Purity in compactly generated derivators and t-structures with Grothendieck hearts

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Abstract

We study t-structures with Grothendieck hearts on compactly generated triangulated categories $\mathcal{T}$ that are underlying categories of strong and stable derivators. This setting includes all algebraic compactly generated triangulated categories. We give an intrinsic characterisation of pure triangles and the definable subcategories of $\mathcal{T}$ in terms of directed homotopy colimits. For a left nondegenerate t-structure $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ on $\mathcal{T}$, we show that $\mathcal{V}$ is definable if and only if $\mathbf{t}$ is smashing and has a Grothendieck heart. Moreover, these conditions are equivalent to $\mathbf{t}$ being homotopically smashing and to $\mathbf{t}$ being cogenerated by a pure-injective partial cosilting object. Finally, we show that finiteness conditions on the heart of $\mathbf{t}$ are determined by purity conditions on the associated partial cosilting object.

1 Introduction.

Triangulated categories arising in representation theory, algebraic geometry and topology often come with the additional structure of a t-structure [6], allowing one to carry out homological algebra with respect to this t-structure. A t-structure $\mathbf{t}$ on a triangulated category $\mathcal{T}$ is a torsion pair satisfying additional properties ensuring that there exists an abelian subcategory $\mathcal{G}$ of $\mathcal{T}$, called the heart, and a cohomological functor from $\mathcal{T}$ to $\mathcal{G}$.

The question of when the heart of a t-structure is a Grothendieck category is a natural one and has been pursued by many authors using a variety of techniques. Given the scarcity of limits and colimits in $\mathcal{T}$, a necessary ingredient in solving this problem is a method for understanding how direct limits might look in $\mathcal{G}$. The various approaches to this problem tend to follow one of two general strategies:

Strategy 1: Consider $\mathcal{T}$ as a subcategory of a Grothendieck category $\mathcal{A}(\mathcal{T})$ and understand direct limits in $\mathcal{G}$ in terms of direct limits in $\mathcal{A}(\mathcal{T})$ ([3], [4], [25] and [35]).

Strategy 2: Consider $\mathcal{T}$ as the underlying category of some higher categorical structure and understand direct limits in $\mathcal{G}$ in terms of directed homotopy colimits ([24], [33]).

Strategy 1 is most effective in the setting where $\mathcal{T}$ is generated by its subcategory of compact objects $\mathcal{T}^{\mathrm{c}}$. In this case $\mathcal{A}(\mathcal{T})$ is the category Mod-$\mathcal{T}^{\mathrm{c}}$ and we have the well-developed theory of purity available to us (see, for example, [21, 5]).

In this paper we combine Strategies 1 and 2: we consider the case where $\mathcal{T}$ is compactly generated and is also the underlying category of a strong and stable derivator. By [12, Thm. 1.36] and [17], this includes all algebraic compactly generated triangulated categories. We observe that, in this setting, the strategies above are completely compatible: the image in Mod-$\mathcal{T}^{\mathrm{c}}$ of each directed homotopy colimit corresponds to an appropriate direct limit in Mod-$\mathcal{T}^{\mathrm{c}}$ (see Remark 3.5).

From this starting point, we are able to characterise important notions from the theory of purity in terms of certain homotopy colimits, which we call coherent ultrapowers, inspired

by their use in model theory; see Section 2.2 for details. These results are analogous to well-known characterisations of purity in module categories (see, for example, [30, Thm. 4.2.18, Thm. 16.1.16] and [10, Sec. 2.3]) and generalise [22, Thm. 7.5].

Theorem (Proposition 3.7, Theorem 3.11)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category and suppose that $\mathcal{T}$ is the underlying category of a strong and stable derivator. Then the following statements hold.*

1. *A triangle $\delta\colon X \to Y \to Z \to X[1]$ is a pure triangle if and only if there is some coherent ultrapower of δ that is a split triangle.*

2. *A full subcategory of $\mathcal{T}$ is definable if and only if it is closed under direct products, directed homotopy colimits and pure subobjects.*

Using the interaction between purity and homotopy colimits, we are able to show that the natural class of (left nondegenerate) t-structures with Grothendieck hearts considered via Strategy 1 (those cogenerated by pure-injective cosilting objects) coincides with the class considered via Strategy 2 (homotopically smashing t-structures). That is, we introduce the notion of partial cosilting t-structures and prove the following theorem (which specialises to the case of nondegenerate and cosilting t-structures).

Theorem (Theorem 4.6)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category and suppose that $\mathcal{T}$ is the underlying category of a strong and stable derivator. Let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a left nondegenerate t-structure on $\mathcal{T}$. Then the following statements are equivalent:*

1. *$\mathbf{t}$ is a partial cosilting t-structure with a pure-injective partial cosilting object C.*

2. *$\mathcal{V}$ is definable.*

3. *$\mathbf{t}$ is homotopically smashing.*

4. *$\mathbf{t}$ is smashing and the heart $\mathcal{G}$ is a Grothendieck category.*

In order to place this work in context, let us give a brief summary of the preceding results concerning Grothendieck hearts. In [27, 28] the authors consider t-structures in the derived category $\mathrm{D}(\mathcal{H})$ of a Grothendieck category $\mathcal{H}$ induced by torsion pairs in $\mathcal{H}$ i.e. Happel-Reiten-Smalø (HRS) t-structures [14]. They show that the heart of a HRS t-structure $\mathbf{t}$ is Grothendieck if and only if the torsion-free class in $\mathcal{H}$ is closed under direct limits. When $\mathcal{H}$ is a module category, these are exactly the torsion-free classes of the form $\mathrm{Cogen}(C)$ for cosilting modules C (see [1, Cor. 3.9]).

The question of when the heart of a t-structure is a Grothendieck category has also been considered in the context of silting theory. The t-structure induced by a large tilting module T has a Grothendieck heart exactly when T is pure-projective [4, Thm. 7.5] and the t-structure induced by a large cotilting module always has a Grothendieck heart [35]. A more general version of these results can be found in [3, Thm. 3.6, Thm. 3.7] in the context of compactly generated triangulated categories: the authors show that a nondegenerate cosmashing (respectively smashing) t-structure has a Grothendieck heart if and only if it is (co)generated by a pure-projective (respectively pure-injective) silting (respectively cosilting) object. The result related to cosilting can also be found in [26, Prop. 2] in the case where the t-structure is assumed to be smashing and cosmashing. Related to the silting t-structures is the case of compactly generated t-structures; these have been shown to have Grothendieck hearts in various settings, for example the homotopy category of a combinatorial stable model category [33, Cor. D] [7, Thm. 0.2].

Finally, in both [33] and [24], the authors consider t-structures such that the coaisle is closed under homotopy colimits, we shall refer to such t-structures as homotopically smashing. In [33, Thm. B], it is shown that for any (strong and stable) derivator this assumption is enough to ensure the heart has exact direct limits. When, in addition, the t-structure is on the homotopy category of a stable combinatorial model category, the authors prove that the heart is a Grothendieck category. In a similar vein, Lurie considers homotopically smashing t-structures with accessible aisles in the context of presentable stable ∞-categories [24, Rem. 1.3.5.23] and observes that also this implies that the heart is a Grothendieck category.

The question of when hearts satisfy various finiteness conditions has also been considered in the literature. For example in [32] HRS t-structures with locally coherent hearts are characterised and in [9] locally noetherian 1-cotilting t-structures are shown to be induced by Σ-pure-injective 1-cotilting modules. In the final section we consider the case where the heart $\mathcal{G}$ has an internal notion of purity and we investigate how the purity in $\mathcal{T}$ interacts with the purity in $\mathcal{G}$. As an application, we generalise the above results by characterising finiteness conditions on $\mathcal{G}$ in terms of purity assumptions on the corresponding partial cosilting object.

Theorem (Proposition 5.6, Proposition 5.10, Theorem 5.12)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a partial cosilting t-structure on $\mathcal{T}$ with partial cosilting object C. Then the following statements hold:*

1. *The heart $\mathcal{G}$ is locally noetherian if and only if C is Σ-pure-injective.*
2. *If C is an elementary cogenerator, then the heart $\mathcal{G}$ is locally coherent.*

The converse of (2) holds when $\mathcal{T}$ is the underlying category of a strong and stable derivator and C is contained in $\mathcal{G}$.

We end this introduction with a summary of the content of the paper. In Section 2, we introduce the basic definitions and notation related to the theory of derivators, as well as the definition of the coherent reduced products and coherent ultraproducts (see Section 2.2). The construction and proof that coherent reduced products exist is contained in Appendix B. In Section 3 we consider purity in strong and stable derivators whose underlying category is compactly generated; we prove that pure triangles can be detected using coherent ultraproducts and that definable subcategories can be characterised via closure conditions. Section 4 is concerned with t-structures whose hearts are Grothendieck categories. We introduce partial cosilting t-structures and homotopically smashing t-structures and in Theorem 4.6 we show that the left nondegenerate t-structures with these properties coincide. We end the section with an example of a t-structure satisfying the equivalent statements of Theorem 4.6. The final section is dedicated to understanding how purity in the triangulated category relates to purity in the heart. We use this to characterise finiteness conditions on the heart in terms of properties of the cosilting object.

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2 Derivators.

The main concept we will use from the theory of derivators is that of homotopy limits and colimits. Their definition is a generalisation of limits and colimits in a category $\mathcal{C}$. That is, for a small category A, the functors $\lim_A$ and colim_A arise as right and left adjoints to the constant diagram functor $\Delta_A \colon \mathcal{C} \to \mathcal{C}^A$. The diagram categories $\mathcal{C}^A$ are the values of the contravariant 2-functor $y_{\mathcal{C}}$ from the 2-category $\mathcal{C}at$ of small categories to the 2-category $\mathcal{C}AT$ of all categories defined on objects by $A \mapsto \mathcal{C}^A$. Accordingly, we define a **prederivator** to be a 2-functor $\mathbb{D} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$, the values of which will be referred to as **coherent diagram categories**. Moreover, the functors Δ_A are the values $y_{\mathcal{C}}(\pi_A)$ where π_A is the unique functor from A to the category $\mathbb{1}$ with a single object and only the identity morphism. A **derivator** is then a prederivator $\mathbb{D} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ satisfying the axioms (Der1)-(Der4) (see Section A.1). Crucially, the axiom (Der3) implies that the functors $\mathbb{D}(\pi_A)$ always have a right adjoint holim_A and a left adjoint $\operatorname{hocolim}_A$.

2.1 A brief introduction to derivators.

In this section we will give an overview of the definitions related to derivators that we will need in later sections. In order to make the definitions more concrete, we will use the derivator associated to the unbounded derived category of a ring (described below in Example 2.1) to illustrate each one. The axioms (Der1)-(Der4) and an explanation of shifted derivators are contained in Appendix A.

2.1.1 Basic terminology.

Throughout the paper we will use the following basic terminology and notation for a prederivator $\mathbb{D} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$:

- The categories $\mathbb{D}(A)$ for each object A in $\mathcal{C}at$ will be referred to as **coherent diagram categories** and the objects of $\mathbb{D}(A)$ will be referred to as **coherent diagrams of shape** A. The objects of the category $\mathbb{D}(\mathbb{1})^A$ will be referred to as the **incoherent diagrams of shape** A.
- Let $[1]$ denote the free category generated by the poset $\{0 < 1\}$. The category $\mathbb{D}(\mathbb{1})^{[1]}$ consists of the morphisms in $\mathbb{D}(\mathbb{1})$ and shall be referred to as the **category of incoherent morphisms**. The category $\mathbb{D}(\mathbb{1})$ will be referred to as the category of **coherent morphisms**.
- The functors $\mathbb{D}(u)$ for each $u \colon A \to B$ in $\mathcal{C}at$ will be referred to as **restriction functors** and will be denoted by u^*. Similarly, the natural transformations $\mathbb{D}(\alpha)$ will be denoted by α^*.
- Let $\mathbb{1}$ denote the category with a single object and its identity morphism. This is a terminal object in $\mathcal{C}at$ and, for each small category A, we will denote the unique arrow $A \to \mathbb{1}$ by π_A (or π if it is unambiguous). The category $\mathbb{D}(\mathbb{1})$ will be referred to as the **underlying category of** $\mathbb{D}$.

Example 2.1. Let R be a ring and let $\mathcal{A}$ denote the category Mod-R of right R modules. There is a prederivator $\mathbb{D}_R \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ such that:

1. For each small category A, this category $\mathbb{D}_R(A)$ is the unbounded derived category $\mathrm{D}(\mathcal{A}^A)$.

2. For each functor $u\colon A \to B$, the functor $u^*\colon \mathrm{D}(\mathcal{A}^B) \to \mathrm{D}(\mathcal{A}^A)$ is induced from the exact functor $\mathcal{A}^B \to \mathcal{A}^A$.

This is a 2-functor since every natural transformation $\alpha\colon u \Rightarrow v$ induces a natural transformation $\alpha^*\colon \mathbb{D}_R(u) \Rightarrow \mathbb{D}_R(v)$. The underlying category $\mathbb{D}_R(\mathbb{1})$ of $\mathbb{D}_R$ is equivalent to $\mathrm{D}(\text{Mod-}R)$.

For each functor $u\colon A \to B$, if the restriction functor $u^*\colon \mathbb{D}(B) \to \mathbb{D}(A)$ has a right adjoint, then it will be denoted $u_*\colon \mathbb{D}(A) \to \mathbb{D}(B)$. Similarly, if it has a left adjoint, then it will be denoted $u_!\colon \mathbb{D}(A) \to \mathbb{D}(B)$. These are referred to as right and left **Kan extensions**.

2.1.2 Underlying diagram functors.

Next we will describe some distinguished restriction functors that will be used frequently in the subsequent sections:

- Let a be an object in a small category A and let $a\colon \mathbb{1} \to A$ denote the unique functor mapping the object in $\mathbb{1}$ to a. Then the restriction functor $a^*\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$ is called the **evaluation functor at** a. For an object X in $\mathbb{D}(A)$, the image of X under a^* is called the **value of** X **at** a and will be denoted by X_a.

- For every morphism $f\colon a \to b$ in A, let $f\colon a \Rightarrow b$ denote the natural transformation from a to b with the unique component given by f. For each object X in $\mathbb{D}(A)$ we will denote the component of $f^*\colon a^* \Rightarrow b^*$ at X by $X_f\colon X_a \to X_b$; this is called the **value of** X **at** f.

We define the **underlying diagram functor** $\mathrm{dia}_A\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})^A$ for each small category A in the following way:

- For each object X in $\mathbb{D}(A)$ we assign the object $\mathrm{dia}_A(X)\colon A \to \mathbb{D}(\mathbb{1})$ of $\mathbb{D}(\mathbb{1})^A$ such that $a \mapsto X_a$ and $f \mapsto X_f$.

- For each morphism $g\colon X \to Y$ in $\mathbb{D}(A)$ we assign the morphism $\mathrm{dia}_A(g)\colon \mathrm{dia}_A(X) \to \mathrm{dia}_A(Y)$ in $\mathbb{D}(\mathbb{1})^A$ given by $\mathrm{dia}_A(g)_a := g_a\colon X_a \to Y_a$ for each object a in A.

Example 2.2. Let X be an object of $\mathbb{D}_R(A)$ for a small category A. We may consider an object X' in $\mathrm{Ch}(\mathcal{A}^A) \simeq \mathrm{Ch}(\mathcal{A})^A$ that maps to X under the localisation functor. Consider X' as an A-shaped diagram in $\mathrm{Ch}(\mathcal{A})^A$, we may postcompose with the localisation functor to obtain an object of $\mathbb{D}_R(\mathbb{1})^A$. This assignment is well-defined and extends to the functor $\mathrm{dia}_A\colon \mathbb{D}_R(A) \to \mathbb{D}_R(\mathbb{1})^A$.

In general, the category $\mathbb{D}_R(\mathbb{1})^A$ of incoherent diagrams is not equivalent to the category $\mathbb{D}_R(A)$ of coherent diagrams. For example, if k is a field then $\mathbb{D}_k(\mathbb{1})$ is abelian and hence $\mathbb{D}_k(\mathbb{1})^A$ is abelian for any A. Let $[1]$ denote the totally ordered set $\{0 < 1\}$ considered as a small category. In contrast, $\mathbb{D}_k([1])$ is the derived category of k-representations of the quiver $\bullet \to \bullet$, which is clearly not abelian.

2.1.3 Homotopy limits and homotopy colimits.

For every small category A consider the unique functor $\pi = \pi_A\colon A \to \mathbb{1}$. We will denote the left adjoint $\pi_!$ of π^* by $\mathrm{hocolim}_A\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$ and refer to it as the **homotopy colimit functor**. Similarly, we will denote the right adjoint π_* of π^* by $\mathrm{holim}_A\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$ and refer to it as the **homotopy limit functor**. For an object X in $\mathbb{D}(A)$, we will refer to $\mathrm{hocolim}_A(X)$ as the **homotopy colimit of** X and to $\mathrm{holim}_A(X)$ as the **homotopy limit of** X.

Example 2.3. Consider a directed category I and the unique functor $\pi\colon I \to \mathbb{1}$. Following [35, Prop. 6.6], we can describe the left Kan extension $(\pi_I)_!$ with respect to $\mathbb{D}_R$ explicitly. The colimit functor $\varinjlim\colon \mathrm{Ch}(R)^I \to \mathrm{Ch}(R)$ and the constant diagram functor $\Delta_I\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)^I$ form an adjoint pair of exact functors. Hence they induce a pair of adjoint functors $\varinjlim\colon \mathbb{D}_R(I) \to \mathbb{D}_R(\mathbb{1})$ and $\pi^*\colon \mathbb{D}_R(\mathbb{1}) \to \mathbb{D}_R(I)$. Left adjoint functors are unique up to equivalence and so we conclude that $\varinjlim_I \cong \mathrm{hocolim}_I$.

As products are exact in Mod-R, similar reasoning may be applied to the direct product functor on $\mathrm{Ch}(R)$ to obtain an example of a homotopy limit functor.

2.1.4 Strong and stable derivators.

For small categories A and B, we define the **partial underlying diagram functor**

$$\mathrm{dia}_{B,A}\colon \mathbb{D}(A \times B) \to \mathbb{D}(A)^B$$

to be the underlying diagram functor $\mathrm{dia}_B^{\mathbb{D}^A}\colon \mathbb{D}^A(B) \to \mathbb{D}^A(\mathbb{1})^B$ with respect to the shifted derivator $\mathbb{D}^A$. A derivator is called **strong** if the partial underlying diagram functor $\mathrm{dia}_{F,A}$ is full and essentially surjective for every small category A and every finite free category F.

For the purposes of this paper we will be concerned with the following consequences of the definition of a **stable derivator** rather than the definition itself. For a full definition we refer the reader to [12, Def. 4.1] (note that we do not include strong in the definition of stable).

Example 2.4. The derivator $\mathbb{D}_R$ is strong and so the functor $\mathrm{dia}_{[1]}\colon \mathbb{D}_R([1]) \to \mathbb{D}_R(\mathbb{1})^{[1]}$, described in Example 2.2, is full and essentially surjective. We may therefore replace any incoherent morphism with the underlying diagram of a coherent morphism.

Another important property of $\mathbb{D}_R$ is that for every small category A, the category $\mathbb{D}_R(A)$ is a triangulated category. This is a consequence of $\mathbb{D}_R$ being a strong and stable derivator.

We call an additive functor $F\colon \mathcal{T} \to \mathcal{T}'$ between triangulated categories **exact** if there exists a natural isomorphism $\eta\colon F \circ [1] \to [1] \circ F$ and for every triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{T}$ we have that

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\eta_X \circ Fh} (FX)[1]$$

is a triangle in $\mathcal{T}'$. The following statements may be found in [12, Thm. 4.16, Cor. 4.19], or in more detail in [13].

Proposition 2.5. *Let $\mathbb{D}$ be a strong and stable derivator. Then the following statements hold.*

1. *For any small category A, the category of coherent diagrams $\mathbb{D}(A)$ is triangulated.*
2. *For any functor $u\colon A \to B$ in $\mathcal{C}at$, the functors u^*, u_* and $u_!$ are exact functors.*

2.2 Coherent reduced products.

In Section 3 we give intrinsic characterisations of both pure-exact triangles and definable subcategories in the underlying category $\mathbb{D}(\mathbb{1})$ of a derivator $\mathbb{D}$ (assuming that $\mathbb{D}(\mathbb{1})$ is compactly generated). These characterisations mimic those given for a locally finitely presented category $\mathcal{C}$ in [30, Thm. 16.1.16] and [20, Cor. 4.6]. A key tool in both of these cases is the reduced product with respect to a proper filter (see Construction 2.6). In this section we will show that, for any derivator $\mathbb{D}$ and any proper filter, there is a coherent diagram whose underlying diagram is isomorphic to the direct system described in Construction 2.6.

Let S be a non-empty set and let $\mathcal{P}(S)$ denote the power set of S. Then a **(proper) filter on** S is a non-empty collection $\mathcal{F} \subset \mathcal{P}(S)$ of subsets of S (not containing $\emptyset$) such that if $P, Q \in \mathcal{F}$

then $P \cap Q \in \mathcal{F}$ and if $P \in \mathcal{F}$ and $P \subseteq Q$ then $Q \in \mathcal{F}$. Moreover, a proper filter $\mathcal{F}$ is called an **ultrafilter** if, for every $P \in \mathcal{P}(S)$, either $P \in \mathcal{F}$ or $S \setminus P \in \mathcal{F}$.

Construction 2.6. Let $\mathcal{C}$ be a complete category with direct limits and let $X = \{X_s\}_{s \in S}$ be a set of objects in $\mathcal{C}$. Given a proper filter $\mathcal{F}$ on S, we may define a directed system in $\mathcal{C}$ consisting of objects $\{\prod_{p \in P} X_p \mid P \in \mathcal{F}\}$ and morphisms $\{\phi_{PQ} \mid Q \subseteq P\}$ where $\phi_{PQ} \colon \prod_{p \in P} X_p \to \prod_{q \in Q} X_q$ denotes the canonical projection. The **reduced product of** X (with respect to $\mathcal{F}$) is the direct limit

$$\prod_{s \in S} X_s/\mathcal{F} := \varinjlim_{P \in \mathcal{F}} \left(\prod_{p \in P} X_p \right).$$

If $\mathcal{F}$ is an ultrafilter, then $\prod_{s \in S} X_s/\mathcal{F}$ is called the **ultraproduct of** X (with respect to $\mathcal{F}$). If $\mathcal{F}$ is an ultrafilter and Y^I is the I-indexed product of copies of Y, then $Y^I/\mathcal{F}$ is called the **ultrapower of** Y (with respect to $\mathcal{F}$).

If $\mathcal{F}$ is a proper filter on a set S, then we can consider $\mathcal{F}$ as a poset with the relation $P \leq Q$ if and only if $Q \subset P$. We will identify $\mathcal{F}$ with the free category generated by this poset and we will denote the morphisms corresponding to $Q \subseteq P$ by $f_{PQ} \colon P \to Q$. We will also identify the set S with the discrete category with objects S. Note that we have an equivalence of categories $\mathbb{D}(S) \simeq \mathbb{D}(\mathbb{1})^S$ by (Der1).

Proposition 2.7. *Let $\mathbb{D}$ be a derivator and let $\mathcal{F}$ be a proper filter on a set S. Then there exists a functor*

$$\mathrm{Red}_{\mathcal{F}} \colon \mathbb{D}(S) \to \mathbb{D}(\mathcal{F})$$

such that, for each diagram X in $\mathbb{D}(S)$, the following statements hold:

1. *The value of $\mathrm{Red}_{\mathcal{F}}(X)$ at P is isomorphic to $\prod_{p \in P} X_p$ for each $P \in \mathcal{F}$.*
2. *The value of $\mathrm{Red}_{\mathcal{F}}(X)$ at f_{PQ} is isomorphic to the canonical projection ϕ_{PQ} for each $Q \subseteq P$ in $\mathcal{F}$.*

Since the proof of Proposition 2.7 is reasonably long and technical, we will postpone it until Appendix B. We will refer to $\mathrm{Red}_{\mathcal{F}}(X)$ as the **coherent reduced product diagram** of X and we define the **coherent reduced product** of X to be $X_{\mathcal{F}} := \mathrm{hocolim}_{\mathcal{F}}(\mathrm{Red}_{\mathcal{F}}(X))$. If $\mathcal{F}$ is an ultrafilter then we will use the terms **coherent ultraproduct** and **coherent ultrapower** where appropriate.

Example 2.8. Let $\mathbb{D}_R$ be the derivator described in Example 2.1. Let $\mathcal{F}$ be a filter on a set S and let X be in $\mathbb{D}_R(S)$. Using the same notation for X when considering it as an object of $\mathrm{Ch}(R)^S$, it follows from Example 2.3 that $X_{\mathcal{F}} \cong \prod_{s \in S} X_s/\mathcal{F}$, where the reduced product on the right is taken in $\mathrm{Ch}(R)$ and then considered as an object in $\mathbb{D}_R(\mathbb{1})$.

Corollary 2.9. *If $\mathbb{D}$ is a strong and stable derivator, then the functors $\mathrm{Red}_{\mathcal{F}}$ and $(-)_{\mathcal{F}}$ are exact functors.*

Proof. This is immediate from the proof of Proposition 2.7, as well as Proposition 2.5. □

Remark 2.10. For any small category A, we may define the shifted derivator $\mathbb{D}^A \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ (see Appendix A.2). So, for any proper filter $\mathcal{F}$ on a set S and any X in $\mathbb{D}(A)^S$, we may define the coherent reduced product of X by considering it as an object of $\mathbb{D}^A(S)$.

Moreover, for each object a in A, the value of the coherent reduced product of X at a is given by the coherent reduced product of the value of X at a. That is, by applying [12, Prop. 2.5], we have natural isomorphisms

$$\begin{array}{ccccc} \mathbb{D}^A(S) & \xrightarrow{\mathrm{Red}_{\mathcal{F}}} & \mathbb{D}^A(\mathcal{F}) & \xrightarrow{\mathrm{hocolim}_{\mathcal{F}}} & \mathbb{D}^A(\mathbb{1}) \\ \downarrow{\scriptstyle (a\times \mathrm{id}_S)^*} & \cong & \downarrow{\scriptstyle (a\times \mathrm{id}_{\mathcal{F}})^*} & \cong & \downarrow{\scriptstyle a^*} \\ \mathbb{D}(S) & \xrightarrow[\mathrm{Red}_{\mathcal{F}}]{} & \mathbb{D}(\mathcal{F}) & \xrightarrow[\mathrm{hocolim}_{\mathcal{F}}]{} & \mathbb{D}(\mathbb{1}) \end{array}$$

where the coherent reduced product and homotopy colimit on the top row take place with respect to $\mathbb{D}^A$ and the coherent reduced product and homotopy colimit on the bottom row take place with respect to $\mathbb{D}$.

For the sake of clarity, we note that this means that for each X in $\mathbb{D}^A(S)$ we have isomorphisms

$$(\mathrm{Red}_{\mathcal{F}}^{\mathbb{D}^A}(X))_a^{\mathbb{D}^{\mathcal{F}}} \cong \mathrm{Red}_{\mathcal{F}}^{\mathbb{D}}(X_a^{\mathbb{D}^S}) \quad \text{and} \quad (X_{\mathcal{F}}^{\mathbb{D}^A})_a^{\mathbb{D}} \cong (X_a^{\mathbb{D}^S})_{\mathcal{F}}^{\mathbb{D}}$$

where the superscripts indicate which derivator each evaluation, reduced product and homotopy colimit is taken with respect to. Note that we may consider X as an object $\{X_s\}_{s\in S}$ of $\mathbb{D}(A)^S$ and $X_a^{\mathbb{D}^S}$ corresponds to the object $\{(X_s)_a^{\mathbb{D}}\}_{s\in S}$ in $\mathbb{D}(\mathbb{1})^S$.

3 Purity in a compactly generated derivator.

In this section we will use the construction of coherent reduced products in Section 2.2 to characterise purity in the underlying category of a compactly generated derivator.

3.1 Purity in compactly generated triangulated categories.

We will focus on strong and stable derivators $\mathbb{D}$ for which $\mathbb{D}(\mathbb{1})$ is a compactly generated triangulated category. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. An object X in $\mathcal{T}$ is called **compact** if the Hom-functor $\mathrm{Hom}_{\mathcal{T}}(X,-)\colon \mathcal{T}\to \mathbf{Ab}$ commutes with arbitrary coproducts. Then $\mathcal{T}$ is **compactly generated** if the full subcategory $\mathcal{T}^{\mathrm{c}}$ of compact objects in $\mathcal{T}$ is skeletally small and $\mathcal{T}^{\mathrm{c}}$ **generates** $\mathcal{T}$ i.e. for every non-zero object Y in $\mathcal{T}$ there exists some X in $\mathcal{T}^{\mathrm{c}}$ such that $\mathrm{Hom}_{\mathcal{T}}(X,Y)\neq 0$.

Next we will summarise some of the basic notions of purity in a compactly generated category $\mathcal{T}$. Consider the category Mod-$\mathcal{T}^{\mathrm{c}}$ of contravariant additive functors from $\mathcal{T}^{\mathrm{c}}$ to the category **Ab** of abelian groups. We will denote the full subcategory of finitely presented functors by mod-$\mathcal{T}^{\mathrm{c}}$.

Let $\mathbf{y}\colon \mathcal{T}\to \text{Mod-}\mathcal{T}^{\mathrm{c}}$ denote the **restricted Yoneda functor** which is defined to be

$$\mathbf{y}X := \mathrm{Hom}_{\mathcal{T}}(-,X)|_{\mathcal{T}^{\mathrm{c}}} \quad \text{and} \quad \mathbf{y}f := \mathrm{Hom}_{\mathcal{T}}(-,f)|_{\mathcal{T}^{\mathrm{c}}}$$

for objects X and morphisms f in $\mathcal{T}$. The functor $\mathbf{y}$ is not always fully faithful but it enables us to consider triangles in $\mathcal{T}$ in terms of exact sequences in Mod-$\mathcal{T}^{\mathrm{c}}$:

- A triangle $\delta\colon X\xrightarrow{f} Y\xrightarrow{g} Z\to X[1]$ is called a **pure triangle** if the sequence

$$\mathbf{y}\delta\colon 0\longrightarrow \mathbf{y}X\xrightarrow{\mathbf{y}f}\mathbf{y}Y\xrightarrow{\mathbf{y}g}\mathbf{y}Z\longrightarrow 0$$

  is exact in Mod-$\mathcal{T}^{\mathrm{c}}$. In this case we refer to f as a **pure monomorphism** and to g as a **pure epimorphism**.

- An object E is called **pure-injective** if every pure monomorphism of the form $E \to X$ is split. An object P is called **pure-projective** if every pure epimorphism of the form $X \to P$ is split.

In fact, the following proposition shows that there is a strong relationship between the injective objects in Mod-$\mathcal{T}^{\mathrm{c}}$ and pure-injective objects in $\mathcal{T}$.

Proposition 3.1 ([21, Thm. 1.8])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category. The following statements are equivalent for an object E in $\mathcal{T}$:*

1. *E is pure-injective.*
2. *$\mathbf{y}E$ is an injective object of* Mod-$\mathcal{T}^{\mathrm{c}}$.
3. *The map $\mathrm{Hom}_{\mathcal{T}}(X, E) \to \mathrm{Hom}_{\text{Mod-}\mathcal{T}^{\mathrm{c}}}(\mathbf{y}X, \mathbf{y}E)$ induced by the functor $\mathbf{y}$ is an isomorphism for all objects X in $\mathcal{T}$.*

Similarly, the following statements are equivalent for an object P in $\mathcal{T}$:

1. *P is pure-projective.*
2. *$\mathbf{y}P$ is a projective object of* Mod-$\mathcal{T}^{\mathrm{c}}$.
3. *The map $\mathrm{Hom}_{\mathcal{T}}(P, X) \to \mathrm{Hom}_{\text{Mod-}\mathcal{T}^{\mathrm{c}}}(\mathbf{y}P, \mathbf{y}X)$ induced by the functor $\mathbf{y}$ is an isomorphism for all objects X in $\mathcal{T}$.*

3.2 Pure triangles in terms of coherent reduced products.

We will call a strong and stable derivator $\mathbb{D}$ **compactly generated** if $\mathbb{D}(\mathbb{1})$ is a compactly generated triangulated category. In this section we give an intrinsic characterisation of pure triangles (and hence of pure injective objects) in the underlying category $\mathbb{D}(\mathbb{1})$ of a compactly generated derivator $\mathbb{D}$.

Lemma 3.2. *Let $\mathbb{D}$ be a compactly generated derivator. For any small category A, the category $\mathbb{D}(A)$ of coherent diagrams is compactly generated.*

Proof. It suffices to show that $\mathbb{D}(A)$ has a set of compact generators. We prove that the set

$$\mathcal{Y} := \{Y \mid Y \cong a_!C \text{ for some } a \text{ in } A \text{ and } C \text{ in } \mathbb{D}(\mathbb{1})^{\mathrm{c}}\}$$

is such a set.

To see that $\mathcal{Y}$ is a generating set, let Z be an object of $\mathbb{D}(A)$ such that $\mathrm{Hom}_{\mathbb{D}(A)}(Y, Z) = 0$ for all $Y \in Y$. Then we have that $\mathrm{Hom}_{\mathbb{D}(\mathbb{1})}(C, Z_a) \cong \mathrm{Hom}_{\mathbb{D}(A)}(a_!C, Z) = 0$ for all a in A and C in $\mathbb{D}(\mathbb{1})^{\mathrm{c}}$. Since $\mathbb{D}(\mathbb{1})$ is compactly generated, we have that $Z_a \cong 0$ for all a in A. It follows from (Der2) that Z is a zero object in $\mathbb{D}(A)$.

Next we show that the objects in $\mathcal{Y}$ are compact. For an object a in A and C in $\mathbb{D}(\mathbb{1})^{\mathrm{c}}$, we have $\mathrm{Hom}_{\mathbb{D}(\mathbb{1})}(C, \left(\bigoplus_{s\in S} X_s\right)_a) \cong \mathrm{Hom}_{\mathbb{D}(\mathbb{1})}(C, \bigoplus_{s\in S}(X_s)_a) \cong \bigoplus_{s\in S} \mathrm{Hom}_{\mathbb{D}(\mathbb{1})}(C, (X_s)_a)$ because a^* is a left adjoint. Thus $\mathrm{Hom}_{\mathbb{D}(A)}(a_!C, \bigoplus_{s\in S} X_s) \cong \bigoplus_{s\in S} \mathrm{Hom}_{\mathbb{D}(A)}(a_!C, X_s)$ as required. □

Corollary 3.3. *Let $\mathbb{D}$ be a compactly generated derivator. For any small category A, the shifted derivator $\mathbb{D}^A$ is compactly generated.*

Proof. By Proposition 2.5, the derivator $\mathbb{D}^A$ is strong and stable and so the statement is immediate from Lemma 3.2. □

Let $\mathbb{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ be a derivator. An object X in $\mathbb{D}(\mathbb{1})$ is called **homotopically finitely presented** if the canonical morphism

$$\varinjlim_{i\in I} \mathrm{Hom}_{\mathbb{D}(\mathbb{1})}(X, Y_i) \to \mathrm{Hom}_{\mathbb{D}(\mathbb{1})}(X, \mathrm{hocolim}_I(Y))$$

is an isomorphism for every small directed category I and every object Y in $\mathbb{D}(I)$.

Proposition 3.4 ([33, Prop. 5.4])**.** *Let $\mathbb{D}$ be a strong and stable derivator. Then an object C in $\mathbb{D}(\mathbb{1})$ is compact if and only if it is homotopically finitely presented.*

Remark 3.5. The preceding result can be summarised following observation: Let $\mathcal{T} \simeq \mathbb{D}(\mathbb{1})$ be the underlying category of a compactly generated derivator $\mathbb{D}$. Then, for any small directed category I and any X in $\mathbb{D}(I)$, we have that

$$\varinjlim_{i\in I} \mathbf{y}X_i \cong \mathbf{y}\mathrm{hocolim}_I(X).$$

Combining this with the construction described in Section 2.2, we have that any reduced product of representable functors in Mod-$\mathcal{T}^{\mathrm{c}}$ is representable. That is, for any proper filter $\mathcal{F}$ on a set S and any X in $\mathbb{D}(S) \simeq \mathcal{T}^S$, we have that

$$\prod_{s\in S} \mathbf{y}X_s/\mathcal{F} \cong \mathbf{y}X_{\mathcal{F}}.$$

Remark 3.6. Reduced products and ultraproducts are ubiquitous in model theory, see for example [8, Chap. 4]. In [11], the authors introduce a language $\mathcal{L}_{\mathcal{T}}$ for a compactly generated triangulated category $\mathcal{T}$ such that the models of $\mathcal{L}_{\mathcal{T}}$ coincide with the objects of Mod-$\mathcal{T}^{\mathrm{c}}$. Moreover, the reduced product in the sense of [8, Prop. 4.1.6] coincides with the reduced product in Mod-$\mathcal{T}^{\mathrm{c}}$. We may consider the objects of $\mathcal{T}$ as models of $\mathcal{L}_{\mathcal{T}}$ via the functor $\mathbf{y}$. Remark 3.5 says that the collection of objects of $\mathcal{T}$ (considered as models of $\mathcal{L}_{\mathcal{T}}$) is closed under taking reduced products.

Proposition 3.7. *Let $\mathcal{T}$ be the underlying category of a compactly generated derivator $\mathbb{D}$. Let $X \to Y \to Z \to X[1]$ be a triangle in $\mathcal{T}$. Then the following statements are equivalent:*

1. *The sequence $X \to Y \to Z \to X[1]$ is a pure triangle.*
2. *There exists an ultrafilter $\mathcal{F}$ on a set S such that the coherent ultrapower*

$$(X^S)_{\mathcal{F}} \to (Y^S)_{\mathcal{F}} \to (Z^S)_{\mathcal{F}} \to (X^S)_{\mathcal{F}}[1]$$

is a split triangle.

Proof. By definition, the statement (1) is equivalent to the sequence $0 \to \mathbf{y}X \to \mathbf{y}Y \to \mathbf{y}Z \to 0$ being an exact sequence in Mod-$\mathcal{T}^{\mathrm{c}}$. As $\mathbf{y}X$ is fp-injective [21, Lem. 1.6], every exact sequence in Mod-$\mathcal{T}^{\mathrm{c}}$ starting with $\mathbf{y}X$ is pure-exact (see Section 5.3 for the definition of fp-injective). By [30, Thm. 16.1.16], this is equivalent to there existing an ultrafilter $\mathcal{F}$ on a set S such that $0 \to (\mathbf{y}X)^S/\mathcal{F} \to (\mathbf{y}Y)^S/\mathcal{F} \to (\mathbf{y}Z)^S/\mathcal{F} \to 0$ is a split exact sequence in Mod-$\mathcal{T}^{\mathrm{c}}$. Moreover, the terms in the split exact sequence can be taken to be (pure-)injective and so, by Remark 3.5, this is equivalent to (2). □

Corollary 3.8. *Let A be a small category. If $X \to Y \to Z \to X[1]$ is a pure triangle in $\mathbb{D}(A)$ then $X_a \to Y_a \to Z_a \to X_a[1]$ is a pure triangle in $\mathcal{T}$ for each $a \in A$.*

Proof. Note that, by Proposition 2.5, the evaluation functor $a^*\colon \mathbb{D}(A) \to \mathbb{D}(\mathbb{1})$ is an exact functor. Combining this with Remark 2.10, the statement is an immediate consequence of Proposition 3.7. □

Remark 3.9. A similar approach was taken in [22] without requiring the presence of a derivator by using the notion of a homology colimit instead of homotopy colimit; pure triangles are characterised as homology colimits of split triangles.

3.3 Definable subcategories in terms of coherent reduced products.

A full subcategory $\mathcal{D}$ of a compactly generated triangulated category $\mathcal{T}$ is called **definable** if it is of the form

$$\mathcal{D} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(F_i, \mathbf{y}X) = 0 \text{ for all } i \in I\}$$

where $\{F_i\}_{i\in I}$ is a family of functors in mod-$\mathcal{T}^c$. For a class of objects $\mathcal{E}$ in $\mathcal{T}$ we will denote by $\mathrm{Def}_{\mathcal{T}}(\mathcal{E})$ the **smallest definable subcategory containing** $\mathcal{E}$. The subcategory $\mathrm{Def}_{\mathcal{T}}(\mathcal{E})$ always exists and is given by

$$\mathrm{Def}_{\mathcal{T}}(\mathcal{E}) = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(F, \mathbf{y}X) = 0 \text{ for all } F \in \mathcal{Y}_{\mathcal{E}}\}$$

where $\mathcal{Y}_{\mathcal{E}} = \{F \in \text{mod-}\mathcal{T}^c \mid \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(F, \mathbf{y}M) = 0 \text{ for all } M \in \mathcal{E}\}$. If $\mathcal{E} = \{M\}$ then we denote the definable subcategory generated by $\mathcal{E}$ by $\mathrm{Def}_{\mathcal{T}}(M)$.

We will need the following lemma, which summarises close relationship between the definable subcategories of $\mathcal{T}$ and the definable subcategories of Mod-$\mathcal{T}^c$ (see [20] for the definition of definable subcategories of Mod-$\mathcal{T}^c$).

Lemma 3.10 ([3, Cor. 4.4])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{E}$ be a class of objects in $\mathcal{T}$. Then* $\mathrm{Def}_{\mathcal{T}}(\mathcal{E}) = \{M \in \mathcal{T} \mid \mathbf{y}M \in \mathrm{Def}_{\text{Mod-}\mathcal{T}^c}(\mathbf{y}\mathcal{E})\}$.

Let $\mathcal{X}$ be a class of objects in $\mathbb{D}(\mathbb{1})$. We will say that $\mathcal{X}$ is **closed under directed homotopy colimits** if, for all directed categories I and all objects X in $\mathbb{D}(I)$ such that $X_i \in \mathcal{X}$ for every i in I, we have $\mathrm{hocolim}_I(X) \in \mathcal{X}$. The following theorem is a triangulated version of [20, Cor. 4.6]:

Theorem 3.11. *Let $\mathcal{T}$ be the underlying category of a compactly generated derivator $\mathbb{D}$. Then, for a full subcategory $\mathcal{D}$ of $\mathcal{T}$, the following statements are equivalent:*

1. *$\mathcal{D}$ is definable;*
2. *$\mathcal{D}$ is closed under products, pure subobjects and directed homotopy colimits;*
3. *$\mathcal{D}$ is closed under pure subobjects and coherent reduced products.*

Proof. First note that if $\mathcal{D}$ is a definable subcategory, then $\mathcal{D}$ is closed under products and pure subobjects (this is immediate from [22, Thm. A]).

(1) ⇒ (2): Let I be a small directed category and let X be an object in $\mathbb{D}(I)$ with X_i contained in $\mathcal{D}$ for all objects i in I. By Lemma 3.10, we have that $\mathrm{hocolim}_I(X)$ is contained in $\mathcal{D}$ if and only if $\mathbf{y}\mathrm{hocolim}_I(X)$ is contained in $\mathrm{Def}_{\text{Mod-}\mathcal{T}^c}(\mathbf{y}\mathcal{D})$. But the latter holds because $\mathbf{y}\mathrm{hocolim}_I(X) \cong \varinjlim_{i\in I} \mathbf{y}X_i$ and $\mathrm{Def}_{\text{Mod-}\mathcal{T}^c}(\mathbf{y}\mathcal{D})$ is closed under direct limits.

(2) ⇒ (3): Follows immediately from the definition of coherent reduced products.

(3) ⇒ (1): We will show that $\mathrm{Def}_{\mathcal{T}}(\mathcal{D}) \subseteq \mathcal{D}$ and hence that $\mathcal{D} = \mathrm{Def}_{\mathcal{T}}(\mathcal{D})$. First let $E \in \mathrm{Def}_{\mathcal{T}}(\mathcal{D})$ be pure-injective. Then, since $\mathbf{y}E \in \mathrm{Def}_{\text{Mod-}\mathcal{T}^c}(\mathbf{y}\mathcal{D})$, there exists a proper filter $\mathcal{F}$ on a set S such that there exists a pure monomorphism $\mathbf{y}E \to \prod_{s\in S} \mathbf{y}D_s/\mathcal{F}$ for some set of objects $\{D_s\}_{s\in S}$ in $\mathcal{D}$ (see [20, Cor. 4.10]). By Remark 3.5, we have $\prod_{s\in S} \mathbf{y}D_s/\mathcal{F} \cong \mathbf{y}D_{\mathcal{F}}$

where D is the object in $\mathbb{D}(S)$ corresponding to $\{D_s\}_{s\in S}$. Since $\mathbf{y}E$ is injective, this is a split monomorphism and it follows that there is a pure epimorphism $D_{\mathcal{F}} \to E$. By assumption $D_{\mathcal{F}} \in \mathcal{D}$ and so $E \in \mathcal{D}$. Consider an arbitrary object $X \in \mathrm{Def}_{\mathcal{T}}(\mathcal{D})$. Then X is a pure subobject of a pure-injective object in $\mathrm{Def}_{\mathcal{T}}(\mathcal{D})$ and so $X \in \mathcal{D}$. □

The following is a triangulated version of [20, Cor. 4.10] and a generalisation of [22, Thm. 7.5]:

Corollary 3.12. *Let $\mathcal{S}$ be a set of objects in $\mathcal{T}$. Then $\mathrm{Def}_{\mathcal{T}}(\mathcal{S})$ consists of the collection of pure subobjects of coherent reduced products of objects in $\mathcal{S}$.*

Proof. It follows from [20, Prop. 4.8] (and the fact that products and direct limits are exact in Mod-$\mathcal{T}^{\mathrm{c}}$) that the collection of pure subobjects of reduced products of objects in $\mathcal{S}$ is closed under pure subobjects and coherent reduced products. □

Corollary 3.13. *Let $\mathcal{V}$ be a definable subcategory of $\mathcal{T}$ and let A be a small category. Then*

$$\mathcal{V}_A := \{X \mid X_a \in \mathcal{V} \text{ for all objects } a \text{ in } A\}$$

is a definable subcategory of $\mathbb{D}(A)$.

Proof. By Remark 2.10 and Corollary 3.8, the class $\mathcal{V}_A$ satisfies the closure conditions (3) in Theorem 3.11. □

4 Smashing t-structures with Grothendieck hearts.

In this section we will consider two kinds of smashing t-structure with Grothendieck hearts: the homotopically smashing t-structures [33] and the pure-injective cosilting t-structures [3]. We will follow the style of the definitions given in [3]; in particular, a t-structure will be a torsion pair in $\mathcal{T}$.

Let $\mathcal{T}$ be a triangulated category. A **t-structure** [6] on $\mathcal{T}$ is a pair $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ of full subcategories satisfying the following conditions:

(t1) $\mathrm{Hom}_{\mathcal{T}}(U, V) = 0$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$;

(t2) $\mathcal{U}[1] \subseteq \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{V}[1]$;

(t3) For each object X in $\mathcal{T}$, there exists a triangle $U \to X \to V \to U[1]$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

The **heart** of $\mathbf{t}$ is defined to be $\mathcal{G} := \mathcal{U}[-1] \cap \mathcal{V}$. By [18], the inclusion $\mathcal{U} \to \mathcal{T}$ (respectively $\mathcal{V} \to \mathcal{T}$) has a right adjoint $\tau_{\mathcal{U}} \colon \mathcal{T} \to \mathcal{U}$ (respectively has a left adjoint $\tau_{\mathcal{V}} \colon \mathcal{T} \to \mathcal{V}$). These are called the **truncation functors** and the triangles in (**t3**) are given by:

$$\tau_{\mathcal{U}}(X) \to X \to \tau_{\mathcal{V}}(X) \to \tau_{\mathcal{U}}(X)[1].$$

The associated cohomological functor to the heart $H^0_{\mathbf{t}} \colon \mathcal{T} \to \mathcal{G}$ is defined to be

$$H^0_{\mathbf{t}} := \tau_{\mathcal{V}}(\tau_{\mathcal{U}}(X[1])[-1]) = (\tau_{\mathcal{U}}(\tau_{\mathcal{V}}(X)[1]))[-1].$$

We say that $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ is **left nondegenerate** (respectively **right nondegenerate**) if $\bigcap_{i\in\mathbb{Z}} \mathcal{U}[i] = \{0\}$ (respective if $\bigcap_{i\in\mathbb{Z}} \mathcal{V}[i] = \{0\}$). If $\mathbf{t}$ is both right and left nondegenerate then we say that $\mathbf{t}$ is **nondegenerate**.

We say that $\mathbf{t}$ is **smashing** if the class $\mathcal{V}$ is closed under coproducts. For any smashing t-structure, the associated cohomological functor $H^0_{\mathbf{t}} \colon \mathcal{T} \to \mathcal{G}$ takes preserves coproducts in $\mathcal{T}$ (see [3, Lem. 3.3]).

4.1 Homotopically smashing t-structures.

The notion of a homotopically smashing t-structure was introduced in [33]. Let $\mathbb{D}$ be a strong and stable derivator and let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathbb{D}(\mathbb{1})$. Then $\mathbf{t}$ is called **homotopically smashing** with respect to $\mathbb{D}$ if $\mathcal{V}$ is closed under directed homotopy colimits.

Remark 4.1. Similar t-structures have been considered in the context of presentable stable ∞-categories. See [24, Rem. 1.3.5.23].

Theorem 4.2 ([33, Thm. A])**.** *Let $\mathbb{D}$ be a strong and stable derivator and let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathbb{D}(\mathbb{1})$. If $\mathbf{t}$ is homotopically smashing, then the heart $\mathcal{G}$ has exact direct limits.*

In fact, it is shown in [33] that a homotopically smashing t-structure on the homotopy category of a combinatorial model category, then the heart is a Grothendieck category. Next we show that if the derivator in question is compactly generated, then we do not need these additional assumptions to obtain the generators.

Lemma 4.3. *Let $\mathbb{D}$ be a compactly generated derivator and let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathbb{D}(\mathbb{1})$. If $\mathbf{t}$ is homotopically smashing then the heart $\mathcal{G}$ is a Grothendieck category.*

Proof. Since $\mathbb{D}(\mathbb{1})$ has arbitrary coproducts, it follows that $\mathcal{G}$ has arbitrary coproducts. Also, follows from the previous theorem that $\mathcal{G}$ has exact direct limits and so it remains to show that $\mathcal{G}$ has a set of generators. Consider the set

$$\mathcal{C} := \{C \mid C \cong \bigoplus_{i \in I} H^0_{\mathbf{t}}(C_i) \text{ where } \{C_i\}_{i \in I} \text{ is a set of compact objects } \}$$

of objects in $\mathcal{G}$. We will show that $\mathcal{C}$ generates $\mathcal{G}$. Let X be an object in $\mathcal{G}$ and consider $\mathbf{y}X$ in Mod-$\mathcal{T}^{\mathrm{c}}$. Since $\{\mathbf{y}C \mid C \text{ in } \mathcal{T}^{\mathrm{c}}\}$ is a generatoring set for Mod-$\mathcal{T}^{\mathrm{c}}$, there exists an epimorphism $\gamma\colon \bigoplus_{i \in I} \mathbf{y}C_i \to \mathbf{y}X \to 0$ for some set $\{C_i\}_{i \in I}$ of compact objects. But then $\bigoplus_{i \in I} \mathbf{y}C_i \cong \mathbf{y}\left(\bigoplus_{i \in I} C_i\right)$ and $\bigoplus_{i \in I} C_i$ is pure-projective so $\gamma = \mathbf{y}g$ for some pure epimorphism $g\colon \bigoplus_{i \in I} C_i \to X$ in $\mathbb{D}(\mathbb{1})$. Since $\mathcal{G}$ has exact direct limits and $H^0_{\mathbf{t}}$ preserves coproducts, we have that $H^0_{\mathbf{t}}$ sends pure epimorphisms to epimorphisms in $\mathcal{G}$ [21, Cor. 2.5]. Thus $H^0_{\mathbf{t}}(g)\colon \bigoplus_{i \in I} H^0_{\mathbf{t}}(C_i) \to X$ is an epimorphism and we have shown that $\mathcal{C}$ generates $\mathcal{G}$. □

4.2 Partial cosilting t-structures.

The cosilting t-structures can be described as particular perpendicular classes of an object. Given an object M in $\mathcal{T}$ and a subset I of $\mathbb{Z}$, we define perpendicular classes as follows:

$$^{\perp_I}M := \{Y \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(Y, M[i]) = 0 \text{ for all } i \in I\}$$

and

$$M^{\perp_I} := \{Y \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(M, Y[i]) = 0 \text{ for all } i \in I\}.$$

In what follows we will represent the set $\{i \in \mathbb{Z} \mid i < 0\}$ by the symbol < 0; similarly for ≤ 0, ≥ 0 and > 0. Also, if $I = \{i\}$ we will simply write $\perp_i$. This notation also applies to objects in an abelian category where Hom-spaces should be replaced by Ext-groups in the obvious way.

An object C in $\mathcal{T}$ is called **cosilting** if $(^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ defines a t-structure (which implies, in particular, that $C \in {}^{\perp_{>0}}C$). A t-structure of the form $(^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ will be referred to as a **cosilting t-structure**. We define an object C in $\mathcal{T}$ to be **partial cosilting** if ${}^{\perp_{>0}}C$ is a coaisle and $C \in {}^{\perp_{>0}}C$; the corresponding t-structure $\mathbf{t} = (\mathcal{U}, {}^{\perp_{>0}}C)$ is called a **partial cosilting t-structure**. Given a (partial) cosilting object C we will denote the heart of the corresponding

t-structure by $\mathcal{G}_C$. A (partial) cosilting object C is called **(partial) cotilting** if $C \in \mathcal{G}_C$. We say that two partial cosilting objects are **equivalent** if they give rise to the same t-structure.

In [3, Thm. 3.6], the authors show that, the cosilting t-structures with pure-injective cosilting object parametrise the nondegenerate smashing t-structures with Grothendieck hearts. In Theorem 4.6, we will make use of the following modification:

Lemma 4.4. *Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a left nondegenerate t-structure on $\mathcal{T}$. If $\mathbf{t}$ is smashing and the heart is a Grothendieck category, then $\mathbf{t}$ is partial cosilting for a pure-injective partial cosilting object C.*

Proof. Let E be an injective cogenerator of $\mathcal{G}$. By the proof of [3, Thm. 3.6], the functor $\mathrm{Hom}_{\mathcal{G}}(H^0_{\mathbf{t}}(-), E)$ is naturally isomorphic to the functor $\mathrm{Hom}_{\mathcal{T}}(-, C)$ for a pure-injective object C. Since $\mathbf{t}$ is left nondegenerate, we have that $\mathcal{V} = \{X \in \mathcal{T} \mid H^0_{\mathbf{t}}(X[i]) = 0 \text{ for all } p < 0\}$ and so $\mathcal{V} = {}^{\perp_{>0}}C$. To see that $C \in {}^{\perp_{>0}}C$, let $U \in \mathcal{U}$. Then $\mathrm{Hom}_{\mathcal{T}}(U, C) \cong \mathrm{Hom}_{\mathcal{G}}(H^0_{\mathbf{t}}(U), E)$. But $H^0_{\mathbf{t}}(U) = 0$ and so $C \in \mathcal{U}^{\perp_0} = {}^{\perp_{>0}}C$. □

It follows from the next lemma that the coaisle of a partial cosilting t-structure with a pure-injective partial cosilting object is definable. In [3, Thm. 4.9], this is shown with the additional assumption that there exists an adjacent co-t-structure.

Lemma 4.5. *Let C be a pure-injective object in the underlying category $\mathcal{T} \simeq \mathbb{D}(\mathbb{1})$ of a compactly generated derivator $\mathbb{D}$. If $\mathcal{V} := {}^{\perp_{>0}}C$ is closed under products, then $\mathcal{V}$ is a definable subcategory.*

Proof. We show that $\mathcal{V}$ has the closure properties given in Theorem 3.11. By assumption $\mathcal{V}$ is closed under products.

Next we show that $\mathcal{V}$ is closed under pure subobjects. So suppose $Y \in \mathcal{V}$ and consider a pure monomorphism $X \to Y$. As $C[j]$ is pure-injective for all $j > 0$, we have an induced epimorphism $\mathrm{Hom}_{\mathcal{T}}(Y, C[j]) \to \mathrm{Hom}_{\mathcal{T}}(X, C[j]) \to 0$ in $\mathbf{Ab}$. Then, since we have $\mathrm{Hom}_{\mathcal{T}}(Y, C[j]) = 0$, it follows that $X \in {}^{\perp_{>0}}C = \mathcal{V}$.

Finally we must show that $\mathcal{V}$ is closed under directed homotopy colimits. Suppose I is a small directed category and let X be an object in $\mathbb{D}(I)$ with $X_i \in \mathcal{V}$ for all $i \in I$. Then $\mathrm{hocolim}_I(X) \in \mathcal{V}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(\mathrm{hocolim}_I(X), C[j]) = 0$ for each $j > 0$. Note that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(\mathrm{hocolim}_I(X), C[j]) &\cong \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(\mathbf{y}\mathrm{hocolim}_I(X), \mathbf{y}C[j]) \\ &\cong \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(\varinjlim_{i\in I} \mathbf{y}X_i, \mathbf{y}C[j]).\end{aligned}$$

Since we have $0 = \mathrm{Hom}_{\mathcal{T}}(X_i, C[j]) \cong \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(\mathbf{y}X_i, \mathbf{y}C[j])$ for all $i \in I$, we also have $\mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(\coprod_{i\in I} \mathbf{y}X_i, \mathbf{y}C[j]) = 0$. Applying the functor $\mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(-, \mathbf{y}C[j])$ the exact sequence

$$\coprod_{i\in I} \mathbf{y}X_i \to \varinjlim_{i\in I} \mathbf{y}X_i \to 0$$

we conclude that

$$\mathrm{Hom}_{\mathcal{T}}(\mathrm{hocolim}_I(X), C[j]) \cong \mathrm{Hom}_{\text{Mod-}\mathcal{T}^c}(\varinjlim_{i\in I} \mathbf{y}X_i, \mathbf{y}C[j]) = 0$$

as desired. □

4.3 Smashing t-structures with Grothendieck hearts.

Collecting together Lemmas 4.3, 4.4, 4.5 with [3, Thm. 3.6], we have proved the following theorem:

Theorem 4.6. *Let $\mathcal{T}$ be the underlying category of a compactly generated derivator $\mathbb{D}$ and consider a (left) nondegenerate t-structure $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ on $\mathcal{T}$. Then the following statements are equivalent:*

1. *$\mathbf{t}$ is a (partial) cosilting t-structure with a pure-injective (partial) cosilting object C.*
2. *$\mathcal{V}$ is definable.*
3. *$\mathbf{t}$ is homotopically smashing.*
4. *$\mathbf{t}$ is smashing and the heart $\mathcal{G}$ is a Grothendieck category.*

4.4 Example via glued t-structures.

Let $\mathcal{H}$ be a locally noetherian Grothendieck category such that $\mathrm{D}(\mathcal{H})$ is compactly generated. Krause proves in [23] that there exists a recollement

$$\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H})) \overset{I}{\underset{I_\rho}{\overset{\longrightarrow}{\underset{\longleftarrow}{}}}} \mathrm{K}(\mathrm{Inj}(\mathcal{H})) \overset{Q}{\underset{Q_\rho}{\overset{\longrightarrow}{\underset{\longleftarrow}{}}}} \mathrm{D}(\mathcal{H}), \qquad I_\lambda \dashv I, \quad Q_\lambda \dashv Q$$

where $\mathrm{K}(\mathrm{Inj}(\mathcal{H}))$ is the homotopy category of the injective objects in $\mathcal{H}$, I is the inclusion of the full subcategory $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}))$ of acyclic complexes in $\mathrm{K}(\mathcal{H})$ that are contained in $\mathrm{K}(\mathrm{Inj}(\mathcal{H}))$ and Q is the composition of the inclusion $\mathrm{K}(\mathrm{Inj}(\mathcal{H})) \to \mathrm{K}(\mathcal{H})$ with the canonical localisation $\mathrm{K}(\mathcal{H}) \to \mathrm{D}(\mathcal{H})$.

We will consider a cosilting object C in $\mathrm{D}(\mathcal{H})$, e.g. the injective cogenerator of $\mathcal{H}$, and show that it induces a t-structure on $\mathrm{K}(\mathrm{Inj}(\mathcal{H}))$ that satisfies the conditions of Theorem 4.6.

Lemma 4.7. *Let C be a cosilting object in $\mathrm{D}(\mathcal{H})$. Then $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H})) = {}^{\perp_{\mathbb{Z}}}Q_\rho(C)$.*

Proof. As C cogenerates $\mathrm{D}(\mathcal{H})$, we have that $X \in {}^{\perp_{\mathbb{Z}}}Q_\rho(C)$ if and only if $\mathrm{Hom}_{\mathrm{D}(\mathcal{H})}(Q(X), C[i]) = 0$ for all $i \in \mathbb{Z}$ if and only if $Q(X) = 0$ if and only if $X \in \ker Q = \mathrm{im} I = \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}))$. □

Proposition 4.8. *Let C be a cosilting object in $\mathrm{D}(\mathcal{H})$ and consider the t-structure $(\mathcal{U}, \mathcal{V})$ obtained by gluing the trivial t-structure $(0, \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}))$ on $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}))$ with the cosilting t-structure $({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ on $\mathrm{D}(\mathcal{H})$. Then $(\mathcal{U}, \mathcal{V})$ is a partial cosilting t-structure with partial cosilting object $Q_\rho(C)$.*

Proof. We know that $\mathcal{V}$ is (by definition) the collection of objects Y in $\mathrm{K}(\mathrm{Inj}(\mathcal{H}))$ such that there exists a triangle

$$X \to Y \to Z \to X[1]$$

with X acylic and $Z \cong Q_\rho(M)$ for $M \in {}^{\perp_{>0}}C$. So consider $Y \in \mathcal{V}$. Then, by the above, we have that $X \in {}^{\perp_{\mathbb{Z}}}Q_\rho(C) \subseteq {}^{\perp_{>0}}Q_\rho(C)$. Moreover, as Q_ρ is fully faithful, it follows that $Z \in {}^{\perp_{>0}}Q_\rho(C)$ and so $Y \in {}^{\perp_{>0}}Q_\rho(C)$. Conversely, for any $Y \in {}^{\perp_{>0}}Q_\rho(C)$, there is a triangle of the desired form given by the counit of (I, I_λ) and the unit of (Q_ρ, Q). So $\mathcal{V} = {}^{\perp_{>0}}Q_\rho(C)$ is a coaisle and clearly $Q_\rho(C) \in \mathcal{V}$, so $Q_\rho(C)$ is partial cosilting. □

In [26, Def. 2] the authors give a definition of a partial silting object which dualises to give a notion of partial cosilting. On the face of it, their definition differs from the definition in this article, not least because our definition yields left nondegenerate t-structures and the dual of [26, Def. 2] yields right nondegenerate t-structures. However in the next proposition we show that, in the setting of this example, the dual of the conditions given in [26, Def. 2] hold for $Q_\rho(C)$. Note that the dual of the second condition given in [26, Def. 2] holds for $Q_\rho(C)$ since the same condition holds for C in $\mathrm{D}(\mathcal{H})$.

Proposition 4.9. *Let C be a cosilting object in $\mathrm{D}(\mathcal{H})$ and consider the t-structure $(\mathcal{U},\mathcal{V})$ obtained by gluing the trivial t-structure $(\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}),0)$ on $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}))$ with the cosilting t-structure $({}^{\perp}{}_{\leq 0}C, {}^{\perp}{}_{>0}C)$ on $\mathrm{D}(\mathcal{H})$. Then the aisle $\mathcal{U}$ coincides with ${}^{\perp}{}_{\leq 0}Q_\rho(C)$.*

Proof. The coaisle $\mathcal{V}$ is given by $Q_\rho({}^{\perp}{}_{>0}C)$. We will show that Q_ρ restricts to an equivalence of full subcategories $Q_\rho\colon {}^{\perp}{}_{>0}C \longrightarrow ({}^{\perp}{}_{\leq 0}Q_\rho(C))^{\perp}{}_0$. That is, we identify $({}^{\perp}{}_{\leq 0}Q_\rho(C))^{\perp}{}_0$ with $\mathcal{V}$ and it then follows that $\mathcal{U} = {}^{\perp}{}_{\leq 0}Q_\rho(C)$.

Let $M \in {}^{\perp}{}_{>0}C$ and let $X \in {}^{\perp}{}_{\leq 0}Q_\rho(C)$. By using the adjuction (Q, Q_ρ), we have that $Q(X) \in {}^{\perp}{}_{\leq 0}C$ and so, using the adjunction again, we have that $Q_\rho(M) \in ({}^{\perp}{}_{\leq 0}Q_\rho(C))^{\perp}{}_0$. So Q_ρ restricts to a well-defined functor and this restriction is clearly fully faithful. It remains to show that the restricted Q_ρ is dense. As $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H})) = {}^{\perp}{}_{\mathbb{Z}}Q_\rho(C) \subseteq {}^{\perp}{}_{\leq 0}Q_\rho(C)$, it follows that $({}^{\perp}{}_{\leq 0}Q_\rho(C))^{\perp}{}_0 \subseteq \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}(\mathcal{H}))^{\perp}{}_0 = \mathrm{im}Q_\rho$. Therefore, for $Y \in ({}^{\perp}{}_{\leq 0}Q_\rho(C))^{\perp}{}_0$, we have that $Q_\rho Q(Y) \cong Y$ and so it suffices to prove that $Q(Y) \in {}^{\perp}{}_{>0}C$. Let $X \in {}^{\perp}{}_{\leq 0}C$. Then $Q_\rho(X) \in Q_\rho({}^{\perp}{}_{\leq 0}C) \subseteq {}^{\perp}{}_{\leq 0}Q_\rho(C)$ and so $\mathrm{Hom}_{\mathrm{K}(\mathrm{Inj}(\mathcal{H}))}(Q_\rho(X), Y) = 0$. But then also $\mathrm{Hom}_{\mathrm{K}(\mathrm{Inj}(\mathcal{H}))}(X, Q(Y)) = 0$ and $Q(Y) \in {}^{\perp}{}_{>0}C$ as required. □

5 Purity and finiteness conditions on the heart.

Let $\mathcal{G}$ denote the heart of a t-structure $\mathbf{t}$. In many interesting examples, the heart $\mathcal{G}$ is locally finitely presented and so has an internal definition of purity (see [20] for more details). Moreover, the purity in $\mathcal{G}$ is intimately linked with finiteness conditions on $\mathcal{G}$. In this section we will investigate the connection between purity in $\mathbb{D}(\mathbb{1})$ and purity in the heart $\mathcal{G}$ of a t-structure satisfying the equivalent conditions of Theorem 4.6.

5.1 Purity in the heart.

Throughout this section let $\mathbb{D}$ be a compactly generated derivator and let $\mathbf{t} = (\mathcal{U},\mathcal{V})$ be a t-structure satisfying the equivalent conditions of Theorem 4.6 with heart $\mathcal{G}$.

Lemma 5.1. *For a directed system $\{X_i\}_{i\in I}$ in the heart $\mathcal{G}$, we have*

$$\mathbf{y}\left(\varinjlim_{i\in I} X_i\right) \cong \varinjlim_{i\in I} \mathbf{y}X_i.$$

Proof. This follows directly from [33, Thm. A] and Remark 3.5. □

Proposition 5.2. *Suppose that the heart $\mathcal{G}$ is locally finitely presented. If a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{G}$ is pure-exact, then the corresponding triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ is a pure triangle in $\mathbb{D}(\mathbb{1})$.*

Proof. The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is the direct limit of a directed system $\{0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to 0\}_{i\in I}$ of split exact sequences in $\mathcal{G}$. There is then a directed system of split (and

hence pure) triangles $\{X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to X_i[1]\}_{i\in I}$. Direct limits in Mod-$\mathcal{T}^{\mathrm{c}}$ are exact so

$$0 \to \varinjlim_{i\in I} \mathbf{y}X_i \to \varinjlim_{i\in I} \mathbf{y}Y_i \to \varinjlim_{i\in I} \mathbf{y}Z_i \to 0$$

is exact. But this is $0 \to \mathbf{y}X \xrightarrow{\mathbf{y}f} \mathbf{y}Y \xrightarrow{\mathbf{y}g} \mathbf{y}Z \to 0$ by Lemma 5.1 and so $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ is pure. □

Remark 5.3. The product (in $\mathcal{G}$) of a set $\{X_s\}_{s\in S}$ of objects in $\mathcal{G}$ is given by $H^0_{\mathbf{t}}(\prod_{s\in S} X_s)$ where the symbol $\prod$ is denotes the product in $\mathbb{D}(\mathbb{1})$. It follows that, in general, the definable subcategories of $\mathcal{G}$ may not be definable subcategories of $\mathbb{D}(\mathbb{1})$.

Corollary 5.4. *Let $\mathcal{D} \subseteq \mathcal{G}$ be a definable subcategory of $\mathcal{T} \simeq \mathbb{D}(\mathbb{1})$. If $\mathcal{G}$ is locally finitely presented, then $\mathcal{D}$ is a definable subcategory of $\mathcal{G}$.*

Proof. We use the closure conditions (2) of Theorem 3.11 to show that $\mathcal{D}$ is closed under pure subobjects, products and direct limits in $\mathcal{G}$. It follows immediately from Proposition 5.2 that $\mathcal{D}$ is closed under pure subobjects in $\mathcal{G}$. For closure under products in $\mathcal{G}$, let $\{X_s\}_{s\in S}$ be a set of objects in $\mathcal{D}$. Then $\prod_{s\in S} X_s \in \mathcal{D} \subseteq \mathcal{G}$ and so the product taken in $\mathcal{G}$ is $H^0_{\mathbf{t}}(\prod_{s\in S} X_s) \cong \prod_{s\in S} X_s$ and hence is contained in $\mathcal{D}$ as required. Finally let $\{X_i\}_{i\in I}$ be a directed system in $\mathcal{D}$. Then $\mathbf{y}\varinjlim_{i\in I} X_i \cong \varinjlim_{i\in I} \mathbf{y}X_i$ is contained in $\mathrm{Def}_{\text{Mod-}\mathcal{T}^{\mathrm{c}}}(\mathbf{y}\mathcal{D})$. Thus we have that $\varinjlim_{i\in I} X_i \in \mathcal{D}$ by Lemma 3.10. □

5.2 Locally noetherian hearts.

In this section, we will work in a compactly generated triangulated category $\mathcal{T}$ that we do not require to be the underlying category of a derivator. An object in a Grothendieck category $\mathcal{H}$ is called **noetherian** if the set of its subobjects satisfies the ascending chain condition. The category $\mathcal{H}$ is called **locally noetherian** if there is a set of noetherian generators.

We will need the properties of a partial cosilting object C stated in the next lemma. A proof for each statement essentially already exists in the literature in some form but we adapt the statements to fit in the current setting.

Lemma 5.5. *Let C be a partial cosilting object in $\mathcal{T}$ and let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ denote the associated t-structure. Then the following statements hold.*

1. $\mathrm{Prod}(C) = \mathcal{V} \cap (\mathcal{V}[-1])^{\perp_0}$.
2. *The cohomological functor $H^0_{\mathbf{t}}$ restricts to an equivalence*

$$H^0_{\mathbf{t}} \colon \mathrm{Prod}(C) \xrightarrow{\sim} \mathrm{Prod}(H^0_{\mathbf{t}}(C)).$$

3. *$H^0_{\mathbf{t}}(C)$ is an injective cogenerator for $\mathcal{G}_C$.*

Proof. By definition $\mathcal{V} = {}^{\perp_{>0}}C$. (1) This follows by the same argument as in [31, Lem. 4.5]; (2) This follows by the same argument as in [3, Lem. 2.8]; (3) The fact that the image $H^0_{\mathbf{t}}(\mathrm{Prod}(C))$ consists of injective objects is the dual of [26, Lem. 2(1)]. To show that $H^0_{\mathbf{t}}(C)$ is a cogenerator, let M be an object contained in $\mathcal{G}_C$ such that $\mathrm{Hom}_{\mathcal{T}}(M, C) = 0$. Then, in fact, $M \in {}^{\perp_{\geq 0}}C = \mathcal{V}[-1]$. So $M \in \mathcal{U}[-1] \cap \mathcal{V}[-1] = 0$. So for all non-zero M in $\mathcal{G}_C$, we have $\mathrm{Hom}_{\mathcal{T}}(M, C) \neq 0$. As $C \in \mathcal{V}$ we may consider the following triangle obtained by shifting and rotating the truncation triangle for $C[1]$: $\tau_{\mathcal{V}}(C[1])[-2] \to H^0_{\mathbf{t}}(C) \to C \to \tau_{\mathcal{V}}(C[1])[-1]$. Applying $\mathrm{Hom}_{\mathcal{T}}(M, -)$ and using that $M \in \mathcal{U}[-1] = {}^{\perp_0}(\mathcal{V}[-1])$ we obtain that $\mathrm{Hom}_{\mathcal{T}}(M, C) \cong \mathrm{Hom}_{\mathcal{G}_C}(M, H^0_{\mathbf{t}}(C))$ and so $H^0_{\mathbf{t}}(C)$ is indeed an injective cogenerator for $\mathcal{G}_C$. □

A pure-injective object N is **Σ-pure-injective** if $N^{(I)}$ (i.e. the I-indexed direct sum of copies of N) is pure-injective for every set I. Similarly, an injective object E is **Σ-injective** if $E^{(I)}$ is injective for every set I.

Proposition 5.6. *Let C be a pure-injective partial cosilting object. The heart $\mathcal{G}_C$ is locally noetherian if and only if C is Σ-pure-injective.*

Proof. First note that $\mathcal{G}_C$ is locally noetherian if and only if $\mathcal{G}_C$ has a Σ-injective cogenerator (see, for example, [15, Cor. 3]). By Lemma 5.5(2), we have that $H^0_{\mathbf{t}}(C)$ is an injective cogenerator of $\mathcal{G}_C$ and it follows from Lemma 5.5(1) that C is Σ-pure-injective in $\mathcal{T}$ if and only if $H^0_{\mathbf{t}}(C)$ is Σ-injective in $\mathcal{G}_C$. □

5.3 Locally coherent hearts.

Let $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ be a partial cosilting t-structure with heart $\mathcal{G}_C$ such that C is pure-injective. In this section we address the question of when $\mathcal{G}_C$ is locally coherent. A Grothendieck category $\mathcal{H}$ is called **locally coherent** if the subcategory $\mathcal{H}^{\mathrm{fp}}$ is an abelian exact subcategory of $\mathcal{H}$ and every object is a direct limit of objects in $\mathcal{H}^{\mathrm{fp}}$.

In [3], the authors show that $\mathcal{G}_C$ is equivalent to a localisation of Mod-$\mathcal{T}^{\mathrm{c}}$. Given Lemma 5.5, we observe that the proof of [3, Thm. 3.6] extends to partial cosilting objects. That is, we have the following equivalence of categories:

$$\mathcal{G}_C \simeq \text{Mod-}\mathcal{T}^{\mathrm{c}}/{}^{\perp_0}\mathbf{y}C$$

where ${}^{\perp_0}\mathbf{y}C$ is a hereditary torsion class in Mod-$\mathcal{T}^{\mathrm{c}}$ with torsion-free class given by $\mathrm{Cogen}(\mathbf{y}C) := \{F \in \text{Mod-}\mathcal{T}^{\mathrm{c}} \mid F \hookrightarrow \mathbf{y}C^I \text{ for some set } I\}$.

The category Mod-$\mathcal{T}^{\mathrm{c}}$ is locally coherent and so, in order to determine when $\mathcal{G}_C$ is locally coherent, we are particularly interested in localisations that preserve finiteness conditions. A torsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{L}$ is called **finite type** if $\mathcal{B}$ is closed under direct limits.

Proposition 5.7 ([19, Thm. 2.6], [16, Thm. 2.16])**.** *Let $\mathcal{L}$ be a locally coherent Grothendieck category and suppose $(\mathcal{A}, \mathcal{B})$ is a hereditary pair of finite type in $\mathcal{L}$. Then both $\mathcal{A}$ and $\mathcal{L}/\mathcal{A}$ are locally coherent.*

The previous proposition indicates that, if we understand which hereditary torsion pairs in Mod-$\mathcal{T}^{\mathrm{c}}$ are of finite type, then we will be closer to understanding when $\mathcal{G}_C$ is locally coherent. The following proposition relates this question to definable subcategories of $\mathcal{T}$.

Proposition 5.8 ([22])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category. Then there is a bijective correspondence between the following sets:*

- *The set of definable subcategories $\mathcal{D}$ of $\mathcal{T}$.*
- *The set of hereditary torsion pairs $(\mathcal{A}, \mathcal{B})$ of finite type in* Mod-$\mathcal{T}^{\mathrm{c}}$.

with mutually inverse bijections given as follows:

$$\mathcal{D} \mapsto \mathcal{A} = \{F \in \text{Mod-}\mathcal{T}^{\mathrm{c}} \mid \mathrm{Hom}_{\text{Mod-}\mathcal{T}^{\mathrm{c}}}(F, \mathbf{y}D) = 0 \text{ for all } D \in \mathrm{Pinj}(\mathcal{D})\}$$

$$\text{and } \mathcal{A} \mapsto \mathcal{D} = \{D \in \mathcal{T} \mid \mathrm{Hom}_{\text{Mod-}\mathcal{T}^{\mathrm{c}}}(F, \mathbf{y}D) = 0 \text{ for all } F \in \mathcal{A} \cap \text{mod-}\mathcal{T}^{\mathrm{c}}\}.$$

A pure-injective object X in $\mathcal{T}$ is called an **elementary cogenerator** if $\mathrm{Def}_{\mathcal{T}}(X) = \mathrm{Cogen}_*(X)$ where $\mathrm{Cogen}_*(X)$ denotes the class of all pure subobjects of products of copies of X. Similarly, we will use the notation $\mathrm{Def}_{\mathcal{G}}(\mathcal{E})$ for the definable subcategory generated by a set of objects $\mathcal{E}$ in a locally finitely presented Grothendieck category $\mathcal{G}$. Moreover, a pure-injective object X in $\mathcal{G}$ is called an **elementary cogenerator** if $\mathrm{Def}_{\mathcal{G}}(X) = \mathrm{Cogen}_*(X)$.

Remark 5.9. Elementary cogenerators were first considered in the context of model theory of modules (see, for example, [29, Sec. 9.4]). The condition is equivalent to say that X is an injective cogenerator in the localisation of Mod-$\mathcal{T}^{\mathrm{c}}$ at the hereditary torsion pair of finite type corresponding to $\mathrm{Def}_{\mathcal{T}}(C)$ via the bijection in Proposition 5.8.

Proposition 5.10. *Let $\mathcal{T}$ be a compactly generated triangulated category and let $(\mathcal{U}, \mathcal{V})$ be a partial cosilting t-structure with heart $\mathcal{G}_C$. If C is an elementary cogenerator, then $\mathcal{G}_C$ is locally coherent.*

Proof. If C is an elementary cogenerator, then the hereditary torsion pair $({}^{\perp_0}\mathbf{y}C, \mathrm{Cogen}(C))$ is the image of $\mathrm{Def}_{\mathcal{T}}(C)$ under the correspondence in Proposition 5.8. In particular, the torsion pair $({}^{\perp_0}\mathbf{y}C, \mathrm{Cogen}(C))$ is of finite-type and so $\mathcal{G}_C$ is locally coherent by Proposition 5.7. □

The following example relates elementary cogenerators to the class of fp-injective objects in $\mathcal{G}$. An object X in locally finitely presented Grothendieck category $\mathcal{G}$ is called **fp-injective** if $\mathrm{Ext}^1_{\mathcal{G}}(F, X) = 0$ for all objects F in $\mathcal{G}^{\mathrm{fp}}$. An object X is fp-injective if and only if X is **absolutely pure** i.e. every exact sequence of the form $0 \to X \to Y \to Z \to 0$ is pure-exact.

Example 5.11. Let $\mathcal{G}$ be a locally finitely presented category and suppose E is an injective cogenerator. Then $\mathrm{Cogen}_*(E)$ is the class of fp-injective objects in $\mathcal{G}$. By combining [32, Prop. 3.5] and [34, Thm. 3.2], we have that the category $\mathcal{G}$ is locally coherent if and only if $\mathrm{Cogen}_*(E)$ is closed under direct limits if and only if E is an elementary cogenerator in $\mathcal{G}$.

As our final result, we will show that, in the case where C is a partial cotilting object of $\mathbb{D}(\mathbb{1})$ for a compactly generated derivator $\mathbb{D}$, the converse of Proposition 5.10 also holds.

Theorem 5.12. *Let $\mathcal{T}$ be the underlying category of a compactly generated derivator $\mathbb{D}$ and consider a partial cotilting t-structure $\mathbf{t} = (\mathcal{U}, \mathcal{V})$ on $\mathcal{T}$ with pure-injective partial cotilting object C. Then $\mathcal{G}_C$ is locally coherent if and only if C is an elementary cogenerator.*

Proof. One direction is Proposition 5.10, we prove the converse. The composition and product of pure monomorphisms is a pure monomorphism, so it is clear that $\mathrm{Cogen}_*(C)$ is closed under pure subobjects and products. By Theorem 3.11, it remains to show that $\mathrm{Cogen}_*(C)$ is closed under directed homotopy colimits. Let I be a small directed category and let $X \in \mathbb{D}(I)$ with $X_i \in \mathrm{Cogen}_*(C)$. Then there exists a pure monomorphism $X_i \xrightarrow{f_i} C^{I_i}$ for each $i \in I$. Denote by $a_{ij} \colon X_i \to X_j$ the morphisms in $\mathrm{dia}_I X$. This induces a directed system $\{\mathbf{y}X_i\}_{i \in I}$ in Mod-$\mathcal{T}^{\mathrm{c}}$, and since C is pure-injective, we obtain a directed system of monomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbf{y}X_i & \xrightarrow{\mathbf{y}f_i} & \mathbf{y}C^{I_i} \\ & & \downarrow{\scriptstyle \mathbf{y}a_{ij}} & & \downarrow{\scriptstyle \beta_{ij}} \\ 0 & \longrightarrow & \mathbf{y}X_j & \xrightarrow[\mathbf{y}f_j]{} & \mathbf{y}C^{I_j} \end{array}$$

and $\beta_{ij} \cong \mathbf{y}b_{ij}$ for some $C^{I_i} \xrightarrow{b_{ij}} C^{I_j}$ in $\mathbb{D}(\mathbb{1})$.

Then $0 \to \varinjlim_{i\in I} \mathbf{y}X_i \xrightarrow{\varinjlim_{i\in I} \mathbf{y}f_i} \varinjlim_{i\in I} \mathbf{y}C^{I_i}$ is a monomorphism. By Lemma 5.1, we have that

$$\mathbf{y}\left(\varinjlim_{i\in I} C^{I_i}\right) \cong \varinjlim_{i\in I} \mathbf{y}C^{I_i}.$$

Moreover, since $\mathcal{G}_C$ is locally coherent we may apply Example 5.11, so the object $\varinjlim_{i\in I} C^{I_i}$ is fp-injective. That is, there exists a pure monomorphism $\varinjlim_{i\in I} C^{I_i} \xrightarrow{h} C^J$ for some set J. Composing $\mathbf{y}h$ with $\varinjlim_{i\in I} \mathbf{y}f_i$ we obtain a monomorphism $\mathbf{y}\mathrm{hocolim}_I(X) \cong \varinjlim_{i\in I} \mathbf{y}X_i \to \mathbf{y}C^J$ in Mod-$\mathcal{T}^{\mathrm{c}}$. As C^J is pure-injective, this is induced by a pure monomorphism $\mathrm{hocolim}_I(X) \to C^J$. That is $\mathrm{hocolim}_I(X) \in \mathrm{Cogen}_*(C)$. □

Example 5.13. Consider the compactly generated derivator $\mathbb{D}_R$ from Example 2.1. We may consider elementary cogenerators in the heart $\mathcal{G} \simeq$ Mod-R of the standard t-structure in $\mathbb{D}_R(\mathbb{1})$. As this is a definable subcategory of $\mathbb{D}_R(\mathbb{1}) \simeq \mathrm{D}(\text{Mod-}R)$ (defined by the functors $\mathrm{Hom}_{\mathbb{D}_R(\mathbb{1})}(R[i], -)$ for $i \neq 0$), a cotilting module (in the sense of [2]) is an elementary cogenerator in Mod-R if and only if it is an elementary cogenerator in D(Mod-R).

A Appendix: The axioms (Der1)-(Der4) and shifted derivators.

A.1 The axioms.

We will now state the axioms defining a derivator. In order to state (Der4) we will need the following definition. Let $u\colon A \to B$ be a morphism in $\mathcal{C}at$ and b be an object in B. Then we may form the **comma category** u/b as follows: the objects of u/b are given by pairs (a, f) with a an object in A and $f\colon u(a) \to b$. The morphisms $(a, f) \to (a', f')$ in u/b are given by morphisms $g\colon a \to a'$ in A such that $f = f' \circ u(g)$. Let $p\colon u/b \to A$ be the obvious projection functor. We may perform the dual construction to obtain the comma category b/u and projection functor $q\colon b/u \to A$.

A prederivator $\mathbb{D}$ is a **derivator** if it has the following properties.

(Der1) For every small family $\{A_i\}_{i\in I}$ of small categories, the canonical functor

$$\mathbb{D}(\coprod_{i\in I} A_i) \to \prod_{i\in I} \mathbb{D}(A_i)$$

is an equivalence of categories.

(Der2) For every small category A, a morphisms $f\colon X \to Y$ in $\mathbb{D}(A)$ is an isomorphism if and only if $f_a\colon X_a \to Y_a$ is an isomorphism for every object a in A.

(Der3) For all functors $u\colon A \to B$, the restriction functor $u^*\colon \mathbb{D}(B) \to \mathbb{D}(A)$ has a left adjoint $u_!\colon \mathbb{D}(A) \to \mathbb{D}(A)$ and a right adjoint $u_*\colon \mathbb{D}(A) \to \mathbb{D}(B)$.

(Der4) For all functors $u\colon A \to B$ and all objects b in B, there are canonical isomorphisms $\pi_! p^* \to b^* u_!$ and $b^* u_* \to \pi_* q^*$.

The functor π_* is a homotopy limit functor (see the next subsection) and so (Der4) means that, for all functors $u\colon A \to B$, the image of the right Kan extension u^* can be expressed point-wise in terms of these simpler right Kan extensions. A similar statement may be made for left Kan extensions.

The canonical isomorphisms arising in (Der4) are instances of canonical mate transformations. Many of the proofs in the later sections of this paper will refer to the calculus of canonical mates and the existence of homotopy exact squares. For a systematic treatment of these techniques, we refer the reader to [12, Sec. 1.2].

A.2 Shifted derivators.

Let B be a small category and consider the 2-functor $B\times -\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}at^{\mathrm{op}}$ taking each A to the product $B \times A$. Then the **shifted derivator** $\mathbb{D}^B$ is defined to be the derivator $\mathbb{D}$ precomposed with $B \times -$. This is clearly a 2-functor and in [12, Thm. 1.25] it is shown that $\mathbb{D}^B$ is a derivator.

The following definitions describe the restriction functors and Kan extensions in the shifted derivator. We have added decorations to indicate which derivator they have been taken with respect to. We will also use this notation in later sections when necessary:

- For each small category A, we have that $\mathbb{D}^B(A) := \mathbb{D}(B \times A)$;
- For each functor $u\colon A \to C$ in $\mathcal{C}at$, we have that $u^*_{\mathbb{D}^B} := (\mathrm{id}_B \times u)^*_{\mathbb{D}}$, $u_*^{\mathbb{D}^B} := (\mathrm{id}_B \times u)_*^{\mathbb{D}}$ and $u_!^{\mathbb{D}^B} := (\mathrm{id}_B \times u)_!^{\mathbb{D}}$;
- The evaluation functors and the functors $\mathrm{hocolim}_A^{\mathbb{D}^B}$, $\mathrm{holim}_A^{\mathbb{D}^B}$ and $\mathrm{dia}_A^{\mathbb{D}^B}$ are all defined as in Sections 2.1.2 and 2.1.3 using the above definitions.
- By [12, Prop. 2.5] we have that $\mathrm{hocolim}_A^{\mathbb{D}}(X_b^{\mathbb{D}^A}) \cong \mathrm{hocolim}_A^{\mathbb{D}^B}(X)_b^{\mathbb{D}}$ and $\mathrm{holim}_A^{\mathbb{D}}(X_b^{\mathbb{D}^A}) \cong \mathrm{holim}_A^{\mathbb{D}^B}(X)_b^{\mathbb{D}}$ for all X in $\mathbb{D}(B \times A)$ and b in B.

Proposition A.1 ([12, Prop. 4.3])**.** *Let $\mathbb{D}$ be a strong and stable derivator. For any small category A, the shifted derivator $\mathbb{D}^A$ is strong and stable.*

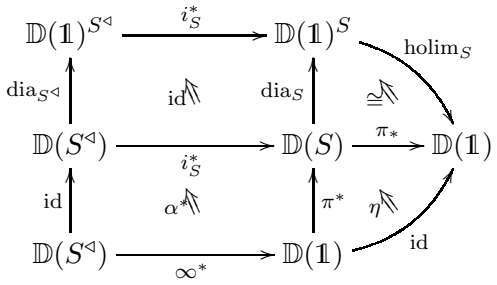
Example A.2. Let k be a field and let Q be a finite quiver. Then we can consider the free category generated by Q and so we can also consider $\mathbb{D}_k(Q)$. Unravelling the definitions, we have that $\mathbb{D}_k(Q)$ is equivalent to the derived category $\mathrm{D}(\text{Mod-}kQ)$ of modules over the path algebra kQ. Now, for every small category A, we have that $(\text{Mod-}k)^{Q\times A} \cong (\text{Mod-}kQ)^A$ and so $\mathbb{D}_{kQ}(A) \cong \mathbb{D}_k(Q \times A)$. The derivator $\mathbb{D}_{kQ}$ is therefore the shifted derivator $\mathbb{D}_k^Q$.

B Appendix: Proof of Proposition 2.7.

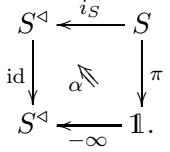
For the proof of Proposition 2.7, we will require the following lemma, which was shared with the author by Moritz Groth. For a small category A, let $A^{\triangleleft}$ denote the category obtained from A by adding a new initial object $-\infty$ and let $i_A\colon A \to A^{\triangleleft}$ be the canonical inclusion. As in [13], we will call an object X in $\mathbb{D}(A^{\triangleleft})$ a **limiting cone** if it is in the essential image of $(i_A)_*$.

Lemma B.1. *Let $\mathbb{D}$ be a derivator and let S be a discrete category. An object X in $\mathbb{D}(S^{\triangleleft})$ is a limiting cone if and only if the underlying diagram $\mathrm{dia}_{S^{\triangleleft}}(X)$ is a product cone i.e. $\mathrm{dia}_{S^{\triangleleft}}(X)$ exhibits $X_{-\infty}$ as the product of the objects $\{X_s\}_{s\in S}$ in $\mathbb{D}(\mathbb{1})$.*

Proof. Consider the diagram



where the top right triangle is a natural isomorphism by [12, Prop. 1.7]; the natural transformation in the bottom right triangle is the unit of the adjunction (π^*, π_*) and α^* is induced by the square



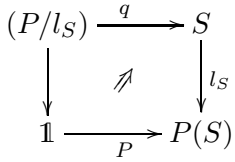
Note that $\mathrm{dia}_S \circ \pi^*$ is the constant diagram functor Δ_S, and so the vertical pasting of the triangles on the right is the diagonal map $Y \to \mathrm{holim}_S(\Delta_S(Y)) = \prod_{s\in S} Y$ for each object Y in $\mathbb{D}(\mathbb{1})$. The vertical pasting of the squares on the left yields a natural transformation $\Delta_S(X_{-\infty}) \to i_S^*(\mathrm{dia}_{S^\triangleleft}(X))$ induced by the structure maps of X. The pasting of the entire diagram therefore gives rise to the map $X_{-\infty} \to \prod_{s\in S} X_s$ produced by the universal property of the product applied to $\mathrm{dia}_{S^\triangleleft}(X)$. So $\mathrm{dia}_{S^\triangleleft}(X)$ is exhibiting $X_{-\infty}$ as the product if and only if this morphism is an isomorphism. Since the top row is inhabited by invertible natural transformations, we have that the total pasting is a natural isomorphism whenever the pasting of the bottow row is a natural isomorphism. By [13, Prop. 2.6], this occurs exactly when X is a limiting cone. □

Proof of Proposition 2.7. We first define a small category $P(S)$ containing each proper filter on S as a full subcategory and show that there exists $\tilde{X}$ in $\mathbb{D}(P(S))$ satisfying the conditions of the theorem. Later we will restrict to the filter $\mathcal{F}$ in particular.

Let $P(S)$ be the small category with objects $\emptyset \neq P \in \mathcal{P}(S)$ and morphisms $f_{PQ}\colon P \to Q$ if and only if $Q \subseteq P$. Consider the functor $l_S\colon S \to P(S)$ defined by $s \mapsto \{s\}$ and the right Kan extension $(l_S)_*\colon \mathbb{D}(S) \to \mathbb{D}(P(S))$ along l_S. For each X in $\mathbb{D}(S)$, define

$$\tilde{X} := (l_S)_*(X).$$

Step 1: Show that for each $P \in P(S)$, the value $\tilde{X}_P$ of $\tilde{X}$ at P is isomorphic to $\prod_{p\in P} X_p$: Consider the slice square



By (Der4), the associated canonical mate transformation $P^*(l_S)_* \to \mathrm{holim}_{(P/l_S)} q^*$ is an isomorphism. Note that the comma category (P/l_S) is equivalent to the discrete category P and q is the canonical embedding of $P \subseteq S$. By [12, Prop. 1.7], we have that

$$\tilde{X}_P = (l_S)_*(X)_P \cong \mathrm{holim}_{(P/l_S)} q^*(X) \cong \prod_{p\in P} X_p.$$

Step 2: Show that the value $\tilde{X}_{f_{PQ}}\colon \tilde{X}_P \to \tilde{X}_Q$ of $\tilde{X}$ at f_{PQ} is the canonical projection ϕ_{PQ} where P is in $P(S)$ and $Q = \{p\}$ for $p \in P$: Consider the fully faithful functor $v_P\colon P^{\triangleleft} \to P(S)$ where $p \mapsto \{p\}$ and $-\infty \mapsto P$. By Lemma B.1, it suffices to show that $v_P^*\tilde{X}$ is a limiting cone. Consider the square

$$\begin{array}{ccc} P & \xrightarrow{i_P} & P^{\triangleleft} \\ {\scriptstyle j_P}\downarrow & \swarrow_{id} & \downarrow{\scriptstyle v_P} \\ S & \xrightarrow[l_S]{} & P(S) \end{array}$$

where j_P is the embedding of P into S. If this square is homotopy exact then $v_P^*(l_S)_*(X) \cong (i_P)_* j_P^*(X)$ as desired. The square can be expressed as the following vertical pasting

$$\begin{array}{ccc} P & \xrightarrow{i_P} & P^{\triangleleft} \\ {\scriptstyle id}\downarrow & \swarrow_{id} & \downarrow{\scriptstyle v_P} \\ P & \xrightarrow[l_P]{} & P(P) \\ {\scriptstyle j_P}\downarrow & \swarrow_{id} & \downarrow{\scriptstyle j_{P(P)}} \\ S & \xrightarrow[l_S]{} & P(S). \end{array}$$

By [13, Lem. 2.12], the top square is homotopy exact and so, by [12, Lem. 1.14], it suffices to show that the bottom square is homotopy exact.

Since j_P and $j_{P(P)}$ are fully faithful, it follows from [13, Lem. 2.12] that it is enough to show that the canonical mate transformation $(j_P)_! l_P^* \to l_S^*(j_{P(P)})_!$ is an isomorphism for all $s \in S \setminus P$. Let Y be an object in $\mathbb{D}(P(P))$ and note that $l_S^*(j_{P(P)})_!(Y)_s \cong (j_{P(P)})_!(Y)_{\{s\}}$. The functors $j_{P(P)}$ and j_P are both cosieves. Since $\{s\}$ is not in the image of $j_{P(P)}$and s is not in the image of j_P, it follows from [12, Prop. 1.23] that both $(j_{P(P)})_!(Y)_{\{s\}}$ and $(j_P)_! l_P^*(Y)_s$ are isomorphic to initial objects in $\mathbb{D}(\mathbb{1})$. Thus $(j_P)_! l_P^*(Y)_s \to l_S^*(j_{P(P)})_!(Y)_s$ is the unique isomorphism between initial objects. It follows that the bottom square is homotopy exact as required.

Step 3: Show that $\tilde{X}_{f_{PQ}}\colon \tilde{X}_P \to \tilde{X}_Q$ is the canonical projection ϕ_{PQ} for each $Q \subseteq P$ in $P(S)$: Let $k_{Q^{\triangleleft}}\colon (Q^{\triangleleft})^{\triangleleft} \to P(S)$ be the functor defined by $q \mapsto \{q\}$ for all $q \in Q$, $-\infty \mapsto Q$ and $-\infty - 1 \mapsto P$. Then, by Step 3, the underlying diagram $\mathrm{dia}_{(Q^{\triangleleft})^{\triangleleft}}(k_{Q^{\triangleleft}}^*\tilde{X})$ is isomorphic to the incoherent diagram consisting of commutative triangles

$$\begin{array}{ccc} \prod_P X_p & \xrightarrow{u} & \prod_Q X_q \\ & {\scriptstyle \pi_{P\{q\}}}\searrow & \downarrow{\scriptstyle \pi_{Q\{q\}}} \\ & & X_q \end{array}$$

for each $q \in Q$. By the universal property of the product, the morphism u must be the canonical projection.

Step 4: Restrict to $\mathcal{F}$: Let $u\colon \mathcal{F} \to P(S)$ be the fully faithful functor mapping each $P \in \mathcal{F}$ to itself. Then let $\mathrm{Red}_{\mathcal{F}} := u^* \circ (l_S)_*$. It follows from the above steps that $\mathrm{Red}_{\mathcal{F}}(X)$ has the desired properties. □

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