM(g'')$. Since $g'' \in IM(g)$ and g' and g'' are adjacent, we have $g' \in IM(g)$.

Suppose now that g is $p-$pairwise stable with $(1 - \phi(p)) \cdot \#(g^N) = 1$. We will show that g is $p-$stable. Take g' such that $0 \leq d(g', g) \leq 1 - \phi(p)$ (i.e. networks that have up to one link different of g). Since $g' \in IM(g)$, which means that all improving of maximum length from g' go directly to g, we have that player i and player j do not want to delete the link ij such that $ij \in g$ and $ij \in g'$, and they do not want to add the link ij such that $ij \notin g$ and $ij \notin g'$. Thus, g is $p-$stable with $(1 - \phi(p)) \cdot \#(g^N) = 1$. Suppose now that g is $p-$pairwise stable with $(1 - \phi(p)) \cdot \#(g^N) = 2$. We will show that g is $p-$stable. Take g' such that $0 \leq d(g', g) \leq 1 - \phi(p)$ (i.e. networks that have up to two links different of g). Since $g' \in IM(g)$, which means that all improving of maximum length from g' go directly to g, we have that player i and player j do not want to delete the link ij such that $ij \in g$ and $ij \in g'$, and they do not want to add the link ij such that $ij \notin g$ and $ij \notin g'$. Thus, g is $p-$stable with $(1 - \phi(p)) \cdot \#(g^N) = 2$; and so on. ■

References

[1] Aumann, R. and R. Myerson, "Endogenous Formation of Links between Players and Coalitions: an Application of the Shapley Value," in The Shapley value, A. Roth Ed., Cambridge University Press, 175-191 (1988).

[2] Bala, V. and S. Goyal, "Self-Organization in Communication Networks," *Econometrica* 68, 1131-1230 (2000).

[3] van Damme, E. and J. Weibull, "Evolution in Games with Endogenous Mistake Probabilities," *Journal of Economic Theory* 106, 296-315 (2002).

[4] Durieu, J. Solal, P. and O. Tercieux, "Adaptive Learning and Curb Set Selection," Mimeo. (2003).

[5] Dutta, B. and M.O. Jackson, "The Stability and Efficiency of Directed Communication Networks," *Review of Economic Design* 5, 251-272 (2000).

[6] Ellison, G., "Learning, Local interaction, and Coordination," *Econometrica* 61, 1047-1071 (1993).

[7] Freidlin, M. and A. Wentzell, Random Perturbations of Dynamical Systems. Springer-Verlag, New York (1984).

[8] Garey, R.G. and D.S. Johnson, Computers and Intractability. A Guide to the Theory of NP-Completeness. W. H Freeman and Company (1979)

[9] Harsanyi, J. and R. Selten, A General Theory of Equilibrium in Games, Cambridge: MIT Press (1988).

[10] Jackson, M.O., "The Stability and Efficiency of Economic and Social Networks," in Networks and Groups: Models of Strategic Formation, edited by B. Dutta and M.O. Jackson, Springer-Verlag: Heidelberg (2003).

[11] Jackson, M.O., "A Survey of Models of Network Formation: Stability and Efficiency," in Group Formation in Economics: Networks, Clubs and Coalitions, edited by G. Demange and M. Wooders, Cambridge University Press (2005).

[12] Jackson, M.O. and A. van den Nouweland, "Strongly Stable Networks," *Games and Economic Behavior* 51, 420-444 (2005).

[13] Jackson, M.O. and A. Watts, "The Evolution of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74 (2002).

[14] Jackson, M.O. and A. Wolinsky, "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74 (1996).

[15] Kandori, M., G. Mailath, and R. Rob "Learning, Mutation, and Long Run Equilibria in Games," *Econometrica*, 61, pp.29-56 (1993).

[16] Maruta, T., "On the Relationship Between Risk-Dominance and Stochastic Stability," *Games and Economic Behavior* 19, 221-234 (1997).

[17] Maruta, T., "Binary Games with State Dependent Stochastic Choice," *Journal of Economic Theory* 103, 351-376 (2002).

[18] Tercieux, O, "$p-$Best Response Set," forthcoming in *Journal of Economic Theory* (2005)

[19] Watts, A., "A Dynamic Model of Network Formation," *Games and Economic Behavior* 34, 331-341 (2001).

[20] Young, P., "The Evolution of Conventions," *Econometrica* 61, 57-64 (1993).