A Characterization of Utilitarism without the transitivity Axiom

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Abstract

We show that d'Aspremont and gevers' (1977) characterization of utilitarism can be strenghened in the following way. First, the condition of transitivity of the social preference relation is dropped and the Anonymity condition is slightly strengthened. (Our Anonymity condition is weaker than the combination of transitivityand d'Aspremont and Gever's Anonimity condition). Second, the Strong Pareto Principle is the Weak Pareto Principle.

1. Introduction

Under certain conditions, a social preference relation over a set of alternatives is equivalent to a social ranking of vectors of utility levels, $u = (u_1, u_2, ..., u_n)$. A reflexive, complete and transitive social relation R over the set of utility vectors is called a Social Welfare Ordering. d'Aspremont and Gevers (1977) have shown that the only Social Welfare Ordering that satisfies Anonymity, the Strong Pareto Principle and the condition of Cardinal Unit Comparability is utilitarism.

The d'Aspremont and Gevers characterization of utilitarism requires that the social preference relation R be transitive. The purpose of this note is to provide an alternative characterization that uses essentially the same axioms as d'Aspremont and Gevers' but drops the transitivity axiom altogether. Specifically, we show that utilitarism is the only reflexive and complete social preference relation R over the n-dimensional Euclidean space E^n (where n is the number of individuals) that satisfies the Weak Pareto Principle, Cardinal Unit Comparability and an appropriate version of the Anonymity condition.

This claim may appear surprising, since it is well known that even without using any kind of interpersonal comparison there are collective choice rules that satisfy the Pareto Principle and Anonymity, like the majority rule (May, 1952) or the Pareto rule (Sen, 1969). In the standard Arrovian framework, transitivity of the social preference relation is crucial to exclude these methods and precipitate the dictatorship result. Clearly, allowing interpersonal utility comparisons cannot rule these methods out, and therefore one might expect that transitivity be crucial in the characterization of utilitarism, too.

This argument, however, does not survive a closer scrutiny. As a matter of fact, in the analysis of Social Welfare Orderings the Anonymity axiom is often formulated in a way that already excludes the majority rule. For instance, d'Aspremont's(1986) Anonymity condition says that two utility vectors that are identical up to a permutation of the n individuals must be socially indifferent. To see that this condition conflicts with the majority rule, consider two utility vectors $u = (6, 3, 2)$ and $v = (3, 2, 6)$. By the majority rule, uPv yet v is obtained by u through a permutation of individuals. What this example shows is that the Anonymity principle can be formalized in different ways which are equivalent in the presence of the transitivity axiom but otherwise are not. The majority rule obeys some versions of the Anonymity principle but violates others and therefore can be eliminated by using an appropriate Anonymity condition. The same is true of the Pareto rule and of other rules that *prima facie* qualify as counterexamples to our claim.

Notice also that in our characterization the Strong Pareto Principle used by d'Aspremont and Gevers is replaced by the Weak Pareto Principle.

The rest of the paper is organized as follows. After presenting our notation in section 2, we discuss some alternative ways of formalizing the Anonymity principle in section 3. The characterization result is stated and proved in section 4.

2. Notation and Definitions

Let E^n be the n-dimensional Euclidean space, where n is the (finite) number of individuals in the society. A reflexive and complete relation R on E^n is called a Social Welfare Relation. A Social Welfare Ordering is a transitive Social Welfare Relation. We denote by P strict social preference and by I social indifference.

Let σ be a permutation of the set $N = \{1, 2, ..., n\}$ of individuals. Then we denote $\sigma u \equiv (u_{\sigma(1)}, ..., u_{\sigma(n)})$.

A Social Welfare Relation satisfies the Strong Pareto Principle if $\forall u, v \in$

$E^n, u > v$ implies uPv. It satisfies the Weak Pareto Principle if $u \gg v$ implies uPv. (Vector inequalities: $u \geq v$ means $u_i \geq v_i$ for all i; $u > v$ means $u \geq v$ but $u \neq v$; $u \gg v$ means $u_i > v_i$ for all i.)

A Social Welfare Relation satisfies Cardinal Unit Comparability if $\forall u, v \in E^n, uRv$ if and only if

$$(a_1 + bu_1, a_2 + bu_2, ..., a_n + bu_n)R(a_1 + bv_1, a_2 + bv_2, ..., a_n + bv_n).$$

where $a_1, a_2, ..., a_n$ and $b > 0$ are any real numbers. Under Cardinal Unit Comparability, welfare gains are interpersonally comparable but welfare levels are not.

We now define some well known Social Welfare Relations.

The utilitarian rule: $\forall u, v \in E^n, uR^U v$ if and only if $\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{n} v_i$.

The Pareto rule: $\forall u, v \in E^n, uP^P v$ if and only if $u > v$.

The majority rule: $\forall u, v \in E^n, uR^M v$ if and only if the number of individuals with $u_i \geq v_i$ is not lower than the number of individuals with $u_i \leq v_i$.

The Suppes rule: $\forall u, v \in E^n, uR^S v$ if and only if there exists a permutation σ of N such that $\sigma u \geq \sigma v$.

3. Anonymity

In this section we formulate three versions of the Anonymity axiom and discuss their mutual relationships.

Anonymity is a condition of symmetry among individuals. Loosely speaking, it says that to rank two utility vectors all that matters is the unordered list of utility levels, and it is irrelevant whom they pertain to. One way to express this idea formally is the following.

Axiom 1. *Let u and v be any two elements of E^n and σ be any permutation of N. A Social Welfare Relation R satisfies A_1 if uRv holds whenever $\sigma uR\sigma v$.*

This corresponds to May's (1952) definition of Anonymity in his characterization of the majority rule. Obviously, the majority rule satisfies A_1.

d'Aspremont (1986) uses a different condition. This condition is also called Suppes Indifference (see Sen, 1986).

Axiom 2. *A Social Welfare Relation R satisfies A_2 if $\forall u \in E^n$ and all permutations σ of N, $uI\sigma u$.*

We have:

Lemma 1. *For Social Welfare Orderings, A_1 and A_2 are equivalent. For Social Welfare Relations, neither condition implies the other.*

Proof. To prove that A_2 implies A_1 if the social preference relation is transitive, suppose uRv; then by A_2 we have σuIu and σvIv and thus by transitivity $\sigma uR\sigma u$, and viceversa. To prove the converse, notice first of all that any permutation of N can be obtained through a finite sequence of pairwise permutations, i.e. permutations involving pairs of individuals. For a pairwise permutation, it is clear that $\sigma\sigma u = u$ for all u. Now suppose σ is a pairwise permutation and $v = \sigma u$. By A_1, uRv implies $\sigma uR\sigma v$, but since $v = \sigma u$ and $u = \sigma v$ this implies vRu whence uIv follows. By repeated use of this argument, A_2 is obtained.

As for the second part of the lemma, we have already seen that the majority rule obeys A_1 but violates A_2. A Social Welfare Relation that satisfies A_2 but violates A_1 is the following: $\forall u, v \in E^n$ let uIv if there is a permutation σ of N such that $v = \sigma u$; otherwise, let uPv if $u_1 > v_1$ or $u_1 = v_1$ and $u_2 > v_2$, etc. (Obviously this Social Welfare Relation is not transitive.) ■

d'Aspremont and Gevers (1977, lemma 4) have proved the following result.

Proposition 1. (d'Aspremont and Gevers) *The utilitarian rule R^U is the only Social Welfare Ordering satisfying the Strong Pareto Principle, Cardinal Unit Comparability and the Anonymity condition A_2.*

Though A_2 is already sufficient to rule out the majority rule without invoking transitivity, the transitivity axiom cannot be dropped in the d'Aspremont and Gevers characterization result leaving the other axioms unaltered, as the following example shows.

Example. The Pareto rule R^P satisfies the Strong Pareto Principle (obviously), Cardinal Unit Comparability (it actually does not use any interpersonal comparison) and A_2. To prove this, suppose that $v = \sigma u$. This implies $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i$, so that it cannot be the case that $u > v$ or $v > u$; by the definition of the Pareto rule, this implies that uIv.

Now consider the following strengthening of A_2.

Axiom 3. *A Social Welfare Relation R satisfies A_3 if $\forall u, v \in E^n$, $v = \sigma u$ for some permutations σ of N implies that $\forall t \in E^n$ we have uRt if and only if vRt.*

That is, if v is obtained from u through a permutation of the set of individuals, then v should be ranked exactly like u *vis-a-vis* any other utility vector. The Anonymity condition A_3 is stronger than A_2; however, it coincides with A_2 when the social preference relation is transitive.

Lemma 2. *For Social Welfare Relations, A_3 implies A_2 but the converse is not true. For Social Welfare Orderings, A_2 also implies A_3.*

Proof. To prove that A_3 implies A_2 just set $t = v$ in the definition of A_3. A Social Welfare Relation that satisfies A_2 but violates A_3 is the Pareto rule R^P. We have already seen that the Pareto rule satisfies A_2. To prove that it violates A_3 it suffices to consider the following example: let $t = (6, 3); u = (4, 2)$ and $v = (2, 4)$. We have $tP^P u$ and $tI^P v$, though clearly there is a permutation σ of N such that $v = \sigma u$.

The second part of the lemma is obvious. ■

As we have just seen, A_2 and transitivity imply A_3. We now show that the converse is not true. This means that A_3 is a weaker condition than the combination of A_2 and transitivity of social preference.

Lemma 3. *A_3 does not imply transitivity of social preference.*

Proof. An example will suffice. The Suppes rule satisfies A_3 (and hence A_2) but is not transitive, as the following example shows. Consider $u = (8, 3); v = (6, 2)$ and $t = (5, 4)$. We have $uP^S v, vI^S t$ and $uI^S t$. ■

We conclude this section with a lemma that describes the relationships between A_3 and A_1.

Lemma 4. *For Social Welfare Orderings, A_1 and A_3 are equivalent. For Social Welfare Relations, neither condition implies the other.*

Proof. The first part of the lemma follows from lemmas 1 and 3. The proof of the second part is identical to the proof of the second part of lemma 1. ■

4. Characterization of the Utilitarian Rule

We are now ready to prove the characterization result. The technique of proof is fairly standard; one has just to check that the transitivity axiom is never used and to adapt the method of proof so as to make use of the Weak (instead of Strong) Pareto Principle.

Proposition 2. *The utilitarian rule R^U is the only Social Welfare Relation satisfying the Weak Pareto Principle, Cardinal Unit Comparability and the Anonymity condition A_3.*

Proof. That the utilitarian rule R^U satisfy the Weak Pareto Principle, Cardinal Unit Comparability and A_3 is obvious. We now show that it is the only Social Welfare Relation that satisfies these conditions.

Suppose $\sum_{i=1}^{n} u_i > \sum_{i=1}^{n} v_i$ and let $\varepsilon < \frac{1}{n}(\sum_{i=1}^{n} u_i - \sum_{i=1}^{n} v_i)$ be a positive number. Permute u and v in such a way that $u_1 \geq u_2 \geq ... \geq u_n$ and $v_1 \geq v_2 \geq ... \geq v_n$. By A_3, the relative ranking of u and v after the permutation must be the same as the original ranking. Now if $u_i \geq v_i + \varepsilon$ deduct v_i from both u_i and v_i; if instead $u_i < v_i + \varepsilon$ deduct $u_i - \varepsilon$. This is a permitted transformation under Cardinal Unit Comparability and therefore the relative ranking of the two utility vectors after the transformation must be the same as the ranking of u and v. By repeated application of this procedure it can be easily shown that the relative ranking of u and v must be the same as the relative ranking of $u' = (u'_1, u'_2, ..., u'_n)$ and $v' = (0, 0, ...0)$ where $u'_i \geq \varepsilon > 0$ for $i = 1, ..., n$. By the Weak Pareto Principle, $u'Pv'$ and therefore uPv.

If instead $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i$, after permuting the individuals deduct from both u_i and v_i the minimum of $\{u_i, v_i\}$. Repeatedly using Cardinal Unit Comparability and A_3 it can be shown that the relative ranking of u and v must be the same as the relative ranking of $u' = v' = (0, 0, ...0)$. By reflexivity of R we have $u'Iv'$ and therefore uIv. ■

We conclude showing that our three axioms are logically independent and that no one of them can be dispensed with in the characterization of utilitarism. A rule that satisfies Cardinal Unit Comparability and the Weak Pareto Principle (but not A_3) is the dictatorial rule (there is an individual d such that uPv whenever $u_d > v_d$). The null rule (which declares that all utility vectors are socially indifferent) satisfies A_3 and Cardinal Unit Comparability, but obviously violates the Weak Pareto Principle. The Suppes rule satisfies the Weak Pareto Principle and A_3 but violates Cardinal Unit Comparability.

5. Concluding Remarks

There is a large literature on the consequences of relaxing the transitivity axiom in the framework of Arrovian social choice theory. By way of contrast, the literature

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