Ostrowski type inequalities for functions whose derivatives are $h$-convex in absolute value

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Abstract

Some new inequalities of Ostrowski type for functions whose derivatives are $h$-convex in modulus are given. Applications for midpoint inequalities are provided as well.

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1 Introduction

1.1 Ostrowski Type Inequalities

Comparison between functions and integral means are incorporated in Ostrowski type inequalities as follows.

The first result in this direction is due to Ostrowski [38].

Theorem 1.1. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $(a,b)$ with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then

$$|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \left( \frac{x-a}{b-a} \right)^2 \right] (b-a) M \quad (1.1)$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following results for absolutely continuous functions hold (see [29] – [31]).

Theorem 1.2. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:


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\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \begin{cases} 
\left[ \frac{1}{4} + \left( \frac{x-a+b}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty [a,b] ; \\
\frac{1}{(\alpha+1)^{\frac{1}{r}}} \left[ \left( \frac{x-a}{b-a} \right)^{\alpha+1} + \left( \frac{b-x}{b-a} \right)^{\alpha+1} \right]^{\frac{1}{\alpha}} \times (b-a)^{\frac{1}{\alpha}} \|f'\|_\beta & \text{if } f' \in L_\beta [a,b], \\
\left[ \frac{1}{2} + \left| \frac{x-a+b}{b-a} \right| \right] \|f'\|_1 & \text{if } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \alpha > 1; 
\end{cases}
\]

where \(\|\cdot\|_{[a,b],r} (r \in [1, \infty])\) are the usual Lebesgue norms on \(L_r [a,b]\), i.e., we recall that

\[
\|g\|_{[a,b],\infty} := \text{ess sup}_{t \in [a,b]} |g(t)|
\]

and

\[
\|g\|_{[a,b],r} := \left( \int_a^b |g(t)|^r \, dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).
\]

The constants \(\frac{1}{4}\), \(\frac{1}{(\alpha+1)^{\frac{1}{r}}}\) and \(\frac{1}{2}\) respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [33] on choosing \(n = 1\) and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that \(f\) is Hölder continuous, then one may state the result (see for instance [21] and the references therein for earlier contributions):

**Theorem 1.3.** Let \(f : [a,b] \to \mathbb{R}\) be of \(r-H\)-Hölder type, i.e.,

\[
|f(x) - f(y)| \leq H |x-y|^r, \quad \text{for all } x, y \in [a,b],
\]

where \(r \in (0,1]\) and \(H > 0\) are fixed. Then, for all \(x \in [a,b]\), we have the inequality:

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \frac{H}{r+1} \left[ \left( \frac{b-x}{b-a} \right)^{r+1} + \left( \frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r. \quad (1.4)
\]

The constant \(\frac{1}{r+1}\) is also sharp in the above sense.

Note that if \(r = 1\), i.e., \(f\) is Lipschitz continuous, then we get the following version of Ostrowski’s inequality for Lipschitzian functions (with \(L\) instead of \(H\)) (see for instance [13])

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \left( \frac{x-a+b}{2b-a} \right)^2 \right] (b-a) \frac{L}{2}, \quad (1.5)
\]
where $x \in [a,b]$. Here the constant $\frac{1}{2}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [15]).

**Theorem 1.4.** Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation and denote by $V_a^b(f)$ its total variation. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{2} + \left| \frac{x-a+b}{b-a} \right| \right] V_a^b(f)$$

for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about $f$, i.e., $f$ is monotonically increasing, then the inequality (1.6) may be improved in the following manner [12] (see also the monograph [28]).

**Theorem 1.5.** Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a,b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{b-a} \left\{ (2x-(a+b)) f(x) + \int_a^b \text{sgn}(t-x) f(t) \, dt \right\}$$

$$\leq \frac{1}{b-a} \left\{ (x-a)[f(x)-f(a)] + (b-x)[f(b)-f(x)] \right\}$$

$$\leq \left[ \frac{1}{2} + \left| \frac{x-a+b}{b-a} \right| \right] [f(b) - f(a)].$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [18]:

**Theorem 1.6.** Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in (a,b)$ one has the inequality

$$\frac{1}{2} \left[ (b-x)^2 f'_+ (x) - (x-a)^2 f'_- (x) \right]$$

$$\leq \int_a^b f(t) \, dt - (b-a) f(x)$$

$$\leq \frac{1}{2} \left[ (b-x)^2 f'_- (b) - (x-a)^2 f'_+ (a) \right].$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x = a$ or $x = b$.

For other Ostrowski’s type inequalities for the Lebesgue integral, see [3]-[13] and [19].

Inequalities for the Riemann-Stieltjes integral may be found in [14], [16] while the generalization for isotonic functionals was provided in [17].

For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [20]
1.2 The Case of Derivatives that are Convex in Modulus

In [17], the author pointed out the following identity in representing an absolutely continuous function. Due to the fact that we use it throughout the paper we give here a short proof.

**Lemma 1.7.** Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous function on \([a, b]\). Then for any \( x \in [a, b] \), one has the equality:

\[
f(x) = \frac{1}{b - a} \int_a^b f(t) \, dt + \frac{1}{b - a} \int_a^b (x - t) \left( \int_0^1 f'[(1 - \lambda) x + \lambda t] \, d\lambda \right) \, dt. \tag{1.9}
\]

**Proof.** For any \( t, x \in [a, b] \), \( x \neq t \), one has

\[
\frac{f(x) - f(t)}{x - t} = \frac{1}{x - t} \int_t^x f'(u) \, du = \int_0^1 f'[(1 - \lambda) x + \lambda t] \, d\lambda,
\]

showing that

\[
f(x) = f(t) + (x - t) \int_0^1 f'[(1 - \lambda) x + \lambda t] \, d\lambda \tag{1.10}
\]

for any \( t, x \in [a, b] \).

If we integrate (1.10) over \( t \) on \([a, b]\) and divide by \((b - a)\), we deduce the desired identity (1.9).

Q.E.D.

Using the above lemma the following result can be pointed out improving Ostrowski’s inequality [4].

**Theorem 1.8.** Let \( f : [a, b] \to \mathbb{C} \) be an absolutely continuous function on \([a, b]\) so that \(|f'|\) is convex on \((a, b)\).

(i) If \( f' \in L_\infty[a, b] \), then for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2} \left[ \frac{1}{4} + \left( \frac{x - a + b}{2b - 2a} \right)^2 \right] (b - a) \| f' \|_\infty. \tag{1.11}
\]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller quantity.

(ii) If \( f' \in L_p[a, b] \), \( p > 1 \), then for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2 (q + 1)^\frac{1}{q}} \left[ \left( \frac{b - x}{b - a} \right)^{q+1} + \left( \frac{x - a}{b - a} \right)^{q+1} \right]^\frac{1}{q} (b - a)^\frac{2}{q} \| f' \|_p. \tag{1.12}
\]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller quantity.
(iii) If \( f' \in L_1[a, b] \), then for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| 
\leq \frac{1}{2} \left[ \frac{1}{2} + \frac{x - \frac{a+b}{2}}{b-a} \right] \left[ (b-a) |f'(x)| + \|f'\|_1 \right].
\]

In order to extend this result for other classes of functions, we need the following preparatory section.

2 \( h \)-Convex Functions

2.1 Some Definitions

We recall here some concepts of convexities that are well known in the literature. Let \( I \) be an interval in \( \mathbb{R} \).

**Definition 2.1** ([32]). We say that \( f : I \to \mathbb{R} \) is a Godunova-Levin function or that \( f \) belongs to the class \( Q(I) \) if \( f \) is non-negative and for all \( x, y \in I \) and \( t \in (0, 1) \) we have

\[
f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y).
\]

Some further properties of this class of functions can be found in [24], [25], [27], [37], [40] and [41]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

**Definition 2.2** ([27]). We say that a function \( f : I \to \mathbb{R} \) belongs to the class \( P(I) \) if it is nonnegative and for all \( x, y \in I \) and \( t \in [0, 1] \) we have

\[
f(tx + (1-t)y) \leq f(x) + f(y).
\]

Obviously \( Q(I) \) contains \( P(I) \) and for applications it is important to note that also \( P(I) \) contain all nonnegative monotone, convex and quasi convex functions, i.e. nonnegative functions satisfying

\[
f(tx + (1-t)y) \leq \max\{f(x), f(y)\}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

For some results on \( P \)-functions see [27] and [39] while for quasi convex functions, the reader can consult [26].

**Definition 2.3** ([6]). Let \( s \) be a real number, \( s \in (0, 1] \). A function \( f : [0, \infty) \to [0, \infty) \) is said to be \( s \)-convex (in the second sense) or Breckner \( s \)-convex if

\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]

for all \( x, y \in [0, \infty) \) and \( t \in [0, 1] \).
For some properties of this class of functions see [1], [2], [6], [7], [22], [23], [34], [35] and [43].

In order to unify the above concepts, S. Varošanec introduced the concept of $h$-convex functions as follows.

Assume that $I$ and $J$ are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions $h$ and $f$ are real non-negative functions defined in $J$ and $I$, respectively.

**Definition 2.4 ([46]).** Let $h : J \to [0, \infty)$ with $h$ not identical to 0. We say that $f : I \to [0, \infty)$ is an $h$-convex function if for all $x, y \in I$ we have

$$f(t x + (1 - t) y) \leq h(t) f(x) + h(1 - t) f(y)$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [46], [5], [36], [44], [42] and [45].

### 2.2 Inequalities of Hermite-Hadamard Type

In [42] the authors proved the following Hermite-Hadamard type inequality for integrable $h$-convex functions.

**Theorem 2.5.** Assume that $f : I \to [0, \infty)$ is an $h$-convex function, $h \in L[0, 1]$ and $f \in L[a,b]$ where $a, b \in I$ with $a < b$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq [f(a) + f(b)] \int_0^1 h(t) \, dt.$$  

(HH)

If we write (HH) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write it for the case of $P$-type functions, i.e., $h(t) = 1$, then we get the inequality

$$\frac{1}{2} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq f(a) + f(b),$$  

(2.1)

provided $f \in L[a,b]$, that has been obtained in [27].

If $f$ is integrable on $[a,b]$ and Breckner $s$-convex on $[a,b]$, for $s \in (0, 1)$, then by taking $h(t) = t^s$ in (HH) we get

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{s + 1}$$  

(2.2)

that was obtained in [22].

Since for the case of Godunova-Levin class of function we have $h(t) = \frac{1}{t}$, which is not Lebesgue integrable on $(0, 1)$, we cannot apply the left inequality in (HH).

We can introduce now another class of functions.

**Definition 2.6.** We say that the function $f : I \to [0, \infty)$ is of $s$-Godunova-Levin type, with $s \in [0, 1]$, if

$$f(t x + (1 - t) y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1 - t)^s} f(y),$$  

(2.3)

for all $t \in (0, 1)$ and $x, y \in I$. 
We observe that for \( s = 0 \) we obtain the class of \( P \)-functions while for \( s = 1 \) we obtain the class of Godunova-Levin. If we denote by \( Q_s (I) \) the class of \( s \)-Godunova-Levin functions defined on \( I \), then we obviously have

\[
P (I) = Q_0 (I) \subseteq Q_{s_1} (I) \subseteq Q_{s_2} (I) \subseteq Q_1 (I) = Q (I)
\]

for \( 0 \leq s_1 \leq s_2 \leq 1 \).

We have the following Hermite-Hadamard type inequality.

**Theorem 2.7.** Assume that the function \( f: I \to [0, \infty) \) is of \( s \)-Godunova-Levin type, with \( s \in [0, 1) \). If \( f \in L [a, b] \) where \( a, b \in I \) and \( a < b \), then

\[
\frac{1}{2s+1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f (t) dt \leq \frac{f (a) + f (b)}{1 - s}.
\]  

(2.4)

We notice that for \( s = 1 \) the first inequality in (2.4) still holds and was obtained for the first time in [27].

### 3 Inequalities for Functions Whose Derivatives are \( h \)-Convex in Modulus

#### 3.1 The Case of \(|f'|\) is \( h \)-Convex

The following result holds:

**Theorem 3.1.** Let \( f: [a, b] \to \mathbb{C} \) be an absolutely continuous function on \([a, b]\) so that \(|f'|\) is \( h \)-convex on \((a, b)\) with \( h \in L [0, 1] \).

(i) If \( f' \in L_\infty [a, b] \), then for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - a + b}{b - a} \right)^2 \right] (b - a) [||f'(x)|| + ||f'||_\infty] \int_0^1 h (t) dt.
\]  

(3.1)

(ii) If \( f' \in L_p [a, b] \), \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), then for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{1}{(q + 1)^{\frac{1}{q}}} \left[ \left( \frac{b - x}{b - a} \right)^{q+1} + \left( \frac{x - a}{b - a} \right)^{q+1} \right] \frac{1}{q} \times (b - a)^{\frac{1}{q}} [||f'(x)|| + ||f'||_p] \int_0^1 h (t) dt.
\]

(iii) If \( f' \in L_1 [a, b] \), then for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{2} + \left| \frac{x - a + b}{b - a} \right| \right] \times [(b - a) [||f'(x)|| + ||f'||_1] \int_0^1 h (t) dt.
\]  

(3.3)
Proof. (i). Using (1.9) and taking the modulus, we have
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| |f'((1-\lambda) x + \lambda t)| d\lambda dt
\]
Utilizing the $h$-convexity of $|f'|$ we have
\[
K \leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| [h(1-\lambda) |f'(x)| + h(\lambda) |f'(t)|] d\lambda dt
\]
\[
= \frac{1}{b-a} \int_a^b |x-t| \left[ |f'(x)| \int_0^1 h(1-\lambda) d\lambda + |f'(t)| \int_0^1 h(\lambda) d\lambda \right] dt
\]
\[
= \frac{1}{b-a} \int_0^1 h(\lambda) d\lambda \sup_{t \in [a,b]} [ |f'(x)| + |f'(t)|] \int_a^b |x-t| d\lambda
\]
\[
= \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] [||f'(x)|| + ||f'||_\infty] \int_0^1 h(\lambda) d\lambda
\]
\[
= \left[ \frac{1}{4} + \left( \frac{x-a+b}{2} \right)^2 \right] (b-a) [||f'(x)|| + ||f'||_\infty] \int_0^1 h(\lambda) d\lambda,
\]
for any $x \in [a,b]$, and the inequality (3.1) is proved.

(ii). As above, we have
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \int_a^b |x-t| [||f'(x)|| + ||f'(t)||] dt := M(x) \int_0^1 h(\lambda) d\lambda.
\]
Using Hölder’s integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we get that
\[
M(x) \leq \frac{1}{b-a} \left( \int_a^b |x-t|^q dt \right)^{\frac{1}{q}} \left( \int_a^b (||f'(x)|| + ||f'(t)||)^p dt \right)^{\frac{1}{p}}
\]
\[
= \frac{1}{b-a} \left[ \frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1} \right] ||f'(x)|| + ||f'||_p
\]
and the inequality (3.2) is proved.
(iii). We also have that

\[
M(x) \leq \sup_{t \in [a,b]} |x - t| \frac{1}{b-a} \int_a^b \left[ |f'(x)| + |f'(t)| \right] dt
\]

\[
= \frac{1}{b-a} \max (x - a, b - x) \left[ (b - a) |f'(x)| + \int_a^b |f'(t)| dt \right]
\]

\[
= \left[ \frac{1}{2} + \frac{x - a + b}{b - a} \right] \left[ (b - a) |f'(x)| + \|f'\|_1 \right]
\]

and the inequality (3.3) is proved. Q.E.D.

The following particular case is interesting.

**Corollary 3.2.** With the assumptions of Theorem 3.1, we have the midpoint inequality

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} (b - a) \left[ \left| f' \left( \frac{a+b}{2} \right) \right| + \|f'\|_{\infty} \right] \int_0^1 h(t) dt,
\]

provided \( f' \in L_{\infty}[a,b] \).

If \( f' \in L_p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), then, we have,

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} (b - a)^{\frac{1}{q}} \left( \int_a^b \left[ \left| f' \left( \frac{a+b}{2} \right) \right| + |f'(t)| \right] p dt \right)^{\frac{1}{p}} \int_0^1 h(t) dt.
\]

If \( f' \in L_1[a,b] \), then

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} \left[ (b - a) \left| f' \left( \frac{a+b}{2} \right) \right| + \int_a^b |f'(t)| dt \right] \int_0^1 h(t) dt.
\]

**Remark 3.3.** We observe that if \( |f'| \) is convex on \((a,b)\), then Theorem 3.1 reduces to Theorem 1.8.

Assume that \( |f'| \) is Breckner \( s \)-convex on \([a,b] \), for \( s \in (0,1) \).
(a) If $f' \in L_\infty[a,b]$, then for any $x \in [a,b],$

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \frac{1}{s+1} \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'(x)\| + \|f''\|_\infty.
$$

(b) If $f' \in L_\infty[a,b]$, then for any $x \in [a,b],$

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \frac{1}{1-s} \left[ \frac{1}{2} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'(x)\| + \|f''\|_\infty.
$$

(b) If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a,b],$

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \frac{1}{(s+1)(q+1)^{\frac{1}{q}}} \left[ \left( \frac{b-x}{b-a} \right)^{q+1} + \left( \frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a) \|f'(x)\| + \|f''\|_p.
$$

(b) If $f' \in L_1[a,b]$, then for any $x \in [a,b],$

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \frac{1}{1-s} \left[ \frac{1}{2} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'(x)\| + \|f''\|_\infty.
$$

Assume that $|f'|$ is of s-Godunova-Levin type, with $s \in [0,1)$.
3.2 The Case of $|f'|^p$ is $h$-Convex

The following result also holds:

**Theorem 3.4.** Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|^p$ with $p > 1$ is $h$-convex on $(a, b)$ and $h \in L[0, 1]$.

(i) If $f' \in L_{\infty}[a, b]$, then for any $x \in [a, b]$,

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \left[ \frac{1}{4} + \left( \frac{x - a + b}{b-a} \right)^2 \right] (b-a) \times \left[ |f'(x)|^p + \|f'||^p_{\infty} \right]^{1/p} \left( \int_0^1 h(t) dt \right)^{1/p}.
$$

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{1/q}} \left[ \left( \frac{b-x}{b-a} \right)^{q+1} + \left( \frac{x-a}{b-a} \right)^{q+1} \right]^{1/q} \times \left[ (b-a) |f'(x)|^p + \|f'||^p_{p} \right]^{1/p} \left( \int_0^1 h(t) dt \right)^{1/p}.
$$

(iii) If $f' \in L_p[a, b]$, then for any $x \in [a, b]$,

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq \left[ \frac{1}{2} + \left| \frac{x - a + b}{b-a} \right| \right] \times \left( (b-a) |f'(x)|^p + \|f'||^p_{p} \right)^{1/p} \left( \int_0^1 h(t) dt \right)^{1/p}.
$$

**Proof.** As in the proof of Theorem 3.1 we have

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| = \frac{1}{b-a} \left| \int_a^b \int_0^1 (x-t) f'[(1-\lambda)x+\lambda t] \, d\lambda dt \right|
$$

$$
\leq \frac{1}{b-a} \int_a^b \left| x - t \right| \left( \int_0^1 |f'[(1-\lambda)x+\lambda t]| \, d\lambda \right) \, dt := K
$$

for any $x \in [a, b]$.

By Hölder’s integral inequality we have

$$
\int_0^1 |f'[(1-\lambda)x+\lambda t]| \, d\lambda \leq \left( \int_0^1 1^q \, d\lambda \right)^{1/q} \left( \int_0^1 |f'[(1-\lambda)x+\lambda t]|^p \, d\lambda \right)^{1/p}
$$

$$
= \left( \int_0^1 |f'[(1-\lambda)x+\lambda t]|^p \, d\lambda \right)^{1/p}
$$
for any \( x \in [a, b] \), where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p > 1 \).

Since \( |f''|^p \) is \( h \)-convex on \( (a, b) \) with \( h \in L[0, 1] \), then

\[
\int_{0}^{1} |f'[(1 - \lambda) x + \lambda t]|^p d\lambda \leq \left[ |f'(x)|^p + |f'(t)|^p \right] \int_{0}^{1} h(\lambda) d\lambda,
\]

for any \( x \in [a, b] \).

Therefore

\[
K \leq \frac{1}{b - a} \left( \int_{0}^{1} h(\lambda) d\lambda \right)^{1/p} \int_{a}^{b} \left| x - t \right| \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} dt
\]

(3.16)

for any \( x \in [a, b] \).

(i). Now, if \( f' \in L_{\infty}[a, b] \) then

\[
\int_{a}^{b} \left| x - t \right| \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
\leq \text{ess sup}_{t \in [a, b]} \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} \int_{a}^{b} \left| x - t \right| dt \\
= \left[ |f'(x)|^p + \|f''\|_{\infty}^p \right]^{1/p} \frac{1}{2} \left( (b - a)^2 + (b - x)^2 \right)
\]

for any \( x \in [a, b] \), and utilizing (3.16), the inequality (3.13) is proved.

(ii). If \( f' \in L_p[a, b], p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), then by Hölder’s inequality we have

\[
\int_{a}^{b} \left| x - t \right| \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
\leq \left( \int_{a}^{b} \left| x - t \right|^q dt \right)^{1/q} \left( \int_{a}^{b} \left[ |f'(x)|^p + |f'(t)|^p \right]^{p} dt \right)^{1/p} \\
= \left( \frac{b - x)^{q+1} + (x - a)^{q+1}}{q + 1} \right)^{1/q} \left[ (b - a) |f'(x)|^p + \|f''\|_{p}^p \right]^{1/p} \\
= \frac{(b - a)^{1+\frac{q}{p}}}{(q + 1)^{1/q}} \left[ \left( \frac{b - x}{b - a} \right)^{q+1} + \left( \frac{x - a}{b - a} \right)^{q+1} \right]^{1/q} \times \left[ (b - a) |f'(x)|^p + \|f''\|_{1}^p \right]^{1/p}
\]

for any \( x \in [a, b] \), and by (3.16) we deduce the desired inequality (3.14).
(iii). If \( f' \in L_p[a,b] \), then by Hölder’s inequality we also have

\[
\int_a^b |x-t| \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
\leq \sup_{t \in [a,b]} |x-t| \int_a^b \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
= \max \{x-a, b-x\} \int_a^b \left[ |f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
= (b-a) \left[ \frac{1}{2} + \frac{x-a+b}{b-a} \right] \left[ |f'(x)|^p + |f'|^p \right]^{1/p} \\
\leq (b-a) \left[ \frac{1}{2} + \frac{x-a+b}{b-a} \right] \left( \int_a^b \left[ |f'(x)|^p + |f'(t)|^p \right] dt \right)^{1/p} \\
= (b-a) \left[ \frac{1}{2} + \frac{x-a+b}{b-a} \right] \left( (b-a) |f'(x)|^p + \|f'\|_p^p \right)^{1/p}
\]

for any \( x \in [a,b] \). Q.E.D.

The following midpoint type inequalities are of interest.

**Corollary 3.5.** With the assumptions of Theorem 3.4, we have the inequality

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} (b-a) \left[ |f' \left( \frac{a+b}{2} \right) |^p + \|f'\|_p^p \right]^{1/p} \left( \int_0^1 h(t) dt \right)^{1/p},
\]

provided \( f' \in L_{\infty}[a,b] \).

If \( f' \in L_p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), then we have

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2 (q+1)^{1/q}} (b-a)^{1/q} \times \left( (b-a) |f' \left( \frac{a+b}{2} \right) |^p + \|f'\|_p^p \right)^{1/p} \left( \int_0^1 h(t) dt \right)^{1/p}.
\]
If \( f' \in L^p[a,b] \), then

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} \left\| f' \left(\frac{a+b}{2}\right) \right\|^p + \left\| f'' \right\|^p \left( \int_0^1 h(t) \ dt \right)^{1/p} \\
\leq \frac{1}{2} \left( b-a \right) \left( f' \left(\frac{a+b}{2}\right) \right)^p + \left\| f'' \right\|^p \left( \int_0^1 h(t) \ dt \right)^{1/p}.
\]

\[(3.19)\]

**Remark 3.6.** The interested reader can state the corresponding particular inequalities for different \( h \)-convex functions. However the details are omitted.

**References**


[16] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_{a}^{b} f(t) \, du(t)$ where $f$ is of Hölder type and $u$ is of bounded variation and applications, *J. KSIAM*, 5(1) (2001), 35-45.


