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Simulation of the Measurer of the Time of Appearance and the Average Power of the Random Pulse Signal

Oleg V. Chernoyarov
International Laboratory of Statistics of Stochastic Processes and Quantitative Finance
National Research Tomsk State University
Tomsk, Russia
Dept. of Higher Mathematics and System Analysis
Maikop State Technological University
Maikop, Russia
e-mail: chernoyarovov@mpei.ru

Alexandra V. Salnikova
International Laboratory of Statistics of Stochastic Processes and Quantitative Finance
National Research Tomsk State University
Tomsk, Russia
Dept. of Building Engineering and Urban Planning
University of Zilina
Zilina, Slovakia
e-mail: amicus.lat@yandex.ru

Alexander A. Makarov
Dept. of Electronics and Nanoelectronics
National Research University “Moscow Power Engineering Institute”
Moscow, Russia
e-mail: o_v_ch@mail.ru

Abstract—The maximum likelihood measurer is considered of the time of appearance and the average power of the fast fluctuating Gaussian band pulse against Gaussian white noise. The possibilities of its practical implementation are demonstrated and its accuracy characteristics are determined. By statistical simulation methods, the experimental values of biases and variances of the resulting estimates are found. The error ranges of the theoretical formulas describing the measurer performance are established. There have been determined the conditions of high a posteriori accuracy for the measurer operation, that is, such signal-to-noise ratios above which the anomalous errors in estimating the pulse time parameter are practically non-existent.

Keywords—maximum likelihood estimate; random pulse; unknown time of appearance, average power; bias; variance; probability of anomalous error; statistical simulation

I. INTRODUCTION

The problem of estimating the parameters of the random pulse signals has a wide application in analyzing the operation of the various radio engineering devices [1-3]. One of the adequate models of a random pulse is the mathematical model of the form

\[ s(t) = \xi(t) I[(t - \lambda_0)/\tau], \]

where \( I(x) = 1, \) if \( |x| \leq 1/2, \) and \( I(x) = 0, \) if \( |x| > 1/2 ; \) \( \xi(t) \) is the stationary centered Gaussian random process; \( \lambda_0 \) is the time of appearance and \( \tau \) is the duration of the pulse. The spectral density of the process \( \xi(t) \) is described by the expression [1-3]

\[ G(\omega) = \left( \pi D_0 / \Omega \right) \left\{ |(\Omega - \omega)/\Omega| + |(\omega + \Omega)/\Omega| \right\}, \]

here the designations are: \( \Omega \) is the band center, \( \Omega \) is the bandwidth of the spectral density, and \( D_0 \) is the average power (dispersion) of the process \( \xi(t) \). It is presupposed that the pulse (1) duration \( \tau \) is much longer than the correlation time \( 2\pi/\Omega \) of the random substructure \( \xi(t) \) (the process \( \xi(t) \) fluctuations are “fast”), so that the following condition is satisfied: \( \mu = \tau \Omega/2\pi \gg 1 \).

Examples of the signal (1) include the reflected radar signals, the signals in noise carrier communication systems, the pulses describing the optical noise flash, explosive noise in transistors, etc. [1-3].

Let the signal (1) be observed against Gaussian white noise \( n(t) \) with the one-sided spectral density \( N_0 \). By the observable realization

\[ x(t) = s(t, \lambda_0, D_0) + n(t), \]

the parameters \( \lambda_0 \) and \( D_0 \) have to be measured taking the values from the prior intervals \([\Lambda_1, \Lambda_2]\) and \([0, \infty)\), respectively. Thus, the boundaries of the observation interval \([T_1, T_2]\) are chosen according to the conditions \( T_1 < \Lambda_1 - \tau/2 < \Lambda_2 + \tau/2 \leq T_2 \), i.e. the pulse (1) is always located within the observation interval.

II. THE ESTIMATION ALGORITHM

In order to estimate the parameters of the random pulse (1), the maximum likelihood method is applied. According to [4, 5], the logarithm of the functional of the likelihood
ratio (FLR) $L(\lambda, D)$, as the function of the current values $\lambda, D$ of the unknown parameters $\lambda_0, D_0$, can be represented in the form of

$$L(\lambda, D) = \left[ DM(\lambda)/(D + E_N) - \tau_0 E_N \ln(1 + D/E_N) \right]/N_0,$$

(2)

$$M(\lambda) = \frac{\lambda^2 + 2\lambda}{2\lambda - D}.\]$$

here $h(t)$ is the pulse response of the filter whose transfer function $H(o)$ satisfies the condition $|H(o)|^2 = I[\{9 - \omega\}/\Omega] + I[\{9 + \omega\}/\Omega]$, and $E_N = N_0 \Omega/2\pi$ is the average power of the noise $n(t)$ within the bandwidth $\Omega$ of the process $\xi(t)$.

Then, the joint maximum likelihood estimates (MLEs) $\lambda_m$ and $D_m$ of the time of appearance $\lambda_0$ and the dispersion $D_0$ are written down as follows [4]

$$\hat{\lambda}_m = \arg \sup_{\lambda \in [\lambda_1, \lambda_2]} L(\hat{\lambda}, D),$$

$$\hat{D}_m = \arg \sup_{D \geq 0} L(\hat{\lambda}_m, D) = \Gamma(\hat{\lambda}_m),$$

(3)

where $\Gamma(\lambda) = \max[0; M(\lambda)/\tau - E_N]$, while $M(\lambda)$ is determined from (2).

According to (3), the maximum likelihood measurer of the time of appearance and the average power can be implemented in the form shown in Fig. 1. Here the designations are: 1 is the switch that is open for the time $[\lambda_1 - \tau/2, \lambda_2 + \tau/2]$; 2 is the filter with the transfer function $H(o)/\sqrt{\tau}$ (2); 3 is the squarer; 4 is the delay line for the time $\tau$; 5 is the integrator; 6 is the extremator that fixes the location of the greatest maximum of the signal as the estimate $\lambda_m$ (3) of the time of appearance; 7 is the nonlinear element with the characteristic $f(x) = \max[0, x]$; 8 is the gating unit generating the signal sample at the point of time $\lambda_m$. The sample magnitude at the output of the gating unit 8 is the estimate $D_m$ (3).

III. THE CHARACTERISTICS OF THE MAXIMUM LIKELIHOOD ESTIMATES

Considering the characteristics of the joint MLEs of the time of appearance $\lambda_m$ and the average power $D_m$ is our next task. From (3), it follows that the structure of the algorithm of the estimate $\lambda_m$ is invariant with respect to the unknown average power $D_0$. Therefore, by applying the results of [6], for the conditional bias (systematic error) and variance (mean square error) of MLE $\lambda_m$, one gets:

$$b(\lambda_m|\lambda_0) = P_0 b_0(\lambda_m|\lambda_0) + (1 - P_0)(\lambda_2 + \lambda_1)/2 - \lambda_0,$$

(4)

and

$$V(\lambda_m|\lambda_0) = P_0 V_0(\lambda_m|\lambda_0) + (1 - P_0)\times$$

$$\times \left[\lambda_2 + \lambda_1 \lambda_2 + \lambda_1^2/3 - (\lambda_2 + \lambda_1)^2/\lambda_0 + \lambda_0^2\right],$$

where $P_0$ is the probability of a reliable estimate, while $b_0(\lambda_m|\lambda_0)$ and $V_0(\lambda_m|\lambda_0)$ denote the conditional bias and variance of a reliable estimate, respectively. As a reliable estimate [6, 7], the estimate found under the assumption that $|\lambda_m - \lambda_0| < \tau$ is considered.

From [6], it follows that

$$P_0 \approx 2\psi \exp\left(\frac{\psi z^2}{2} + \psi z^2\right) \int_{(q_0 + q_0)} \int_0^{\frac{q_n(1 + q_0)}{\sqrt{2\pi}}} \psi^2 \exp\left(-\frac{\psi^2 z^2}{2}\right)\left[\exp(-\psi z)\Phi[x - z(\psi + 1)] - \exp\left(\frac{\psi^2 z^2}{2} + \psi z(2z - 2x)\right)\Phi[x - z(2\psi + 1)]\right] dx,$$

(5)

$$b_0(\lambda_m|\lambda_0) \approx 0, \quad V_0(\lambda_m|\lambda_0) \approx 13\psi^2\left[1 + (1 + q_0)^2\right]/8\mu^2 q_0^4,$$

where
\[ \psi = 2(1 + q_0)^2 \left[ 1 + (1 + q_0)^2 \right], \quad q_0 = D_0 / E_N, \]  
\[ m = (\Lambda_2 - \Lambda_1) / \tau, \quad z^2 = \mu q_0^2 / (1 + q_0)^2 \]
is the power signal-to-noise ratio (SNR), and 
\[ \Phi(x) = \int_x^\infty \exp(-t^2/2) \, dt / \sqrt{2\pi} \]is the probability integral.

The accuracy of the formulas (4), (5) increases with \( m, z, m \).

The characteristics of the estimate \( D_m \) (3), while the time of appearance \( \lambda_0 \) is unknown, are found in [4]. Under \( m > 1 \), the expressions for the conditional bias and variance of MLE \( D_m \), while taking into account the possible anomalous errors [7] in estimate \( \lambda_m \) of the parameter \( \lambda_0 \), can be written as

\[ b(D_m|D_0) = \{ D_m \} - D_0, \quad V(D_m|D_0) = \{ D_m^2 \} - 2D_0 \{ D_m \} + D_0^2, \]  
(7)

\[ \{ D_m \} = \frac{D_0}{z} \left[ \int_0^\infty [1 - F(x)] \, dx \right], \quad \{ D_m^2 \} = \frac{2D_0^2}{z^2} \left[ \int_0^\infty [x - F(x)]^2 \, dx \right], \]
where

\[ F(x) = F_S(x)F_N[x(1 + q_0)], \]
\[ F_S(x) = \Phi(x - z) - 2\exp \left[ \frac{w_2^2 - 2 - \psi z(x - z)}{2} \right] \Phi(x - z(\psi + 1)) + \exp \left[ 2\psi^2 z^2 + 2\psi z(x - z) \right] \Phi(x - z(2\psi + 1)), \]  
(8)
\[ F_N(x) = \left\{ \begin{array}{ll} \exp \left[ \left( mx / \sqrt{2\pi} \right) \exp(-x^2/2) \right], & x \geq 1, \\ 0, & x < 1, \end{array} \right. \]
and \( z, \psi, m \) are determined from (6).

Under \( m \leq 1 \), when the estimate of the time of appearance \( \lambda_m \) are the reliable one, the formulas (7) are simplified and take the form of [4]

\[ b(D_m|D_0) = D_0 \left\{ \left( 1 + \frac{3}{2\psi^2z^2} \right) \Phi(z) + \frac{1}{\sqrt{2\pi z^2}} \exp \left( -\frac{z^2}{2} \right) \right\} - \frac{1}{\psi^2z^2} \exp \left( \frac{w_2^2}{2} + \psi^2 z^2 \right) \left[ 1 - \Phi(\psi(z + 1)) \right] - \frac{1}{2\psi^2z^2} \exp \left( 2\psi^2 z^2 + 2\psi z^2 \right) \left[ 1 - \Phi(2\psi(z + 1)) \right]. \]  
(9)

The accuracy of the expressions (7), (8) increases with \( m, z, m \) while the accuracy of the expressions (9) – with \( m, z \) [4].

IV. THE RESULTS OF THE STATISTICAL SIMULATION

Analytical calculation of the error ranges of the formulas specified above is very difficult. Therefore, it is of interest to study the noise immunity of the maximum likelihood measurer and the limits of applicability of the approximate expressions (4), (5) and (7)-(9) for the characteristics of the joint MLEs \( \lambda_m \) and \( D_m \) by the methods of the statistical computer simulation. To reduce the amount of computer time required for the simulation, the representation is used of the response of the narrowband filter \( h(t) \) (2) through its low-frequency quadratures [6]. This allows forming the decision statistics (2) as the sum of the two independent random processes as follows

\[ M(\lambda) = \frac{1}{2} [M_1(\lambda) + M_2(\lambda)], \]
\[ M_i(\lambda) = \int y_i^2(t) \, dt, \quad y_i(t) = \sqrt{\psi} \chi_i(t) - r(t), \quad \chi_i(t) = \xi_i(t) \sqrt{\lambda_i(t - \lambda_0(t))/\tau} + n_i(t), \quad i = 1, 2, \]
where \( \xi_i(t) \) and \( n_i(t) \) are the statistically independent Gaussian random processes with the spectral densities \( G_{\xi_i}(\omega) = (2\pi D_0 / \Omega) I(\omega_0 / \Omega) \) and \( G_n(\omega) = N_0 \), respectively, while \( h_i(t) \) is the function whose spectrum \( H_i(\omega) \) satisfies the condition \( |H_i(\omega)|^2 = I(\omega_0 / \Omega) \).

During the simulation within the interval \([\Lambda_1, \Lambda_2]\), \( \Lambda_i = \Lambda_1 / \tau, \quad i = 1, 2 \) with the discretization step \( \Delta \), the samples were formed of the realizations of the random processes \( y_i(t) \) (10). It allowed us to obtain the stepwise approximation of the decision statistics of the form of

\[ M(\lambda) = \frac{1}{2} \sum_{k=1}^{k=\infty} (y_{1k}^2 + y_{2k}^2) \Delta, \]  
(11)
here \( k_{\text{min}} = \text{int}\{l - 1/2)/\Delta\} \), \( k_{\text{max}} = \text{int}\{l + 1/2)/\Delta\} \), \( l = \lambda/\tau \) is the normalized current value of the time of appearance, \( \text{int}\{\} \) is an integer. In case when \( \Delta = 0.05/\mu \) and \( \Delta = 0.01 \) (\( \Delta \) is the discretization step along the variable \( l \)), the mean square error of the step approximation (11) of the continuous realization (10) does not exceed 10 %. The samples of the processes \( y_{ik}, i = 1, 2 \) are generated in terms of the sequence of independent Gaussian random numbers by a moving summation method \([6]\) as follows:

\[
y_{ik} = \frac{1}{\Delta} \sum_{m=k-p}^{k+p} \alpha_{im} H_{k,m} + \sum_{m=\max(n(m_{\text{init}}),k-p)}^{\min(m_{\text{max}},k+p)} \varepsilon_{im} H_{k,m},
\]

where \( m_{\text{min}} = \text{int}\{l_0 - 1/2)/\Delta\} \), \( m_{\text{max}} = \text{int}\{l_0 + 1/2)/\Delta\} \), \( l_0 = \lambda_0/\tau \), \( H_{k,m} = \sin[2\pi \mu \Delta(k - m)]/\pi(k - m) \), and \( \alpha_{im}, \beta_{im} \) are independent Gaussian random numbers with zero mathematical expectations and unit dispersions.

In the sums (12), the number of summands corresponds to the value \( p = 50 \) providing a relative deviation of the generated sample dispersion from the simulated process dispersion to be no more than 5 %. Formation of the Gaussian numbers \( \alpha_{im}, \beta_{im} \) with the parameters \((0,1)\) has been implemented based on the sequences of the independent random numbers \( \phi_n, \varphi_n \) uniformly distributed within the interval \([0,1]\) by the Cornish-Fisher method \([6, 8]\):

\[
\zeta_i = Z_i + \frac{Z_i^3 - 3Z_i}{20N}, \quad Z_i = \sqrt{\frac{12}{N} \sum_{n=1}^{N} \left[ D_N(n-1) + n - 0.5 \right]},
\]

where \( \zeta_i \) is one of the sequences \( \alpha_{im}, \beta_{im} \), and \( \theta_n \) is sequence \( \phi_n, \varphi_n \) corresponding to it. The number of summands \( N \) in the sum (13), following \([6, 8]\), has been chosen as equal to 5.

By the realization of the process \( M(l) \) obtained with the help of the formulas (11), (12), according to (3), the normalized estimates \( \lambda_m = \lambda_m/\tau \), \( g_m = D_m/E_N \) are determined and the variances of these estimates are found. Some results of the statistical simulation are presented in Figs. 2-6 where the corresponding theoretical dependences are also shown. Each experimental value has been obtained as a result of processing of no less than \( 10^4 \) realizations of \( M(l) \) under \( \tilde{\lambda}_1 = 0 \), \( \tilde{\lambda}_2 = m \), \( l_0 = (\tilde{\lambda}_2 + \tilde{\lambda}_1)/2 \). Thus, with the probability of 0.9, the confidence intervals boundaries deviate from the experimental values no more than by 10...15 %.

In Fig. 2, there is presented the theoretical dependence (5) of the normalized variance \( V_{(l)} = V_0(\lambda_m/l_0)^2 \) of the reliable estimate \( \lambda_m \) under \( m = 1 \).

In Fig. 3 one can see the analogous dependence (4) of the normalized variance \( V_{(l)} = V_0(\lambda_m/l_0)^2 \) of the estimate \( \lambda_m \) with the anomalous errors taken into account under \( m = 20 \). The solid lines depict the results of the calculations when \( \mu = 50 \), while the dashed lines demonstrate them under \( \mu = 100 \) and the dash-dotted lines – under \( \mu = 200 \). The corresponding experimental values of the variances \( V_{(l)} \) and \( V_{(l)} \) are designated by rectangles, crosses and diamonds under \( \mu = 50 \), \( \mu = 100 \) and \( \mu = 200 \), respectively.

In Fig. 4, there are shown the theoretical and experimental dependences of the probability of the anomalous error \( P_\alpha = P[|\lambda_m - \lambda_0| > \tau_0] = 1 - P_0 \) (5). In Figs. 5, 6, one can see the theoretical and experimental dependences (9) and (7), (8) of the normalized variances \( V_q = V(D_m/E_N)^2 \) of the estimate \( D_m \) under \( m = 1 \) (when MLE \( \lambda_m \) is reliable) and \( m = 20 \) (when the anomalous errors are possible in estimating the pulse (1) time of appearance), respectively. The designations in Figs. 4-6 correspond to those given in Fig. 3.
V. Conclusion

Based on the results obtained, the following conclusions can be drawn. As follows from Figs. 2, 3, the theoretical dependences (5) for the variance of the reliable estimate \( \lambda_m \) well approximate the experimental data under SNR \( z > 1.5 \ldots 2 \), while the theoretical dependences (4) for the variance of the estimate \( \hat{\lambda}_m \) with the anomalous errors taken into account agree generally with the experimental data under \( \mu \gtrsim 50 \) and \( z \gtrsim 0.5 \). If \( z < 1.5 \), then the theoretical dependences (5) deviate from the experimental values, as the formula (5) for the variance of the reliable estimate of the time of appearance does not take into account the finite length of the prior interval \([\Lambda_1, \Lambda_2]\) of the possible values of the parameter \( \lambda_0 \). As a result, when the variance becomes comparable with or greater than the value \( (\Lambda_2 - \Lambda_1)^2/12 \), the accuracy of the formulas (5) deteriorates significantly.

The deviation of the theoretical dependences \( V_0(\hat{\lambda}_m|\lambda_0) \) (5), \( V(\hat{\lambda}_m|\lambda_0) \) (4) from the experimental values is also observed in case of the large SNRs, when \( q_0 > 2 \ldots 3 \). This is due to the fact that the formula (5) for the variance of the reliable estimate of the time of appearance has been obtained in neglecting the estimation errors of the order of the correlation time of the random process \( \xi(t) \) [6]. Therefore, when the normalized variance decreases up to the value of the order of \( \mu^{-2} \), the error of the formulas (4), (5) becomes significant.

If the SNR is not large enough (\( z < 4 \ldots 5 \)) and the reduced length of the prior interval is \( m >> 1 \), then it is necessary to take into account the anomalous errors in estimating the time of appearance. In this case, the accuracy of the MLE \( \hat{\lambda}_m \) can significantly deteriorate. Under \( z > 5 \) and \( m \leq 10 \ldots 20 \), when the probability of the anomalous errors can be neglected, the values of the variances of the estimate \( \lambda_m \) obtained by the formulas (4) and (5) almost coincide.

According to Figs. 5, 6, the formulas (7), (8) and (9) are consistent satisfactorily with the experimental values of the variance of the estimated dispersion, if \( z > 3 \ldots 4 \). Under \( z > 5 \), when the probability of anomalous errors in estimating the parameter \( \lambda_0 \) is sufficiently small (the estimate of the time of appearance is reliable), the variances \( V(D_m|\lambda_0) \) (7), (8) and (9) of the estimated dispersion coincide.

It may be noted that, as it is stated in [9, 10], the accuracy of the discontinuous parameter (time of appearance) can be increased by 20 percent (under big SNRs), approximately, by applying the Bayesian method to obtain the estimates. However, in this case the structure of the measurer becomes more complex.
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