TOMOGRAPHIC TRAVELTIME INVERSION FOR LINEAR INHOMOGENEITY AND ELLIPTICAL ANISOTROPY

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by

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Abstract

A velocity model is described in which we assume that velocity increases linearly with depth and varies elliptically with propagation direction. That is, we consider a linearly inhomogeneous elliptically anisotropic model. The variation of velocity with depth is given in terms of parameters a and b, and the elliptical anisotropy is given in terms of parameter χ. An analytical traveltime expression is then derived to account for the direct traveltime between an offset source and a receiver in a well; such as in a vertical seismic profile (VSP) setting. A method of inverting traveltime observations to estimate parameters a, b and χ is derived. The application of this method is exemplified using a data set from the Western Canada Basin. The parameter estimation also includes a statistical analysis. In the above case, we obtain a good agreement between the field data and the model. Furthermore, the inclusion of elliptical anisotropy is validated by showing that an isotropic

model is outside of the confidence interval for χ. Once a, b and χ are known, a further application is considered; namely, we use the model to calculate the possible reflection points, collectively referred to as the zone of illumination, for a VSP experiment with a given source–receiver geometry. Such modelling is useful for both data analysis and survey design. Two computer codes are given using Maple®. The first code is for the estimation of the parameters and the second one is for the calculation of the zone of illumination.

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It is rare that a student begins a Masters degree program without first completing a Bachelors degree. I am no exception; a bachelors degree in applied mathematics formed the basis for my Masters degree. When I think about my undergraduate studies, one person above all comes to mind; for sharing his love of mathematics I thank Dr. Richard Charron.

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List of Symbols

$=$	equality
$\approx$	approximation
$:=$	definition
$\cdot$	inner or dot product
σ_{ij}	component of the stress tensor
c_{ijkl} or C_{mn}	component of elasticity tensor
ε_{kl}	component of strain tensor
p_i	component of the slowness vector, $\mathbf{p}$
$\lvert\mathbf{p}\rvert^2$ or $\sum p_i^2$	squared magnitude of slowness vector
ϑ	wavefront angle (a.k.a, phase angle) in direction of wavefront normal with respect to the vertical
v_i	component of the wavefront velocity vector, $\mathbf{v}$

$v_{(i)}$	one of several possible wavefront velocities
v_{H}, v_{V}	wavefront velocity in horizontal, vertical directions, respectively
V	ray velocity
χ	ellipticity parameter
V_{d}, V_{u}	ray velocity of downgoing, upgoing ray, respectively, in a reflection geometry
θ	ray angle with respect to the vertical
θ_{d}, θ_{u}	ray angle of downgoing,upgoing ray, respectively, in a reflection geometry
$\mathcal{H}$	ray–theory Hamiltonian
d	total derivative
∂	partial derivative
ρ	mass density
δ_{ij}	Kronecker's delta
ψ	eikonal function
ζ	scaling factor
$\times$	multiplication

Chapter 1

Introduction

An important part of the study of the earth's interior, particularly in petroleum exploration, involves the acquisition and processing of seismic data. To acquire the data a seismic crew will visit a site and set up receiving devices along the surface of the area to be studied. An energy source is then used to generate a signal to be received by the geophones. From the received data, an image of the subsurface can then be constructed.

It is common in today's seismic surveys to place the geophones below the surface within a well. This particular type of experimental setup is referred to as vertical seismic profiling or as a VSP. We note that VSP could also refer to the case of a source in the well and receivers at the surface as in the case of taking measurements while drilling (MWD). However we shall not deal with this case and will not mention it further. VSPs provide a means of extracting additional geophysical information as compared to surface seismic data. VSPs provide a well–constrained data set with high quality of data as signals spend less time traveling within the earth and are less affected by the near–surface material. Hence, they are less affected by attenuation due to both absorption and geometrical spreading.

One purpose of acquiring VSP data is to use the field measurements to generate a tomographical image of the subsurface providing both physical and geometrical information about it. This information can then used by structural geologists and sedimentologists who interpret the results to identify potential sites of economically viable reserves.

One of the early steps in the processing of seismic data is the construction of a velocity model. From the velocity model one can attempt to derive a traveltime expression which can be used to invert field observations to estimate the model parameters. There are two dominating factors to be considered when constructing models for the velocity field of a given medium; they are inhomogeneity and, often to a lesser extent, anisotropy. Inhomogeneity refers to the changes in the velocity of signal propagation from point to point within the medium. In the context of this thesis we will consider inhomogeneity to be a function of depth alone. Anisotropy refers to the variation in propagation velocity with direction at a given point. For the sake of simplicity, discussions and texts often describe velocity by an average, be it true average, root-mean-square average or normal-moveout velocity. Simplifications in velocity modelling turn up in practice as well. Velocity modeling frequently ignores one of the two major influences described above or deals with each separately. Over the past few decades, advances in the quality of seismic data justify in today's physical models the inclusion of both inhomogeneity and anisotropy.

In this thesis we will attempt to do that. In other words, we will account for both inhomogeneity and anisotropy in a single model. To do so, we first propose a velocity model which includes inhomogeneity and anisotropy. Then, the corresponding, closed-form traveltime expression is derived. Based on the traveltime expression a method is derived for the estimation of the model parameters by a regression analysis of the observed VSP traveltimes. The process of using traveltimes in this manner to estimate model parameters is commonly referred to as "traveltime inversion". Traveltime inversion is then applied to a real seismic data set. Finally, an application of the closed form traveltime expression is given. Specifically, the traveltime expression is used to determine the zone of illumination possible for a given set of sources and receivers in the VSP experiment.

Chapter 2

Background

The approach to the study of seismic phenomena taken in this thesis is based largely on concepts of continuum mechanics. Since the focus is not continuum mechanics itself, we will not go into details of it beyond some definitions which we require to develop the velocity and subsequent traveltime models. For a detailed discussion of continuum mechanics as it applies to seismic ray theory the reader is referred to the book "Seismic waves and rays in elastic media" by Slawinski[1] (2003), particularly part I of the book. For our present purposes, it is sufficient to say that when modelling the propagation of seismic energy, we do not think of the interaction of individual granules in the medium (as we would say when watching a billiards game), but rather

[1] Slawinski, M.A., (2003) Seismic waves and rays in elastic media: Pergamon, pp. 7-124.

consider the medium to be a smooth continuous mass. To quote Slawinski, "The concept of continuum allows us to consider materials in such a way that their descriptive functions are continuous and differentiable".[2] It is this continuity that allows us to use the power of calculus in subsequent formulations.

For the remainder of this thesis, we will assume the medium to be a linearly elastic continuum. This means we will only consider solids in which stress is linearly proportional to strain. Such a solid is also referred to as a Hookian solid after Robert Hooke and his work with springs. Recall Hooke's Law which states that the spring constant k is such that if a spring is stretched a distance Δx, the restoring force is given by $F = -k\Delta x$. In a three–dimensional linearly elastic solid the same rule applies only it is a little more complicated due to the extra dimensions that must be accounted for. Specifically we must replace the vectors and scalars with tensors. Thus we have

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} \varepsilon_{kl}, \ \ i,j \in \{1, 2, 3\}, \tag{2.1}$$

where σ_{ij} are components of the stress-tensor (F in 1D), c_{ijkl} are components

[2] Ibid, p. 8.

of the elasticity tensor (k in 1D), and where ε_{kl} are components of the strain–tensor (Δx in 1D). More specifically, if a material is deformed in such a way that point $\mathbf{x}$ is moved by vector $\mathbf{u}(\mathbf{x})$ then the strain tensor components are defined by

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right). \tag{2.2}$$

Since the strain–tensor is defined in terms of the partial derivatives of the displacement vector, expression (2.1) can also be written as

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} \frac{1}{2} c_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad i,j \in \{1,2,3\}\,. \tag{2.3}$$

Although we will not deal directly with the components of the elasticity tensor in this thesis, expression (2.3) will be used in the formulation of the Christoffel equations, and the eikonal equation. Suggestions for future work in Chapter 12 make direct use of expression (2.3) and so we include it here.

2.1 Cauchy equations of motion

In this section we consider the balance of linear momentum as it applies to a volume of the continuum. By considering the solvability condition of the resultant expression we formulate the Cauchy equations of motion.

The balance of linear momentum requires that the total force acting upon a volume be the sum of its surface forces, due to elasticity, and its body forces, due to gravity. Consider a constant volume of the continuum which changes position with time. That is, assume $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. For example, we may think of a water balloon tumbling in a river; it's density and total volume will not change but it's shape, and hence surface normal, will. Recalling that force is equal to mass times acceleration, we can write the balance of linear momentum in component form as

$$\iiint_{V(t)} \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2} \mathrm{d}V = \iint_{S(t)} \left(\sum_{j=1}^{3} \sigma_{ji} n_j \right) \mathrm{d}S + \iiint_{V(t)} f_i \mathrm{d}V, \quad i \in \{1, 2, 3\}. \tag{2.4}$$

In this case, the displacement, $\mathbf{u}(\mathbf{x}, t)$, is a function of position and time, $\sum_{j=1}^{3} \sigma_{ji} n_j$ is the surface force per unit area expressed in terms of the stress tensor and the surface normal, and f_i represents the component of the body force. It can be shown by considering the gravitational potential and Poisson's equation that for a wave propagating with frequency greater than $1Hz$ the effect of body force is negligible.[3] Since in exploration seismological studies, the frequencies of the signals which are studied are significantly greater than $1Hz$ we can simplify expression (2.4) by neglecting the body force.

Applying the divergence theorem[4] to the surface integral we can combine the integrals to get

$$\iiint_{V(t)} \left(\sum_{j=1}^{3} \frac{\partial \sigma_{ji}}{\partial x_j} - \rho \frac{\partial^2 u_i}{\partial t^2} \right) \mathrm{d}V = 0, \; i \in \{1, 2, 3\}. \tag{2.5}$$

In order for expression (2.5) to be satisfied for any volume V then it must be true that the integrand vanishes, namely,

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ji}}{\partial x_j} = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}, \; i \in \{1, 2, 3\}. \tag{2.6}$$

Expression (2.6) is referred to as Cauchy's equations of motion.

[3] see for example Udias, A., (1999) Principles of seismology: Cambridge, pp. 39–40.

[4] The divergence theorem can be found in nearly any calculus book. For example, Lang, S., (1987) Calculus of Several Variables: Springer-Verlag, pp. 345-350.

2.2 Christoffel equations

In this section we will consider Cauchy's equations of motion for an inhomogeneous medium. Combining these with the stress-strain relation of expression (2.3) we will express the Christoffel equations for which the solvability conditions lead to the eikonal equation.

Since we wish to deal with inhomogeneous media we must express Cauchy's equations of motion (2.6), such that both the density, ρ, and the components of elasticity, c_{ijkl}, are functions of position. Thus we get

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ji}(\mathbf{x})}{\partial x_j} = \rho(\mathbf{x}) \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}, \; i \in \{1, 2, 3\}, \tag{2.7}$$

where

$$\sigma_{ij}(\mathbf{x}) = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{1}{2} c_{ijkl}(\mathbf{x}) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \; i, j \in \{1, 2, 3\}. \tag{2.8}$$

Before we can proceed further we need some idea as to the form of $\mathbf{u}(\mathbf{x}, t)$. A common practice is to assume a trial solution of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \, f(\eta) = \mathbf{A}(\mathbf{x}) \, f(v_0 [\psi(\mathbf{x}) - t]) \tag{2.9}$$

where $\mathbf{A}(\mathbf{x})$ is the amplitude of the disturbance, which varies with position,

v_0 is a constant velocity, and $\psi(\mathbf{x})$ gives the moving wavefront at time t. The term "wavefront" is defined here as the locus of points for which the medium goes from being undisturbed to disturbed. The trial solution then is such that the "function f gives the waveform as a function of time with $\mathbf{A}$ being the spatially variable amplitude of this waveform".[5] Substituting expressions (2.8) and (2.9) into expression (2.7) results in an expression of the form

$$a(\mathbf{x})\, f + b(\mathbf{x}) \frac{\partial f}{\partial \eta} + c(\mathbf{x}) \frac{\partial^2 f}{\partial \eta^2} = 0. \tag{2.10}$$

Functions $a(\mathbf{x})$, $b(\mathbf{x})$ and $c(\mathbf{x})$ define solvability conditions since in order for expression (2.10) to be true for an arbitrary $f(\eta)$, each coefficient must be identically zero. The coefficients of the differentials are of particular interest. The expression corresponding to $b(\mathbf{x})$ is commonly referred to as the transport equation and is important when one wishes to study amplitudes. For our purposes we will be focusing on the coefficient $c(\mathbf{x})$ which in expanded form, for a given $i \in \{1, 2, 3\}$, is

$$\sum_{k=1}^{3} \left(\sum_{j=1}^{3} \sum_{l=1}^{3} c_{ijkl}(\mathbf{x}) \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_l} - \rho(\mathbf{x})\, \delta_{ik} \right) A_k(\mathbf{x}) = 0, \tag{2.11}$$

[5] Slawinski, M.A., (2003) Seismic waves and rays in elastic media: Pergamon, p. 165.

where δ_{ik} is the Kronecker delta. Since $\psi(\mathbf{x})$ describes the wavefront with level sets giving the wavefront at time t, the partial derivatives $\partial\psi/\partial x_j$ are in the direction of the wavefront normal and have units of slowness. Therefore, we define the "slowness vector", $\mathbf{p} := \nabla\psi$, and will use its components given by

$$p_i = \frac{\partial \psi}{\partial x_i}. \tag{2.12}$$

By definition the slowness vector will be equal in magnitude to the inverse of the wavefront velocity, v. Since all points on the wavefront are of the same phase, $\mathbf{p}$ is sometimes referred to as the phase-slowness vector. Consequently, the angle that this vector makes with the vertical is sometimes referred to as the phase angle. Since in this thesis we will not make direct reference to any other type of slowness, we will take all references to "slowness" to mean phase-slowness, without confusion. Using the definition of slowness we can write expressions (2.11) as

$$\sum_{k=1}^{3}\left(\sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\, p_j p_l - \rho(\mathbf{x})\,\delta_{ik}\right) A_k(\mathbf{x}) = 0,\ i \in \{1,2,3\}, \tag{2.13}$$

which are commonly referred to as the Christoffel equations for an inhomogeneous anisotropic medium.

2.3 Eikonal equation

Now we consider the solvability condition for the Christoffel equations and use it to derive the eikonal equation.

In order for equation (2.13) to have a solution it must be true that the 3×3 matrix with ik entries given by

$$\sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\, p_j p_l - \rho(\mathbf{x})\, \delta_{ik},\ i,k \in \{1,2,3\}, \tag{2.14}$$

have a determinant equal to zero. That is,

$$\det\left[\sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\, p_j p_l - \rho(\mathbf{x})\, \delta_{ik}\right] = 0, \quad i,k \in \{1,2,3\}. \tag{2.15}$$

By defining $p^2 := |\mathbf{p}|^2$, we can rewrite this expression as

$$\left(p^2\right)^3 \det\left[\sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}(\mathbf{x})\, \frac{p_j p_l}{p^2} - \frac{\rho(\mathbf{x})}{p^2}\delta_{ik}\right] = 0, \quad i,k \in \{1,2,3\}. \tag{2.16}$$

From the factor leading the determinant we see that this expression is a polynomial of degree three in p^2. As such, we know that this polynomial can be factored as

$$\left(p^2 - \frac{1}{v_{(1)}^2(\mathbf{x},\mathbf{p})}\right)\left(p^2 - \frac{1}{v_{(2)}^2(\mathbf{x},\mathbf{p})}\right)\left(p^2 - \frac{1}{v_{(3)}^2(\mathbf{x},\mathbf{p})}\right) = 0, \tag{2.17}$$

where we have denoted the three roots (not the components) by $1/v_{(i)}^2(\mathbf{x},\mathbf{p})$ which are functions of the position, $\mathbf{x}$, and the direction of the slowness vector, $\mathbf{p}$, and where we have denoted $v^2 := |\mathbf{v}|^2$. The fact that there are three roots (eigenvalues) indicates that there are three wave types possible in inhomogeneous anisotropic media each with a wavefront velocity, $v_{(i)}$. For each velocity, or eigenvalue, there is also an associated eigenvector. This eigenvector gives the direction of the displacement of the medium as the disturbance propagates in the direction of the wavefront normal indicated by the direction of $\mathbf{p}$.

Since herein we will be using field observations of first arrival traveltimes that correspond to the fastest waves, the longitudinal waves, we will not discuss further the other two roots corresponding to transverse waves, S_1 and S_2. For a given wavefront then, we have

$$p^2 = \frac{1}{v^2(\mathbf{x},\mathbf{p})}, \tag{2.18}$$

which is commonly referred to as the "eikonal equation". Since it depends both on position $\mathbf{x}$ and propagation direction, indicated by $\mathbf{p}$, the eikonal equation applies to inhomogeneous anisotropic media. An important property of this equation can be seen in expression (2.16) above. Although the eikonal equation depends on the direction of the slowness vector $\mathbf{p}$, it does not depend on the magnitude of slowness. Since

$$p^2 = |\mathbf{p}|^2 = |\mathbf{p}| \cdot |\mathbf{p}|, \tag{2.19}$$

we can write

$$\frac{p_j p_l}{p^2} = \frac{p_j}{|\mathbf{p}|}\frac{p_l}{|\mathbf{p}|} = n_j n_l,$$

where n_j is the jth–component of the unit normal in the direction of $\mathbf{p}$. Thus we can rewrite equation (2.16) as

$$\left(p^2\right)^3 \det\left[\sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}\left(\mathbf{x}\right) n_j n_l - \rho\left(\mathbf{x}\right) v^2 \delta_{ik}\right] = 0, \; i, k \in \{1, 2, 3\}. \tag{2.20}$$

Returning to equation (2.18), we can write

$$p^2 = \frac{1}{v^2\left(\mathbf{x}, \mathbf{p}\right)} = \frac{1}{v^2\left(\mathbf{x}, \mathbf{n}\right)}, \tag{2.21}$$

where $\mathbf{n}$ is the unit vector normal to the wavefront, which means that $\mathbf{n}$ and

$\mathbf{p}$ have the same direction. This equation indicates the dependence of v on the orientation of $\mathbf{p}$ but not its magnitude.

Definition 2.1 *A function, $v(\mathbf{x}, \mathbf{p})$ is said to be homogeneous of degree r in $\mathbf{p}$ if*

$$v(\mathbf{x}, c\mathbf{p}) = c^r v(\mathbf{x}, \mathbf{p})$$

for every real c.

Since we have stated that the magnitude of the wavefront velocity function v does not depend on the magnitude of the vector giving the orientation of the wavefront, the wavefront velocity function must be homogeneous of degree zero in $\mathbf{p}$. That is,

$$v(\mathbf{x}, c\mathbf{p}) = c^0 v(\mathbf{x}, \mathbf{p}) . \tag{2.22}$$

2.4 Hamilton ray equations

2.4.1 Formulation

Recalling expression (2.12) we see that the eikonal equation (2.18), namely,

$$p^2 = \sum_{i=1}^{3} \left(\frac{\partial \psi (\mathbf{x})}{\partial x_i} \right)^2 = \frac{1}{v^2 (\mathbf{x}, \mathbf{p})},$$

is a first-order nonlinear partial differential equation (PDE) with solution surface given by $\mathbf{p}(\mathbf{x})$. In this section we will seek to construct the solution surface by means of a union of curves. The curves when projected into physical space are the rays which define the paths of energy propagation from source to receiver. The equations defining the rays are known as the Hamilton ray equations, which we aim to derive.

To begin we define the function $F(\mathbf{x}, \mathbf{p})$ for which the solution to the eikonal equation (2.18) forms a level set. That is, we define

$$F(\mathbf{x}, \mathbf{p}(\mathbf{x})) = p^2 v^2 (\mathbf{x}, \mathbf{p}(\mathbf{x})) = 1. \tag{2.23}$$

Since $F(\mathbf{x}, \mathbf{p}(\mathbf{x}))$ is constant, its differential must be zero. Hence we can write

$$\mathrm{d}F\left(\mathbf{x},\mathbf{p}\left(\mathbf{x}\right)\right)=\sum_{i=1}^{3}\left[\frac{\partial F}{\partial x_i}+\sum_{j=1}^{3}\frac{\partial F}{\partial p_j}\frac{\partial p_i}{\partial x_j}\right]\mathrm{d}x_i=0.$$

For each i we seek a non-trivial solution and so we disregard $\mathrm{d}x_i=0$ and it follows that

$$\frac{\partial F}{\partial x_i}+\sum_{j=1}^{3}\frac{\partial F}{\partial p_j}\frac{\partial p_i}{\partial x_j}=0,\ i\in\{1,2,3\}.$$

For a given i we can write this expression as an inner product, namely,

$$\left(\frac{\partial F}{\partial p_1},\frac{\partial F}{\partial p_2},\frac{\partial F}{\partial p_3},-\frac{\partial F}{\partial x_i}\right)\cdot\left(\frac{\partial p_i}{\partial x_1},\frac{\partial p_i}{\partial x_2},\frac{\partial p_i}{\partial x_3},-1\right)=0, \tag{2.24}$$

which indicates that these two vectors are orthogonal to one another.

Since the solution of the eikonal equation is a surface given by $\mathbf{p}\left(\mathbf{x}\right)$, for each i we seek to construct the solution surface by finding curves $p_i\left(\mathbf{x}\right)$ with normal vector components

$$\mathbf{n}_i=\left(\frac{\partial p_i}{\partial x_1},\frac{\partial p_i}{\partial x_2},\frac{\partial p_i}{\partial x_3},-1\right),\ i\in\{1,2,3\}. \tag{2.25}$$

For expression (2.24) above we stated that the vector

$$\left(\frac{\partial F}{\partial p_1}, \frac{\partial F}{\partial p_2}, \frac{\partial F}{\partial p_3}, -\frac{\partial F}{\partial x_i}\right) \tag{2.26}$$

was orthogonal to the vector

$$\left(\frac{\partial p_i}{\partial x_1}, \frac{\partial p_i}{\partial x_2}, \frac{\partial p_i}{\partial x_3}, -1\right),$$

which by expression (2.25) we know is normal to the curve $p_i(\mathbf{x})$ in the solution surface. Therefore we can conclude that the vector on the left in expression (2.24) is tangential to the solution surface. As such, if we parameterize the curve by s (i.e., $[x_1(s), x_2(s), x_3(s), p_i(s)]$) then we solve for it by solving

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}s} = \zeta \dfrac{\partial F}{\partial p_i} \\ \\ \dfrac{\mathrm{d}p_i}{\mathrm{d}s} = -\zeta \dfrac{\partial F}{\partial x_i} \end{cases}, \; i \in \{1, 2, 3\}, \tag{2.27}$$

for some scaling factor ζ. We note that system (2.27) is a set of ordinary differential equations (ODE). Since we would want all of our formulations parameterized with respect to time we will need to find the appropriate scaling factor.

2.4.2 Parameterization in terms of time

Recall expression (2.9), namely

$$\mathbf{u}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})\, f\left(v_0\left[\psi(\mathbf{x}) - t\right]\right), \tag{2.28}$$

which indicates that the eikonal function, $\psi(\mathbf{x})$, has units of time. Therefore we require

$$\mathrm{d}\psi(\mathbf{x}) = \mathrm{d}t.$$

In terms of parameter s we require

$$\frac{\mathrm{d}\psi(\mathbf{x}(s))}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}, \tag{2.29}$$

which, if s is to be equivalent to time t, means that

$$\frac{\mathrm{d}\psi(\mathbf{x})}{\mathrm{d}s} = \sum_{i=1}^{3} \frac{\partial\psi(\mathbf{x})}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}s} = 1,$$

or equivalently in view of definition (2.12),

$$\frac{\mathrm{d}\psi(\mathbf{x})}{\mathrm{d}s} = \sum_{i=1}^{3} p_i \frac{\mathrm{d}x_i}{\mathrm{d}s} = 1. \tag{2.30}$$

Substituting expression (2.23) into the first ODE of system (2.27) and expanding we get

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}s} &= \zeta \frac{\partial}{\partial p_i}\left(p^2 v^2\right) \\ &= \zeta\left(2p_i v^2 + 2p^2 v \frac{\partial v}{\partial p_i}\right) \\ &= 2\zeta\left(p_i v^2 + p^2 v \frac{\partial v}{\partial p_i}\right). \end{aligned} \tag{2.31}$$

Substituting expression (2.31) into expression (2.30) we get

$$\begin{aligned} 1 = \sum_{i=1}^{3} p_i \frac{\mathrm{d}x_i}{\mathrm{d}s} &= \sum_{i=1}^{3} 2\zeta p_i \left(p_i v^2 + p^2 v \frac{\partial v}{\partial p_i}\right) \\ &= 2\zeta\left(p^2 v^2 + p^2 v \sum_{i=1}^{3} p_i \frac{\partial v}{\partial p_i}\right). \end{aligned} \tag{2.32}$$

Recall from section 2.3 where it was shown that the velocity function is homogeneous of degree zero in p_i. This means that we can now apply Euler's homogeneous-function theorem[6] which tells us that

$$\sum_{i=1}^{3} p_i \frac{\partial v}{\partial p_i} = 0.$$

Thus solving for ζ in expression (2.32) we find that in order for the parameterization of system (2.27) to be of time, we require

$$\zeta = \frac{1}{2\left(p^2 v^2\right)} = \frac{1}{2}.$$

Hence we have

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = \dfrac{1}{2}\dfrac{\partial F}{\partial p_i} \\ \\ \dfrac{\mathrm{d}p_i}{\mathrm{d}t} = -\dfrac{1}{2}\dfrac{\partial F}{\partial x_i} \end{array} \right. , \; i \in \{1, 2, 3\}. \tag{2.33}$$

Defining the Hamiltonian,

$$\mathcal{H} := F/2 = \frac{1}{2} p^2 v^2 \left(\mathbf{x}, \mathbf{p}\right), \tag{2.34}$$

[6] for a description of Euler's Theorem see Taylor, A.E., (1955) Advanced calculus: Blaisdell Publishing Company, pp.184-185.

we have

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \\ \dfrac{\mathrm{d}p_i}{\mathrm{d}t} = -\dfrac{\partial \mathcal{H}}{\partial x_i} \end{cases}, \ i \in \{1,2,3\} \tag{2.35}$$

which are referred to as Hamilton's ray equations.

Given a velocity model $v(\mathbf{x}, \mathbf{p})$, we could now use one of many tools developed for solving ODEs and solve the equations of system (2.35). It turns out that if we were to solve the wave equation,

$$\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = \frac{1}{v^2(\mathbf{x}, \mathbf{p})} \frac{\partial^2 u}{\partial t^2},$$

by twice using the method of characteristics[7], we would find that the curves $\mathbf{x}(t)$ are characteristics of the eikonal equation. Furthermore, it could be verified that it is along these curves that the energy of the signal propagates. As such, these are the rays in the physical space, hence the name, ray equations.

[7] For an introduction to the use of characteristics to solve PDE's see McOwen, R.C., (2003) Partial differential equations: Methods and applications, 2ed.: Pearson Education, pp. 11-24.

Chapter 3

Ray velocity in terms of wavefront properties

In this chapter we take the first step in formulating a traveltime expression in terms of ray properties. We will begin by expressing the ray velocity in terms of the wavefront velocity and angle. This is done by substituting the Hamiltonian (2.34) into the first of Hamilton's ray equations and using the result to express the ray velocity. In subsequent chapters we will replace wavefront properties with ray properties until we arrive at an expression for ray velocity in terms of ray quantities alone.

The solution of the first of Hamilton's ray equations results in the curve $\mathbf{x}(t)$ parameterized by time. This curve is referred to as the ray. It is along this curve that the signal propagates. It follows then that the vector $\dot{\mathbf{x}}(t)$ is tangent to the signal trajectory and gives its direction at time t. The magnitude of this vector is equal to the velocity of the energy propagation in the given direction. Hence this vector is called the ray velocity vector and is denoted by

$$V = |\dot{\mathbf{x}}(t)| = \left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right| = \sqrt{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}. \tag{3.1}$$

To derive the expression for $\dot{\mathbf{x}}(t)$ we recall the first of Hamilton's ray equations along with the Hamiltonian $\mathcal{H} = p^2 v^2(\mathbf{x}, \mathbf{p})$ to get

$$\begin{aligned} \dot{x}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ &= p_i v^2 + p^2 v \frac{\partial v(\mathbf{x}, \mathbf{p})}{\partial p_i} \\ &= p_i v^2 + \frac{1}{v} p^2 v^2 \frac{\partial v(\mathbf{x}, \mathbf{p})}{\partial p_i}, \end{aligned} \tag{3.2}$$

which can be simplified with the eikonal equation (2.18) to give

$$\dot{x}_i = p_i v^2(\mathbf{x}, \mathbf{p}) + \frac{1}{v} \frac{\partial v(\mathbf{x}, \mathbf{p})}{\partial p_i}. \tag{3.3}$$

Therefore the square of the ray velocity vector, expression (3.1), can be written as

$$V^2 = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = \sum_{i=1}^{3} \left[v^4 p_i^2 + 2vp_i \frac{\partial v}{\partial p_i} + \frac{1}{v^2} \left(\frac{\partial v}{\partial p_i} \right)^2 \right]. \tag{3.4}$$

Applying the linearity of the summation we can rewrite this as

$$V^2 = v^4 \sum_{i=1}^{3} p_i^2 + 2v \sum_{i=1}^{3} p_i \frac{\partial v}{\partial p_i} + \frac{1}{v^2} \sum_{i=1}^{3} \left(\frac{\partial v}{\partial p_i} \right)^2. \tag{3.5}$$

Since we established earlier that $v(\mathbf{x}, \mathbf{p})$ was homogeneous of degree zero in $\mathbf{p}$, we now apply Euler's homogeneous–function theorem to expression (3.5).

Theorem 3.1 *Euler's homogeneous–function theorem. If the function* $v(\mathbf{x}, \mathbf{p})$ *is homogeneous of degree* r *in* $\mathbf{p}$ *then*

$$\sum_{i=1}^{n} p_i \frac{\partial v}{\partial p_i} = rv(\mathbf{x}, \mathbf{p}).$$

Proof. *In view of Definition 2.1, we write*

$$v(\mathbf{x}, c\mathbf{p}) = c^r v(\mathbf{x}, \mathbf{p})$$

for some $c \in \mathbb{R}$*. Differentiating with respect to* c *results in*

$$\sum_{i=1}^{n} \frac{\partial v}{\partial (cp_i)} p_i = rc^{r-1} v(\mathbf{x}, \mathbf{p}).$$

Letting $c = 1$*, we obtain*

$$\sum_{i=1}^{n} p_i \frac{\partial v}{\partial p_i} = rv(\mathbf{x}, \mathbf{p}).$$

■

Given that $v(\mathbf{x}, \mathbf{p})$ is homogeneous of degree zero in $\mathbf{p}$, the second summation of expression (3.5) is zero and with the help of the eikonal equation, expression (3.5) is reduced to

$$V(\mathbf{x}, \mathbf{p}) = \sqrt{v^2(\mathbf{x}, \mathbf{p}) + \frac{1}{v^2(\mathbf{x}, \mathbf{p})} \sum_{i=1}^{3} \left(\frac{\partial v(\mathbf{x}, \mathbf{p})}{\partial p_i} \right)^2}. \tag{3.6}$$

Expression (3.6) gives the ray velocity in an inhomogeneous anisotropic medium as a function of the position, $\mathbf{x}$, and the orientation of slowness vector $\mathbf{p}$.

Chapter 4

Transversely isotropic media

We now define a particular symmetry class of media which are often justified in the context of petroleum exploration and lead to some convenient simplifications in the mathematical formulations which follow. In particular, we define the transversely isotropic media.

We recall expression (2.1) in which we stated that in a three–dimensional linearly elastic solid, stress is proportional to strain, i.e.,

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}\varepsilon_{kl}, \quad i,j \in \{1,2,3\}.$$

Initially it appears that to fully determine the elastic properties of a given medium we must find the values of 81 components. It turns out, however, that because of the symmetry of the stress tensor ($\sigma_{ij} = \sigma_{ji}$), due to the balance of angular momentum, and because of the symmetry of the strain tensor ($\varepsilon_{ij} = \varepsilon_{ji}$), by its very definition, the 81 components are reduced to 36 independent components. Furthermore, since we assume that all of the energy required to deform the medium is stored as potential energy, the strain-energy function, namely,

$$2W = \sum_{i,j,k,l=1}^{3} c_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \tag{4.1}$$

requires[1] that $c_{ijkl} = c_{klij}$ and so the number of independent components necessary to completely define the elasticity of any medium is further reduced to 21.

The symmetries discussed above are intrinsic and apply to all media. If the medium under study possesses additional "material symmetries" then we are able to further reduce the number of independent components. By a material symmetry we mean that for a given property under investigation, the results of physical measurements made with respect to a number of specific coordinate systems (orientations of the experimental setup) are

[1]This is verified by rewriting the strain-energy function as

$$2W = \sum c_{klij}\varepsilon_{ij}\varepsilon_{kl}$$

and applying the equality of mixed partial derivatives. For further details see Love, A.E.H., (1944) A treatise on the mathematical theory of elasticity, 4ed: Dover, p. 99.

indistinguishable. In such a case, the elasticity tensor would be invariant to specific orthogonal transformations of the coordinate system thus reducing the number of independent components while defining a specific symmetry class.[2]

One type of invariance, or symmetry class, that is of particular interest to geoscientists is that of rotational invariance or transverse isotopy with a vertical symmetry axis, commonly referred to as VTI. For VTI media, experimental measurements are unaffected by the azimuthal orientation of the setup. For example, the observed traveltimes of a signal from a source at the surface to a receiver with a fixed offset and depth would be the same regardless of the orientation of the vertical plane containing the source and the receiver; be it N, W, or NNE.

When we consider that hydrocarbons most commonly form and collect in areas of sedimentary deposition, we immediately see the interest of VTI media in the field of petroleum exploration. As sediments settle they typically do so in a layered fashion. Since the lower sediments will be more compacted due to the pressure exerted by the overburden, vertical planes will be have an inhomogeneity that varies with depth while the horizontal planes contained

[2]Readers interested in how this reduction of components of elasticity is done for the different symmetry classes are referred to Bos, L., et al, (2004) Classes of anisotropic media: A tutorial: Studia geophysica et geodaetica, **48**, pp. 265-287.

in the layers will be homogeneous. Furthermore, since within the horizontal planes the grains will not have any preferred alignment, such a layered medium will be transversely isotropic because as Helbig[3] states "a random assembly of crystallites is equivalent to an isotropic medium". Across layers however, the orientation of the granules with respect to the vertical plane will generate an anisotropic effect. For these reasons we see why the assumption of VTI is sometimes justified in the modelling of a section of the subsurface and why "The vast majority of existing studies of seismic anisotropy are performed for a transversely isotropic (TI) medium...".[4]

As a result of the rotational invariance of VTI media, the number of independent components is reduced from 21 to just five[5]. Also, another convenient consequence of VTI is the fact that because of the inherent lateral homogeneity, as the signal propagates it will not move out of the vertical plane containing the source and receiver. As a result, the rays found by solving the first of Hamilton's ray equations (2.35) can be expressed in two dimensions which simplifies subsequent manipulation of equations.

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[3] Helbig, K., (1994) Foundations of anisotropy for exploration seismics: Pergamon, p. 8.

[4] Tsvankin, I., (2001) Seismic signatures and analysis of reflection data in anisotropic media: Pergamon, p. 11.

[5] Bos, L., et al, (2004) Classes of anisotropic media: A tutorial: Studia geophysica et geodaetica, **48**, pp. 281-284.

Chapter 5

Relation between ray angle and wavefront angle

In this chapter we continue to replace wavefront properties in the ray velocity expression (3.6) with ray properties. Specifically, we derive the relation between the ray angle and the wavefront angle, and use this relation to express the ray velocity in terms of the wavefront velocity and ray angle.

Considering an infinitesimal time increment along the ray, $\mathbf{x}(t)$, in a VTI medium described above, and illustrated in Figure 5.1, we see that the ray angle, θ, is given by

$$\tan\theta = \frac{\mathrm{d}x_1}{\mathrm{d}x_3} = \frac{\mathrm{d}x_1/\mathrm{d}t}{\mathrm{d}x_3/\mathrm{d}t} = \frac{\dot{x}_1}{\dot{x}_3}. \tag{5.1}$$

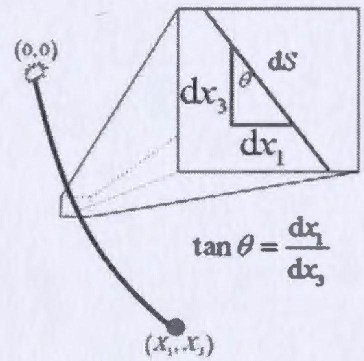

Figure 5.1: Plot of infinitesimal section of the ray. Ray angle indicated in figure.

Recalling expression (3.3) this can be rewritten as

$$\tan\theta = \frac{p_1 v^2 + \dfrac{1}{v}\dfrac{\partial v}{\partial p_1}}{p_3 v^2 + \dfrac{1}{v}\dfrac{\partial v}{\partial p_3}}. \tag{5.2}$$

We see that to express the ray angle in terms of the wavefront angle we must first express the differential operator in terms of the wavefront angle. Since the wavefront angle is related to the slowness by

$$\tan\vartheta = \frac{p_1}{p_3}, \tag{5.3}$$

we apply the identity

$$\frac{\mathrm{d}\tan^{-1}(f(x))}{\mathrm{d}x} = \frac{1}{1+f^2}\frac{\mathrm{d}f}{\mathrm{d}x} \tag{5.4}$$

and recall that $p^2 = |\mathbf{p}|^2$ to get

$$\frac{\partial}{\partial p_1} = \frac{\partial\vartheta}{\partial p_1}\frac{\partial}{\partial\vartheta} = \frac{p_3}{p_1^2+p_3^3}\frac{\partial}{\partial\vartheta} = p_3 v^2\frac{\partial}{\partial\vartheta} \tag{5.5}$$

and

$$\frac{\partial}{\partial p_3} = \frac{\partial\vartheta}{\partial p_3}\frac{\partial}{\partial\vartheta} = -\frac{p_1}{p_1^2+p_3^3}\frac{\partial}{\partial\vartheta} = -p_1 v^2\frac{\partial}{\partial\vartheta}. \tag{5.6}$$

Expression (5.2) then becomes

$$\tan\theta = \frac{p_1 v^2 + \dfrac{1}{v}\left(p_3 v^2 \dfrac{\partial v}{\partial\vartheta}\right)}{p_3 v^2 + \dfrac{1}{v}\left(-p_1 v^2\dfrac{\partial v}{\partial\vartheta}\right)}$$

$$= \frac{\dfrac{p_1}{p_3} + \dfrac{1}{v}\dfrac{\partial v}{\partial\vartheta}}{1 - \dfrac{p_1}{p_3}\dfrac{1}{v}\dfrac{\partial v}{\partial\vartheta}}. \tag{5.7}$$

Using expression (5.3) above we can simplify to get

$$\tan\theta = \frac{\tan\vartheta + \dfrac{1}{v}\dfrac{\partial v}{\partial\vartheta}}{1 - \dfrac{\tan\vartheta}{v}\dfrac{\partial v}{\partial\vartheta}} \tag{5.8}$$

which is the expression for the corresponding ray angle in a two dimensional medium given in terms of the wavefront angle and velocity.

At this point we can also apply expressions (5.5) and (5.6) to expression (3.6) to express the ray velocity in terms of the wavefront angle. Doing so gives

$$V^2(\mathbf{x},\vartheta) = v^2(\mathbf{x},\vartheta) + \left(\frac{\partial v}{\partial\vartheta}\right)^2, \tag{5.9}$$

where by using expression (5.8) we can replace the wavefront angle with the ray angle, $\theta(\vartheta)$.

To further illustrate the relationship between the ray and wavefront angles consider Figure 5.2.

From the figure we see that

$$\tan\theta = \tan(\vartheta + \xi),$$

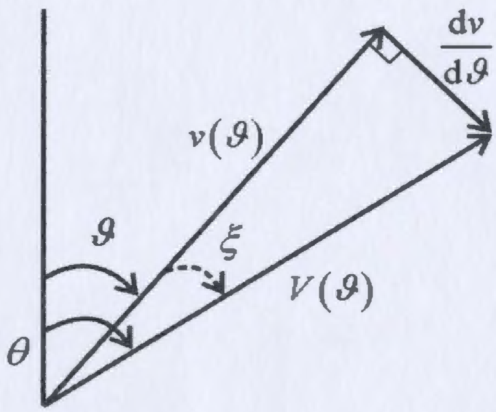

Figure 5.2: Graphical illustration of the relationship between the ray and wavefront angles. The ray angle is denoted by θ and the wavefront angle is denoted by ϑ. The difference between the two is denoted by ξ. The ray velocity is denoted by vector V and the wavefront velocity by vector v.

which by identity can be written as

$$\tan\theta = \frac{\tan\vartheta + \tan\xi}{1 - \tan\vartheta\tan\xi}.$$

Since

$$\tan\xi = \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\vartheta},$$

we get

$$\tan\theta = \frac{\tan\vartheta + \dfrac{1}{v}\dfrac{\mathrm{d}v}{\mathrm{d}\vartheta}}{1 - \dfrac{\tan\vartheta}{v}\dfrac{\mathrm{d}v}{\mathrm{d}\vartheta}}$$

as expected.

Chapter 6

Elliptical anisotropy

Up to this point we have not made any assumptions about the dependence of velocity on propagation direction. In this chapter we introduce the concept of elliptical anisotropy. In the second section, we introduce the ellipticity parameter which we use to formulate the ray velocity expression in terms of ray angle and depth with no explicit dependence on wavefront properties.

6.1 Definition

In order to continue our analytic work we are required to explicitly solve expression (5.8) for the wavefront angle in terms of the ray angle. It has been asserted that such a solution is only possible if the wavefront velocity

v in expression (3.3) is quadratic[1] in the p_i. That is, only when the slowness curve is elliptical (parabolic and hyperbolic curves are not possible since slowness must be a closed curve and both parabolic and hyperbolic would imply directions of infinite velocity!). Therefore, from this point on we restrict ourselves to elliptical anisotropy.

The assumption of elliptical anisotropy means that at a fixed point, if we represent[2] the horizontal velocity by v_{H} and the vertical velocity by v_{V}, then the wavefront velocity at intermediate directions is given by

$$v(\vartheta) = \sqrt{v_{\mathrm{H}}^2 \sin^2(\vartheta) + v_{\mathrm{V}}^2 \cos^2(\vartheta)}. \tag{6.1}$$

Expression (6.1) is not an expression for an ellipse. However, when we consider that the magnitude of slowness is the inverse of the magnitude of wavefront velocity, we can write

$$p(\vartheta) = \frac{1}{\sqrt{v_{\mathrm{H}}^2 \sin^2(\vartheta) + v_{\mathrm{V}}^2 \cos^2(\vartheta)}}, \tag{6.2}$$

for which a polar plot results in an ellipse. Polar plots of expressions (6.1)

---

[1] Slawinski, M.A., (2003) Seismic waves and rays in elastic media: Pergamon, p. 205. See also discussion in section 12.2.

[2] Note that we take v_{H} and v_{V} to be the magnitudes of the wavefront velocity along the x_1- and x_3-axes, respectively. In other words, $v_{\mathrm{H}} = v(\pi/2)$ and $v_{\mathrm{V}} = v(0)$.

and (6.2) are shown in Figure 6.1. We can see that the $v(\vartheta)$–curve is not

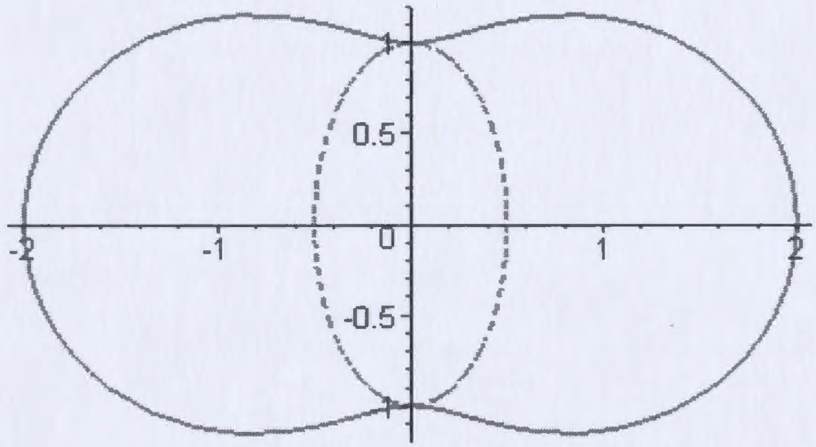

Figure 6.1: Polar plots of expressions (6.1) and (6.2) (i.e., wavefront velocity (solid curve) and slowness (dashed curve), respectively). For graphical emphasis, ellipticity assumed to be more pronounced than expected in most geophysical experiments.

an ellipse while the $p(\vartheta)$–curve is. Additionally, under the assumption of elliptical anisotropy given a point source in a homogeneous medium the expression for the resultant wavefront will be

$$\frac{x_1^2}{v_{\mathrm{H}}} + \frac{x_3^2}{v_{\mathrm{V}}} = t^2, \tag{6.3}$$

which is again an ellipse (if we assume the medium to be inhomogeneous than the above expression will hold only for infinitesimal time increments). As will be shown in the next section, when we consider the ray velocity as a function of the ray angle, it too will form an ellipse in a polar plot.

We note that when dealing with a medium consisting of isotropic layers, it has been shown that longitudinal waves cannot have elliptical wavefronts.[3] However, the assumption of elliptical anisotropy is justified by the resultant simplifications in the model and also, *a posteriori*, by the accurate modelling of field observations. For SH-waves however the wavefronts in such a case are elliptical and thus SH-wave data in layered media would be exactly described by a model based on an assumption of elliptical anisotropy using the expressions which follow.

In Chapter 4 material symmetries were mentioned. The definition of elliptical anisotropy however is not directly related to any symmetry of the medium. The use of the term "elliptical anisotropy" refers to a specific wave and its slowness curve. As was shown in section 2.3, for an anisotropic medium, there are three different waves possible, each with a different velocity expression. Although one particular wave may generate elliptical wavefronts, at least for infinitesimal time increments, the others may not. Thus the labelling of a medium as being elliptically anisotropic depends on the

[3] Helbig, K., (1983) Elliptical anisotropy - Its significance and meaning: Geophysics, **48**, pp. 825-832.

type of wave being studied.

6.2 Ray velocity for elliptical anisotropy and inhomogeneity

We now derive the ray velocity expression for an inhomogeneous medium under the assumption of elliptical anisotropy in terms of depth and ray angle.

Recall expression (5.9) which gives the ray velocity V in terms of the wavefront velocity v and angle ϑ, namely,

$$V^2(\mathbf{x},\vartheta) = v^2(\mathbf{x},\vartheta) + \left(\frac{\partial v(\mathbf{x},\vartheta)}{\partial\vartheta}\right)^2, \tag{6.4}$$

which is valid for an inhomogeneous, anisotropic medium. In seismological surveys, we often measure the first arrival times of the seismic disturbance. In view of Fermat's principle we take the measured traveltime of the disturbance to be along the ray that makes the traveltime stationary. For our purposes we can assume that the traveltime is always minimized. Since our field measurements are better related to the ray than to the wavefront, whenever we attempt to solve an inverse problem based on field data we would like to have expressions involving only ray quantities. Hence we need to replace the wavefront angle in expression (6.4) with the ray angle.

First, let us assume that the wavefront velocity varies spatially as a function of depth. Such an assumption would be consistent with a VTI medium in which compaction, and hence density, increases with depth resulting in an increase of velocity. Such a medium is said to be vertically inhomogeneous. Next, we assume an elliptical velocity dependence on direction. Defining the ellipticity parameter

$$\chi := \frac{v_{\mathrm{H}}^2 - v_{\mathrm{V}}^2}{2v_{\mathrm{V}}^2},$$

which we assume to be constant, we can express the wavefront velocity expression (6.1) as

$$v\left(x_3, \vartheta\right) = v_{\mathrm{V}}\left(x_3\right)\sqrt{1 + 2\chi \sin^2 \vartheta}. \tag{6.5}$$

Substituting the above expression into expression (6.4) we get

$$\begin{aligned} V^2\left(x_3, \vartheta\right) &= v^2\left(x_3, \vartheta\right) + \left(\frac{\partial v\left(x_3, \vartheta\right)}{\partial \vartheta}\right)^2 \\ &= v_{\mathrm{V}}^2\left(x_3\right)\left(1 + 2\chi \sin^2 \vartheta\right) + 4\chi^2 v_{\mathrm{V}}^2\left(x_3\right)\frac{\sin^2 \vartheta \cos^2 \vartheta}{1 + 2\chi \sin^2 \vartheta} \\ &= v_{\mathrm{V}}^2\left(x_3\right)\left(\frac{1 + 4\chi\left(1 + \chi\right)\sin^2 \vartheta}{1 + 2\chi \sin^2 \vartheta}\right). \end{aligned}$$

Dividing both the numerator and denominator by $\cos^2\vartheta$ we get

$$V^2(x_3,\vartheta) = v_V^2(x_3)\frac{\sec^2\vartheta + 4\chi(1+\chi)\tan^2\vartheta}{\sec^2\vartheta + 2\chi\tan^2\vartheta} \tag{6.6}$$

$$= v_V^2(x_3)\frac{1+(1+2\chi)^2\tan^2\vartheta}{1+(1+2\chi)\tan^2\vartheta}. \tag{6.7}$$

To proceed further, we must now solve expression (5.8) for ϑ. Substituting expression (6.5) into expression (5.8) and simplifying we get

$$\tan\theta = \frac{\tan\vartheta + \dfrac{1}{v_V(x_3)\sqrt{1+2\chi\sin^2\vartheta}}\left(2\chi v_V(x_3)\dfrac{\sin\vartheta\cos\vartheta}{\sqrt{1+2\chi\sin^2\vartheta}}\right)}{1 - \dfrac{\tan\vartheta}{v_V(x_3)\sqrt{1+2\chi\sin^2\vartheta}}\left(2\chi v_V(x_3)\dfrac{\sin\vartheta\cos\vartheta}{\sqrt{1+2\chi\sin^2\vartheta}}\right)}$$

$$= (1+2\chi)\tan\vartheta. \tag{6.8}$$

Returning to expression (6.7) we now write

$$
\begin{aligned}
V^2\left(x_3,\theta\left(\vartheta\right)\right) &= v_{\mathrm{V}}^2\left(x_3\right)\left(\frac{1+\left(1+2\chi\right)^2\tan^2\vartheta}{1+\left(1+2\chi\right)\tan^2\vartheta}\right) \\
&= v_{\mathrm{V}}^2\left(x_3\right)\left(\frac{1+\left(1+2\chi\right)^2\dfrac{\tan^2\theta}{\left(1+2\chi\right)^2}}{1+\left(1+2\chi\right)\dfrac{\tan^2\theta}{\left(1+2\chi\right)^2}}\right) \\
&= v_{\mathrm{V}}^2\left(x_3\right)\frac{\left(1+2\chi\right)\sec^2\theta}{\left(1+2\chi\right)+\tan^2\theta},
\end{aligned}
$$

and so we obtain

$$
V\left(x_3,\theta\right) = v_{\mathrm{V}}\left(x_3\right)\sqrt{\frac{1+2\chi}{1+2\chi\cos^2\theta}}. \tag{6.9}
$$

Expression (6.9) gives the ray velocity in terms of the ray angle and depth for an elliptical velocity dependence and vertical inhomogeneity. The ellipticity is characterized by χ which remains constant and the inhomogeneity is such that velocity increases with depth.

Chapter 7

Ray parameter

In this chapter we introduce the concept of a conserved quantity along the ray. Specifically we derive the expression for the ray parameter in terms of the ray velocity and ray angle. Then we will simplify further to write the expression of the ray parameter in terms of a point on the ray.

In Figure 7.1 we see that the total traveltime along the ray can be written in integral form as

$$t = \int \frac{\mathrm{d}s}{V} = \int \frac{\sqrt{(\mathrm{d}x_1)^2 + (\mathrm{d}x_3)^2}}{V(x_3, \theta)}. \tag{7.1}$$

If we now define

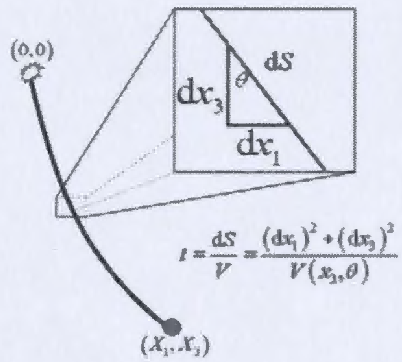

Figure 7.1: Plot of infinitesimal section of the ray. Traveltime along dS is indicated in figure.

$$x_3' := \frac{\mathrm{d}x_3}{\mathrm{d}x_1} = \cot\theta, \tag{7.2}$$

we can rewrite to get

$$t = \int\limits_{x_1=0}^{X_1} \frac{\sqrt{1 + (x_3')^2}}{V\left(x_3, x_3'\right)} \mathrm{d}x_1, \tag{7.3}$$

where X_1 is the total horizontal offset of the receiver from the source. As stated earlier, the traveltime of seismic wave propagation is stationary. Such a condition is fundamental in the branch of mathematics known as the calculus of variations. Within the context of the calculus of variations it can be concluded that in order for an integral equation to be stationary, the integrand, denoted F, must satisfy the Beltrami identity[1], namely

$$\frac{\partial F\left(x_1, x_3, x_3'\right)}{\partial x_1} + \frac{\mathrm{d}}{\mathrm{d}x_1}\left(x_3' \frac{\partial F\left(x_1, x_3, x_3'\right)}{\partial x_3'} - F\left(x_1, x_3, x_3'\right)\right) = 0. \qquad (7.4)$$

In our particular case we note that the integrand of expression (7.3) does not *explicitly* depend on x_1. Hence when we substitute this integrand into expression (7.4) the first partial derivative will be zero. Furthermore, this implies that the expression in parenthesis must be constant with respect to x_1. Therefore we can write,

$$p = F\left(x_3, x_3'\right) - x_3' \frac{\partial F\left(x_3, x_3'\right)}{\partial x_3'}, \qquad (7.5)$$

where F denotes the integrand of expression (7.3). The parameter p represents a conserved quantity along the ray and is commonly referred to as the ray parameter. In our particular case (VTI media) what is conserved is the horizontal component of wavefront slowness, hence the use of p to denote the constant. Substituting the integrand of expression (7.3) into (7.5) above

[1] for details see Marion, J.B., (1970) Classical dynamics of particles and systems, 2ed.: Academic Press, p. 184–189.

we get

$$p = \frac{\sqrt{1 + (x_3')^2}}{V(x_3, x_3')} - x_3' \frac{\partial}{\partial x_3'} \left[\frac{\sqrt{1 + (x_3')^2}}{V(x_3, x_3')} \right]$$

$$= \frac{\sqrt{1 + (x_3')^2}}{V(x_3, x_3')} - x_3' \frac{\partial \left(\frac{1}{V} \right)}{\partial x_3'} \sqrt{1 + (x_3')^2} - \frac{(x_3')^2}{V\sqrt{1 + (x_3')^2}},$$

and so we get

$$p = \frac{1}{V\sqrt{1 + (x_3')^2}} - x_3' \frac{\partial \left(\frac{1}{V} \right)}{\partial x_3'} \sqrt{1 + (x_3')^2}. \tag{7.6}$$

To express the ray parameter in terms of the ray angle, θ, we note that

$$\frac{\partial}{\partial x_3'} = \frac{\partial \theta}{\partial x_3'} \frac{\partial}{\partial \theta}.$$

Recalling expression (7.2), namely,

$$x_3' := \frac{\mathrm{d}x_3}{\mathrm{d}x_1} = \cot\theta \Rightarrow \theta = \operatorname{arccot}(x_3'),$$

we can make the change of variables in the differential operator by

$$\frac{\partial \theta}{x_3'} = \frac{\partial \operatorname{arccot}(x_3')}{x_3'} = \frac{-1}{1+(x_3')^2} = \frac{-1}{1+\cot^2(\theta)} = \frac{-1}{\csc^2(\theta)} = -\sin^2(\theta)$$

to get

$$\frac{\partial}{\partial x_3'} = \frac{\partial \theta}{\partial x_3'}\frac{\partial}{\partial \theta} = -\sin^2(\theta)\frac{\partial}{\partial \theta}. \tag{7.7}$$

Therefore we can express the ray parameter as a function of the depth and the ray angle in a VTI medium as

$$p = \frac{\sin\theta}{V(x_3,\theta)} + \cos\theta\frac{\partial}{\partial\theta}\left(\frac{1}{V(x_3,\theta)}\right). \tag{7.8}$$

We note here that in the case of an isotropic medium, there would be no dependence of ray velocity on the ray angle. Hence the partial derivative in expression (7.8) would be zero and we would be left with the commonly known form of Snell's law,

$$p \equiv \frac{\sin\vartheta}{v(x_3)} = \frac{\sin\theta}{V(x_3)}, \tag{7.9}$$

where in an isotropic medium, there is no distinction between wavefront and ray quantities. It is clear then that Snell's law gives the horizontal component of the wavefront slowness in an isotropic medium.

Substituting the ray velocity expression for an elliptically anisotropic, vertically inhomogeneous medium, namely expression (6.9), into ray parameter expression (7.8) we get

$$p = \frac{\sin\theta}{v_V(x_3)\sqrt{\dfrac{1+2\chi}{1+2\chi\cos^2\theta}}} + \cos\theta \frac{\partial}{\partial\theta}\left(v_V(x_3)\sqrt{\frac{1+2\chi}{1+2\chi\cos^2\theta}}\right)^{-1} \tag{7.10}$$

which simplifies to

$$p = \frac{\sin\theta}{v_V(x_3)} \frac{1}{\sqrt{1+2\chi}} \frac{1}{\sqrt{1+2\chi\cos^2\theta}}, \tag{7.11}$$

and gives the ray parameter for an elliptically anisotropic vertically inhomogeneous medium as a function of depth and ray angle.

We have stated that the ray parameter is a conserved quantity along the ray and in the case of VTI it is the horizontal component of slowness. To have a value of the ray parameter, it will be necessary to express p in terms of a known point on the ray. In our case the point that we will know for certain is the receiver position. To express p in terms of a point on the ray[2] we first divide both the numerator and denominator of the right side of expression (7.11) by $\cos\theta$ and then by recalling that $\tan\theta = \mathrm{d}x_1/\mathrm{d}x_3$, we get

$$\frac{p v_{\mathrm{V}}(x_3)(1+2\chi)}{\sqrt{1-p^{2v_{\mathrm{V}}(x_3)2}(1+2\chi)}}\mathrm{d}x_3 = \mathrm{d}x_1. \tag{7.12}$$

Now by integrating each side from the source at $(0,0)$ to the receiver at (X_1, X_3) we get

$$\int_0^{X_3} \frac{p v_{\mathrm{V}}(x_3)(1+2\chi)}{\sqrt{1-p^2 v_{\mathrm{V}}(x_3)^2(1+2\chi)}}\mathrm{d}x_3 = \int_0^{X_1} \mathrm{d}x_1. \tag{7.13}$$

To carry out the integration we need to specify $v_{\mathrm{V}}(x_3)$. As was stated in Chapter 4, VTI media are most commonly the result of sedimentary deposition. As the layer of sediment builds, the lower layers become increasingly compacted and hence, more dense. Therefore, we expect the velocity of signal propagation through such a medium to increase with depth. As stated by Slotnick[3] "Experience has shown that the velocity of seismic wave propagation in Tertiary basins can be closely approximated by expressing it as a linear function of depth". In other words, we assume inhomogeneity is in

[2] For complete details see Appendix A.

[3] Slotnick, M.M., (1959) Lessons in seismic computing: Society of exploration geophysicists, p. 205.

the form of a constant velocity gradient where

$$v_V(x_3) = a + bx_3, \tag{7.14}$$

where a gives the vertical velocity at zero depth and where b is the gradient which defines the rate of increase in vertical velocity with depth. Therefore we simplify expression (7.13) to get

$$\int_0^{X_3} \frac{p\,(a+bx_3)\,(1+2\chi)}{\sqrt{1-p^2\,(a+bx_3)^2\,(1+2\chi)}}\mathrm{d}x_3 = \int_0^{X_1} \mathrm{d}x_1$$
$$\frac{1}{pb}\left[\sqrt{1-p^2a^2\,(1+2\chi)} - \sqrt{1-p^2\,(a+bx_3)^2\,(1+2\chi)}\right] = X_1. \tag{7.15}$$

Solving expression (7.15) for p we obtain

$$p = \frac{2X_1}{\sqrt{(X_1^2+(1+2\chi)X_3^2)[(2a+bX_3)^2\,(1+2\chi)+b^2X_1^2]}}, \tag{7.16}$$

which gives the value of the ray parameter in terms of the model parameters, a, b and χ and the receiver position, (X_1, X_3). We note that the value of the ray parameter is the same for all points along the ray so in general, (X_1, X_3) can be replaced by (x_1, x_3) for any point on the ray. The problem however is that initially, the only point on the ray that we will know for certain is the receiver position and so we would not in general make the substitution.

Chapter 8

Traveltime expression for elliptically anisotropic vertically inhomogeneous media

In chapter 7 we derived the expression for the ray parameter in terms of ray quantities. In doing so we specified the ray velocity expression for a vertically inhomogeneous, elliptically anisotropic medium. In this chapter we will use the ray parameter expression to give the ray angle as a function of position. Having such an expression for the ray angle we will then derive the traveltime expression for signal propagation between a source and a receiver at (X_1, X_3).

Recall Figure 7.1 in which we consider an infinitesimal section of a ray. Let us assume as above that the medium is VTI with an elliptical velocity dependence and is vertically inhomogeneous. We see that the total traveltime along the ray from a source at the origin to a receiver at a depth of X_3, can be expressed as

$$t = \int \frac{\mathrm{d}s}{V} = \int \frac{\mathrm{d}x_3}{V_V} = \int\limits_{x_3=0}^{x_3=X_3} \frac{\mathrm{d}x_3}{V(x_3, \theta)\cos\theta}, \tag{8.1}$$

where $V(x_3, \theta)$ is the ray velocity as a function of depth, x_3, and ray angle, θ. Inserting ray velocity expression (6.9) we write

$$t = \int\limits_0^{X_3} \frac{\mathrm{d}x_3}{(a + bx_3)\sqrt{\dfrac{1+2\chi}{1+2\chi\cos^2\theta}}\cos\theta}. \tag{8.2}$$

To evaluate this integral we must find an expression for $\cos\theta$ in terms of depth. We do so by solving expression (7.11) to get

$$\cos\theta = \sqrt{\frac{1 - p^2(a+bx_3)^2(1+2\chi)}{1 + 2\chi p^2 (a+bx_3)^2(1+2\chi)}}. \tag{8.3}$$

Substitution of expression (8.3) into the traveltime expression (8.2) gives

$$t = \int\limits_{x_3=0}^{X_3} \frac{\mathrm{d}x_3}{(a + bx_3)\sqrt{1 - p^2 (a + bx_3)^2 (1 + 2\chi)}}. \tag{8.4}$$

Carrying out this integration[1] we obtain

$$t = \frac{1}{b} \ln \left[\frac{a + bX_3}{a} \frac{1 + \sqrt{1 - p^2 a^2 (1 + 2\chi)}}{1 + \sqrt{1 - p^2 (a + bX_3)^2 (1 + 2\chi)}} \right], \tag{8.5}$$

which is the direct–arrival traveltime expression between a source at the origin, $(0,0)$, and a receiver located at (X_1, X_3) in an elliptically anisotropic medium with inhomogeneity given by a constant velocity gradient. The value of p is given by expression (7.16).

[1]For the details of this integration, see Appendix A.2.

Chapter 9

Parameter estimation

In this chapter we will examine a procedure that can be used to determine the values of the model parameters, a, b and χ from a given data set. The procedure will involve a regression analysis for which we will begin by defining the objective of the regression and then specify the procedure for carrying it through.

9.1 Objective

Now that we have an analytic traveltime expression we would like to determine the values of a, b and χ which best account for VSP traveltime in a set of field observations. Due to measurement errors in the experimental observations we must write the observed traveltime as

$$T_i = t\left(\mathbf{X}_i; \mathbf{P}\right) + \epsilon_i = t_i\left(\mathbf{P}\right) + \epsilon_i, \tag{9.1}$$

where T_i indicates the observed traveltime, t_i indicates the modeled traveltime for receiver located at $\mathbf{X}_i = \left[X_1, X_3\right]_i$ and where $\mathbf{P} = [a, b, \chi]$ is the vector composed of the model parameters to be estimated. Solving for the residual we get the expression

$$\epsilon_i = T_i - t_i\left(\mathbf{P}\right). \tag{9.2}$$

Our objective in estimating vector $\mathbf{P}$ will be to minimize the sum of squared residuals given by

$$\Delta\left(\mathbf{P}\right) = \sum_{i=1}^{n} \left[T_i - t_i\left(\mathbf{P}\right)\right]^2, \tag{9.3}$$

for n observations. Numerous techniques are available to find this value of $\mathbf{P}$, also known as the least-squares estimator $\hat{\mathbf{P}}$. In our case we choose to proceed by means of Gauss-Newton minimization.

9.2 Gauss-Newton method

By examining expression (8.5) we see that the traveltime expression is a nonlinear function with respect to the parameters a, b and χ given by

$$t_i = t\left(\mathbf{X}_i; a, b, \chi\right), \tag{9.4}$$

for a given receiver, $\mathbf{X}_i = [X_1, X_3]_i$. Since Gauss-Newton optimization requires a linear model with respect to the parameters to be estimated, we must first linearize expression (8.5) about some initial guess for those values. We denote the initial parameter estimates by $\mathbf{P}^0$. Linearization is done by taking the first-order Taylor series expansion of the model centered on $\mathbf{P}^0$. That is

$$\begin{aligned}
\varepsilon_i &\approx T_i - \left[\begin{array}{r} t_i(\mathbf{P}^0) + \left(\left.\dfrac{\partial t}{\partial a}\right|_{\mathbf{P}^0}\right)\left(a - a^0\right) + \left(\left.\dfrac{\partial t}{\partial b}\right|_{\mathbf{P}^0}\right)\left(b - b^0\right) \\ + \left(\left.\dfrac{\partial t}{\partial \chi}\right|_{\mathbf{P}^0}\right)\left(\chi - \chi^0\right) \end{array}\right] \\
\widetilde{\varepsilon}_i &= \left[T_i - t_i(\mathbf{P}^0)\right] - \sum_{j=1}^{3}\left(J_{ij}|_{\mathbf{P}^0}\right)\left(P_j - P_j^0\right) \\
&= \varepsilon_i\left(\mathbf{P}^0\right) - \sum_{j=1}^{3}\left(J_{ij}|_{\mathbf{P}^0}\right)\left(P_j - P_j^0\right)
\end{aligned} \tag{9.5}$$

where the partial derivatives are collected as elements of the Jacobian matrix, J_{ij}. Here, $j \in \{1, 2, 3\}$ since there are three parameters and $i \in \{1 ... n\}$

to account for n observations. Considering all n observations we write the objective, which we have chosen to be minimizing the sum of the squared residuals, in matrix form as

$$\min_P \left\| \boldsymbol{\varepsilon}\left(\mathbf{P}^0\right) - \left(\mathbf{J}|_{\mathbf{P}^0}\right)\left(\mathbf{P} - \mathbf{P}^0\right) \right\|^2 .$$

To proceed further, we must orthogonalize the Jacobian matrix. One way to this is by means of a Householder **QR**-decomposition.[1] That is, we solve for matrices **Q** and **R** such that

$$\mathbf{J} = \mathbf{QR}$$

where

- **Q** is an orthogonal $n \times n$ matrix (i.e., $\mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$ and $\|\mathbf{Qx}\| = \|\mathbf{x}\|$).
- **Q** can be written as

$$\mathbf{Q}_{n\times n} = \left[\mathbf{Q}_{1[n\times j]}, \mathbf{0}_{[n\times(n-j)]}\right] + \left[\mathbf{0}_{[n\times j]}, \mathbf{Q}_{[(n-j)\times n]}\right],$$

  where for us $j = 3$ as stated above.

[1] For details on QR-decompostion see for example Golub, G.H., and Van Loan, C.F., (1983) Matrix Computations: The Johns Hopkins University Press, pp. 146–160.

- $\mathbf{R}$ is an uppertriangular $n \times j$ matrix.

- $\mathbf{R}$ can be written as

$$\mathbf{R}_{[n\times j]}=\begin{bmatrix} \mathbf{R}_{1[j\times j]} \\ \mathbf{0}_{[n-j\times j]} \end{bmatrix}$$

  where $\mathbf{R}_1$ is invertible.

Having a **QR**-decomposition of the Jacobian matrix we can write the expression we wish to minimize as

$$\begin{aligned}
&\left\|\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{Q}\mathbf{R}\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2\\
&=\left\|\mathbf{Q}\left[\mathbf{Q}^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{R}\left(\mathbf{P}-\mathbf{P}^0\right)\right]\right\|^2\\
&=\left\|\mathbf{Q}^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{R}\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2\\
&=\left\|\left[\mathbf{Q}_1,\mathbf{0}\right]^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)+\left[\mathbf{0},\mathbf{Q}_2\right]^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\left[\frac{\mathbf{R}_1}{\mathbf{0}}\right]\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2\\
&=\left\|\mathbf{Q}_1^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{R}_1\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2+\left\|\mathbf{Q}_2^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)\right\|^2,
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
&\left\|\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{Q}\mathbf{R}\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2\\
&=\left\|\mathbf{Q}_1^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{R}_1\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2+\left\|\mathbf{Q}_2^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)\right\|^2. \qquad (9.6)
\end{aligned}$$

Now, since the term

$$\left\|\mathbf{Q}_2^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)\right\|^2,$$

on the right-hand side of expression (9.6) does not depend on $\mathbf{P}$, minimizing the squared residuals amounts to setting the left term to zero. That is

$$\begin{aligned}\left\|\mathbf{Q}_1^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{R}_1\left(\mathbf{P}-\mathbf{P}^0\right)\right\|^2&=0\\ \mathbf{Q}_1^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)-\mathbf{R}_1\left(\mathbf{P}-\mathbf{P}^0\right)&=0\\ \mathbf{R}_1\left(\mathbf{P}-\mathbf{P}^0\right)&=\mathbf{Q}_1^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right).\end{aligned}$$

Hence, for a given estimate of the parameter vector $\mathbf{P}^0$ we find that the squared residuals for $\mathbf{P}^0$ are minimized by setting

$$\mathbf{P}=\mathbf{R}_1^{-1}\mathbf{Q}_1^{\mathrm{T}}\varepsilon\left(\mathbf{P}^0\right)+\mathbf{P}^0. \tag{9.7}$$

If we now use this value of $\mathbf{P}$ as the starting point of a new minimization we define an iterative technique to solve for the parameter values that lead to a global minimum of the sum of the squared residuals. In other words, solving for $\mathbf{P}$, setting $\mathbf{P}^0=\mathbf{P}$ and repeating in this manner until a prescribed tolerance level for the difference between two estimates is met gives the final estimate of $\mathbf{P}$, denoted $\hat{\mathbf{P}}$.

Chapter 10

Example: Western Canada Basin

In this chapter we apply the model of Chapter 8 along with the method of parameter estimation of Chapter 9 to a real data set. In addition to calculating the parameter estimates we also introduce the concept of confidence intervals which we also calculate. To illustrate the improvement in traveltime modelling of the model developed herein, the model residuals are compared to those of an isotropic, homogeneous model and also to those of an anisotropic, homogeneous model.

10.1 Parameter estimation

We now give an example of the application of an $ab\chi$ model to a real data set. Listed in Table 10.1 are the observed first arrival times for a VSP seismic survey in the Western Canada Basin taken with a 1ms sampling rate.Since in this area, layering is strongly horizontal, an assumption of

Table 10.1: Traveltime data from VSP survey in the Western Canada Basin

Depth, Z	Receiver Offset $X = 39$ m	Receiver Offset $X = 635$ m
	Observed traveltime	Observed traveltime
950 m	0.358 s	0.422 s
1025 m	0.381 s	0.441 s
1100 m	0.404 s	0.460 s
1175 m	0.426 s	0.479 s
1250 m	0.449 s	0.500 s
1325 m	0.471 s	0.520 s
1400 m	0.492 s	0.538 s
1445 m	0.504 s	0.548 s

lateral homogeneity appears to be reasonable. Further, due to increasing compaction of the sediments with depth it also seems reasonable to expect the signal velocity to increase with depth. Therefore the applicability of an $ab\chi$ model appears justified and so we proceed.

To begin the approximation technique described above, we set the initial parameter estimates to

$$\mathbf{P}^0 = \begin{bmatrix} a^0 \\ b^0 \\ \chi^0 \end{bmatrix} = \begin{bmatrix} 2000 \text{ m/s} \\ 0.30 \text{ s}^{-1} \\ 0.25 \end{bmatrix}.$$

Carrying out the Gauss-Newton method we get the following estimates of the parameter values,

$$\hat{\mathbf{P}} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{\chi} \end{bmatrix} = \begin{bmatrix} 2271 \text{ m/s} \\ 0.879 \text{ s}^{-1} \\ 0.039 \end{bmatrix}. \tag{10.1}$$

10.2 Confidence intervals of parameter estimates

Now that we have estimated the parameter values we wish to form an idea of the "reliability" of these estimates. We do so by calculating the associated confidence intervals.

In statistical analysis, confidence intervals are assigned a percentile value with the usual being 95%. What is meant by a 95% confidence interval is that if one were to repeat an experiment 100 times, the resultant parameter estimates would fall somewhere in the confidence interval 95 times out of the 100.

Confidence intervals also serve as an indicator of how well the chosen model applies to the data set. If the confidence interval is large, then we would conclude that the assumption of the $ab\chi$ model for the given data set was not a good choice.

It should not be surprising that if we were to remove a parameter from our modeling, say by removing χ, and perform a parameter estimation, the results would be different from the ones in expression (10.1). Therefore, when estimating confidence intervals we must bear in mind the interaction between parameters. The way to do so is to calculate simultaneous confidence intervals. A common means of doing so for multiparameter models is to assume a Fisher distribution of parameter estimates and apply the formula[1]

$$\begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{\chi} \end{bmatrix} \pm s\sqrt{3F^{k}_{3,16-3}\mathbf{J}^{\mathrm{T}}\left(\hat{\mathbf{P}}\right)\mathbf{J}\left(\hat{\mathbf{P}}\right)}. \tag{10.2}$$

[1]Seber, G.A.F., and Wild, C.J., (1989) Nonlinear Regression: John Wiley & Sons, p. 194.

In expression (10.2), s represents the variance estimate and is given by

$$s^2 = \frac{\left|\left|\mathbf{T} - \mathbf{t}\left(\mathbf{x}; \hat{\mathbf{P}}\right)\right|\right|^2}{16-3}. \tag{10.3}$$

The factor $F^k_{3,16-3}$ is the Fisher distribution value which assumes a $k = 0.05$ quantile (i.e., a 95% confidence interval) with (3,13) degrees of freedom and $\mathbf{J}\left(\hat{\mathbf{P}}\right)$ is the Jacobian matrix evaluated at the final estimate of the parameter values. Thus we get the 95% confidence intervals for a, b and χ calculated simultaneously.

Looking up the appropriate value of the Fisher distribution and substituting in the Jacobian matrix evaluated for the parameter estimates in expression (10.1), we find the simultaneous confidence intervals to be

$$\begin{aligned} 2262 &\leq a \leq 2279 \\ 0.864 &\leq b \leq 0.894 \\ 0.035 &\leq \chi \leq 0.042. \end{aligned} \tag{10.4}$$

Since the confidence intervals do not vary significantly from the estimated values, we can conclude that the observed data do not vary significantly from the proposed model. From the confidence interval for χ we see that, although its value are is small, χ is statistically significant since $\chi = 0$ is not contained in the interval. Furthermore, $\chi = 0$ is a full five interval widths from lower bound of the interval, so the likelihood of getting that value is vanishing small. The same is true of b. This then would imply that assuming a homogeneous, isotropic model would be doubly erroneous. To further illustrate how well the traveltimes are modelled we calculate the traveltime residuals to get the results in Table 10.2.

Table 10.2: Model residuals for Western Canada Basin VSP data. Model parameters are $a = 2271 m/s$, $b = 0.879 s^{-1}$, and $\chi = 0.039$.

Depth, Z	Receiver Offset $X = 39$ m		Receiver Offset $X = 635$ m	
	Observed traveltime	Model residual	Observed traveltime	Model residual
950 m	0.358 s	1.42 ms	0.422 s	-0.63 ms
1025 m	0.381 s	0.74 ms	0.441 s	-0.64 ms
1100 m	0.404 s	0.03 ms	0.460 s	-0.94 ms
1175 m	0.426 s	0.34 ms	0.479 s	-0.98 ms
1250 m	0.449 s	0.14 ms	0.500 s	0.96 ms
1325 m	0.471 s	0.12 ms	0.520 s	1.60 ms
1400 m	0.492 s	0.54 ms	0.538 s	1.42 ms
1445 m	0.504 s	-1.91 ms	0.548 s	-0.04 ms

In Table 10.2 we see that the conclusions above regarding the suitability of an $ab\chi$ model are reaffirmed, since the residuals of the modelled times versus the observed times are, with only two exceptions, less than 1ms. By using the $ab\chi$ model we are able to account for traveltimes to within the sampling rate of 1ms which we assume is representative of the observation error. Therefore, further complications of the model are not justified for this data set.

In order to compare the present model with both a homogeneous isotropic model (i.e., $a \neq 0$, $b = 0$ and $\chi = 0$) and an inhomogeneous isotropic model (i.e., $a \neq 0$, $b \neq 0$ and $\chi = 0$) we refer to Table 10.3.

Table 10.3: Average absolute residuals between observed traveltime and modelled traveltime.

Homogeneous, isotropic model	$(b = 0, \chi = 0)$	8.70 ms
Inhomogeneous, isotropic model	$(b \neq 0, \chi = 0)$	1.81 ms
Inhomogeneous, anisotropic model	$(b \neq 0, \chi \neq 0)$	0.78 ms

In both cases the parameters values (a then a and b, respectively) were calculated independently and the model residuals were calculated based on those independent values.

As was mentioned above, it is now obvious that to assume an isotropic model is doubly erroneous since the average absolute residual for such a model is nearly 9ms. We see that the introduction of a constant velocity gradient reduces the absolute residual to slightly less than 2ms. A notable improvement yet since we expect the sampling rate to be representative of the measurement error we are motivated to extend the model further. Once we introduce anisotropy into the model in the form of χ we see that the average residual is within our measurement error and we accept the model as satisfactory.

A word of caution when attempting to incorporate anisotropy in the modelling. The definition of anisotropy requires the dependence of signal velocity on propagation direction. Therefore, when attempting to apply a model based on any assumption of anisotropy to a given data set, the range of source and receiver positions of the data set must be such that observed traveltimes are recorded for a range of propagation directions. Otherwise, one would be attempting to estimate the variation of velocity with direction based on observations for essentially a single direction! As an example consider a survey performed in a marine setting. It is common practice in a marine survey for a survey ship (the source) to run concentric circles about the borehole while data is recorded in the borehole. Of course all "shots" at a given radius will give results with no variation in the ray. Furthermore, given the large depths of many marine drilling operations, sources at several radii may still not have the required varied directions necessary to draw any conclusions about anisotropy. What is important is the relative range of source and receiver positions to their respective offsets.

In the case of the Western Canada Basin data examined in this chapter, let us for the moment make the rough approximation of rays by assuming them to be straight. Doing so we see that the source at 35m offset with the receiver at 1445m depth would have a ray angle of approximately 1°

and the source at 635m offset with receiver at 950m depth would have a ray with angle of approximately 53°. The sources and receivers between these extremes would then give observations for the intermediate directions. Because of the wide range of observations we understand why we were able to get a good estimate of χ. Without this range, problems can arise.

Chapter 11

Application: Estimation of zone of illumination

Now that we have a model with the estimated parameters, which have been statistically validated, we will now use the model to determine the zone of possible reflection points of an offset VSP for a given range of source and receiver positions. We refer to the collection of possible reflection points as the "zone of illumination". Knowledge of the zone of illumination for given source–receiver geometry is useful in the planning of a seismic survey. Given an area of interest in the subsurface, one can determine the positioning of sources and receivers necessary to image that particular area.

In addition to showing how to determine the zone of illumination we will also derive an expression to trace the trajectory of the signal as it propagates from the source to the receiver. That is, we will find the ray.

Assume we have a receiver located in the borehole at (X_1, X_3), with the source being located at $(0,0)$. For the given receiver position we must determine the location of the reflection points, (x_r, z_r), over a range of depths. Doing so, we will get a curve. The envelope of such curves – calculated for the range of source and receiver positions for a relevant interval of depth – determines the zone of illumination. Since we have an analytical traveltime expression, we achieve this by invoking the principle of stationary traveltime.

In view of lateral homogeneity, the ray is symmetric about the vertical line passing through the reflection point. Examining Figure 11.1 we see that for a receiver located at (X_1, X_3), we can write the traveltime of the reflected signal as

$$T(x_r, z_r) = 2\, t(x_1, x_3)|_{(x_r, z_r)} - t(x_1, x_3)|_{(2x_r - X_1, X_3)}\,. \qquad (11.1)$$

The first term on the right-hand side of expression (11.1) is twice the traveltime from the source to the reflection point and the second term on the right-hand side is the traveltime along the ray from the source to the receiver depth. In the case of our $ab\chi$ model t is given by expression (8.5),

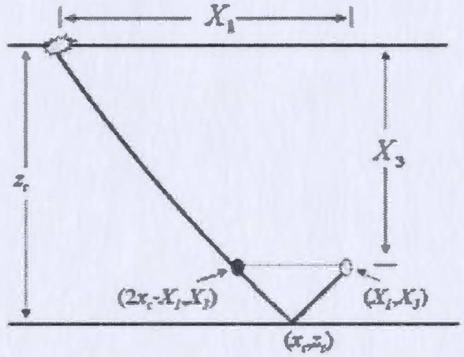

Figure 11.1: VSP reflection geometry. Reflection point at (x_r, z_r) and receiver located at (X_1, X_3). Both positions relative to source at $(0, 0)$. Medium parameters assumed to be $a = 2271m/s$, $b = 0.879s^{-1}$ and $\chi = 0.039$. Raypath plotted using expression (11.11).

namely,

$$t = \frac{1}{b}\ln\left[\frac{a + bx_3}{a}\frac{1 + \sqrt{1 - a^2p^2(1 + 2\chi)}}{1 + \sqrt{1 - (a + bx_3)^2 p^2(1 + 2\chi)}}\right], \tag{11.2}$$

where the value of the ray parameter is given by

$$p = \frac{2X_1}{\sqrt{\left[{X_1}^2 + (1 + 2\chi)X_3^2\right]\left[(2a + bX_3)^2(1 + 2\chi) + b^2X_1^2\right]}}. \tag{11.3}$$

To find the value of x_r corresponding to the stationarity of traveltime, we take the derivative of expression (11.1) with respect to x_r and set this derivative to zero. Then we use numerical methods to solve for x_r. The computer code to do this is given in Appendix D. By varying depth we get the curve of possible reflection points for each source–receiver pair. Taking the envelope of these curves we generate the zone of illumination, as shown in Figure 11.2.

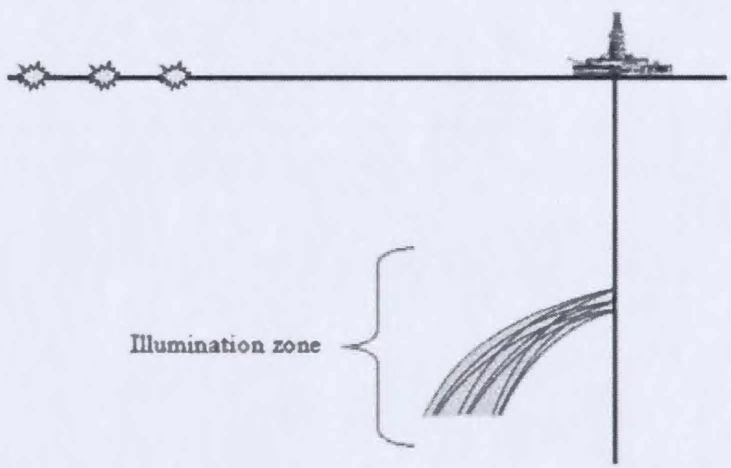

Figure 11.2: Plot of reflection points for three sources and three receivers. Illumination zone indicated by shaded area bounded by the curves of reflection points.

In general, to obtain the illumination zone, we must resort to numerical methods to find x_r. To illustrate the case where we can obtain a closed–form expression for the reflection point in terms of receiver location and reflector depth, consider the case of homogeneity (i.e., $b = 0$). In principle, solving for the reflection point in a homogeneous medium involves finding the limit of expression (11.2) as b tends to zero, then finding the stationary point of the corresponding reflected traveltime expression. The details of doing so are shown in Appendix A.3. However, here we can use the fact that rays in homogeneous media are straight[1]. This allows us to express the ray angle in terms of offset and depth and thus develop a reflected traveltime expression. Alternatively, we could apply Snell's law and simply say that the reflection point must be such that the angle of incidence must be equal to the angle of reflection. However to emphasize the general approach we follow the method used above.

Using the Pythagorean theorem, we get the reflected traveltime as

$$t = \frac{\sqrt{x_r^2 + z_r^2}}{V_{\mathrm{d}}(\theta)} + \frac{\sqrt{(X_1 - x_r)^2 + (X_3 - z_r)^2}}{V_{\mathrm{u}}(\theta)}, \tag{11.4}$$

where V_{d} and V_{u} are the ray velocities of the downgoing and upgoing signals, respectively. To find explicit expressions for V_{d} and V_{u}, we use expression (6.9) for the homogeneous case, namely,

$$V(\theta) = v_{\mathrm{V}} \sqrt{\frac{1 + 2\chi}{1 + 2\chi \cos^2 \theta}}. \tag{11.5}$$

[1]Slawinski, M.A., (2003) Seismic rays and waves in elatic media: Pergamon, p. 194.

For the downgoing ray, the angle is related to the reflection point by

$$\cos^2\theta_{\rm d} = \frac{z_r^2}{x_r^2 + z_r^2}, \tag{11.6}$$

and for the reflected ray,

$$\cos^2\theta_{\rm u} = \frac{(X_3 - z_r)^2}{(X_3 - z_r)^2 + (X_1 + x_r)^2}. \tag{11.7}$$

Thus, for the downgoing and reflected rays the respective velocities are given by

$$V_{\rm d} = a\sqrt{\frac{(x_r^2 + z_r^2)(1 + 2\chi)}{x_r^2 + z_r^2(1 + 2\chi)}} \tag{11.8}$$

and

$$V_{\rm u} = a\sqrt{\frac{\left[(X_1 - x_r)^2 + (X_3 - z_r)^2\right](1 + 2\chi)}{(X_1 - x_r)^2 + (X_3 - z_r)^2(1 + 2\chi)}}. \tag{11.9}$$

Inserting expressions (11.8) and (11.9) into expression (11.4), we can explicitly write the traveltime expression for a reflected signal between the source and a given receiver in a homogeneous medium. Now, we can invoke the principle of stationary traveltime. Taking the derivative of the resulting expression with respect to x_r and setting it to zero, we solve for x_r to obtain

$$x_r = \frac{z_r X_1}{2z_r - X_3}. \tag{11.10}$$

The computer code that implements expression (11.10) is shown in Appendix D.

We note that in deriving expression (11.10) we assumed the medium to be homogenous and elliptically anisotropic. However, since χ disappears in the process of differentiation, expression (11.10) holds for both isotropic- and anisotropic-, homogeneous media. As stated above, this is a result of the fact that in homogeneous media all rays are straight. In other words, in homogenous media, anisotropy does not affect the rays. It does, however, affect the traveltime of the signal along the rays.

Given the parameter values for an $ab\chi$ model, we can determine the illumination zone for a given source and a set of receivers. In particular, our results allow us to obtain — for a given source–receiver pair — the location of the reflection point at a given depth. To see the importance of this relation, consider the following example.

Dividing both the numerator and the denominator of the right-hand side of equation (11.10) by z_r and letting z_r tend to infinity, we see that for very large depths, the reflection point is approximated by the midpoint between the source and receiver. In the context of seismic exploration, however, expression $x_r \approx X/2$ applies to depths that are beyond those of exploration interest. For most cases of interest, reflection points tend to be significantly closer to the receiver than to the source. To illustrate this point, consider an $ab\chi$ model with $a = 2271\text{m/s}$, $b = 0.879\text{s}^{-1}$ and $\chi = 0.039$. Let us assume a receiver located at $(1000\text{m}, 1000\text{m})$ and calculate the reflection point for an interface at a depth of 2000m. The reflection point is found to be 634m from the source, which is a significant distance from the midpoint at 500m.

In finding the reflection points above, we have indirectly used expression (11.3) to trace the ray from the source, to reflection point, to the receiver. To obtain the closed–form expression that traces the ray we replace the receiver coordinates, (X_1, X_3), in expression (11.3) with any point on the ray, (x_1, x_3), and solve for x_1 to obtain

$$x_1(x_3) = \frac{1}{bp}\left[\sqrt{1 - p^2a^2(1+2\chi)} - \sqrt{1 - p^2(a + bx_3)^2(1+2\chi)}\right]. \quad (11.11)$$

As above, the value of p is given by substituting the coordinates of a known point along the ray (e.g., the reflection point or the receiver position) into expression (11.3). The rays shown in Figures 11.1 and 12.1 were traced using expression (11.11).

We note that our formulation does not require the receivers to be in a vertical well. In other words, this formulation can be used in deviated wells since, having the location of the source, all equations require only the location of a given receiver. In other words, these equations are associated with point-to-point problems. In all cases, the illumination zone is bounded by the enveloping curves of the reflection points and by the well.

Chapter 12

Suggested future work

We conclude this thesis by describing several extensions of the present work that may form the basis of future study. In particular, we will begin by considering the problem of extending the present work to a multi-layer model. We will then take a closer look at the conditions necessary for the explicit expression of ray angle in terms of wavefront angle. In the final section, we will discuss how one might apply both traveltime inversion and the inversion of three-component data sets in the direct estimation of the components of elasticity.

12.1 Parameter estimation for multilayer model

The first and most obvious problem is the extension of traveltime inversion to a multilayer problem. Considering the two layer problem in Figure 12.1,we

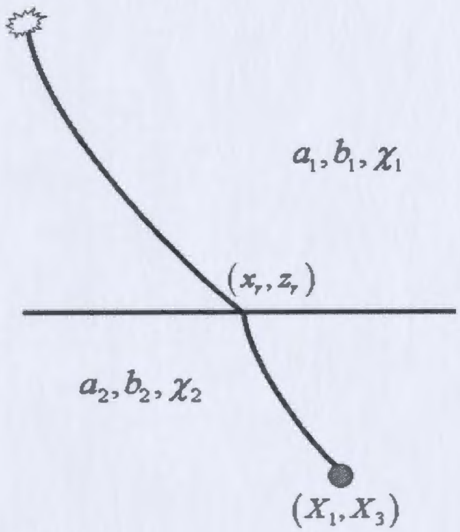

Figure 12.1: Plot of possible ray for two layer problem. Upper medium has parameters a_1, b_1 and χ_1. The lower medium has parameters a_2, b_2 and χ_2. The refraction point is at (x_r, z_r).

apply the methods described previously to determine estimates for a_1, b_1 and χ_1 in the upper layer. When we consider the lower layer we add three new model parameters (a_2, b_2 and χ_2) together with a refraction point that must be solved for each observation. If we assume the interface depth to be known then the horizontal offset, x_r, remains to be found. In such a case the total traveltime is given by

$$T(X_1, X_3; a_2, b_2, \chi_2) = t(x_r, z_r; a_1, b_1, \chi_1) + t(X_1 - x_r, X_3 - z_r; a_2, b_2, \chi_2) \tag{12.1}$$

where t is given by expressions (7.16) and (8.5) with the appropriate substitutions. As stated above, a complication in this case is that we must also find x_r for each field observation. A potential way to deal with this is to find an expression for x_r in terms of the receiver location and the model parameters of the lower layer, that is $x_r\,(X_1, X_3; a_2, b_2, \chi_2)$, with z_r, a_1, b_1 and χ_1 being known. Having such an expression for x_r we could then apply Gauss-Newton optimization to estimate a_2, b_2 and χ_2.

To find such an expression for x_r recall that in a laterally homogeneous medium, the ray parameter is equal to the horizontal component of slowness, which remains constant along the entire ray. Therefore the horizontal component of slowness in the upper medium, denoted by p_1 and given by $p(x_r, z_r; a_1, b_1, \chi_1)$ in expression (7.16), must be equal to the horizontal component of slowness in the lower medium, denoted by p_2 and given by $p(X_1 - x_r, X_3 - z_r; a_2, b_2, \chi_2)$. Thus, we have

$$p_1^2 = p_2^2,$$

which leads to condition

$$\frac{x_r^2}{\left[x_r^2+(1+2\chi_1)z_r^2\right]\left[(2a_1+b_1z_r)^2(1+2\chi_1)+b_1^2x_r^2\right]}$$

$$= \left[\frac{1}{\left[\left(X_1-x_r\right)^2+\left(1+2\chi_2\right)\left(X_3-z_r\right)^2\right]}\right]\times$$
$$\left[\frac{\left(X_1-x_r\right)^2}{\left[(2a_2+b_2\left(X_3-z_r\right))^2(1+2\chi_2)+b_2^2\left(X_1-x_r\right)^2\right]}\right].$$

This is a sixth–degree polynomial in x_r and so the expression for x_r is not unique. However, several conditions may make the problem more constrained.

We know that the refraction point must be a real value that lies between the source and receiver. Secondly, we know that the refraction point must make the total traveltime stationary. Therefore, we can apply the stationarity of traveltime and set the derivative of expression (12.1) with respect to x_r equal to zero, after substituting in the expressions for the ray parameters.

Thus, we get a system of two equations that can be used to determine an expression for x_r, namely

$$\begin{cases} \dfrac{\partial T(X_1, X_3; a_2, b_2, \chi_2)}{\partial x_r} = 0 \\ \\ p_1^2 = p_2^2 \end{cases},$$

with the added condition that x_r must be real and lie between 0 and X_1.

Having an expression for x_r we could then write the total traveltime expression as,

$$\begin{aligned} T(X_1, X_3; a_2, b_2, \chi_2) &= t(x_r\left[X_1, X_3; a_2, b_2, \chi_2, z_r\right], z_r; a_1, b_1, \chi_1) \\ &+ t(X_1 - x_r\left[X_1, X_3; a_2, b_2, \chi_2, z_r\right], X_3 - z_r; a_2, b_2, \chi_2) \end{aligned} \tag{12.2}$$

for which Gauss-Newton optimization could be applied directly to estimate the values of a_2, b_2 and χ_2 assuming z_r, a_1, b_1 and χ_1 are known. Some questions which might be addressed are as follows.

- What is the effect of adding layers on the confidence intervals of parameter estimates?
- What is the maximum number of layers that can be considered for a given set of data points?

- For a particular layer of interest, is it better to consider the overburden as a single layer or would a model consisting of a number of layers better model traveltimes?
- How could dipping interfaces be considered in the modelling?
- Could the model parameters, a, b and χ, be estimated for each layer simultaneously?

12.2 Explicit expression of ray velocity in terms of ray angle

As discussed in Chapter 6, it has been asserted by Slawinski (2003) that expression (5.8), namely,

$$\tan\theta = \frac{\tan\vartheta + \dfrac{1}{v}\dfrac{\partial v}{\partial\vartheta}}{1 - \dfrac{\tan\vartheta}{v}\dfrac{\partial v}{\partial\vartheta}},$$

"can be explicitly solved for ϑ if and only if function v is quadratic in the p_i". This statement is not proven nor is a reference for such a proof given. As another suggestion for future work, one could attempt to prove this statement.

To shed some light on this problem, let us consider the following. We can explicitly write Hamilton's ray equations (2.35) for a medium in which rays propagate in two dimensions as

$$\begin{cases} \dot{x}_1 = \dfrac{\partial \mathcal{H}}{\partial p_1} \\ \dot{x}_3 = \dfrac{\partial \mathcal{H}}{\partial p_3} \\ \dot{p}_1 = -\dfrac{\partial \mathcal{H}}{\partial x_1} \\ \dot{p}_3 = -\dfrac{\partial \mathcal{H}}{\partial x_3} \end{cases},$$

where the Hamiltonian is given by expression (2.34), namely,

$$\mathcal{H} = \frac{1}{2} p^2 v^2 (\mathbf{x}, \mathbf{p}) . \tag{12.3}$$

Since we wish to state the wavefront angle, ϑ, explicitly in terms of the ray angle, θ, recall

$$\tan \vartheta = \frac{p_1}{p_3} \tag{12.4}$$

and

$$\tan \theta = \frac{\dot{x}_1}{\dot{x}_3}. \tag{12.5}$$

To express ϑ in terms of θ, we must be able to solve

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i}, i \in \{1, 3\},$$

for the p_i in terms of the $\dot{x}_i$. Inserting expression (12.3) into this equation we get

$$\frac{1}{2} \frac{\partial}{\partial p_i} \left[p^2 v^2 (x_1, x_3, p_1, p_3) \right] - \dot{x}_i = 0, i \in \{1, 3\}.$$

Setting $i = 1$ we obtain

$$p_1 v^2 + p^2 v \frac{\partial v}{\partial p_1} - \dot{x}_1 = 0, \tag{12.6}$$

and for $i = 3$ we obtain

$$p_3 v^2 + p^2 v \frac{\partial v}{\partial p_3} - \dot{x}_3 = 0. \tag{12.7}$$

If v^2, not v, is a polynomial of degree r in p_i then we see that the left-hand side of equations (12.6) and (12.7) are polynomials of degree at least $2 + r/2 + (r/2 - 1) = 1 + r$ in $p_1$and p_3, respectively.

Since v^2 will always be of even degree in p_i (i.e., $r \in \{2, 4, 6, ...\}$), expressions (12.6) and (12.7) will be of odd degree in p_i (i.e., $\{3, 5, 7, ...\}$). According to Abel's impossibility theorem, in general, an algebraic solution of a polynomial is only possible for those of degree four or less. Therefore, we require $r = 2$. However, there is the added complication to solving for each p_i in the fact that both must be solved simultaneously. Expressions (12.6) and (12.7) each depend on p_1 and p_3. Therefore, when we solve for p_1 in expression (12.6) it will depend on $\dot{x}_1$ and p_3. We must then substitute the expression for p_1 into expression (12.7). This substitution may then complicate the solving for p_3. It remains to rigorously establish that v^2 quadratic in p_i is the only case for which $p_1$and p_3 can be solved explicitly in terms of $\dot{x}_1$ and $\dot{x}_2$, respectively.

To exemplify the case where one can solve for p_i as a function of $\dot{x}_i$, let us use equation (6.1), namely

$$v(\vartheta) = \sqrt{v_{\mathrm{H}}^2 \sin^2 \vartheta + v_{\mathrm{V}}^2 \cos^2 \vartheta}.$$

Following expression (12.4), we get

$$v(\mathbf{p}) = \frac{1}{|\mathbf{p}|} \sqrt{v_{\mathrm{H}}^2 p_1^2 + v_{\mathrm{V}}^2 p_3^2},$$

which is quadratic in $p_1$and p_3. Differentiating with respect to $p_1$we get

$$\frac{\partial v}{\partial p_1} = \frac{1}{v} \frac{p_1 p_3^2 \left(v_{\mathrm{H}}^2 - v_{\mathrm{V}}^2\right)}{\left(p_1^2 + p_3^2\right)^2}$$

and equation (12.6) becomes

$$v_{\mathrm{H}}^2 p_1 - \dot{x}_1 = 0.$$

Solving for p_1, we obtain

$$p_1 = \frac{1}{v_{\mathrm{H}}^2} \dot{x}_1. \tag{12.8}$$

Following the analogous approach, we obtain

$$p_3 = \frac{1}{v_{\mathrm{V}}^2} \dot{x}_3. \tag{12.9}$$

Using expressions (12.8) and (12.9), we get

$$\frac{p_1}{p_3} = \frac{v_{\mathrm{V}}^2}{v_{\mathrm{H}}^2} \frac{\dot{x}_1}{\dot{x}_3}.$$

Invoking expressions (12.4) and (12.5), we get

$$\tan\vartheta = \frac{v_{\mathrm{V}}^2}{v_{\mathrm{H}}^2}\tan\theta.$$

Solving for the wavefront angle, we obtain

$$\vartheta = \arctan\left(\frac{v_{\mathrm{V}}^2}{v_{\mathrm{H}}^2}\tan\theta\right),$$

which is explicitly in terms of the ray angle, as required.

12.3 Determination of elasticity parameters

A third area of study that might be considered involves more extensive work with the stress, strain and elasticity tensors. In particular, it involves using the measurement of the angle of particle displacement at the receiver as the signal impacts upon it and using this measurement to estimate the components of the elasticity tensor. The angle is commonly referred to as the polarization angle and can be determined from data collected with three-component geophones.

Often the polarization angle is mistakenly considered to be equal to the ray angle. Although it often closer to the ray angle than to the wavefront angle, it is fact distinct from both.[1] In a paper by de Parscau[2] (1991) the relationship between the three angles is discussed, and an example is given where even for a weakly anisotropic medium the polarization angle differs by eight degrees from the ray angle. This means that assuming the measured angle in a field observation is the ray angle can lead to errors in the model results.

To obtain the expression for the polarization angle, consider a homogeneous VTI medium, for which the elements of the elasticity tensor are

$$C_{mn} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix},$$

where, for convenience and because of symmetries of the stress and strain

[1] Musgrave, M.J.P., (1970) Crystal acoustics: Holden-Day.

[2] de Parscau, J., (1991) Relationship between phase velocitites and polarization in transversely isotropic media: Geophysics, **56**, pp. 1578-1583.

tensors, we have defined $c_{ijkl} = C_{mn}$ where

$$m = i\delta_{ij} + (1 - \delta_{ij})(9 - i - j),$$

and

$$n = k\delta_{kl} + (1 - \delta_{kl})(9 - k - l).$$

Thus, for example in a VTI medium, $c_{1321} = C_{56} = 0$ and $c_{3322} = C_{32} = C_{13} = c_{1133}$.

Recalling the Christoffel equations (2.13) along with the eikonal equation, the solvability condition implies for a homogeneous medium

$$\det\left[\sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}n_j n_l - \rho v^2 \delta_{ik}\right] = 0, \quad i,k \in \{1,2,3\}. \tag{12.10}$$

By defining matrix Γ with entries given by

$$\Gamma_{ik} := \sum_{j=1}^{3}\sum_{l=1}^{3} c_{ijkl}n_j n_l,$$

expression (12.10) can be written in matrix form as

$$\det\left[\boldsymbol{\Gamma} - \rho v^2 \mathbf{I}\right] = 0. \tag{12.11}$$

Next, solving for v and substituting in the values of C_{mn} corresponding to a VTI medium, we get three expressions for $v(\vartheta)$, the eigenvalues. One of which is

$$v(\vartheta) = \sqrt{\frac{(C_{33} - C_{11})\cos^2\vartheta + C_{11} + C_{44} + \sqrt{\Delta}}{2\rho}}, \tag{12.12}$$

where $n_1^2 + n_2^2 = 1 - n_3^2 = \sin^2\vartheta$ and where

$$\begin{aligned}\Delta = {} & \left[(C_{11} - C_{33})\cos^2\vartheta - C_{11} - C_{44}\right]^2 \\ & - 4[C_{33}C_{44}\cos^4\vartheta - \left[2C_{13}C_{44} - C_{11}C_{33} + C_{13}^2\right]\cos^2\vartheta\sin^2\vartheta \\ & + C_{11}C_{44}\sin^4\vartheta].\end{aligned} \tag{12.13}$$

By calculating the associated eigenvector, we get the expression for the polarization angle, namely

$$\phi = \tan^{-1}\left[\frac{\rho v^2(\vartheta) - C_{44}\sin^2\vartheta - C_{33}\cos^2\vartheta}{(C_{13} - C_{44})\sin\vartheta\cos\vartheta}\right]. \tag{12.14}$$

Expressions (12.12) and (12.14) represent longitudinal wave propagation, which can be verified by considering the special case of a wavefront moving horizontally along the symmetry axis. That is, the case where $\vartheta = \pi/2$ and hence, $\cos\vartheta = 0$. This then means that $\phi = \pi/2$ indicating that the particle displacements are in the direction of propagation as we would expect for a longitudinal wave. A direction in which the polarization angle is equal to the wavefront angle is a special direction commonly referred to as a pure–mode direction. In general, the polarization angle of particle displacement is distinct from both the wavefront propagation angle and the ray angle.

Recalling expression (5.8) and the fact that rays are straight in homogeneous media, we can use the ray angle to get the following expression

$$\frac{X}{Z} = \frac{\tan\vartheta + \dfrac{1}{v}\dfrac{\partial v}{\partial \vartheta}}{1 - \dfrac{\tan\vartheta}{v}\dfrac{\partial v}{\partial \vartheta}} \tag{12.15}$$

where $v(\vartheta)$ is given by expressions (12.12) and (12.13) above. From this expression we can express the wavefront angle in terms of the components of elasticity (i.e., $\vartheta(C_{11}, C_{13}, C_{33}, C_{44})$). Substituting this expression into (12.14) above, we get an expression for the polarization angle in terms of the components of elasticity, namely

$$\phi = \tan^{-1}\left[\frac{\rho v^2(\vartheta) - C_{44}\sin^2\vartheta - C_{33}\cos^2\vartheta}{(C_{13} - C_{44})\sin\vartheta\cos\vartheta}\right], \tag{12.16}$$

where $\vartheta = \vartheta(C_{11}, C_{13}, C_{33}, C_{44})$.

Recalling expression (5.9) for a homogeneous medium, namely

$$V^2(\vartheta) = v^2(\vartheta) + \left(\frac{\partial v}{\partial \vartheta}\right)^2, \tag{12.17}$$

where $\vartheta = \vartheta(C_{11}, C_{13}, C_{33}, C_{44})$, we can also express the ray velocity in terms of the components of elasticity. This leads to an expression for the traveltime, specifically

$$T = \frac{\sqrt{X^2 + Z^2}}{V(\vartheta)}, \tag{12.18}$$

where $\vartheta = \vartheta(C_{11}, C_{13}, C_{33}, C_{44})$.

Combining expressions (12.16) and (12.18) then we have two expressions for four unknowns; C_{11}, C_{13}, C_{33} and C_{44}. For the case of horizontal propagation (i.e., $\vartheta = \pi/2$), Δ is given by

$$
\begin{aligned}
\Delta &= [C_{11} + C_{44}]^2 - 4C_{11}C_{44} \\
&= C_{11}^2 + 2C_{11}C_{44} + C_{44}^2 - 4C_{11}C_{44} \\
&= C_{11}^2 - 2C_{11}C_{44} + C_{44}^2 \\
&= (C_{11} - C_{44})^2
\end{aligned}
$$

and so the wavefront velocity (and also the ray velocity, $\vartheta = \pi/2$ being a pure mode direction) is given by

$$
\begin{aligned}
v_{\mathrm{H}} = v\left(\frac{\pi}{2}\right) &= \sqrt{\frac{C_{11} + C_{44} + \sqrt{(C_{11} - C_{44})^2}}{2\rho}} \\
&= \sqrt{\frac{C_{11} + C_{44} + C_{11} - C_{44}}{2\rho}} \\
&= \sqrt{\frac{C_{11}}{\rho}}. \qquad (12.19)
\end{aligned}
$$

Similarly, if we consider vertical propagation (i.e., $\vartheta = 0$), we get the expression

$$
v_{\mathrm{V}} = v\left(0\right) = \sqrt{\frac{C_{33}}{\rho}}. \qquad (12.20)
$$

Therefore, assuming we can get estimates of the vertical and horizontal velocities along with the material density, say from well logs, we can get estimates of both C_{11} and C_{33}.

Having values for C_{11} and C_{33}, we can then use expressions (12.16) and (12.18) to solve for the remaining elasticity components; C_{13} and C_{44}. Due to errors in measurements, again some sort of optimization would have to be used to estimate parameter values from field data.

Several key issues that must be considered when attempting to determine, *in situ*, elasticity parameters C_{mn} are as follows.

- Sensitivity of optimization to initial parameter estimates. Initial attempts to use the polarization measurements in determining components of the elasticity tensor suggest that conditioning (e.g., normalization of parameters) of the model may be necessary to improve its response to initial parameter estimates.

- While traveltime represents a global measurement over the entire ray, the polarization angle is a local measurement made at the receiver. Therefore, unaccounted inhomogeneities near the receiver can severely affect the resultant measurements. One might explore how the chosen optimization technique responds to inhomogeneities and whether or not a particular optimization algorithm is better at "seeing through" local inhomogeneities to arrive at true parameter estimates.

Chapter 13

Bibliography

The following is a list of publications that were used by the author in the course of study and research that lead to completion of this thesis. In the main body of text, an attempt was made to keep references very specific so that interested readers would benefit directly from the works cited. Included in this list are also sources of information that, although not specifically referenced, are acknowledged by the author for their contributions in general.

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Appendix A

Mathematical details of derivations

For the purposes of clarity and to avoid unnecessary digressions in the main text of the thesis many details of mathematical derivations were omitted. In this appendix we return to some of the key derivations, and for completeness, give the steps involved in detail.

A.1 Ray parameter in terms of ray quantities

This first section outlines the details of solving for the ray parameter in terms in the receiver position, (X_1, X_3), and parameterized with a, b and χ.

Recall the expression for the ray parameter, namely,

$$p = \frac{\sin\theta}{a + bx_3} \frac{1}{\sqrt{1+2\chi}} \frac{1}{\sqrt{1+2\chi\cos^2\theta}},$$

for which we want to replace the ray angle, θ, with the coordinates of the receiver position. We start by squaring both sides of the equation and isolating the ray angle,

$$p^2 (a + bz)^2 (1 + 2\chi) = \frac{\sin^2\theta}{1 + 2\chi\cos^2\theta}.$$

Next we divide both the numerator and the denominator of the right hand side by $\cos^2\theta$ and recall the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$ to get

$$p^2 (a + bx_3)^2 (1 + 2\chi) = \frac{\tan^2\theta}{(1 + 2\chi) + \tan^2\theta}.$$

Solving for $\tan^2\theta$ and taking the square root, we obtain

$$\frac{p\,(a + bx_3)\,(1 + 2\chi)}{\sqrt{1 - p^2 (a + bx_3)^2 (1 + 2\chi)}} = \tan\theta.$$

Now, we recall that $\tan\theta = \mathrm{d}x_1/\mathrm{d}x_3$. Using this expression on the right-hand side of the above equation, we can write

$$\frac{p\,(a+bx_3)\,(1+2\chi)}{\sqrt{1-p^2\,(a+bx_3)^2\,(1+2\chi)}}\mathrm{d}x_3 = \mathrm{d}x_1.$$

Integrating both sides, we get

$$\int\limits_0^{X_3}\frac{p\,(a+bx_3)\,(1+2\chi)}{\sqrt{1-p^2\,(a+bx_3)^2\,(1+2\chi)}}\mathrm{d}x_3 = \int\limits_0^{X_1}\mathrm{d}x_1,$$

where (X_1, X_3) is the receiver position relative to the source at $(0,0)$. Performing the integration, we get

$$\frac{-1}{pb}\left[\sqrt{1-p^2\,(a+bx_3)^2\,(1+2\chi)}\right]_{x_3=0}^{X_3} = X_1,$$

which gives

$$\frac{1}{pb}\left[\sqrt{1-p^2a^2\,(1+2\chi)}-\sqrt{1-p^2\,(a+bX_3)^2\,(1+2\chi)}\right] = X_1.$$

Now we need to solve for p in terms of X_1 and X_3. To make the algebra easier to follow, let us define

$$B := (a + bX_3)^2$$

and

$$C := (1 + 2\chi)\,.$$

We start by squaring both sides,

$$\begin{aligned}\left[\sqrt{1 - p^2a^2C} - \sqrt{1 - p^2BC}\right]^2 &= [pbX_1]^2 \\ 1 - p^2a^2C + 1 - p^2BC - 2\sqrt{1 - p^2a^2C}\sqrt{1 - p^2BC} &= p^2b^2X_1^2 \\ 2 - p^2C\left(a^2 + B\right) - 2\sqrt{1 - p^2a^2C}\sqrt{1 - p^2BC} &= p^2b^2X_1^2.\end{aligned}$$

Next, we isolate the square roots to get

$$2\sqrt{1 - p^2a^2C}\sqrt{1 - p^2BC} = 2 - p^2\left[C\left(a^2 + B\right) + b^2X_1^2\right].$$

Again, we square both sides to get rid of the root signs,

$$4\left[1-p^2a^2C\right]\left[1-p^2BC\right]=\left[2-p^2\left[C\left(a^2+B\right)+b^2X_1^2\right]\right]^2.$$

Thus we can write

$$LHS=p^4\left[4C^2a^2B\right]-p^2\left[4C\left(a^2+B\right)\right]+4$$

and

$$RHS=p^4\left[C\left(a^2+B\right)+b^2X_1^2\right]^2-p^2\left[4C\left(a^2+B\right)+4b^2X_1^2\right]+4.$$

Subtracting, we get

$$\begin{aligned}LHS-RHS&=p^4\left[2a^2BC^2-C^2a^4-C^2B^2-2Ca^2b^2X_1^2-2CBb^2X_1^2-b^4X_1^4\right]\\&\quad+p^2\left[4b^2X_1^2\right]\\0&=p^4\left[-C^2\left(a^4-2a^2B+B^2\right)-2C\left(a^2+B\right)b^2X_1^2-b^4X_1^4\right]\\&\quad+p^2\left[4b^2X_1^2\right]\\0&=p^4\left[-C^2\left(a^2-B\right)^2-2C\left(a^2+B\right)b^2X_1^2-b^4X_1^4\right]+p^2\left[4b^2X_1^2\right].\end{aligned}$$

Since we are only interested in $p^2 \neq 0$, we can divide by p^2 and isolate the remaining p^2 to get

$$p^2 = \frac{4b^2X_1^2}{C^2\left(a^2 - B\right)^2 + 2C\left(a^2 + B\right)b^2X_1^2 + b^4X_1^4}.$$

Converting the symbols B and C back to the original values and simplifying we get

$$p^2 = \frac{4b^2X_1^2}{b^2(X_1^2 + (1+2\chi)X_3^2)[(2a + bX_3)^2\left(1 + 2\chi\right) + b^2X_1^2]}.$$

Taking the square root we get the ray parameter expression for an elliptically anisotropic linearly inhomogeneous medium given in terms of the model parameters, a, b and χ, and the receiver position, (X_1, X_3), namely

$$p = \frac{2X_1}{\sqrt{(X_1^2 + (1+2\chi)X_3^2)[(2a + bX_3)^2\left(1 + 2\chi\right) + b^2X_1^2]}}. \tag{A.1}$$

A.2 Integration of traveltime expression

In this section we give the details of the integration of traveltime expression (8.2), namely

$$t = \int_{x_3=0}^{X_3} \frac{\mathrm{d}x_3}{(a + bx_3)\sqrt{\left(\frac{1+2\chi}{1+2\chi\cos^2\theta}\right)\cos^2\theta}}. \tag{A.2}$$

From expression (8.3) we have

$$\cos^2\theta = \frac{1 - p^2\,(a + bx_3)^2\,(1 + 2\chi)}{1 + 2\chi p^2\,(a + bx_3)^2\,(1 + 2\chi)}. \tag{A.3}$$

Simplifying the expression under the square root of the traveltime expression (A.2), we get

$$\begin{aligned}
&\left(\frac{1+2\chi}{1+2\chi\cos^2\theta}\right)\cos^2\theta \\
&= \frac{1+2\chi}{1+2\chi\left(\dfrac{1-p^2(a+bx_3)^2(1+2\chi)}{1+2\chi p^2(a+bx_3)^2(1+2\chi)}\right)}\frac{1-p^2(a+bx_3)^2(1+2\chi)}{1+2\chi p^2(a+bx_3)^2(1+2\chi)} \\
&= \left[\frac{(1+2\chi)\left(1+2\chi p^2(a+bx_3)^2(1+2\chi)\right)}{\left(1+2\chi p^2(a+bx_3)^2(1+2\chi)\right)+2\chi\left(1-p^2(a+bx_3)^2(1+2\chi)\right)}\right]\times \\
&\qquad \left[\frac{1-p^2(a+bx_3)^2(1+2\chi)}{1+2\chi p^2(a+bx_3)^2(1+2\chi)}\right] \\
&= \frac{(1+2\chi)\left(1-p^2(a+bx_3)^2(1+2\chi)\right)}{(1+2\chi)+2\chi p^2(a+bx_3)^2(1+2\chi)-2\chi p^2(a+bx_3)^2(1+2\chi)} \\
&= 1-p^2(a+bx_3)^2(1+2\chi).
\end{aligned}$$

Thus, the traveltime integral becomes

$$t = \int_{x_3=0}^{X_3} \frac{\mathrm{d}x_3}{(a+bx_3)\sqrt{1-p^2(a+bx_3)^2(1+2\chi)}}.$$

Here we note that the ray parameter, p, is a conserved quantity along the ray and as such it is a constant for every point on the ray. We must not consider expression (A.1) for p a function of X_1 and X_3. It is a constant along the ray and substitution of any point (X_1, X_3) of the ray gives that constant value. Therefore we can rewrite the traveltime integral as

$$t = \frac{1}{p\sqrt{1+2\chi}} \int_{x_3=0}^{X_3} \frac{\mathrm{d}x_3}{(a+bx_3)\sqrt{\left(\frac{1}{p\sqrt{1+2\chi}}\right)^2 - (a+bx_3)^2}}.$$

Defining

$$u := a + bx_3 \qquad \omega := \frac{1}{p\sqrt{1+2\chi}}$$
$$\mathrm{d}u = \frac{1}{b}\mathrm{d}x_3,$$

we write

$$t = \frac{\omega}{b} \int_{a}^{a+bX_3} \frac{\mathrm{d}u}{u\sqrt{\omega^2 - u^2}}.$$

We now recognize the traveltime integral written in this manner as a trigonometric form. Letting

$$u := \omega \sin\beta \qquad \cos\beta = \frac{1}{\omega}\sqrt{\omega^2 - u^2}$$
$$\mathrm{d}u = \omega \cos(\beta)\, \mathrm{d}\beta,$$

we write

$$t = \frac{\omega}{b} \int\limits_{\arcsin(a/\omega)}^{\arcsin((a+bX_3)/\omega)} \frac{\omega \cos(\beta)\, \mathrm{d}\beta}{(\omega \sin\beta)\,(\omega\cos\beta)}$$

$$= \frac{1}{b} \int\limits_{\arcsin(a/\omega)}^{\arcsin((a+bX_3)/\omega)} \frac{\mathrm{d}\beta}{\sin\beta}$$

$$= \frac{1}{b} \ln \left[\tan\frac{\beta}{2}\right]_{\arcsin(a/\omega)}^{\arcsin((a+bX_3)/\omega)},$$

which, by identity

$$\tan\left(\frac{\beta}{2}\right) = \frac{\sin\beta}{1+\cos\beta},$$

can be written as

$$t = \frac{1}{b} \ln \left[\frac{\sin\beta}{1+\cos\beta}\right]_{\arcsin(a/\omega)}^{\arcsin((a+bX_3)/\omega)}$$

$$= \frac{1}{b} \ln \left[\frac{\sin\beta}{1+\sqrt{1-(\sin\beta)^2}}\right]_{\arcsin(a/\omega)}^{\arcsin((a+bX_3)/\omega)}.$$

Inserting the limits of integration, we get

$$t = \frac{1}{b}\ln\left[\frac{\sin\left(\arcsin((a+bX_3)/\omega)\right)}{1+\sqrt{1-\left(\sin\left(\arcsin((a+bX_3)/\omega)\right)\right)^2}}\right] - \frac{1}{b}\ln\left[\frac{\sin\left(\arcsin(a/\omega)\right)}{1+\sqrt{1-\left(\sin\left(\arcsin(a/\omega)\right)\right)^2}}\right]$$

$$= \frac{1}{b}\ln\left[\frac{(a+bX_3)/\omega}{1+\sqrt{1-(a+bX_3)/\omega)^2}}\right] - \frac{1}{b}\ln\left[\frac{a/\omega}{1+\sqrt{1-(a/\omega)^2}}\right]$$
$$= \frac{1}{b}\ln\left[\frac{(a+bX_3)/\omega}{1+\sqrt{1-(a+bX_3)/\omega)^2}}\frac{1+\sqrt{1-(a/\omega)^2}}{a/\omega}\right]$$

$$= \frac{1}{b}\ln\left[\frac{a+bX_3}{a}\frac{1+\sqrt{1-(a/\omega)^2}}{1+\sqrt{1-(a+bX_3)/\omega)^2}}\right].$$

Recalling from above that

$$\omega^2 = \frac{1}{p^2(1+2\chi)}, \tag{A.4}$$

we obtain the desired traveltime expression,

$$t = \frac{1}{b}\ln\left[\frac{a+bX_3}{a}\frac{1+\sqrt{1-p^2a^2(1+2\chi)}}{1+\sqrt{1-p^2(a+bX_3)^2(1+2\chi)}}\right],$$

where

$$p = \frac{2X_1}{\sqrt{(X_1^2 + (1+2\chi)X_3^2)[(2a+bX_3)^2(1+2\chi) + b^2X_1^2]}}.$$

A.3 Illumination zone for homogeneous anisotropic media

In this appendix, we study traveltime expression (11.2) in the context of a homogeneous medium. That is, we find the limit as $b \to 0$. Consider expression (11.2), namely,

$$t = \frac{1}{b}\ln\left[\frac{a+bx_3}{a}\frac{1+\sqrt{1-a^2p^2(1+2\chi)}}{1+\sqrt{1-(a+bx_3)^2p^2(1+2\chi)}}\right], \tag{A.5}$$

where

$$p = \frac{2x_1}{\sqrt{\left[x_1^2 + (1+2\chi)x_3^2\right]\left[(2a+bx_3)^2(1+2\chi) + b^2x_1^2\right]}}. \tag{A.6}$$

We wish to derive the traveltime expression for a homogeneous medium, which corresponds to $b = 0$. Thus, we take the limit of expression (A.5) as $b \to 0$. We note that as $b \to 0$, both the numerator and denominator of expression (A.5) tend to zero. Hence, we can use de l'Hôpital's rule to find the limit. Following this rule, we obtain

$$\lim_{b\to 0} t = \frac{x_3\,(1+2\chi) + \sqrt{(1+2\chi)\left[x_1^2 + x_3^2\,(1+2\chi)\right]}}{a\,(1+2\chi)\left[1+\sqrt{\dfrac{x_3^2\,(1+2\chi)}{x_1^2 + x_3^2\,(1+2\chi)}}\right]}. \tag{A.7}$$

Substituting expression (A.7) into traveltime expression (11.1), namely,

$$T(x_r, z_r) = 2\, t(x_1, x_3)|_{(x_r, z_r)} - t(x_1, x_3)|_{(2x_r - X_1, X_3)}\,, \tag{A.8}$$

and then finding the value of x_r for the stationary point of T, we find the reflection point to be located at

$$x_r = \frac{z_r X_1}{2z_r - X_3}, \tag{A.9}$$

which is expression (11.10), as expected.

Appendix B

Alternative derivation of traveltime expression

In this Appendix we derive the traveltime expression for a constant velocity gradient medium in a manner similar to that of Slotnick (1959). We will then extend the expression to account for elliptical anisotropy by means of a coordinate transformation to obtain expression (8.5).

We begin by assuming that the interval medium between a source located at the surface and a receiver located at some depth, z, in a well, is homogeneous and isotropic (Here to avoid notation confusion, we use $x := x_1$ and $z := x_3$). In such a medium we know the rays will be straight and the wavefront and ray velocities will be equal. If V and θ are the ray velocity and take off angle, respectively, than the receiver location and traveltime are given by

$$x = z \tan(\theta) \tag{B.1}$$

and

$$t = \frac{z}{V \cos(\theta)}, \tag{B.2}$$

respectively. From Snell's law we know that

$$\frac{\sin(\theta)}{V} = p. \tag{B.3}$$

Therefore, we can rewrite expressions (B.1) and (B.2) above in terms of the ray parameter, p, as

$$x = \frac{(pV)\, z}{\sqrt{1 - p^2 V^2}} \tag{B.4}$$

and

$$t = \frac{z}{V\sqrt{1 - p^2 V^2}}. \tag{B.5}$$

Thus equations (B.4) and (B.5) give the ray and traveltime for a single in a constant velocity medium.

Next, if we extend this model to the case of n constant velocity, homogeneous layers, the velocity of each being given by V_i, we simply recall that p is conserved and include summation signs in the equations. Doing so we get

$$x = \sum_{i=1}^{n} \frac{(pV_i)\, z_i}{\sqrt{1 - p^2 V_i^2}} \tag{B.6}$$

and

$$t = \sum_{i=1}^{n} \frac{z_i}{V_i \sqrt{1 - p^2 V_i^2}} \tag{B.7}$$

where z_i is the base of the i^{th}-layer.

B.1 Single layer, velocity dependent on depth

We have formulated the model for a medium that is a series of discreet layers with a constant velocity in each. Now we wish to extend this model to a single continuous layer in which the velocity is some continuous function of depth. We begin by dividing up the medium, as above, into layers in which the thickness of each is denoted $\Delta z_i = z_i - z_{i-1}$. Although within each layer velocity is dependant upon the depth, let us approximate the velocity within each layer by some velocity between the minimum and maximum velocities of that layer and denote this approximation by V_i^*. That is,

$$\forall z \in [z_{i-1}, z_i]\,, \inf V(z) < V_i^* < \sup V(z)\,. \tag{B.8}$$

Note as well, as $\Delta z_i \to 0$, $V_i^* \to V(z)$. Doing so, equations (B.6) and (B.7) become

$$x \approx \sum_{i=1}^{n} \frac{pV_i^* \Delta z_i}{\sqrt{1 - p^2 V_i^{*2}}}, \tag{B.9}$$

and

$$t \approx \sum_{i=1}^{n} \frac{\Delta z_i}{V_i^* \sqrt{1 - p^2 V_i^{*2}}}.$$

Taking the limit as the thickness of each layer goes to zero we get

$$x = \lim_{\Delta z_i \to 0} \sum_{i=1}^{n} \frac{pV_i^* \Delta z_i}{\sqrt{1 - p^2 V_i^{*2}}} = \int_0^z \frac{pV(w)}{\sqrt{1 - p^2 V^2(w)}} \mathrm{d}w \tag{B.10}$$

and

$$t = \lim_{\Delta z_i \to 0} \sum_{i=1}^{n} \frac{\Delta z_i}{V_i^{*2} \sqrt{1 - p^2 V_i^{*2}}} = \int_0^z \frac{1}{V(w)\sqrt{1 - p^2 V^2(w)}} dw, \qquad \text{(B.11)}$$

where w is the dummy variable of integration and where $|pV(z)| < 1$.

B.2 Velocity linear function of depth

Let us now assume a specific velocity function and evaluate the integrals in equations (B.10) and (B.11) to get explicit expressions. To quote Slotnick "experience has shown that the velocity of seismic wave propagation in Tertiary basins can be closely approximated by expressing it as a linear function of depth", and hence we will make the assumption here. Denoting the surface velocity as a and the velocity gradient with depth as b, the linear velocity model for a vertically inhomogeneous medium is then

$$V(z) = a + bz. \qquad \text{(B.12)}$$

Substituting this expression into expression (B.10) we get

$$
\begin{aligned}
x &= \int_0^z \frac{p\,(a+bw)}{\sqrt{1-p^2\,(a+bw)^2}}\mathrm{d}w \\
&= \frac{-1}{pb}\sqrt{1-p^2\,(a+bw)^2}\Bigg|_{w=0}^{z} \\
x &= \frac{1}{pb}\left[\sqrt{1-p^2a^2}-\sqrt{1-p^2\,(a+bz)^2}\right].
\end{aligned}
\tag{B.13}
$$

To get an explicit traveltime expression in terms of the receiver position and the velocity parameters, we will need to solve the above expression for the ray parameter. In anticipation of this we solve expression (B.13) for p. Doing so, we obtain

$$
p = \frac{2x}{\sqrt{(x^2+z^2)\left(b^2x^2+(2a+bz)^2\right)}}. \tag{B.14}
$$

Returning to expression (B.11) we perform the integration to get

$$
\begin{aligned}
t &= \int_0^z \frac{1}{(a+bw)\sqrt{1-p^2\,(a+bw)^2}}\mathrm{d}w \\
&= \frac{1}{b}\ln\left[\frac{p\,(a+bw)}{1+\sqrt{1-p^2\,(a+bw)^2}}\right]_{w=0}^{z}
\end{aligned}
$$

and hence the traveltime expression for an vertically inhomogeneous, isotropic medium is given by

$$t = \frac{1}{b} \ln \left[\left(\frac{(a+bz)}{a} \right) \frac{1 + \sqrt{1 - p^2 a^2}}{1 + \sqrt{1 - p^2 (a+bz)^2}} \right], \qquad \text{(B.15)}$$

where p is given by expression (B.14).

B.3 Traveltime in inhomogeneous anisotropic medium

Let us now include anisotropy in the model. To obtain an explicit traveltime expression, we will first assume that the dependance of velocity on propagation direction is elliptical. Then, by an appropriate transformation of coordinates, we will define a coordinate system in which the wavefront propagation will be the same as in the isotropic case. This will allow us to use expression (B.15) given above in the transformed coordinate system.

Assume an elliptical velocity dependance and let the magnitudes of the horizontal and vertical wavefront velocities be given by

$$v_{\mathrm{H}} = a\sqrt{1 + 2\chi} \qquad \text{(B.16)}$$

and

$$v_{\mathrm{V}} = a, \tag{B.17}$$

respectively. Here we define the parameter of velocity anisotropy, namely,

$$\chi := \frac{v_{\mathrm{H}}^2 - v_{\mathrm{V}}^2}{2v_{\mathrm{V}}^2}, \tag{B.18}$$

which we assume to be a constant. We note that – in the context of ray theory – since v_{H} and v_{V} are the magnitudes of wavefront velocities along the symmetry axes of the ellipse, expressions (B.16) and (B.17) are the same for both the wavefront and ray velocities.

Infinitesimal wavefronts resulting from a point source within the medium are ellipses with axes $(\mathrm{d}t)\, v_{\mathrm{H}}$ and $(\mathrm{d}t)\, v_{\mathrm{V}}$, where $\mathrm{d}t$ is the traveltime increment. We can write such a wavefront as

$$\frac{(\mathrm{d}x)^2}{(\mathrm{d}t)^2\, v_{\mathrm{H}}^2} + \frac{(\mathrm{d}z)^2}{(\mathrm{d}t)^2\, v_{\mathrm{V}}^2} = 1, \tag{B.19}$$

which, using expressions (B.16) and (B.17), we can rewrite as

$$\frac{(\mathrm{d}x)^2}{1 + 2\chi} + (\mathrm{d}z)^2 = (\mathrm{d}t)^2\, a^2. \tag{B.20}$$

Now, we will obtain the desired traveltime expression by a linear transformation of coordinates, which allows us to treat elliptical velocity dependance as an isotropic case in the transformed coordinates. Since v_{H} and v_{V} are the magnitudes of velocities along the x-axis and the z-axis, respectively, we can scale the z-axis by a factor of $\sqrt{1+2\chi}$ to obtain circular wavefronts. In other words, we transform the xz-plane into the $x\zeta$-plane, where

$$\zeta = z\sqrt{1+2\chi}. \tag{B.21}$$

Thus, in view of expression (B.21), we let $z = \zeta/\sqrt{1+2\chi}$ to write expression (B.20) as

$$(\mathrm{d}x)^2 + (\mathrm{d}\zeta)^2 = (\mathrm{d}t)^2\,\alpha^2, \tag{B.22}$$

where

$$\alpha = a\sqrt{1+2\chi} \tag{B.23}$$

is the velocity in the $x\zeta$-plane. Expression (B.22) describes a circular wavefront in the $x\zeta$-plane, which is equivalent to wave propagation in an isotropic medium in xz-space.

To include inhomogeneity of the velocity model, we simply assume that the magnitude of velocity in $x\zeta$-space increases linearly along the ζ-axis, namely, $v(\zeta) = \alpha + b\zeta$.

Since — in the $x\zeta$-plane — we have a wavefront which propagates in the same manner as an isotropic and linearly inhomogeneous wavefront in the xz-plane, we can simply apply traveltime expression (B.15) between the source at $(0,0)$ and the receiver at (X, Ξ), namely,

$$t = \frac{1}{b} \ln \left[\frac{\alpha + b\Xi}{\alpha} \frac{1 + \sqrt{1 - p^2\alpha^2}}{1 + \sqrt{1 - p^2 (\alpha + b\Xi)^2}} \right], \tag{B.24}$$

where

$$p = \frac{2X}{\sqrt{(X^2 + \Xi^2)\left[4\alpha^2 + 4\alpha b\Xi + b^2 (X^2 + \Xi^2)\right]}}. \tag{B.25}$$

However, since our experimental measurements are made in the xz-plane we wish to state expression (B.24) in the coordinates of this plane. We achieve this by substituting expression (B.23) into expressions (B.24) and (B.25), as well as — in view of expression (B.21) — letting $\Xi = Z\sqrt{1 + 2\chi}$, to obtain

$$t = \frac{1}{b}\ln\left[\frac{a+bZ}{a}\frac{1+\sqrt{1-a^2p^2(1+2\chi)}}{1+\sqrt{1-(a+bZ)^2\,p^2(1+2\chi)}}\right], \tag{B.26}$$

where

$$p = \frac{2X}{\sqrt{(X^2+(1+2\chi)Z^2)[(2a+bZ)^2\,(1+2\chi)+b^2X^2]}}. \tag{B.27}$$

Expressions (B.26) and (B.27) are the traveltime and ray parameter expressions that correspond to a linearly inhomogeneous and elliptically anisotropic velocity model and are the same as expressions (8.5) and (7.16), as expected.

Appendix C

Computer Code: Parameter estimation

This appendix gives the Maple®code which invokes the Gauss-Newton method of optimization, outlined in Chapter 9, to invert the field data to estimate the values of a, b and χ. The aim of the optimization is to find the parameter values which minimize the sum of the squared residuals between the modelled and observed traveltimes.

```
[>restart:
[>##################################
[>###          USER INPUT:
[>###     Initial parameter estimates.
```

```
[>####################################
[>ao:= initial estimate of parameter a
[>bo:= initial estimate of parameter b
[>chio:= initial estimate of parameter χ
[>sigfigs:= 0.0001 set the number of significant figures
            in final result to 4
[>maxiter := sets the maximum number of iterations to perform
[>####################################
[>with(linalg): # Must load the linear algebra package
[>####################################
[>###     Ray parameter and traveltime expressions
[>####################################
[>p:= 2*X/((X^2+(1+2*chi)*Z^2)*(a^2*(4+8*chi)+4*a*b*
     (1+2*chi)*Z     +b^2*(X^2+(1+2*chi)*Z^2)))^(1/2):
[>t:= 1/b*ln((a+b*Z)/a*(1+sqrt(1-p^2*a^2-2*p^2*a^2*chi))/
     (1+sqrt(1-p^2*a^2-2*p^2*a^2*chi-2*p^2*a*b*Z-4*
     p^2*a*b*Z*chi-p^2*b^2*Z^2-2*p^2*b^2*Z^2*chi))):
[>####################################
[>###     Read in and assign the data file
[>####################################
```

```
[>data:= readdata("C:/data.txt",3):

[>n:=nops(data):  # Count the number of data points

[>for i from 1 to n do

    X[i] := data[i,1]:

    Z[i] := data[i,2]:

    T[i] := data[i,3]:

end do:

[>P:=Vector(3,[ao,bo,chio]): # Initialize the parameter vector

[>Po:= Vector(3):   # Define the vector to hold parameter estimates

[>err[i]:= Vector(n):   # Define the vecotr to hold model residuals

[>##################################

[>### Open data file to write results of iteration...

[>fd:=fopen("Results",WRITE,TEXT):

[>gzz := 1   # Initialize counter to prevent infinite looping

[>while ( abs(Po[1]-P[1])>sigfigs and abs(Po[2]-P[2])>sigfigs

    and abs(Po[3]-P[3])>sigfigs and gzz<maxiter ) do

    ##################################

    ### Initialize estimates of parameters in iteration...

    Po[1]:=P[1]: Po[2]:=P[2]: Po[3]:=P[3]:

    ##################################
```

```
###                 Based on parameter estimates,
###             define the observation/model errors
##################################
for i from 1 to n do
#mt[i] is the model time_i
mt[i]:=subs({X=X[i], Z=Z[i], a=Po[1], b=Po[2], chi=Po[3]}, t):
err[i]:=T[i]-mt[i]:#Residual between observed and modelled times
end do:
##################################
###     Build the Jacobian martix
##################################
J := matrix(n,3):       # The Jacobian matrix will be nx3
for i from 1 to n do
J[i,1]:=subs({X=X[i], Z=Z[i], a=Po[1], b=Po[2], chi=Po[3]}, diff(t,a)):
J[i,2]:=subs({X=X[i], Z=Z[i], a=Po[1], b=Po[2], chi=Po[3]}, diff(t,b)):
J[i,3]:=subs({X=X[i], Z=Z[i], a=Po[1], b=Po[2], chi=Po[3]}, diff(t,chi)):
end do:
##################################
###     Perform a QR-decomposition
##################################
```

```
    R1 := QRdecom(J,Q='q', fullspan=false):

    Q1 := evalm(q):

    ###################################

    ###     New estimate of P from Gauss-Newton

    ###################################

H := Vector(3,matadd(multiply(inverse(R1),transpose(Q1),err),Po)):

P[1] := H[1]:  P[2] := H[2]:  P[3] := H[3]:

    ###################################

    ###     Print results to file

    fprintf(fd, "%g %g %g %g \n", gzz, P[1], P[2], P[3]):

    gtt:=gzz+1: gzz:=gtt: #Increment the iteration counter

end do:

[>fclose(fd): # Must close the data file.
```

Appendix D

Computer Code: Illumination zone

This appendix gives the Maple® code used to generate the curves bounding the zone of illumination which accounts for inhomogeneity (i.e., $b \neq 0$) and anisotropy (i.e., $\chi \neq 0$) as stated in equation (6.9) with (7.14).

[>restart:

[>###### Required User Input #####

[Model parameters...

[>a:= *vertical velocity at z = 0*

[>b:= *vertical velocity gradient*

[>χ:= *ellipticity parameter*

```
[Relative source-receiver position

[>X:= source-receiver offset

[>zo:=depth of receiver

[Maximum reflector depth...

[>mrd:=maximum reflector depth

[ Data file to write the results...

[>fd:=fopen("C:/Datafiel.txt",WRITE,TEXT):

[>################################

[       DATA FILE GENERATION

[>################################

[>dbr:=Depth of initial reflection below receiver (m)

[>incr:=Incremental depth of reflection calculations (m)

[>################################

[># For a homogeneous medium (b=0) we use...

>if (b=0) then

    # - - - - - - - - - -

    for zr from zo+dbr by incr to mrd do

        xr:=(zr*X)/(2*zr-zo):

        fprintf(fd, "%.2f %.2f \n", xr,zr):

    end do:
```

```
    # - - - - - - - - - -
else
    # - - - - - - - - - -
# For a non-homogeneous medium (b<>0) we use...
##### Traveltime of downgoing signal #####
p1:=(xr,zr)->2*xr/sqrt((xr^2+(1+2*chi)*zr^2)*
    ((2*a+b*zr)^2*(1+2*chi)+b^2*xr^2)):
t1:=(xr,zr)->1/b*ln((a+b*zr)/a*(1+sqrt(1-p(xr,zr)^2*
    a^2*(1+2*chi)))/(1+sqrt(1-p(xr,zr)^2*
    (a+b*zr)^2*(1+2*chi)))):
##### Traveltime of upgoing signal #####
a2:=a+b*zo
p2:=(xr,zr)->2*xr/sqrt((xr^2+(1+2*chi)*zr^2)*
    ((2*a2+b*zr)^2*(1+2*chi)+b^2*xr^2)):
t2:=(xr,zr)->1/b*ln((a2+b*zr)/a2*(1+sqrt(1-p(xr,zr)^2*
    a2^2*(1+2*chi)))/(1+sqrt(1-p(xr,zr)^2*
    (a2+b*zr)^2*(1+2*chi)))):
```

```
##### Total Reflected Traveltime #####

T:=(xr,zr)->2*t(xr,zr)-t(2*xr-X,Z):

##### Minimize traveltime for a given zr by solving... #####

eqn:=diff(T(xr,zr),xr)=0: #Stationarity of Traveltime.

# - - - - - - - - - -

for zr from zo+dbr by incr to md do

    fsolve(eqn,xr):

    xpos:=%:

    fprintf(fd,“%6.2f %6.2f \n”, xpos,zr):

end do:

# - - - - - - - - - -

end if:

[Must Close Data File.

[>fclose(fd):
```