Revisiting borehole stresses in anisotropic elastic media: comparison of analytical versus numerical solutions

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ABSTRACT: We present a rigorous validation of the analytical Amadei solution for the stress concentration around arbitrarily orientated borehole in general anisotropic elastic media. First, we revisit the theoretical framework of the Amadei solution and present analytical insights that show that the solution does indeed contain all special cases of symmetry, contrary to previous understanding, provided that the reduced strain coefficients $\beta_{11}$ and $\beta_{55}$ are not equal. It is shown from theoretical considerations and published experimental data that the $\beta_{11}$ and $\beta_{55}$ are not equal for realistic rocks. Second, we develop a 3D finite-element elastic model within a hybrid analytical-numerical workflow that circumvents the need to rebuild and remesh the model for every borehole and material orientation. Third, we show that the borehole stresses computed from the numerical model and the analytical solution match almost perfectly for different borehole orientations (vertical, deviated and horizontal) and for several cases involving isotropic and transverse isotropic symmetries. It is concluded that the analytical Amadei solution is valid with no restrictions on the borehole orientation or elastic anisotropy symmetry.

1 Introduction

The calculation of stresses and displacements around cavities is required in some of the most important subsurface geotechnical engineering problems such as for boreholes, tunnels and mine excavations. For example, the presence of a borehole in a stressed subsurface rock formation alters the local principal stress directions and magnitudes around the borehole and away from it over a distance of several borehole diameters. For isotropic elastic homogeneous rocks, borehole stresses are given by the classical elastic solution by [1] or its generalized version for nonaligned borehole and stress directions by [2, 3] and [4]. Borehole stresses depend on the far-field stress, the orientation of the borehole with respect to the stress field directions, the wellbore pressure and the material Poisson’s ratio. These solutions are very convenient for practical purposes as all the borehole stress components except the axial component are independent of the material elastic properties, and the axial component is only dependent on the Poisson’s ratio by the virtue of the plane strain assumption. Consequently, these classical elastic solutions are widely used for engineering and research applications.

Most wells drilled for the purpose of natural oil and gas extraction encounter anisotropic shale formations during the drilling process, either in the overburden for conventional reservoirs or in the reservoir itself for unconventional shale reservoirs. For conventional elastic reservoirs, it has been reported that shales constitute about 75% of the clastic fill of sedimentary basins [5] and for unconventional reservoirs, the recent exploration and production of US gas shale reservoirs has put a renewed focus on drilling and hydraulic fracturing in shale formations [6]. Shales are known to exhibit anisotropic properties not only for their elastic behavior [7, 8, 9, 10, 11] but also for their strength due to their laminated structure [12, 13]. [14] gives a thorough review of existing experimental data in shales. Today, most wells drilled in highly deviated or horizontal directions are penetrating strongly transverse isotropic formations or lower symmetries such as orthorhombic or monoclinic if fractures are present. Consequently, in principle, most rock mechanics analysis in such anisotropic environments, for example for
wellbore stability and hydraulic fracturing design, should involve two key steps: first, the calculation of borehole stresses for anisotropic rocks, and second, a stress-related failure criteria for anisotropic rocks.

The fundamentals for the stress analysis in anisotropic media were established by [15]. [16] used this approach based on generalized plane strain assumption to calculate the stress concentration around arbitrarily oriented boreholes in arbitrary anisotropic rock formations. Although well established from a theoretical point of view since more than two decades, this fundamental analytical solution is rarely used for practical applications in anisotropic media except by a few authors [17, 18, 19, 14] or is replaced by numerical computation [20]. Alternative theoretical derivation has also been obtained [21]. We speculate that there are several reasons for the unfortunate low use or acceptance of the Amadei solution: (1) the solution involves three coordinate systems (stress field, borehole and material orientations) and is more complicated in form than the isotropic solution; (2) it has been attributed some “unjustified” severe shortcoming; (3) no numerical verification of the solution has been presented in the literature and (4) in practice, the elastic and strength anisotropic material properties may be difficult to obtain in-situ. From a measurement point of view, modern sonic logging tools [22] can measure three or four out of five elastic constants of transversely isotropic media [23, 24] and can potentially be used for borehole stress computations in such media. From a theoretical point of view, despite a very thorough and comprehensive analysis of the Amadei solution, [18] and [19] have stated that the Amadei solution does not reduce to the Kirsch solution for isotropic media or for transverse isotropic (hereafter called TI) media when the borehole axis coincides with the TI symmetry axis. This apparent shortcoming is severe but unjustified, as shown in this paper.

The purpose of our paper is to present a rigorous validation of the Amadei solution using a numerical finite-element analysis for several cases involving different anisotropic symmetries and well orientations. First, we revisit the theory of borehole stresses in anisotropic elastic media and present analytical insights that show that the Amadei solution does indeed contain all special cases of symmetry, contrary to previous understanding, provided that the reduced strain coefficients $\beta_{11}$ and $\beta_{55}$ are not equal. Second, we develop a 3D finite element elastic model within a hybrid analytical-numerical workflow that circumvents the need to rebuild and remesh the model for every borehole and material orientation. Third, we show that the borehole stresses computed from the numerical model and the analytical solution match almost perfectly for different borehole orientations (vertical, deviated and horizontal) and for several cases involving isotropic and transverse isotropic symmetries.

2 Borehole stresses in anisotropic elastic media

Here, we give the framework within which the stresses around a fluid-filled borehole in anisotropic homogeneous elastic media are derived. We only provide the governing equations and the final result. As this work was pioneered by [15] and [16] we recommend their work for a more detailed understanding of the problem.

2.1 General Assumptions

We consider an infinite formation of arbitrary anisotropy which is homogeneous and continuous in all directions. Internally this body is bounded by a cylindrical borehole of radius $a$.

2.1.1 Geometry and coordinate systems

![Figure 1: Schematic of the geographic and borehole reference frames and the principal stress directions. The geographic reference frame is the north-east-vertical (NEV) frame whose x-axis points to the north, y-axis points to the east, and z-axis points downward in vertical direction. The borehole frame is the top-of-hole (TOH) frame whose z-axis points along the borehole in the direction of increasing depth. The x-axis is in the cross-sectional plane and points to the most upward direction, and the y-axis is found by rotating the x-axis 90° in the cross-sectional plane in a direction dictated by the right-hand rule. The principal stress directions are chosen such that one component is parallel to the vertical NEV axis and the maximum horizontal component is rotated by the angle $\gamma$ with respect to the north axis.](image)

In the far-field an in-situ stress field is applied where the
principal stress tensor takes the form

\[
\sigma = \begin{pmatrix}
\sigma_H & 0 & 0 \\
0 & \sigma_h & 0 \\
0 & 0 & \sigma_v
\end{pmatrix},
\]

(1)

where \(\sigma_H\) and \(\sigma_h\) are the maximum and minimum horizontal stresses, respectively, and \(\sigma_v\) is the vertical stress (see Figure 1). For the sake of simplicity, but without loss of generality, we assume that the vertical stress \(\sigma_v\) is always aligned with the vertical component (V) of the NEV (north-east-vertical) coordinates system. The horizontal stress field can be rotated by an angle \(\gamma\) measured between N (north) and \(\sigma_H\) towards E (east). The regional stress field is first rotated into the NEV frame. For the computation of the borehole stress concentration it is convenient to rotate the stress field in the NEV frame into the top-of-hole borehole coordinate system, hereafter called TOH (see Figure 1 for definition). The orientation of the borehole is defined by the deviation angle \(\alpha_D\) and the azimuth angle \(\alpha_A\). All these coordinate transforms are defined by [25]. Here and in the rest of the paper we assume, for convenience, that the in-situ stress field is aligned with the NEV frame (i.e., \(\gamma = 0\)). The solution of the stress concentration is obtained in the TOH frame in Cartesian coordinates [16]. Due to geometry the borehole problem it is natural to transform the borehole stress components into cylindrical coordinates [see 18, eq. A.15].

2.1.2 Generalized plane strain

For the solution of the stress concentration and displacement problems some assumptions can be made which simplify the general solution but are still reasonable approximations. We can assume that the borehole is infinite and homogeneous in the axial direction; this is referred to as a plane strain formulation which can be applied if solutions are sought far enough from the ends of the borehole as well as interfaces. For an isotropic medium this formulation requires that the strains \(\varepsilon_{zz} = \varepsilon_{xx} = \varepsilon_{yy} = 0\), from which follows that the axial displacement is \(w = 0\). For an anisotropic body this assumption is not valid as the equation of equilibrium and Hooke’s law would not be satisfied [15]. Thus, we have to apply the generalized plain strain formulation [16] which requires the displacement components to be functions of \(x\) and \(y\) only:

\[
\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0.
\]

(2)

From equation 2 it follows that the only strain which is zero is \(\varepsilon_{zz}\). Based on this assumption, we will introduce the basic equations that are necessary to determine the stresses induced by a far-field stress field around a borehole in an arbitrarily anisotropic formation.

2.1.3 Governing equations

For all elastostatic problems, the stress, strain and displacement components must satisfy the constitutive relations, the equations of equilibrium, the equations of compatibility for strains and the strain-displacement relations, as well as the boundary conditions.

The strain components \(\epsilon_{ij}\) are related to the stress components \(\sigma_{ij}\) via the constitutive relation:

\[
\epsilon_{ij} = S_{ijkl}\sigma_{kl},
\]

(3)

where \(S_{ijkl}\) is the compliance tensor. Although the theoretical developments presented here are valid for arbitrary anisotropy symmetry, for the sake of clarity, we consider here only the following symmetries for the compliance tensor: (1) isotropic with two elastic constants (hereafter called ISO) and (2) transverse isotropy with five elastic constants (i.e., TI). In addition, depending on the orientation of the TI symmetry axis, we distinguish two TI symmetries: TI with a vertical axis of symmetry (hereafter called VTI) and TI with tilted axis of symmetry (hereafter called TTI).

As all measurements are obtained in the borehole, it is convenient to rotate the compliance tensor into the TOH frame. This is done by applying two Bond transformations to the \(6\times6\) Voigt notation compliance matrix \(s_{ij}\) giving \(a_{ij}\). The definition of this transformation can be found in [18]. The orientaion of a TTI material is defined by the dip angle \(\beta_D\) and the azimuth \(\beta_A\) (see Figure 2).

At any position around the borehole, the strain is now related to the stress in Cartesian coordinates via the constitutive relation

\[
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
0 \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\
a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\
a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\
a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66}
\end{pmatrix}\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{pmatrix},
\]

(4)

where \(\gamma_{yz} = 2\varepsilon_{yz}\) etc. The \(a_{ij}\) are the components of the compliance tensor \(a\) for a general anisotropic medium in the borehole frame. After rotating the compliance tensor \(s\) into the borehole frame, all components of \(a\) can be nonzero. The only assumption made at this point is that \(\varepsilon_{zz} = 0\). For generalized plain strain problems very often the reduced strain coefficient \(\beta_{ij}\) is used which is defined as

\[
\beta_{ij} = a_{ij} - \frac{a_{33}a_{ij}}{a_{33}} \quad i, j = 1...6.
\]

(5)

In addition to Hooke’s law, we also require the equations of equilibrium which can be found in [see 26, Chap. 5.5]. As there are five strain components but only three displacement components, we need, in addition, two strain compatibility equations which can be found in [see 26, Chap. 2.13].
2.2 General Solution

A general solution for the stresses around a borehole in an anisotropic medium can be found by using the concept of Airy stress functions [26, Chap. 5.7]. The general expressions for the borehole-induced stresses \( \sigma_{bi} \) which can be superimposed onto the corresponding components of the far-field in-situ stress tensor in the TOH frame \( \sigma_{TOH} \) to get the borehole stress tensor \( \sigma_{BH} \) are

\[
\begin{align*}
\sigma_{xx,BH} &= \sigma_{xx,TOH} + \sigma_{xx,bi} \\
\sigma_{yy,BH} &= \sigma_{yy,TOH} + \sigma_{yy,bi} \\
\sigma_{zz,BH} &= \sigma_{zz,TOH} + \sigma_{zz,bi} \\
\tau_{xy,BH} &= \tau_{xy,TOH} + \sigma_{xy,bi} \\
\tau_{xz,BH} &= \tau_{xz,TOH} + \sigma_{xz,bi} \\
\tau_{yz,BH} &= \tau_{yz,TOH} + \sigma_{yz,bi} \\
\end{align*}
\]

where \( \phi'_i \) (i = 1, 2, 3) are the spatial derivatives of three analytic functions which are defined by [18]. The problem can now be fully solved by finding solutions to the analytic functions \( \phi'_i \) by applying the correct boundary conditions in the far-field as well as on the borehole wall. This is described in detail by [16] and [18].

From equation (4), it is obvious due to the generalized plain strain assumption that the axial stress \( \sigma_{zz,BH} \) can be written as

\[
\sigma_{zz,BH} = \sigma_{zz,TOH} - \frac{1}{a_{33}} \left( a_{31} \sigma_{xx,bi} + a_{32} \sigma_{yy,bi} + a_{34} \tau_{yz,bi} + a_{35} \tau_{xz,bi} + a_{36} \tau_{xy,bi} \right). \tag{7}
\]

\( \sigma_{zz,BH} \) can be computed after the other induced stress components are obtained from equation (6).

2.3 Special cases of anisotropy

There are several cases to consider in which a degeneration of the general solution happens [16]: (i) orthorhombic medium with one plane of elastic symmetry perpendicular to the hole axis (the two other planes being parallel to it), (ii) transverse isotropic medium with the plane of isotropy striking parallel to the hole axis, (iii) transverse isotropic medium with the plane of isotropy perpendicular to the hole axis and (iv) isotropic medium. This is the case when the system of coupled differential equations in [see 18, Eq.3.3.3] is decoupled \((L_3 = 0)\). Thus the elastic tensor takes the following form

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\
a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & a_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{66}
\end{pmatrix}. \tag{8}
\]

This is the elastic tensor for an orthorhombic material with nine different elastic components (three Young’s moduli, three shear moduli and three Poisson’s ratios).

Now the problem is decoupled into two problems: a plain strain problem involving \( \sigma_{xx}, \sigma_{yy} \) and \( \tau_{xy} \) and a longitudinal shear problem involving \( \tau_{xz} \) and \( \tau_{yz} \).

After the decoupling, it also follows that the resulting characteristic equation for elastic media in which there is no azimuthal symmetry ((iii) and (iv)), is [18, eq.3.6.4]

\[
l_4 = \beta_1 (\mu^2 + 2 \mu^2 + 1) = 0 \tag{9}
\]

\[
l_2 = \beta_{55} (\mu^2 + 1) = 0. \tag{10}
\]

It has been stated by [18] and [19] that a solution for this problem can only be found for the above cases (i) and (ii), and that for cases (iii) and (iv), no solution can be found due to coincident roots which result in singularities. We would like to show here that this is not the case, and the solution actually works for any symmetry.

The roots of this characteristic equation are identical to \( \mu = i \) only if \( \beta_{11} = \beta_{55} \neq 0 \) [18, eq.3.6.4]. For that case the solution would have singularities due to coinciding roots. We investigated this assumption about the identity
of the reduced strain coefficients $\beta_{11}$ and $\beta_{55}$. The identity $\beta_{11} = \beta_{55}$ would imply, for transversely isotropic media, the relationships $\nu^2 = E_v(1/E_h - 1/G_v)$, i.e. $G_v \geq E_h$ and for isotropic media $\nu^2 = 1 - E/G$, i.e. $G \geq E$; both inequalities are unlikely for realistic rocks. Furthermore, we have calculated the reduced strain coefficients from an extensive ultrasonic velocity and anisotropy data set published by [11]. The data set includes a wide variety of sedimentary rocks (reservoir rocks and seals) from oil and gas fields around the world. We have taken the entire data set into consideration, consisting of 17 brine-saturated shale samples, one gas and brine-saturated coal sample, 8 brine-saturated sands, 12 gas-saturated sands, 32 gas-saturated carbonate samples and 25 brine-saturated carbonate samples. In Figure 3, we have plotted $\beta_{11}$ against $\beta_{55}$ for the entire data set. The dashed line represents the assumption that $\beta_{11} = \beta_{55}$. One can see that $\beta_{11}$ is never equal to $\beta_{55}$, and in the case of the shales $\beta_{55}$ can be up to three times larger than $\beta_{11}$. It is important to note that the numerical instability mentioned by [18] only occurs when $\beta_{11}$ is exactly equal to $\beta_{55}$.

![Figure 3: Reduced strain coefficients $\beta_{11}$ and $\beta_{55}$ for various shales, sandstones and carbonates (gas and brine saturated); obtained from experimental ultrasonic velocities [11].](image)

In the following section we show that the Amadei solution is in agreement with our numerical model for the special cases (iii) and (iv) as the condition $\beta_{11} = \beta_{55}$ is never fulfilled for any realistic combination of elastic parameters. A close inspection of the values of the roots of the characteristic equations for the isotropic cases shows that these roots are very close to the imaginary number $i$ but always have a small real part and an imaginary part slightly different from $i$. We can only speculate that the reason why Ong found limitations to the Amadei solution is that single-precision computation would yield $\mu = i$ when double-precision computation would give $\mu \neq i$. This small difference always gives a stable result for all the special cases considered provided our computations are done with double precision variables.

### 3 Numerical Method

In order to validate the analytical solution described above, we utilized the commercial finite element solver COMSOL [27]. We have developed a 3D finite element model which computes the stresses around a deviated borehole in an arbitrarily anisotropic medium. This problem cannot be modeled as a 2D problem as for 2D the out-of-plane stress components would be neglected, and the far-field boundary condition acting parallel to the borehole could not be applied. The model also cannot be reduced to one quadrant using symmetries in the geometry as is usually done for isotropic simulations because for an arbitrary anisotropic material the symmetries are not known a priori. Thus, in order to compute borehole stresses, we use a cubical model with an edge length of 5 m. The borehole is placed in the center of the model with a radius of 0.1 m. The structured mesh is illustrated in Figure 4 and consists of 11400 hexahedral elements with 294840 degrees of freedom. The computational time for such a model is around three minutes on a workstation with a 3 GHz processor and 32 GB RAM. The mesh is refined in a cylindrical region around the borehole, which has a radius of six borehole diameters (0.6 m). In this region, the mesh density is increased linearly. The size of the innermost elements is 5 times smaller than the size of the outermost elements of the cylindrical region. The Dirichlet boundary conditions are applied as stresses. As stated before, we assumed that the coordinate system of our model is aligned with the in-situ stress $\sigma$ and therefore represents the NEV-frame. This means that for a vertical borehole only normal stress components are applied to the boundaries of the model. The linear solver MUMPS [28] is used.

![Figure 4: View of the mesh of the 3D numerical model and a 2D cross section perpendicular to the borehole axis.](image)
Figure 5: Comparison of borehole stresses computed using finite element and Kirsch solutions for a vertical well in an isotropic medium ($\nu = 0.32$), and a stress field $\sigma_V = 30$ MPa, $\sigma_H = 20$ MPa, $\sigma_h = 10$ MPa with $P_w = 5$ MPa. The radial, tangential and axial stresses are plotted as functions of radial position away from the borehole in the direction of $\sigma_h$ (borehole azimuth $\theta = 0$). The stress concentration due to the borehole abates within the densely meshed region of the numerical model.

The Kirsch solution [1] for a vertical well in an isotropic formation. In Figure 5 the borehole stresses, $\sigma_{rr}$, $\sigma_{zz}$, and $\sigma_{\theta\theta}$, from the numerical model medium are plotted together with the corresponding Kirsch solution radially away from the borehole. It shows a very good agreement between the two models. It also shows that the chosen geometry and mesh give steady results and that the size of the model is big enough to avoid that the stress concentration is influenced by the size of the model.

In order to keep the same geometry and the same mesh for all models of interest we decided to mimic the borehole deviation by applying appropriate stress boundary conditions on the surface of the block Figure 6-a and 6-b. This can be achieved by computing the components of the rotated stress tensor for a given borehole deviation and azimuth. The resulting stress tensor can have up to six different components (three normal and three shear stresses). These components are applied as stress boundary conditions on the surface of the block.

The components of the elastic tensor are influenced by the material symmetry and orientation as well as the borehole orientation. The Bond transformation gives the full elastic tensor for a given material dip angle and azimuth (for example for a TTI medium) as well as borehole deviation and azimuth. We use the rotated tensor in the finite element simulation in order to mimic the material orientation relative to the borehole (Figure 6-a and 6-b).

4 Validation of the Amadei solution

4.1 Boundary Conditions and Material Parameters

Figure 6: Schematic of the stress and elastic tensors transformations required to set up the boundary conditions of a “borehole centric” finite element model mimicking the situation of an arbitrarily oriented borehole in an arbitrarily oriented anisotropic medium. Subfigure (a) depicts the in-situ conditions and (b) the “borehole centric” finite element model with appropriate boundary conditions.

Figure 7: Borehole stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{\theta z}$ around the borehole wall from the analytical (lines) and numerical finite element (dots) solutions for a vertical well in an isotropic medium with Poisson’s ratio $\nu = 0.079$.
We validated the Amadei solution for isotropic, VTI and TTI media in an normal stress regime ($\sigma_v = 30$ MPa, $\sigma_H = 20$ MPa and $\sigma_h = 10$ MPa) and a wellbore pressure of $P_w = 5$ MPa.

For the VTI medium we chose a representative shale (Sample E5) from the experimental data collection published by [11]. The values for the five elastic parameters are listed in Table 1. The elastic tensor for the TTI medium is defined by tilting the VTI medium via a bond transformation with a material dip of $\beta_D = 30^\circ$ and a material azimuth of $\beta_A = 30^\circ$. The various borehole and material orientations are summarized in Table 2.

<table>
<thead>
<tr>
<th>Geomechanics</th>
<th>Geophysics</th>
<th>Stiffness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_h = 31.17$ GPa</td>
<td>$V_p0 = 3340$ m/s</td>
<td>$C_{11} = 45.2$ GPa</td>
</tr>
<tr>
<td>$E_v = 15.42$ GPa</td>
<td>$V_s0 = 1675$ m/s</td>
<td>$C_{33} = 28$ GPa</td>
</tr>
<tr>
<td>$\nu_h = 0.079$</td>
<td>$\epsilon = 0.3065$</td>
<td>$C_{44} = 7.05$ GPa</td>
</tr>
<tr>
<td>$\nu_v = 0.32$</td>
<td>$\gamma = 0.234$</td>
<td>$C_{66} = 14.4$ GPa</td>
</tr>
<tr>
<td>$G_w = 7.05$ GPa</td>
<td>$\delta = 0.5244$</td>
<td>$C_{13} = 19.67$ GPa</td>
</tr>
</tbody>
</table>

Table 1: Anisotropic elastic properties for the VTI and TTI models reported in several notations: rock mechanics (Young’s and shear moduli and Poisson’s ratio), geophysics (velocities and $\epsilon$’s parameters) and stiffness. Values are from shale E5 from [11] where the density is $2.535$ g/cm$^3$ and porosity is 5%.

4.2 Validation Results

Figures 7 to 9 summarize the results of the borehole stresses at the borehole wall and around it for the seven chosen models (Table 3) using both the numerical model and the Amadei solution. We observe that the agreement between the two solutions is excellent. The degeneration of the Amadei solution when a plane of elastic symmetry is perpendicular to the borehole axis (Figure 8a) as described by [19] was not observed. In particular, Figure 7 shows that the Amadei solution is also valid for isotropic media as expected from the theoretical section. Furthermore, we obtained identical results for any borehole orientation in an isotropic medium using the Kirsch and the Amadei solutions; we chose not to plot them here as the curves are indistinguishable. Furthermore, for all these special cases and for arbitrary borehole and material orientations we have found no limitations of the Amadei solution and therefore have validated this closed-form solution up to TI anisotropy.

5 Discussion

Every borehole stability analysis has to address two essential questions; first, what is the stress distribution around the borehole and away from it and, second, at what stress...
Figure 9: Borehole stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{\theta z}$ around the borehole wall from the analytical (lines) and numerical finite element (dots) solutions for a TTI medium (TI plane has $30^\circ$ dip angle and azimuth) and well orientations: (a) vertical, (b) deviated and (c) horizontal. Corresponding material properties are given in Table 1 and orientations angles in Table 2.

Table 2: Summary of the models used for validation with the orientations of the well ($\alpha_D$ and $\alpha_A$) and the TI medium ($\beta_D$ and $\beta_A$) as well as the figure numbers where the results are displayed.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha_D$</th>
<th>$\alpha_A$</th>
<th>$\beta_D$</th>
<th>$\beta_A$</th>
<th>Figure</th>
</tr>
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<tr>
<td>ISO vert</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>VTI vert</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>VTI dev</td>
<td>45</td>
<td>45</td>
<td>0</td>
<td>0</td>
<td>8(a)</td>
</tr>
<tr>
<td>VTI hor</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8(c)</td>
</tr>
<tr>
<td>TTI vert</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>30</td>
<td>8(b)</td>
</tr>
<tr>
<td>TTI dev</td>
<td>45</td>
<td>45</td>
<td>30</td>
<td>30</td>
<td>9(a)</td>
</tr>
<tr>
<td>TTI hor</td>
<td>90</td>
<td>0</td>
<td>30</td>
<td>30</td>
<td>9(c)</td>
</tr>
</tbody>
</table>
pled problems and first steps have been taken in this direction [32], although numerical solutions have also been developed [20]. As soon as the borehole starts to fail and time dependent processes occur, the stresses will be redistributed and static elastic solutions such as the one presented here are no longer valid. Dynamic stress redistributions have to be analyzed using modern numerical methods.

6 Conclusion

We have presented a rigorous validation of the analytical Amadei solution for the stress concentration around arbitrarily orientated boreholes in general anisotropic elastic media. First, the review of the theory provided analytical insights that the solution does indeed contain all special cases of symmetry contrary to previous understanding provided the reduced strain coefficients $\beta_{11}$ and $\beta_{55}$ are not equal. Next, we have shown from theoretical considerations and published experimental data that the $\beta_{11}$ and $\beta_{55}$ are not equal for realistic rocks. Second, we developed an efficient hybrid analytical-numerical workflow using a 3D finite element elastic model that circumvents the need to rebuild and remesh the model for every borehole and material orientation. Third, we have shown that the borehole stresses computed from the numerical model and the analytical solution match almost perfectly for different borehole orientations (vertical, deviated and horizontal) and for several cases involving isotropic and transverse isotropic symmetries. It is concluded that the analytical Amadei solution is valid with no restrictions on the borehole orientation or elastic anisotropy symmetry provided the generalized plane strain assumption is met.

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