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PREFACE

The ever increasing complexity of the systems to be modeled and analyzed, taxes the existing mathematical and numerical techniques far beyond our present day capabilities. By their intrinsic nature, some problems are so difficult to solve that at best we may hope to find a solution to an approximation of the original problem. Stochastic optimization problems, except in a few special cases, are typical examples of this class.

This however raises the question of what is a valid "approximate" to the original problem. The design of the approximation must be such that (i) the solution to the approximate provides approximate solutions to the original problem and (ii) a refinement of the approximation yields a better approximate solution. The classical techniques for approximating functions are of little use in this setting. In fact very simple examples show that classical approximation techniques dramatically fail in meeting the objectives laid out above.

What is needed, at least at a theoretical level, is to design the approximates to the original problem in such a way that they satisfy an epi-convergence criterion. The convergence of the functions defining the problem is to be replaced by the convergence of the sets defined by these functions. That type of convergence has many properties but for our purpose the main one is that it implies the convergence of the (optimal) solutions.

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This article is devoted to the relationship between the epi-convergence and the classical notion of pointwise-convergence. A strong semicontinuity condition is introduced and it is shown to be the link between these two types of convergences. It provides a number of useful criteria which can be used in the design of approximates to difficult problems.
CONVERGENCE OF FUNCTIONS: EQUI-SEMICONTINUITY

Given a space $X$, by $\mathbb{R}^X$ we denote the space of all functions defined on $X$ and with values in $\mathbb{R}$, the extended reals. We are interested in the relationship between various notions of convergence in $\mathbb{R}^X$, in particular between pointwise convergence and that induced by the convergence of the epigraphs. We extend and refine the results of De Giorgi and Franzoni (1975) (collection of "equi-Lipschitzian" functions with respect to pseudonorms) and of Salinetti and Wets (1977) (sequences of convex functions on a reflexive Banach space). The range of applicability of the results is substantially enlarged, in particular the removal of the convexity, reflexivity (Salinetti and Wets 1977) and norm dependence (De Giorgi and Franzoni 1975) assumptions is significant in many applications. The work in this area was motivated by: the search for "valid" approximations to extremal statistical problems, variational inequalities and difficult optimization problems, cf., the above mentioned articles. Also by relying only on minimal properties for the topology of the domain space and for the class
of functions involved, the derivation itself takes on an elementary and insightful character.

By their nature the results are asymmetric; semicontinuity is a one-sided concept. We have chosen to deal with lower semicontinuity and epigraphs rather than upper semicontinuity and hypographs. Every assertion in one setting has its obvious counterpart in the other. This choice however, does condition the addition rule for the extended reals, viz. \((+\infty) + a = +\infty\) for all \(a \in \mathbb{R}\) and \((-\infty) + a = -\infty\) for all \(a \in [-\infty, +\infty]\). Also, note that we are working with the extended reals, thus every collection of elements of \(\mathbb{R}\) has lower and upper bounds in \(\mathbb{R}\); all limits involving extended-real numbers must be interpreted in that sense.

I Limit Functions

Let \((X, \tau)\) be a topological space and \(f\) a generic element of \(\mathbb{R}^X\). The effective domain of \(f\) is

\[
\text{dom } f = \{x \in X | f(x) < +\infty\}
\]

and its epigraph is

\[
\text{epi } f = \{(x, \eta) \in X \times \mathbb{R} | f(x) \leq \eta\}.
\]

The function \(f\) is \(\tau\)-lower semicontinuous (\(\tau\)-l.sc.) if \(\text{epi } f\) is a closed subset of \(X \times \mathbb{R}\), or equivalently if

\[
(d_0) \text{ to each } x \in \text{dom } f \text{ and to each } \varepsilon > 0, \text{ there corresponds a } \tau\text{-neighborhood } V \text{ of } x \text{ such that}
\]

\[
\inf_{y \in V} f(y) \geq f(x) - \varepsilon.
\]
(\neg d_0) to each $x \notin \text{dom } f$ and to each $a \in \mathbb{R}$, there corresponds a $\tau$-neighborhood $V$ of $x$ such that

$$\inf_{y \in V} f(y) \geq a.$$ 

Note that if $\sigma \supseteq \tau$, i.e., $\sigma$ is finer than $\tau$, then $f$ $\tau$-l.sc. implies $f$ $\sigma$-l.sc.

To define limits of collection of functions, i.e., elements of $\mathcal{R}^X$, we adopt the following framework: $N$ is an index space and $\mathcal{H}$ is a filter on $N$. (If $\tau$ has a local countable base at each point, it would be sufficient to consider limits in terms of sequences, unfortunately many interesting functional spaces do not have this property.) The $e_\tau$-limit inferior of a filtered collection of functions $\{f_v, v \in N\}$ is denoted by $\liminf_\tau f_v$, and is defined by

$$\liminf_\tau f_v(x) = \sup_{G \in G_\tau(x)} \sup_{H \in \mathcal{H}} \inf_{v \in H} \inf_{y \in G} f_v(y),$$

where $G_\tau(x)$ is the family of (open) $\tau$-neighborhoods of $x$. The $e_\tau$-limit superior is denoted by $\limsup_\tau f_v$, and is defined similarly,

$$\limsup_\tau f_v(x) = \sup_{G \in G_\tau(x)} \inf_{H \in \mathcal{H}} \sup_{v \in H} \inf_{y \in G} f_v(y).$$

In the literature on $\Gamma$-convergence, these two functions are known respectively as the $\Gamma^-(\tau)$-limit inferior and the $\Gamma^+(\tau)$-limit superior, cf. De Giorgi and Franzoni (1975). By $\mathcal{H}$ we denote the grill associated with the filter $\mathcal{H}$, i.e. the family of subsets of $N$ that meet every set $H$ in $\mathcal{H}$. Given any collection $\{a_v \in \mathcal{R}, v \in N\}$, it is easy to verify the identity

$$\sup_{H \in \mathcal{H}} \inf_{v \in H} a_v = \inf_{H \in \mathcal{H}} \sup_{v \in H} a_v$$

if we observe that $\mathcal{H}$ is the "grill" of $\mathcal{H}$, i.e. the collection of all subsets of $N$ that meet every set in $\mathcal{H}$. From this it follows that
Since \( H \subset H \) it follows directly that

\[
(1.5) \quad \liminf_{\tau} f_\nu \leq \limsup_{\tau} f_\nu .
\]

The collection \( \{f_\nu, \nu \in \mathbb{N}\} \) admits an \( e_\tau \)-limit, denoted by \( \lim_{\tau} f_\nu \), if

\[
\liminf_{\tau} f_\nu = \limsup_{\tau} f_\nu = \lim f_\nu ,
\]

in which case the \( f_\nu \) are said to epi-converge to \( \lim f_\nu \). This terminology is justified by the fact that epi \( \lim f_\nu \) is the limit of the epigraphs of the \( f_\nu \); this is made explicit here below.

The limit inferior \( \liminf C_\nu \) and limit superior \( \limsup C_\nu \) of a filtered collection \( \{C_\nu, \nu \in \mathbb{N}\} \) of subsets of a topological space are defined by

\[
(1.6) \quad \liminf C_\nu = \bigcap_{H \in H} \overline{\bigcup_{\nu \in \mathbb{N}} C_\nu} \quad \text{and}
\]

\[
(1.7) \quad \limsup C_\nu = \bigcap_{H \in H} \overline{\bigcup_{\nu \in \mathbb{N}} C_\nu} \quad \text{and}
\]

Since \( H \subset H \) and thus we always have that

\[
\liminf C_\nu \subset \limsup C_\nu .
\]

The filtered collection \( \{C_\nu, \nu \in \mathbb{N}\} \) is said to have a limit, \( \lim C_\nu \), if the limits inferior and superior coincide, i.e.,

\[
(1.8) \quad \limsup C_\nu = \lim C_\nu = \liminf C_\nu
\]

All these limit sets are closed as follows directly from their definitions.
Proposition 1.9. (Mosco 1969) Suppose that \( \{ f_v, v \in \mathbb{N} \} \subset \mathbb{R}^X \) is a filtered collection of functions. Then

\[
\text{(1.10)} \quad \text{epi} \; \text{li}_\tau f_v = L_s \text{epi} f_v
\]

and

\[
\text{(1.11)} \quad \text{epi} \; L_s f_v = L_l \text{epi} f_v
\]

Proof. We first derive (1.10). From the definition (1.7) of \( L_s \text{epi} f_v \), it follows that \((x,a) \in L_s \text{epi} f_v\) if and only if

\[
(x,a) \in \text{cl}(\bigcup_{v \in H} \text{epi} f_v) \quad \text{for all } H \in H,
\]

or equivalently--because the sets involved are epigraphs--if and only if for all \( \epsilon > 0 \) and \( G \in G_\tau (x) \) such that

\[
G \times [a - \epsilon, +\infty] \cap \bigcup_{v \in H} \text{epi} f_v \neq \emptyset \quad \text{for all } H \in H
\]

or still, if and only if for every \( H \in H \), \( \epsilon > 0 \) and \( G \in G_\tau (x) \)

there correspond \( v \in H \) and \( y \in G \) such that

\[
f_v(y) \geq a - \epsilon.
\]

This holds, if and only if

\[
a \geq \sup_{G \in G_\tau (x)} \sup_{H \in H} \inf_{v \in H} \inf_{y \in G} f_v(y)
\]

and, as follows from (1.1), if and only if \( a \geq (\text{li}_\tau f_v)(x) \) or equivalently, if and only if \((x,a) \in \text{epi} \; \text{li}_\tau f_v\).
In view of (1.4), the proof of (1.11) follows from exactly the same argument with the grill $\tilde{H}$ replacing $H$. 

Corollary 1.12 Given any filtered collection of functions 
\[ \{f_v, v \in \mathbb{N}\} \subset \mathbb{R}^X, \] the functions $\liminf_T f_v$, $\limsup_T f_v$, and $\lim_T f_v$ if it exists, are $\tau$-lower semicontinuous.

Proof. The lower semicontinuity follows directly from (1.10) and (1.11) since they imply that the epigraphs are closed.

We shall be interested in the implications of a change in topology for $X$. In particular, we have the following:

Proposition 1.13. Suppose that $\sigma$ and $\tau$ are two topologies defined on $X$ such that $\sigma \supset \tau$. Then

\begin{equation}
\liminf_T f_v \leq \liminf_\sigma f_v,
\end{equation}

and

\begin{equation}
\limsup_T f_v \leq \limsup_\sigma f_v.
\end{equation}

Proof. This follows from the definitions (1.1) and (1.2) and the fact that $\sigma \supset \tau$ implies that $G_\sigma(x) \supset G_\tau(x)$. 

In some applications, in particular those involving variational inequalities, it is useful to use a stronger notion of limit function. Again, let $\sigma$ and $\tau$ be two topologies defined on $X$, the $e_{T, \sigma}$-limit of a collection of functions $\{f_v, v \in \mathbb{N}\} \subset \mathbb{R}^X$, denoted by $\lim_{T, \sigma} f_v$, exists if
The case of interest is $\sigma \supseteq \tau$, this models the situation when $X$ is a normed linear (functional) space, and $\sigma$ and $\tau$ are respectively the strong and weak topologies; in this setting this limit function is called the Mosco limit, cf. Mosco (1969) and Attouch (1979), for example.

Proposition 1.17. Suppose that $\sigma$ and $\tau$ are two topologies defined on $X$ such that $\sigma \supseteq \tau$. Moreover suppose that $\lim_{\tau, \sigma} f_\nu$ exists. Then

$$\lim_{\sigma} f_\nu = \lim_{\tau, \sigma} f_\nu = \lim_{\tau} f_\nu$$

Proof. This follows directly from Proposition (1.13), inequality (1.5) and the definition (1.16) of $\lim_{\tau, \sigma} f_\nu$. □

II $\tau/\sigma$-EQUI-SEMICONTINUITY

As already indicated in Section I, we are interested in exploring the relationship between the limit functions of a collection of functions $\{f_\nu, \nu \in \mathbb{N}\} \subset \mathbb{R}^X$, when $X$ is equipped with different topologies, say $\sigma$ and $\tau$. The question of the equality between $\lim_{\tau}$ and $\lim_{\sigma}$ was already raised in connection with the existence of the Mosco limit $\lim_{\tau, \sigma}$. Recall also that for variational problems epi-convergence essentially implies the convergence of the solutions, it is thus useful to have conditions that allow us to pass from epi-convergence in a given topology to epi-convergence in a finer topology because of the stronger continuity properties of the solution of the limit problem, consult Attouch (1979), Theorem 2.1, for example. Finally, a special and extreme case is when
\( \sigma = 1 \), the discrete topology. The study of the connections between \( \text{lm} \) and \( \text{lm} \) becomes that of the relationship between epi-convergence and pointwise-convergence. This is particularly useful in the design of approximation schemes for optimization problems. We deal with this special case of pointwise-convergence at the end of this section.

The inequalities (1.14) and (1.15), relating the \( e_\tau \)-limits inferior and superior, become equalities if the family of functions \( \{f_v, v \in \mathbb{N}\} \) is \( \tau/\sigma \)-equi-lower semicontinuous. This property, defined below, is not only sufficient (Theorem 2.3) but is also necessary (Theorem 2.10). It constitutes in fact a sort of compactness condition, this is clarified in Section IV.

**Definition 2.1.** A filtered collection of functions \( \{f_v, v \in \mathbb{N}\} \subset R^X \) is \( \tau/\sigma \)-equi-lower semicontinuous (\( \tau/\sigma \)-eq-l.sc.) if there exists a set \( D \subset X \) such that

\[(d) \text{ given any } x \in D, \text{ to every } \epsilon > 0 \text{ and every } W \subset G_\sigma (x) \text{ there correspond } H \in H \text{ and } V \subset G_\tau (x) \text{ such that for all } v \in H \]

\[ \inf_{y \in V} f_v(y) \geq \inf_{y \in W} f_v(y) - \epsilon , \]

and

\[(\sim d) \text{ given any } x \not\in D, \text{ to every } a \in \mathbb{R} \text{ there correspond } H \in H \text{ and } V \subset G_\tau (x) \text{ such that for all } v \in H \]

\[ \inf_{y \in V} f_v(y) \geq a . \]
We call $D$ the reference set. If $\sigma \subset \tau$, then (d) holds with $V = W$ and $H$ arbitrary, and hence any collection is $\tau/\sigma$-equi-l.sc. with $D = X$. In applications, as far as we can tell, the only case of genuine interest is when $\sigma$ is finer than $\tau$; however, the results are derived for arbitrary topologies.

Proposition 2.2. Suppose that $\sigma_2 \supset \sigma_1$ and $\tau_2 \subset \tau_1$. Then for any collection of functions, $\tau_2/\sigma_2$-equi-lower semi-continuity implies $\tau_1/\sigma_1$-equi-lower semicontinuity.

Proof. Follows simply from the definition (2.1) and the inclusions $G_{\sigma_2}(x) \supset G_{\sigma_1}(x)$ and $G_{\tau_2}(x) \subset G_{\tau_1}(x)$.

Theorem 2.3. Suppose that the filtered collection of functions $\{f_v, v \in \mathbb{N}\} \subset \mathbb{R}^X$ is $\tau/\sigma$-equi-l.sc. Then

(2.4) \[ \liminf_{\sigma} f_v \leq \liminf_{\tau} f_v \]

and

(2.5) \[ \limsup_{\sigma} f_v \leq \limsup_{\tau} f_v \]

Proof. We start with the proof of (2.4). Given $x \in D$ and $\varepsilon > 0$, it follows from the definition of $\liminf_{\sigma} f_v$ that there exists $G_{\varepsilon} \in G_{\sigma}(x)$ and $H_{\varepsilon} \in H$ such that for all $v \in H_{\varepsilon}$

\[ (\liminf_{\sigma} f_v)(x) \leq \left[ \inf_{y \in G_{\varepsilon}} f_v(y) \right] + \varepsilon \]
In turn, (d) guarantees the existence of \( V \in G_\tau(x) \) and \( H' \in H \) such that for all \( \forall \in H' \)

\[
\inf_{y \in G_\varepsilon} f_\forall(y) \leq \inf_{y \in V} f_\forall(y) + \varepsilon,
\]

and hence for all \( \forall \in H' \cap H_\varepsilon \) we have that

\[
(lif_\forall)(x) = \inf_{y \in V} f_\forall(y) + 2\varepsilon.
\]

This yields

\[
(lif_\forall)(x) \leq \sup_{V \in G_\tau(x)} \sup_{H \in H} \inf_{y \in V} f_\forall(y) + 2\varepsilon
= (lif_\forall)(x) + 2\varepsilon.
\]

Since this holds for every \( \varepsilon > 0 \), we have that \( lif_\forall \leq lif_\tau \) on \( D \).

If \( x \notin D \), condition \( \sim d \) implies that for every \( a \in R \), there exists \( V_a \in G_\tau(x) \) and \( H_a \in H \) such that

\[
(lif_\forall)(x) \geq \inf_{y \in H_a} \inf_{y' \in V_a} f_\forall(y) \geq a.
\]

Hence \( lif_\tau(x) = +\infty \) for every \( x \) in \( X \setminus D \) and the inequality \( lif_\forall \leq lif_\tau \) is trivially satisfied.

In view of (1.4), the same argument can be used to derive (2.5) replacing simply \( li \) by \( ls \) and \( H \) by \( \bar{H} \). \( \square \)
Corollary 2.6. Suppose that the filtered collection of functions \( \{ f_v, v \in \mathbb{N} \} \subseteq \overline{\mathbb{R}}^X \) is \( \tau/\sigma\)-equi-\( l.sc. \). Then

\[
(2.7) \quad \text{li}_0 f_v = \text{li}_\tau f_v
\]

and

\[
(2.8) \quad \text{ls}_0 f_v = \text{ls}_\sigma f_v
\]

Moreover dom \( \text{li}_0 f_v = \text{dom} \text{li}_\tau f_v \) is the smallest of all subsets \( D \) of \( X \) with respect to which both \( (d) \) and \( (-d) \) hold for the collection \( \{ f_v, v \in \mathbb{N} \} \), i.e., dom \( \text{li}_0 f_v \) is the smallest possible reference set.

Proof. The equalities follow directly from Theorem 2.5 and the Proposition (1.13). To obtain the last assertion, we note that if \( C \subseteq D \), \( \text{li}_0 f_v = +\infty \) on \( D \setminus C \) and the collection \( \{ f_v, v \in \mathbb{N} \} \) is \( \tau/\sigma\)-equi-\( l.sc. \) with respect to \( D \), it is also \( \tau/\sigma\)-equi-\( l.sc. \) with respect to \( C \). Clearly dom \( \text{li}_0 f_v \) is the smallest such set \( C \) since for any strictly smaller set \( C' \subseteq \text{dom} \text{li}_0 f_v \), \( (-d) \) will fail on \( (\text{dom} \text{li}_0 f_v) \setminus C' \). \( \square \)
Corollary 2.9. (Convergence Theorem). Suppose that $\sigma \supset \tau$ and that the filtered collection of functions $\{f_v, v \in \mathbb{N}\}$ is $\tau/\sigma$-equi-l.sc. then

$$f = \lim_{\tau} f_v$$

if and only if

$$f = \lim_{\sigma} f_v.$$ 

Proof. From $f = \lim_{\tau} f_v$ and Proposition 1.13 it follows that

$$f \leq \lim_{\tau} f_v \leq \lim_{\sigma} f_v.$$ 

On the other hand from the Theorem, more precisely (2.5), the $\tau/\sigma$-equi-l.sc. yields

$$f \geq \lim_{\sigma} f_v \geq \lim_{\tau} f_v,$$

and hence $f = \lim_{\sigma} f_v = \lim_{\tau} f_v = \lim_{\sigma} f_v$ as follows from (1.5).

If $f = \lim_{\sigma} f_v$, then Proposition 1.13 implies that

$$f \geq \lim_{\sigma} f_v \geq \lim_{\tau} f_v.$$

and $\tau/\sigma$-equi-lower semicontinuity yields via (2.4)

$$f \leq \lim_{\sigma} f_v \leq \lim_{\tau} f_v.$$ 

To complete the proof we again appeal to (1.5). $\Box$

The next Theorem shows that $\tau/\sigma$-equi-semicontinuity is a minimal condition that allows to pass from the epi-convergence in one topology to the epi-convergence in another topology.
Theorem 2.10. Suppose that \( \{ f_v, v \in \mathbb{N} \} \subset \mathbb{R}^X \) is a filtered collection of functions such that \( -\infty < \limsup f_v \leq \liminf f_v \). Then the collection \( \{ f_v, v \in \mathbb{N} \} \) is \( \tau/\sigma \)-equi-l.s.c. Moreover if \( \sigma \supset \tau \), then also

\[
\lim_{\sigma} f_v = \lim_{\tau} f_v.
\]

Proof. The equality (2.10) follows from the assumptions via (1.5) and Proposition 1.13. For brevity, let \( f = \lim_{\tau} f_v \). To prove equi-l.s.c. we argue by contradiction. First suppose that \( x \notin \text{dom } f \) and (\( \sim \alpha \) ) fails, i.e., there exists \( a \in \mathbb{R} \) such that for every \( V \in \mathcal{G}_\tau (x) \)

and \( H \in H \) there exists \( v \in H \) and \( y \in V \) with

\[
f_v(y) < a.
\]

Then \( f(x) = (\limf_v)(x) \leq a \), contradicting the hypothesis that \( x \notin \text{dom } f \).

If \( f(x) = (\limf_v)(x) \geq (\limsup f_v)(x) \) is finite and (\( \sim \alpha \) ) fails, it means that there exists \( \varepsilon > 0 \) and \( W \in \mathcal{G}_\sigma (x) \) such that for every \( H \in H \) and \( V \in \mathcal{G}_\tau \)

\[
\varepsilon + \inf_{y \in V} f_v(y) < \inf_{y \in W} f_v(y)
\]

for some \( v \in H \). In particular, this must hold for some \( v' \in H' \) with the pair \( (H', \mathcal{G}_{\tau}) \) constructed as follows. From the definitions of \( \limf \) and \( \limsup \), it follows that

(i) there exist \( G_\varepsilon \in \mathcal{G}_\tau (x) \) and \( H_\varepsilon \in H \) such that

\[
(\limf_v)(x) - \varepsilon/4 \leq \inf_{v \in H_\varepsilon} \inf_{y \in G_\varepsilon} f_v(y),
\]

and
(ii) to \( W \in G_0(x) \), there corresponds \( H_W \in H \) such that

\[
(ls_{t_\nu}^V)(x) + \varepsilon/4 \leq \sup_{V \in H_W} \inf_{y \in W} f_V(y)
\]

Now simply define \( H_\varepsilon \cap H_W = H'(\varepsilon, H) \) and because (d) fails, for some \( \nu' \in H' \)

\[
\varepsilon + \inf_{y \in G_\varepsilon} f_{\nu'}(y) < \inf_{y \in W} f_{\nu'}(y)
\]

and thus

\[
\varepsilon + \inf_{\nu \in H'} \inf_{y \in G_\varepsilon} f_{\nu}(y) < \sup_{\nu \in H'} \inf_{y \in W} f_{\nu}(y)
\]

Hence

\[
f(x) + 3\varepsilon/4 = \varepsilon + (li_{t_\nu}^V)(x) - \varepsilon/4 \leq \varepsilon + \inf_{\nu \in H_\varepsilon} \inf_{y \in G_\varepsilon} f_{\nu}(y)
\]

\[
\leq \varepsilon + \inf_{\nu \in H'} \inf_{y \in G_\varepsilon} f_{\nu}(y) < \sup_{\nu \in H'} \inf_{y \in W} f_{\nu}(y)
\]

\[
\leq \sup_{\nu \in H_W} \inf_{y \in W} f_{\nu}(y) \leq (ls_{t_\nu}^V)(x) + \varepsilon/4 \leq f(x) + \varepsilon/4,
\]

a clear contradiction. \( \Box \)

The pointwise-limit functions of a filtered collection of functions \( \{f_{\nu}, \nu \in \mathbb{N}\} \) are denoted by \( li_{t_\nu}^V \) and \( ls_{t_\nu}^V \) and are defined by

\[
(2.11) \quad li_{t_\nu}^V(x) = \sup_{H \in H} \inf_{\nu \in H} f_{\nu}(x)
\]

and
\[(2.12) \quad \text{ls} \, f_\nu(x) = \inf_{H \in H} \sup_{\nu \in H} f_\nu(x) = \sup_{H \in H} \inf_{\nu \in H} f_\nu(x).\]

The last equality follows from (1.3).

Let \( \tau \) denote the discrete topology on \( X \), then \( G_\tau(x) \) consists of all subsets of \( X \) that contain \( x \). From this it follows that

\[\text{li} \, f_\nu = \text{li}_{\tau} f_\nu \quad \text{and} \quad \text{ls} \, f_\nu = \text{ls}_{\tau} f_\nu,\]

and thus the preceding results also yield the relationship between epi-convergence and pointwise-convergence, for example, (1.14) and (1.15) become

\[(2.13) \quad \text{li}_{\tau} f_\nu \leq \text{li} f_\nu,\]

and

\[(2.14) \quad \text{ls}_{\tau} f_\nu \leq \text{ls} f_\nu.\]

When \( \sigma = \tau \) it is possible to replace (d) by:

\[\text{(d)}_{\tau} \quad \text{given any } x \in D, \text{ to every } \varepsilon > 0 \text{ there corresponds } H \in H \text{ and } \nu \in G_\tau(x) \text{ such that for all } \nu \in H, \]

\[\inf_{y \in V} f_\nu(y) \geq f_\nu(x) - \varepsilon\]

This condition is easier to verify and is in fact equivalent to (d) as we show next. Clearly (d) implies (d) since \( \{x\} \in G_\tau(x) \).

On the other hand, given \( x \in D \), and any \( \varepsilon > 0 \) and \( W \in G_\tau(x) \) (any set containing \( x \)), we always have that
If \((d_p)\) is satisfied, there then exists \(H \in \mathcal{H}\) and \(V \in G_{\tau}(x)\) such that

\[
f_V(x) - \varepsilon \leq \inf_{y \in V} f_V(y)
\]

for all \(v \in H\). Combining the two preceding inequalities we get (d). In this setting, Theorem 2.3 and its corollaries, and Theorem 2.10 become:

Theorem 2.15. Suppose that \(\{f_V, v \in \mathbb{N}\} \subseteq \mathbb{R}^X\) is a filtered collection of functions:

(i) If the collection is \(\tau\)-equi-l.s.c., then

\[
\lim_{\tau} f_V = \lim f_V \quad \text{and} \quad \lim_{\tau} s f_V = \lim s f_V.
\]

Also, \(f = \lim f_V\) if and only if \(f = \lim_{\tau} f_V\).

(ii) If \(-\infty < f = \lim f_V = \lim_{\tau} f_V\), then the collection of functions \(\{f_V, v \in \mathbb{N}\}\) is \(\tau\)-equi-l.s.c.

By means of Proposition 2.2, we obtain as corollaries to the above, a whole slough of convergence results. For example:

Corollary 2.16. Suppose that \(\sigma \subseteq \tau\). If \(f = \lim f_V\) and the filtered collection \(\{f_V, v \in \mathbb{N}\}\) is \(\tau\)-equi-l.s.c., then \(f = \lim_{\tau, \sigma} f_V\). Also, if \(f = \lim_{\sigma, \tau} f_V\) and the collection is \(\tau\)-equi-l.s.c. then \(f = \lim f_V\).
The assertions of Theorem 2.15 remain valid with a weakened version of τ-equi-l.sc., when X is a subset of a linear topological space and the \( \{f_v, v \in \mathbb{N}\} \) are convex functions. For (−d) we substitute the following condition:

(−d\(_C\)) given any \( x \in \text{cl} D \), to every \( a \in \mathbb{R} \) there corresponds \( H \in \mathcal{H} \) and \( V \in \mathcal{G}_\tau(x) \) such that for all \( v \in H \)

\[
\inf_{y \in V} f_v(y) > a.
\]

Obviously (−d) implies (−d\(_C\)), the converse also holds in the "convex" case, but that needs to be argued. To start with, we need the convexity of some limit functions which we obtain as a corollary to the next proposition.

Proposition 2.17. Suppose that \( \{C_v, v \in \mathbb{N}\} \) is a filtered collection of convex subsets of a linear topological space. Then \( \text{Li} \ C_v \) is convex.

Proof. From the definition (1.6) of \( \text{Li} \ C_v \), it follows that \( x \in \text{Li} \ C_v \) if and only if to every neighborhood \( V \) of \( x \), there corresponds \( H \in \mathcal{H} \) such that for all \( v \in H \)

\[
(2.18) \quad C_v \cap V \neq \emptyset.
\]

Now take \( x^0, x^1 \in \text{Li} \ C_v \) and for \( \lambda \in [0,1] \) define

\[
x^\lambda = (1 - \lambda)x^0 + \lambda x^1.
\]

We need to show that if \( V^\lambda \) is a neighborhood of \( x^\lambda \), there exists \( H^\lambda \in \mathcal{H} \) such that \( C_v \cap V \neq \emptyset \) for all \( v \in H^\lambda \). Define

\[
V^0 = V^\lambda - x^\lambda + x^0.
\]
and

$$v^1 = v^\lambda - x^\lambda + x^1.$$  

These are neighborhoods of $x^0$ and $x^1$ and thus there exist $H^0$ and $H^1$ such that (2.18) is satisfied. Let $H^\lambda = H^0 \cap H^1$. Since $H$ is a filter, $H^\lambda \in H$ and clearly for all $v \in H^\lambda$ we have that

$$v^0 \cap C_v \neq \emptyset \quad \text{and} \quad v^1 \cap C_v \neq \emptyset,$$

from which it follows that for all $v \in H^\lambda$

$$v^\lambda \cap C_v \neq \emptyset$$

because all the $C_v$ are convex. □

Corollary 2.19. Suppose that $\{f_v, v \in N\}$ is a filtered collection of convex functions defined on the linear topological space $(X, \tau)$. Then $ls_{\tau} f_v$ is a convex function, and if they exist so are $lm_{\tau} f_v$ and $lm f_v$.

Proof. Recall that a function is convex if and only if its epigraph is convex. Thus the convexity of $ls_{\tau} f_v$ follows from (1.11) and Proposition 2.17 since by assumption all the $\{\text{epi } f_v, v \in N\}$ are convex. The rest follows from the facts that if they exist $lm_{\tau} = ls_{\tau}$ and $lm = lm_{\lambda}$. □
Note however that in general $\lim_{\tau} f_\nu$ is not convex, although the $f_\nu$ are convex. Consider, for example $X = \mathbb{R}$, $\tau$ the natural (or the discrete) topology and for $k = 1, 2, \ldots$ the functions

$$f_{2k}(x) = |x - 1|,$$

and

$$f_{2k-1}(x) = |x + 1|.$$

Then clearly $\lim_{\tau} f_\nu$ is not convex, since

$$\lim_{\tau} f_\nu = \begin{cases} |x + 1| & \text{if } x \leq 0 \\ |x - 1| & \text{if } x \geq 0 \end{cases}.$$
Proposition 2.20. Suppose that \( \{f_v, v \in \mathbb{N}\} \) is a filtered collection of convex functions defined on the linear topological space \( X \). Moreover suppose that either \( -\infty < \lim_{\tau} f_v \) exists or that \( -\infty < \lim f_v \) exists and is \( \tau\)-l.sc.. Then the collection \( \{f_v, v \in \mathbb{N}\} \) is \( \tau\)-equi-l.sc. if and only if it satisfies \( (d_p) \) and \( (\sim d_c) \), with the same reference set \( D \).

Proof. Since \( (\sim d) \) implies \( (\sim d_c) \), the only thing to prove is the converse in the presence of \( (d_p) \), convexity and the existence of a limit function. From the proof of Theorem 2.3, with \( \sigma = I \), we see that \( (d_p) \) implies that \( \text{li} f_v \leq \text{li}_{\tau} f_v \) and that \( \text{ls} f_v \leq \text{ls}_{\tau} f_v \) on \( D \). Similarly that \( (\sim d_c) \) yields the same relations on \( X \setminus \text{cl} D \). Combining these inequalities with (2.13) and (2.14), we have that \( (d_p) \) and \( (\sim d_c) \) imply that

\[
(2.21) \quad \text{li}_{\tau} f_v = \text{li} f_v \quad \text{and} \quad \text{ls}_{\tau} f_v = \text{ls} f_v
\]
on \( X \setminus Q \), where \( Q = \text{cl} D \setminus D \). Moreover, in view of Corollary 2.19, \( \text{ls}_{\tau} f_v \) is always convex and so are \( \text{lim} f_v \) and \( \text{lim}_{\tau} f_v \) if they exist.

If \( -\infty < f = \text{lim} f_v \) exists and is \( \tau\)-l.sc., it follows from the above that \( f = \text{ls}_{\tau} f_v = \text{li}_{\tau} f_v \) on \( X \setminus Q \). Convexity also yields the equality on \( Q \). We argue this by contradiction. Suppose to the contrary that for some \( x^1 \in Q \)

\[
a = (\text{li}_{\tau} f_v)(x^1) < f(x^1) .
\]

Take \( x^0 \in \text{dom} f \subseteq D \), and without loss of generality, assume that \( f(x^0) = 0 \). Given any \( \epsilon > 0 \), \( G \subseteq C_{\tau}(x^1) \), \( H \subseteq H \), the definition of \( \text{li}_{\tau} \) yields \( v_\epsilon \subseteq H \) and \( y_\epsilon \in G \) such that
For \( \lambda \in [0,1] \), define

\[
x_{H,G}^\lambda = (1 - \lambda)x^0 + \lambda y_{\varepsilon}.
\]

The convexity of the \( f_{\varepsilon} \), implies that

\[
f_{\varepsilon}(x_{H,G}^\lambda) \leq (1 - \lambda)f_{\varepsilon}(x^0) + \lambda f_{\varepsilon}(y_{\varepsilon}) \leq (1 - \lambda)f_{\varepsilon}(x^0) + \lambda(\alpha + \varepsilon).
\]

Now note that for any fixed \( \lambda \in [0,1] \), \( x_{H,G}^\lambda = (1 - \lambda)x^0 + \lambda x^1 \) is a limit point of the filtered collection \( \{x_{H,G}^\lambda, (H,G) \in H \times C_\tau(x^1)\} \). Hence, we have that for every \( \lambda \in [0,1] \)

\[
f(\lambda^\lambda) \leq (\liminf_{\tau} f_{\varepsilon})(x^\lambda) \leq (1 - \lambda)f(x^0) + \lambda \alpha = \lambda \alpha.
\]

Let \( \lambda \downarrow 1 \). From the lower semicontinuity of \( f \) we get that \( f(x^1) \leq \alpha \), contradicting our working hypothesis. And thus we have shown that \( \lim_{\tau} f_{\varepsilon} = \lim f_{\varepsilon} = \alpha \) on \( \mathcal{X} \), and hence the collection is \( \tau \)-equi-l.sc. as follows from Theorem 2.10, with \( \sigma = 1 \).

On the other hand, if \( f = \lim_{\tau} f_{\varepsilon} \) exists and the collection of convex functions \( \{f_{\varepsilon}, \nu \in \mathbb{N}\} \) satisfies \( (d_p) \) and \( (d_c) \) with respect to \( \mathcal{D} \) (necessarily containing \( \text{dom} f \)), it follows from (2.21) that on \( \mathcal{X} \setminus \mathcal{Q} \),

\[
\lim_{\tau} f_{\varepsilon} = \liminf_{\nu} f_{\varepsilon} = \limsup_{\nu} f_{\varepsilon}.
\]

Corollary 2.6 implies that \( \mathcal{D} \supset \text{dom} \lim_{\tau} f_{\varepsilon} \) and thus \( \lim_{\tau} f_{\varepsilon} = +\infty \) on \( \mathcal{Q} \). By (2.13), on all of \( \mathcal{X} \) we have that

\[
f = \lim_{\tau} f_{\varepsilon} \leq \liminf_{\nu} f_{\varepsilon} \leq \limsup_{\nu} f_{\varepsilon}.
\]
from which it follows that on $Q$, $f = \lim f_v = 1s f_v = +\infty$. Thus we have shown that on all of $X$, $\lim f_v = f = \lim f_v$. Again with $\sigma = 1$ Theorem (2.10) then yields the $\tau$-equi-l.sc. of the $f_v$. □

When $X$ is a reflexive Banach space and the $\{f_v, v \in N\}$ are convex, the original definition of $\tau$-equi-l.sc., as given in Salinetti and Wets (1977), coincides with this weakened version involving $(d_p)$ and $(\sim d_c)$. Condition (a) of Salinetti and Wets (1977) is precisely $(d_p)$. In general $(\sim d_c)$ implies (γ) of Salinetti and Wets (1977) and because the closed balls of a reflexive space are weakly compact (γ) implies $(\sim d_c)$. Condition (β) of Salinetti and Wets (1977) is automatically satisfied if the functions $f_v$ converge pointwise (Salinetti and Wets, 1977, Lemma 2.ii) and it is implied by $(d_p)$ and $(\sim d_c)$ if the $f_v$ epi-converge. Thus, Theorem 1., 2. and 3. of Salinetti and Wets (1977) are special cases of Theorem 2.15 and Corollary 2.16.

III THE HYPERSPACE OF CLOSED SETS

Let $(Y, \eta)$ be a topological space. In this section we have collected some facts about the (hyper)space of closed subsets of $Y$ equipped with the topology of set-convergence, as defined by (1.8). This turns out to be a variant of the Vietoris finite topology, at least when $(Y, \eta)$ is separated (Hausdorff) and locally compact. The results found in this section can be extracted from articles by Choquet (1947-48), and by Michael (1951) and from the book by Kuratowski (1958).

By $F_Y$, or simply $F$ if no confusion is possible, we denote the hyperspace of closed subsets of $Y$. The topology $T$ on $F$ is generated by the subbase of open sets:
where $K$ and $G$ are the hyperspaces of compact and open subsets of $Y$ respectively, and for any $Q \subseteq Y$.

$$F^Q = \{ F \in F | F \cap Q = \emptyset \}$$

and

$$F_Q = \{ F \in F | F \cap Q \neq \emptyset \}.$$

Proposition 3.1. Suppose that $Y$ is separated and locally compact, $(C_v, v \in N)$ is a filtered collection of subsets of $Y$, and $C \subseteq Y$ is closed. Then

(i) $C \subseteq L C_v$ if and only if to every $G \in G$ such that $C \cap G \neq \emptyset$, there corresponds $H_G \in H$ such that for every $v \in H_G$, $C_v \cap G \neq \emptyset$.

(ii) $C \supseteq L C_v$ if and only if to every $K \in K$ such that $C \cap K = \emptyset$, there corresponds $H_K \in H$ such that for every $v \in H_K$, $C_v \in K = \emptyset$.

Moreover $C = \liminf C_v$ if and only if $C = \limsup C_v$.

Proof. It will be sufficient to prove (i) and (ii) since the last assertion follows immediately from (i) and (ii) and the construction of $T$.

Suppose first that $x \in C$, then $C \cap G \neq \emptyset$ for all $G \in G_n(x)$.

The "if" part of (i), implies that $C_v \cap G \neq \emptyset$ for all $v \in H_G$ with $H_G \in H$. Every $H'$ in $H$ meets every $H \in H$ and hence

$$(U_{v \in H} C_v) \cap G \neq \emptyset$$

for every $H \in H$ and $G \in G_n(x)$. Thus for every $H \in H$, $x \in cl(U_{v \in H} C_v)$ and consequently by (1.5) $x \in L C_v$, i.e., $C \subseteq L C_v$.
If $C \subseteq L \varepsilon C_v$, then $C \cap G \not= \emptyset$ implies that $G \cap \left( \bigcap_{H \in \mathcal{H}} \text{cl}(U_{v \in H} C_v) \right) \not= \emptyset$, i.e., for every $H \in \mathcal{H}$

$$(U_{v \in H} C_v) \cap G \not= \emptyset$$

or equivalently there exists $H \subseteq H$ such that for all $v \in H$, $C_v \cap G \not= \emptyset$, again because $H$ consists of all the subsets of $\mathbb{N}$ that meet every set in $\mathcal{H}$. This completes the proof of (i).

Suppose that $x \in L \varepsilon C_v$, then for every $H \in \mathcal{H}$, $x \in \text{cl}(U_{v \in H} C_v)$, cf. (1.6). If $x \not\in C$, by local compactness of $Y$, there is a compact neighborhood $K$ of $x$ such that $K \cap C = \emptyset$. The "if" part of (ii) then implies that $K \cap \left( \bigcup_{v \in H} C_v \right) = \emptyset$ for some $H \subseteq H$, i.e.,

$x \not\in \text{cl}(U_{v \in H} C_v)$ contradicting the assumption that $x \in L \varepsilon C_v$.

Now suppose that $C \supseteq L \varepsilon C_v$, $C \cap K = \emptyset$, but for every $H \in \mathcal{H}$ we can find $v$ such that $C_v \cap K \not= \emptyset$, i.e., there exists $H' \subseteq H$ such that $C_v \cap K \not= \emptyset$ for every $v \in H'$. Since $K$ is compact, it follows that the $\{C_v \cap K, v \in H'\}$ admit at least one cluster point $x \in K$. Then for every $H \in H$

$$x \in \text{cl}(U_{v \in H} C_v) \cap K$$

and consequently $x \in L \varepsilon C_v \cap K$. But this contradicts the assumption that $C \supseteq L \varepsilon C_v$.

Thus $T$ is indeed the topology of set-convergence as defined in Section I. The next Proposition yields the properties of $(F, T)$ that are needed in the sequel.

Proposition 3.2. Suppose that $Y$ is separated (Hausdorff) and locally compact. Then $(F, T)$ is regular and compact.
Proof. By construction the sets \{F_K; K \in K\} and \{F^G; G \in G\} are the complements of open (base) sets, and thus are closed. In particular, this implies that singletons are closed, since

\[ F = (\bigcap_{y \in F} F_y) \cap F^G, \]

\[ G = Y \setminus F \text{ is open.} \]

To see that \((F, \tau)\) is separated, let \(F_1\) and \(F_2\) be two subsets of \(F\) such that \(F_1 \neq F_2\). Then there is some \(y\) that belongs to \(F_1\) but not to \(F_2\) (or vice-versa). Since \(Y\) is locally compact by assumption and \(F\) is closed, there exists \(K^0\), an open precompact neighborhood of \(y\), such that \(K = \text{cl} K^0\) is disjoint of \(F_2\). Hence

\[ F_1 \in F_{K^0} \quad \text{and} \quad F_2 \in F^K. \]

The compactness of \((F, \tau)\) follows from Alexander's characterization of compactness in terms of the finite intersection property of a subbase of closed (hyper)sets. Suppose that

\[ (\bigcap_{i \in I} F_{K_i}) \cap (\bigcap_{j \in J} F^G_j) = \emptyset \]

where \(K_i \in K\), \(G_j \in G\) and, \(I\) and \(J\) are arbitrary index sets. We must show that the family of sets \(\{K_i, i \in I; G_j, j \in J\}\) contains a finite subfamily that has an empty intersection. Let \(G = \bigcup_{j \in J} G_j\) and note that \(G \in G\). Now observe that (3.3) holds if and only if

\[ \bigcap_{i \in I} (F_{K_i} \cap F^G) = \emptyset \]

or still, if and only if for some \(i_0 \in I, F_{K_{i_0}} \cap F^G = \emptyset\), or
equivalently, if and only if there exists $i_0 \in I$ such that $K_{i_0} \subseteq G$

But $K_{i_0}$ is compact and thus the open cover $\{G_j, j \in J\}$ contains a finite subcover $\{G_{j_1}, \ldots, G_{j_q}\}$. Hence (3.3) holds if and only if

$$F_{K_{i_0}} \cap \bigcap_{i=1}^{q} F_{G_{j_i}} = \emptyset$$

Since $(F, T)$ is compact and separated, it is also regular. $\square$

IV COMPACTNESS CRITERIA FOR SPACES OF SEMICONTINUOUS FUNCTIONS

The relationship between pointwise- and $e_\tau$-limits through equi-semicontinuity suggests a number of compactness criteria for spaces of semicontinuous and continuous functions, the celebrated Arzelà-Ascoli Theorem being a special case of these. Our approach in fact provides an unconventional proof of this classical result.

Although a few of the (weaker) subsequent statements remain valid in a more general setting, we shall assume henceforth that the domain-space $(X, \tau)$ is separated and locally compact. Let $SC(X)$ be the space of $\tau$-l.sc. functions with range $\mathbb{R}$ and domain $X$. The elements of $SC(X)$ are in one-to-one correspondence with the elements of $E$, the hyperspace of epigraphs, i.e. the closed subsets $E$ of $Y = X \times \mathbb{R}$ such that $(x, a) \in E$ implies that $(x, b) \in E$ for all $b \geq a$. Note that $\emptyset \in E$ and corresponds to the (continuous) function $f \equiv +\infty$. $E$ is a subset of $F_Y$, the hyperspace of closed subsets of $Y = X \times \mathbb{R}$. 
Proposition 4.1. Suppose that \((X,\tau)\) is separated and locally compact. Then \(E \subset F_Y\) is compact with respect to the \(\tau\) topology. Moreover the \(\tau\)-relative topology on \(E\) can be generated by the subbase of open sets:

\[
\{E^K,a ; K \in K_X, a \in \mathbb{R}\}
\]

and

\[
\{E_G,a^o ; G \in G_X, a \in \mathbb{R}\},
\]

where for any \(Q \subset X\) and \(a \in \mathbb{R}\)

\[
E^{Q,a} = \{E \in E | E \cap (Q x ] - \infty , a)] = \emptyset \}
\]

and

\[
E_{Q,a^o} = \{E \in E | E \cap (Q x ] - \infty , a[ \neq \emptyset \}
\]

Proof. Suppose \(F \in F_Y \setminus E\), then there exists \(x \in X\) and \(a < b\) such that \((x,a) \in F\) but \((x,b) \notin F\). The local compactness of \(X\) yields an open precompact set \(K^o\) such that

\[
F^{K \times \{b\} \cap F^0 \times \{a-\epsilon, a+\epsilon[},
\]

with \(K = \text{cl} K^0\) and \(0 < \epsilon < b - a\), is an open neighbourhood of \(F\) that does not contain any epigraphs. Thus \(F \setminus E\) is open or equivalently \(E\) is closed. Since \(F\) is compact, so is \(E\).

To see that the \(\tau\)-relative topology on \(E\) can be generated the subbase described above, note that the topological properties of \(Y = X \times \mathbb{R}\) imply that the sets of the type
and

\[ \{ f_{X}^{K}(a,b) : K \in K_{X}, a,b \in \mathbb{R} \} \]

also are a subbase for \( T \) on \( F_{Y} \). The restriction of this subbase to \( E \), yields

\[ E^{K}(a,b) = E^{K},a \]

and

\[ E^{G}(a,b) = E^{G},o \]

Combining Propositions 3.2 and 4.1 we get:

Corollary 4.2. The topological space \((E,T)\) is regular and compact.

From Propositions 1.9, 3.1 and 4.1, with \( e_{T} \) the topology of epi-convergence in \( SC(X) \), we also get:

Corollary 4.3. The topological space \((SC(X), e_{T})\) is regular and compact.

The above implies that any closed subset of \( SC \) is compact. In particular, note that for any \( a \in \mathbb{R} \) and \( D \subset X \), the set

\[ SC^{a}(D) = \{ f \in SC | f \leq a \text{ on } D \} = \bigcap_{x \in D} \{ f \in SC | f(x) \leq a \} \]

is compact. To see this simply observe that \( \{ f \in SC | f(x) \leq a \} \) is closed since it corresponds in \( E \) to the \( T \)-closed set

\[ \{ E \in E | E \cap \{(x)_{X}]-\infty,a]\} \neq \emptyset \]
Also, for any $a \in \mathbb{R}$ and any open $G \subseteq X$, the set

$$SC_a(G) = \{f \in SC | f \geq a \text{ on } G\}$$

is closed since it corresponds in $E$ to the $T$-closed set

$$\{E \in E | E \cap (G \times [-\infty, a]) = \emptyset\}.$$

We have just shown that:

Corollary 4.4. Any bounded collection of $\tau$-$l.sc.$ functions is a compact subset of $(SC(X), e_T)$.

The topological space $(SC, p)$ is the space of $\tau$-$l.sc.$ functions equipped with the topology of pointwise convergence. We already know that neither pointwise nor epigraph-convergence implies the other. However, in view of Theorem 2.15, these topologies coincide on $\tau$-equi-$l.sc.$ subsets of $SC$:

Definition 4.5. A set $A \subseteq SC(X)$ is equi-$l.sc.$ if there exists a set $D \subseteq X$ such that

$$(d_{SC}) \text{ given any } x \in D, \text{ to every } \varepsilon > 0, \text{ there corresponds }$$

$$V \in G_{\tau}(x) \text{ such that for every } f \in A$$

$$\inf_{y \in V} f(y) \geq f(x) - \varepsilon,$$

and

$$(\sim d_{SC}) \text{ given any } x \notin D, \text{ to every } a \in \mathbb{R} \text{ there corresponds }$$

$$V \in G_{\tau}(x) \text{ such that for all } f \in A,$$

$$\inf_{y \in V} f(y) \geq a.$$
Theorem 4.6. Suppose that \((X,\tau)\) is separated and locally compact. Then any \(\tau\)-equi-l.sc. family of \(\tau\)-l.sc. functions contains a (filtered) subfamily converging pointwise to a \(\tau\)-l.sc. function. Moreover, if the family of functions is bounded, it contains a subfamily converging pointwise to a bounded \(\tau\)-l.sc. function.

Proof. As follows from Theorem \((2.15)\), for \(\tau\)-equi-l.sc. subsets of \(SC(X)\), the \(p\)-closure or \(e_\tau\)-closure coincide. The first statement then follows from Corollary 4.3 and the second from Corollary 4.4. \(\Box\)

Every property derived for \((SC(X),e_\tau)\) has its counterpart in \((-SC(X),-e_\tau)\), the space of \(\tau\)-upper semicontinuous functions (\(\tau\)-u.sc.) with the topology \(-e_\tau\) of hypo(graph)-convergence. In particular, \((-SC(X),-e_\tau)\) is compact and any bounded subfamily is precompact. And thus, any \(\tau\)-equi-u.sc. family of (bounded) u.sc. functions contains a subfamily converging pointwise to a (bounded) \(\tau\)-u.sc. function.

Given \(\{f_v, v \in \mathbb{N}\}\) a filtered collection of functions, the \(-e_\tau\)-limit inferior is \(-\text{ls}_{\tau} f_v\) and the \(-e_\tau\)-limit superior is \(-\text{li}_{\tau} f_v\). The hypographs of these functions being precisely \(\text{Ll} \text{ hypo } f_v\) and \(\text{Ls} \text{ hypo } f_v\). We always have that

\[\text{li}_{\tau} f_v \leq \text{li} f_v = -(\text{ls}_{\tau} f_v) \leq -(\text{ls}_{\tau} f_v)\]

and

\[\text{ls}_{\tau} f_v \leq \text{ls} f_v = -(\text{li} f_v) \leq -(\text{li} f_v)\]

In each one of the preceding expressions, the first (second resp.) inequality becomes an equality if the collection is \(\tau\)-equi-l.sc. (\(\tau\)-equi-u.sc. resp.).
Let $\tilde{C}(X) = SC(X) \cap -SC(X)$ be the space of continuous extended-real valued functions, $\iota e_{\tau}$ the join of the two topologies $e_{\tau}$ and $-e_{\tau}$, and again $p$ the topology of pointwise convergence. In general $(\tilde{C}(X), \iota e_{\tau})$ is not compact but as we shall see, its equi-continuous subsets are precompact. A subset $A \subseteq \tilde{C}(X)$ is equi-continuous if it is both $\tau$-equi-l.sc. and $\tau$-equi-u.sc. with the same reference set $D$ being used in the verification of the equi-sc. conditions. (Note that necessarily $D$ must be open.)

**Proposition 4.7.** Suppose that $X$ is separated and locally compact. Then $A \subseteq \tilde{C}(X)$ is precompact (with respect to $\iota e_{\tau}$) if and only if it is equi-continuous.

**Proof.** If $A$ is equi-continuous, it is equi-l.sc. and hence every subset of $A$ contains a filtered family $\{f_v, v \in \mathbb{N}\}$ such that $\lim_{\tau} f_v = \lim f_v$, but by assumption the $\{f_v, v \in \mathbb{N}\}$ are also equi-u.sc. and thus contain a subfamily (a finer filter on $\mathbb{N}$) such that

$$\lim_{\tau} f_v = \lim f_v = -(\lim_{\tau} -f_v)$$

from it follows that $A$ is precompact.

On the other hand, if $A$ is not equi-continuous, then assume for example, that $\tau$-equi-lower semicontinuity fails. This means that for some collection of functions $\{f_v, v \in \mathbb{N}\}$ and some $x$, we have that

$$(\lim_{\tau} f_v)(x) < (\lim f_v)(x) = -(\lim_{\tau} -f_v)(x) < -(\lim_{\tau} -f_v)(x).$$

Hence there is obviously no subcollection of the $\{f_v\}$ whose hypographs converge to $\lim_{\tau} f_v$, since at $x$ the $-e_{\tau}$-limit inferior of the $\{f_v\}$ is strictly larger than $\lim_{\tau} f_v(x)$. Thus $A$ cannot be precompact. □
Finally, we consider the space $C(X)$ of continuous real-valued functions with the topologies $\tau_e$, $p$ and $\| \cdot \|$, the last one being the sup-norm topology induced by the pseudo-norm defined by

$$\| f \| = \sup_{x \in X} |f(x)| .$$

This pseudo-norm induces a topology on $C$. The fundamental system of neighborhoods of an element $f$ is defined by the sets

$$\{ g \in C \mid \| f - g \| < a \} \text{ with } a > 0. \text{ Note that if } X \text{ is compact, then } \| \cdot \| \text{ is a norm on } C(X) \text{ and the topology } \| \cdot \| : C \to \tau \text{ as can easily be verified. In general however these two topologies are not comparable.}$$

**Theorem 4.8.** Suppose that $X$ is separated and locally compact and $A \subseteq C(X)$ is equi-continuous and bounded. Then $A$ is $\tau_e$-precompact.

**Proof.** This follows from the fact that bounded subsets of $SC(X)$ and $-SC(X)$ are $e_\tau$ and $-e_\tau$-compact respectively, cf. Corollary 4.4. As in Proposition 4.7 equi-continuity providing the link between the limit functions. □

**Corollary 4.9.** (Arzelà-Ascoli) Suppose that $X$ is separated and compact. Then $A$ is precompact, with respect to the $\tau_e$-topology, and consequently with respect to the $\| \cdot \|$ topology, if and only if $A$ is equi-continuous and bounded.

Sufficiency follows from Theorem 4.8. The necessity of equi-continuity is argued as in Proposition 4.7. Finally, if $A$ is unbounded, there exist $\{ f_v, v \in \mathbb{N} \}$ and $\{ x_v, v \in \mathbb{N} \}$ such that $f_v(x_v) \to -\infty$ (or $+\infty$). The compactness of $X$ implies that the family $\{ x_v, v \in \mathbb{N} \}$ admits an accumulation point, say $x$. Then $(\liminf f_v)(x) = -\infty$ (or $-(\limsup f)(x) = +\infty$) and hence the $\tau_e$-closure of $A$ can not be in $C(X)$ if $A$ is unbounded. □
APPENDIX

There is an intimate connection between the semicontinuity properties of multifunctions and the convergence of (filtered) families of sets. The appendix is devoted to clarifying these relations; most of this can be found in one form or another in Choquet (1947-1948) or Kuratowski (1958).

A map $\Gamma$ with domain $Y$ and whose values are subsets of $X$ (possibly the empty set) is called a multifunction. The graph of $\Gamma$ is

$$\text{grph } \Gamma = \{ (y,x) \in Y \times X | x \in \Gamma(y) \} .$$

We recall that the image of $A \subset Y$ is $\Gamma A = \bigcup_{y \in A} \Gamma(y)$ and the preimage of $B \subset X$ is $\Gamma^{-1} B = \{ y \in Y | \Gamma(y) \cap B = \emptyset \} .$

A neighborhood base $\mathcal{B}(y_0)$ of $y_0 \in Y$ is a filter base on $Y.$ A multifunction $\Gamma$ is said to be upper semicontinuous (u.s.c.) at $y_0$ whenever

$$\text{(Ls } \Gamma)(y_0) = \bigcap_{W \in \mathcal{B}(y_0)} \text{cl } W \subset \Gamma(y_0).$$
or equivalently if to each \( x^0 \not\in \Gamma(y^0) \) we can associate neighborhoods \( Q \) of \( x^0 \) and \( W \) of \( y^0 \) such that \( \Gamma W \cap Q = \emptyset \). Note that \( \Gamma \) is u.sc. (at every \( y \)) if and only if \( \text{grph } \Gamma \) is closed.

In the literature one can find a couple of closely connected definitions of upper semicontinuity. A multifunction \( \Gamma \) is said to be \( K\)-u.sc. at \( y^0 \), if to each closed set \( F \) disjoint of \( \Gamma(y^0) \) there corresponds a neighborhood \( W \) of \( y_0 \) such that \( \Gamma W \cap F = \emptyset \), or equivalently if to each open set \( G \) that includes \( \Gamma(y^0) \) there corresponds a neighborhood \( W \) of \( y_0 \) such that \( \Gamma W \subseteq G \). If \( X \) is regular, then \( \Gamma \) closed-valued and \( K\)-u.sc. at \( y_0 \) implies \( \Gamma \) u.sc. at \( y_0 \). If \( X \) is compact and \( \Gamma \) is closed-valued at \( y_0 \) then both notions coincide.

A multifunction is said to be \( C\)-u.sc. at \( y_0 \), if to each compact set \( K \) disjoint of \( \Gamma(y_0) \) there corresponds \( V \) a neighborhood of \( y_0 \) such that \( \Gamma V \cap K = \emptyset \). Obviously u.sc. implies \( C\)-u.sc. . The converse can be obtained with anyone of the following assumptions

(i) \( X \) is locally compact,

(ii) \( \Gamma^{-1} \) is \( K\)-u.sc. at every \( x_0 \) (for example, if \( f : Y \to X \) is a continuous function and \( \Gamma = f^{-1} \), then \( \Gamma^{-1} \) is \( K\)-u.sc.),

(iii) \( X \) is metizable, \( y_0 \) has a countable neighborhood base and \( \Gamma y_0 \) is closed, cf. Dolecki (1980).

The proof of the last assertion proceeds as follows: Suppose that \( \Gamma \) is not u.sc. at \( y_0 \). Then there exists \( x_0 \not\in \Gamma y_0 \) and neighborhood bases \( \{ Q_v, v = 1, 2, \ldots \} \) of \( x_0 \) and \( \{ W_v, v = 1, 2, \ldots \} \) of \( y_0 \) such that for all \( v \),
\[ \Gamma y_0 \cap Q_v \neq \emptyset \]

because \( \Gamma y_0 \) is closed, and for all \( v \)

\[ \Gamma W_v \cap Q_v \neq \emptyset \]

because \( \Gamma \) is not u.s.c. at \( y_0 \). For every \( v \), pick \( x_v \in \Gamma W_v \cap Q_v \).

The set \( K = \{x_1, x_2, \ldots, x_0\} \subset X \) is compact (every subsequence converges to \( x_0 \)) and disjoint of \( \Gamma y_0 \) but meets every \( \Gamma W \). This contradicts the C-u.s.c. of \( \Gamma \) at \( y_0 \).

A multifunction is lower semicontinuous (l.s.c.) at \( y_0 \) if

\[ \Gamma(y_0) \subset (L \Gamma)(y_0) = \cap_{V \in \mathcal{B}(y_0)} \text{cl} \Gamma V \]

where \( \mathcal{B}(y_0) \) is the grill associated to the filter base \( \mathcal{B}(y_0) \), or equivalently if \( \Gamma^{-1}G \) is a neighborhood of \( y_0 \) whenever \( G \) is an open set that meets \( \Gamma(y_0) \).

For a given set \( X \), we denote by \( P(X) \) the power set of \( X \), i.e., the hyperspace containing all subsets of \( X \), by \( F(X) = \mathcal{F} \) the hyperspace of closed subsets of \( X \), and \( \mathcal{O} = \mathcal{F} \setminus \{\emptyset\} \). We now consider the multifunction \( \Lambda \) from \( P(X) \) into \( X \) defined by \( \Lambda Q = Q \).

We have that \( \Lambda^{-1}_A = \{Q|Q \cap A \neq \emptyset\} \) and \( (\Lambda^{-1}_A)^C = \{F|F \cap A^C\} \).

We restrict \( \Lambda \) to \( \mathcal{F} \). The sets \( \{A^{-1}G, G \text{ open}\} \) form a subbase for a topology on \( \mathcal{O}F \) (but not for \( F \)). Similarly, the collection \( \{(A^{-1}K)^C, K \text{ compact}\} \) constitutes a subbase for another topology on \( F \). The supremum of these two topologies yields a topology \( T \) on \( F \). It is the coarsest topology for which \( \Lambda \) is both l.s.c. and C-u.s.c. The topology \( V \), the Vietoris topology, on \( F \) has a subbase consisting of the collections \( \{A^{-1}G, G \text{ open}\} \) and \( \{(A^{-1}F)^C, F \text{ closed}\} \). It is the coarsest topology for which the multifunction \( \Lambda : F \rightharpoonup X \) is l.s.c. and K-u.s.c.
1. When convergence in the $\tau$ topology can be defined in terms of sequential convergence, the limit functions can also be obtained as follows: let $N = \{1, 2, \ldots\}$, then

$$
(l_{\tau} f_v)(x) = \inf_{\{v_{\mu}\} \subseteq N} \liminf_{\mu \in N} f_{v_{\mu}}(x_{\mu})
$$

and

$$
(l_{\sigma} f_v)(x) = \inf_{\{x_{v_{\mu}} \to x\}} \limsup_{\nu} f_{v_{\nu}}(x_{\nu}),
$$

where in the first expression the infimum is over all sub-sequences of functions $\{f_{v_{\mu}}, \mu \in N\}$ and all sequences $\{x_{\mu}, \mu \in N\}$ converging to $x$.

2. A function $f$ from $X$ to $\bar{R}$ is $\tau/\sigma$-$l.sc.$ if (d) and ($\sim$d) hold with $\mathcal{V} = \text{dom } f$ and $f_{v_{\nu}} = f$ for all $\nu \in N$. If $\tau \supset \sigma$ the concept is essentially meaningless since then any function $f \in \bar{R}^X$ is then $\tau/\sigma$-$l.sc.$. If $\sigma \supset \tau$, then $f$ is $\tau/\sigma$-$l.sc.$ if and only if $\tau\text{-cl}(\sigma\text{-cl epi } f) = \sigma\text{-cl epi } f$. In particular if $\sigma = 1$ then $\tau/1$-$l.sc.$ corresponds to the usual notion of $\tau$-$l.sc.$.
REFERENCES


