Mathematical Catastrophe Revisited

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ABSTRACT

Different models of mathematical catastrophes of single variable have been studied, and finally from the substance we have found a mathematical analysis of mathematical catastrophe. Two mathematical catastrophes viz, fold catastrophe and cusp catastrophe are broadly discussed in order to find some results newly in both the cases.

Key words: *Degenerate critical point, extremum, maximum, minimum, catastrophe, increasing function, decreasing function.*

1. Introduction:

A mathematical catastrophe is a point in a model of an input-output system, where a diminutive change in the input can produce a large change in the output.

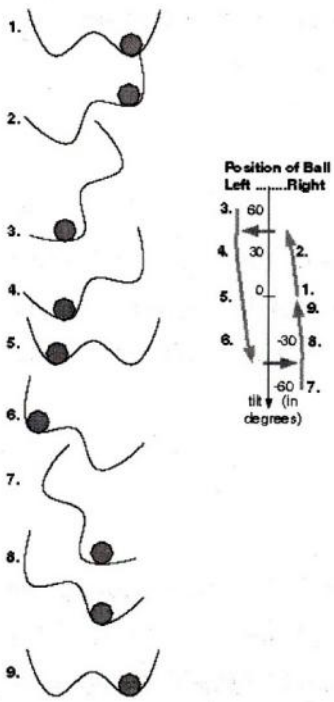

Figure 1.1: A ball free to roll under gravity in a double-well

For instance the system where a ball, free to roll under gravity in a double-well container, may be a mathematical catastrophe if there is a tilt from one side to the other. Here the input is the tilt of the container, and the output is the position of the ball.

There is one catastrophe just beyond the pattern labeled 2 (figure 1.1). This is the point where the ball is just poised to fall to the left, but is still balanced on the right side of the well. If the ball is exactly at that point, the tiniest additional tilt will cause a large displacement of the ball which is shown in the figure by the arrow. A symmetrical catastrophe is labeled 6 (Figure 1.1).

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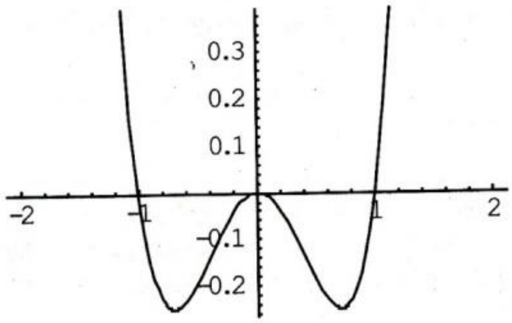
The catastrophe in this system is a one-parameter catastrophe, i.e. one controlling variable 'tilt of the well'. In other mathematical catastrophes the output is determined by a mechanism that seeks the lowest possible position compatible with the constraint, i.e. gravity in this case [AMC 2007] .

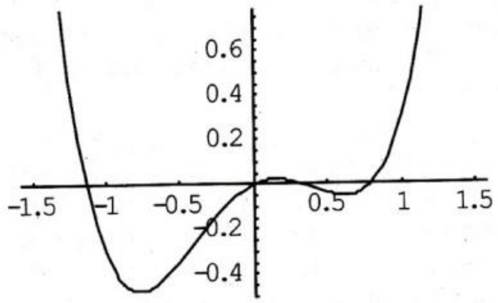
2. An Algebraic Version of the Double Well

It is inept to calculate precisely the angle at which the catastrophe happens in the double-well.



In figure 2.1 the ball is constrained to roll on the graph of the function $y = x^4 - x^2$. Instead of tilting the graph we perturb the function by adding a linear term ax so as to raise one well and lower the other.

Figure 2.1 : Graph of the function $y = x^4 - x^2$



The graph of of $y = x^4 - x^2 + .3x$ is shown in the figure 2.2 where if the ball had started on the right, it would still be on the right.

Figure 2.2 : Graph of the function $y = x^4 - x^2 + .3x$

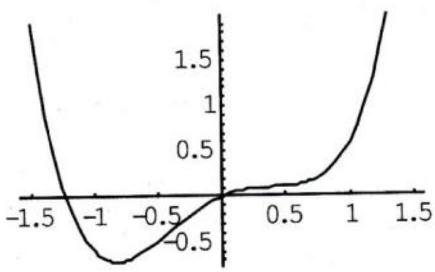
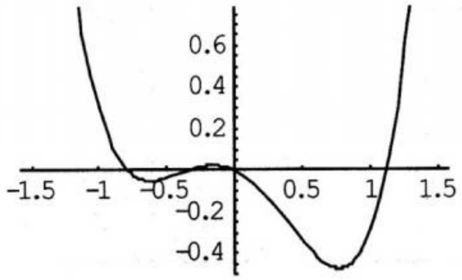


Figure 2.3 is representing the graph of $y = x^4 - x^2 + .6x$. The ball would have rolled over to the left. The exact point at which this happens can be reckoned to be $a = \frac{4}{3}\sqrt{\frac{1}{6}} = .5443......$

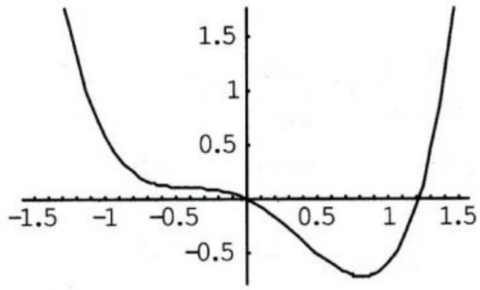
Figure 2.3 : Graph of the function $y = x^4 - x^2 + .6x$

The corresponding negative values a=-.3, a=-.6 give graphs where the left well grows higher than the right.



In figure 2.4 the graph of $y = x^4 - x^2 - .3x$ stands at a= -.3 where if the ball had started on the left, it would still be on the left.

Figure 2.4 : Graph of the function $y = x^4 - x^2 - .3x$



Here is the graph of $y = x^4 - x^2 - .6x$. The ball would have rolled over to the right. The exact point at which it happens can be reckoned to be

$$a = \frac{4}{3}\sqrt{\frac{1}{6}} = .5443......$$

Figure 2.5 : Graph of the function $y = x^4 - x^2 - .6x$

These values and the initial a=0 generate a family of figures exactly analogous from configurations 1 to configurations 9 in figure 1.1 of the original double well. The perturbation parameter plays the role of the angle of tilt.

The catastrophes take place when $a = \frac{4}{3}\sqrt{\frac{1}{6}} = .5443$ ......and when $x = \sqrt{\frac{1}{6}} = 0.4082$.

3. Cusp Catastrophe

The Cusp Catastrophe corresponds to the perturbation of $y = x^4$ by the addition of a quadratic function: $y = x^4 + ax^2 + bx$.

Case I: b=0 If a= 0, the graph has only one minumum (local and global too) at x = 0
If a > 0 this extremum remains unchanged

If a < 0 this extremum changes into three - one local maximum at x = 0 and two local minima and graphs look like a " W " symmetric about the y - axis.

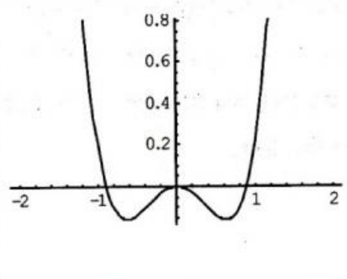
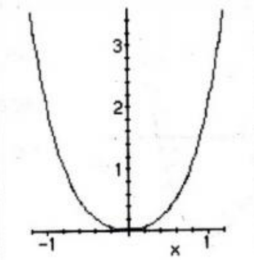
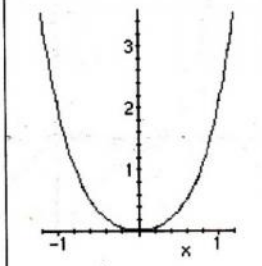
	a = - 0.8	a = 0	a = 1
b=0			

Figure3.1: Graph of $y = x^4 + ax^2 + bx$ at b=0

Case II : a = 0

If b= 0, the graph has only one minumum (local and global too) at x = 0

For b > 0 the minimum goes to the left side of the y - axis.

For b < 0 the minimum goes to the right side of the y - axis.

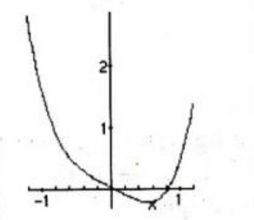
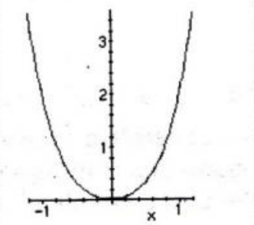
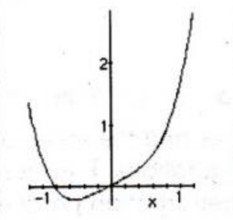
	b = - 0.6	b = 0	B = 0.6
a = 0			

Figure 3.2: Graph of $y = x^4 + ax^2 + bx$ at a=0

Case III : $a \neq 0$ *and* $b \neq 0$

$y = x^4 + ax^2 + bx$

For extrema $\frac{dy}{dx} = 0$ or $4x^3 + 2ax + b = 0$ ....(1)

But at this stage we are interested in values of a and b, i.e. we need to solve a and b in terms of x and want to find the relation between them.

A parametric solution of the equation (1) is obtained when $a = -6x^2,\ b = -8x^3$

wherefrom we get $$\begin{cases} a = (-3.375)^{1/3}\, b^{2/3} \\ b = \sqrt{\dfrac{1}{3.375}}(-a)^{3/2}, \text{ exists only when } a < 0 \end{cases}$$

Therefore, we conclude: in $y = x^4 + ax^2 + bx$, a must be negative if we expect any extremum.

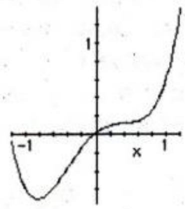
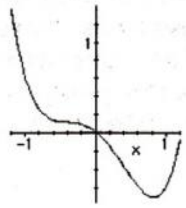
	b = - 0.6	b = 0.6
a = -1		

Figure 3.3: Graph of $y = x^4 + ax^2 + bx$ at a= -1 and b varies.

But then if keeping a at -1, b is moved towards negative values, the missed bifurcation will manifest itself in a "catastrophic" jump to the right-hand local minimum.

A little calculus shows that the system has a single minimum unless $-0.5443(-b)^{3/2} < a < 0.5443(-b)^{3/2}$. When the discontinuity locus $a = \pm 0.5443(-b)^{3/2}$ is graphed in the control space, with coordinates a and b, the plot shows the cusp shape characteristic of this catastrophe

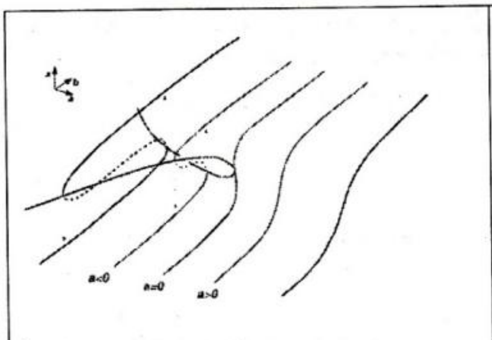


Figure 3.4: Diagram of Cusp Catastrophe

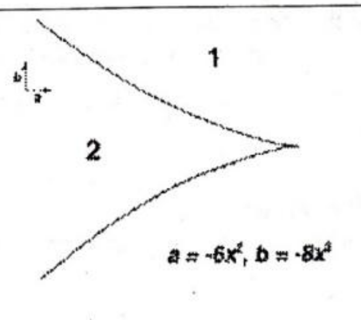


Figure 3.5: Locus of Cusp bifurcations.

The above figure 3.4 is showing curves of x satisfying for parameters (a,b), drawn for parameter b continuously varied, for several values of parameter a. Outside the cusp locus of bifurcations (blue), for each point (a,b) in parameter space there is only one extremising value of x. Inside the cusp, there are two different values of x giving local minima of y(x) for each (a,b), sefparated by a value of x giving a local maximum [FSF 2002].

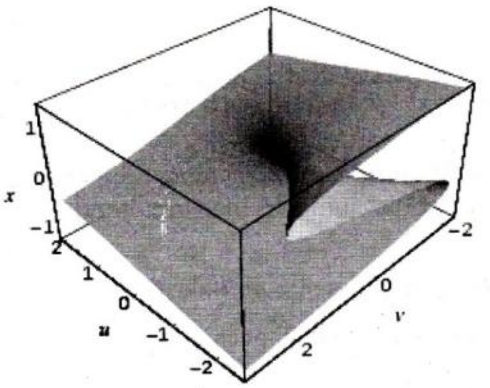


Figure 3.6 : *3D picture of cusp catastrophe*

But the bifurcation curve loops back on itself, giving a second branch where this alternate solution itself loses stability, and will make a jump back to the original solution set.

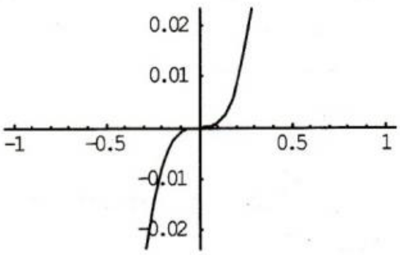
By repeatedly increasing b and then decreasing it, one can therefore observe hysteresis loops, as the system alternately follows one solution, jumps to the other, follows the other back, then jumps back to the first.

However, this is only possible in the region of parameter space $a<0$. As a is increased, the hysteresis loops become smaller and smaller, until above $a=0$ they disappear altogether (the cusp catastrophe), and there is only one stable solution.

One can also consider what happens if one holds b constant and varies a. In the symmetrical case b=0, one observes a pitchfork bifurcation as a is reduced, with one stable solution suddenly splitting into two stable solutions and one unstable solution as the physical system passes to a<0 through the cusp point a=0, b=0 (an example of spontaneous symmetry breaking). Away from the cusp point, there is no sudden change in a physical solution being followed: when passing through the curve of fold bifurcations, all that happens is an alternate second solution becomes available [2].

In figure 12 a 3D picture of cusp catastrophe is shown which can occur for two control factors and one behavior axis.

4. The Fold Catastrophe



Germ: $y = x^3$

It has no extremum but a degenerate critical point at (0,0). This point is also aninflexion point.

Figure 4.1 : graph of $y = x^3$

Potential: $y = x^3 + ax^2; \quad a \neq 0$

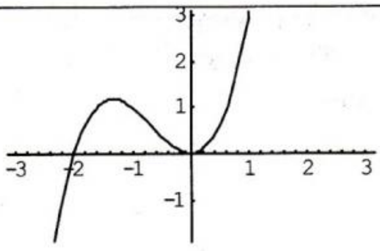


Figure 4.2: $y = x^3 - 2x^2$

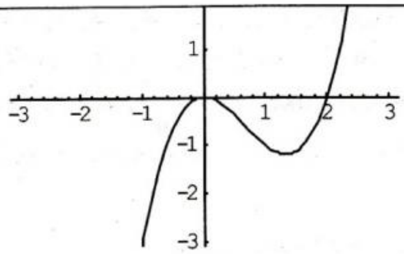


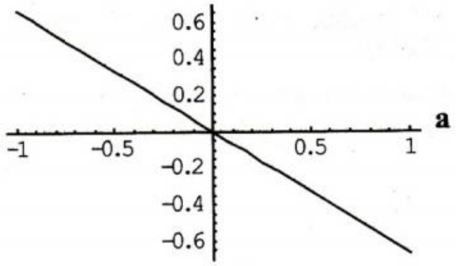
Figure 4.3: $y = x^3 + 2x^2$

Case I: a>0

The inflexion point bifurcates into one local minimum at the origin and one local maximum in the 2nd quadrant (in figure 4.2)

Case II: $a<0$

The inflexion point bifurcates into a local maximum at the origin and a local maximum in the 4th quadrant (in figure 4.3).



The most interesting thing is to note the behavior of the point (0,0). It remains as a local maximum so long as a remains negative, but becomes local minimum if a becomes positive, and vice versa.

Fig 4.4 : $x = -\frac{2}{3}a$

To investigate the other extrimum we have

$$\frac{dy}{dx} = 3x^2 + 2ax = 0$$

$$\Rightarrow 3x + 2ax = 0 \quad \Rightarrow x = -\frac{2}{3}a$$

i.e, x becomes a decreasing linear function of a.

Thus the abscissa of the extremum changes its position with that of parameter 'a' but its value always bears a constant ratio with the value of 'a'. Hence this situation can't be termed a catstrophe. We observe that for any $y = x^n + ax^{n-1}; \quad n = 3,5,7,...,2n+1$, similar conditions hold. But if n is very very big, we can take $x = -\frac{n-1}{n}a \approx a$ For example,
$y = x^{5003} - 27x^{5002}$ will have an extrimum when $x \approx 27$.

Potential : $y = x^3 + ax; \quad a \neq 0$

Putting $\frac{dy}{dx} = 3x^2 + 2ax = 0$ we get the parametric solution $a = -3x^2 \;\; or \;\; x = \pm\frac{\sqrt{-a}}{\sqrt{3}}$

which confirms that 'a' must be -'ve if the potential should possess an extrimum.

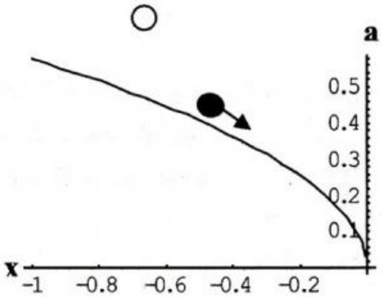


Figure4.5: $x = \dfrac{\sqrt{-a}}{\sqrt{3}}$

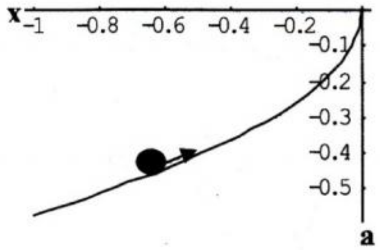


Figure 4.6: $x = -\dfrac{\sqrt{-a}}{\sqrt{3}}$

In figure 4.5, if any particle is pulled backwards along the curve and then released, it will turn back to its previous position and then always tends to go to the position (0, 0). It means that the particle is at its minimum point, and this extremum cannot be called a catastrophe.

On the other hand, in figure 4.6, if a particle is pulled along the curve in the -'ve direction of 'a' and then released, it never comes back. It means that the particle is at the maximum point, and this extremum is certainly a catastrophe, or we may say that there always exists a change of a catstrophe.

Example: $y = x^3 - 12.25x$, In this case, a = -12.25 and therefore, an extrimum does exist at which

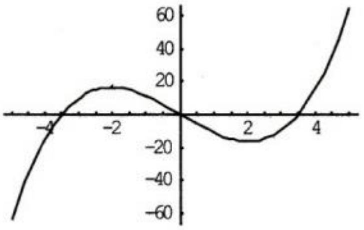


Fig 4.7 : $y = x^3 - 12.25x$

$$x = \frac{1}{\sqrt{3}}\sqrt{-a} \quad ; \quad -\frac{1}{\sqrt{3}}\sqrt{-a}$$

$$= \frac{1}{\sqrt{3}}\sqrt{12.25} \quad ; \quad -\frac{1}{\sqrt{3}}\sqrt{12.25}$$

$$= \frac{3.5}{\sqrt{3}} \quad ; \quad -\frac{3.5}{\sqrt{3}}$$

$$y = \frac{2}{3}(-12.25)\frac{3.5}{\sqrt{3}} \; ; \; \frac{2}{3}(-12.25)\left(\frac{-3.5}{\sqrt{3}}\right)$$

$$= \frac{-85.75}{3\sqrt{3}} \quad ; \quad \frac{85.75}{3\sqrt{3}}$$

According to the argument given above the point $\left(\dfrac{-3.5}{\sqrt{3}}, \dfrac{85.75}{3\sqrt{3}}\right)$ is a fold catastrophe.

Here we can see the reason for the name "Fold Catastrophe." If the graph is projected onto the y-axis, the catastrophe corresponds to the appearance or disappearance of a pair of folds [1].

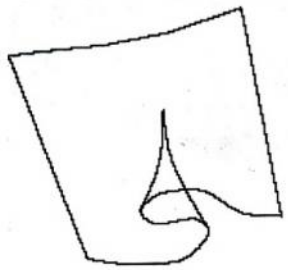


Figure 4.8: A fold catastrophe

As in the fold case, the system will have a discontinuous response at points where a local minimum appears or disappears. [2]. When the right-hand fold is approached from the left, or vice-versa, the output is forced to jump to the other sheet. The membrane in the middle corresponds to the local maximum and is inaccessible.

The figure 4.8 showing a 3D picture of fold catastrophe which can occur for one control factor and one behavior axis.

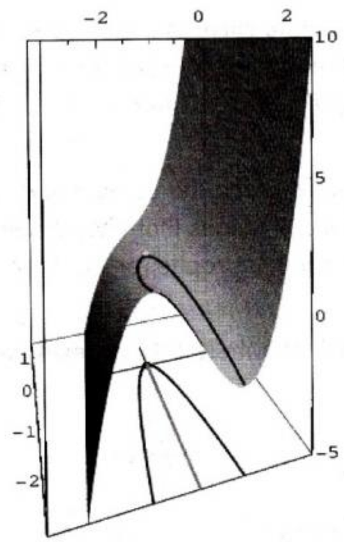


Figure 4.9: 3D picture of Fold Catastrophe

5. Swallowtail catastrophe

The swallowtail Catastrophe corresponds to the perturbation of $y = x^5$ b by the addition of a 3rd degree function: $y = x^5 + ax^3 + bx^2 + cx$

The control parameter space is three dimensional. The bifurcation set in parameter space is made up of three surfaces of fold bifurcations, which meet in two lines of cusp bifurcations, which in turn meet at a single swallowtail bifurcation point.

As the parameters go through the surface of fold bifurcations, one minimum and one maximum of the potential function disappear. At the cusp bifurcations, two minima and one maximum are replaced by one minimum; beyond them the fold bifurcations disappear. At the swallowtail point, two minima and two maxima all meet at a single value of x. For values of $a>0$, beyond the swallowtail, there is either one maximum-minimum pair, or none at all, depending on the values of b and c. Two of the surfaces of fold bifurcations, and the two lines of cusp bifurcations where they meet for $a<0$, therefore disappear at the swallowtail point, to be replaced with only a single surface of fold bifurcations remaining [Thomson 1982].

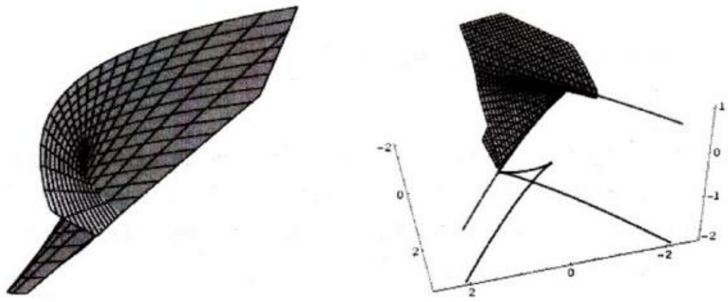
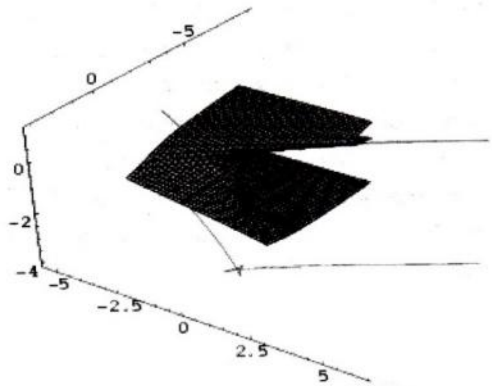


Figure 5.1: *3D picture of swallowtail Catstrophe*

A catastrophe which can occur for three control factors and one behavior axis. The swallowtail catastrophe is the universal unfolding of singularity with codimension 3, i.e. in three unfolding parameters

6. Butterfly catastrophe



The butterfly Catastrophe corresponds to the perturbation of $y = x^6$ by the addition of a fourth degree function : $y = x^6 + ax^4 + bx^3 + cx^2 + dx$:

Figure 6.1: *3D picture of butterfly Catastrophe*

A catastrophe which can occur for four control factors and one behavior axis. The butterfly catastrophe is the universal unfolding of the singularity $f(x) = x^6$ of codimension 4, i.e. with four unfolding parameters. It has the form $F(x,u,v,w,t) = x^6 + ux^4 + vx^3 + wx^2 + tx$.

Depending on the parameter values, the potential function may have three, two, or one different local minima, separated by the loci of fold bifurcations. At the butterfly point, the different 3-surfaces of fold bifurcations, the 2-surfaces of cusp bifurcations, and the lines of swallowtail bifurcations all meet up and disappear, leaving a single cusp structure remaining when a>0 [Thomas 1989].

7. Utility of Mathematical Catastrophy

It has been observed that mathematical catastrophe has its application in explaining happenings in nature even in the behaviour of animals some of which are summarised in what follows.

Small changes in certain parameters of a nonlinear system can cause equilibria to appear or disappear, or to change from attracting to repelling and vice versa, leading to large and sudden changes of the behaviour of the system. However, examined in a larger parameter space, catastrophe theory reveals that such bifurcation points tend to occur as part of well-defined qualitative geometrical structures [Poston 1998] .

In mathematics, catastrophe theory is a branch of bifurcation theory in the study of dynamical systems; it is also a particular special case of more general singularity theory in geometry .

Fold bifurcations and the cusp geometry are by far the most important practical consequences of catastrophe theory. They are patterns which reoccur again and again in physics, engineering and mathematical modelling .

A famous suggestion is that the cusp catastrophe can be used to model the behaviour of a stressed dog, which may respond by becoming cowed or becoming angry. The suggestion is that at moderate stress ($a>0$), the dog will exhibit a smooth transition of response from cowed to angry, depending on how it is provoked. But higher stress levels correspond to moving to the region ($a<0$). Then, if the dog starts cowed, it will remain cowed as it is irritated more and more, until it reaches the 'fold' point, when it will suddenly, discontinuously snap through to angry mode. Once in 'angry' mode, it will remain angry, even if the direct irritation parameter is considerably reduced [Postle 1980].

To be very practical, Bangladesh is a land of disasters. Floods, hurricane, abrupt rainfalls often occur here. Sudden inflexion, big-scale rise and fall in share markets are very common features. Proper mathematical modelling will help analyse these catastrophes.

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