ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **4** (2015), no.2, 161–174
ISSN 1857-8365 UDC: 517.546:517.51]:517.58

THE ROLE OF SPECIAL FUNCTIONS AND GENERALIZED FRACTIONAL CALCULUS IN STUDYING CLASSES OF UNIVALENT FUNCTIONS

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Presented at the 11th International Symposium
GEOMETRIC FUNCTION THEORY AND APPLICATIONS
24-27 August 2015, Ohrid, Republic of Macedonia

Abstract. In this survey we aim to emphasize the efficient role of the special functions and their properties for obtaining some mapping, distortion and other characterization properties of the operators of the generalized fractional calculus, when acting on the class of the univalent functions in the unit disk and some of its subclasses. Thus we provide an unified approach to attack similar problems for all particular cases of our operators of generalized fractional integration. The results surveyed here extend the corresponding ones for many known linear integral operators considered in geometric function theory by various authors, as well as our previous results.

1. Introduction

Let $A(n)$, $A := A(1)$ denote the classes of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1,2,3,\dots\}), \qquad f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (n=1), \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. By $S(n) \subset A(n)$ it is denoted the subclass of *univalent functions* in U. In geometric function theory various their subclasses have been studied. We give the denotations of those touched in this survey:

– $T(n) \subset S(n)$: functions with *negative* coefficients,

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0;$$

– $S^*_\alpha(n) \subset S(n)$: functions *starlike of order* α, $0 \leq \alpha < 1$, iff

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad \text{for } \alpha = 0: \ S^*_0(n), \quad \text{or} \quad S^*(\alpha), \text{ resp. } S^* \text{ if } n = 1;$$

– $K_\alpha(n) \subset S(n)$: functions *convex of order* α, $0 \leq \alpha < 1$, iff

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad \text{for } \alpha = 0: \ K_0(n), \quad \text{or} \quad K(\alpha), \text{ resp. } K \text{ if } n = 1;$$

– $T_\alpha(n) := S_\alpha \bigcap T(n)$; $L_\alpha(n) := K_\alpha(n) \bigcap T(n)$;
– note that $f(z) \in K_\alpha(n)$ if and only if $zf'(z) \in S^*_\alpha(n)$;
– note that for any $0 \leq \alpha < 1$: $S^*_\alpha(n) \subseteq S^*_0(n)$, $K_\alpha(n) \subseteq K_0(n)$, $K_\alpha(n) \subset S^*_\alpha(n)$.

2010 *Mathematics Subject Classification.* 30C45, 26A33, 33C20, 33C60.
Key words and phrases. Classes of univalent functions, special functions, generalized hypergeometric functions, operators of fractional calculus.

In the geometric function theory (GFT), also the notion of *Hadamard product (convolution)* of two analytic functions f, q in U is used, defined as:

$$f(z)=\sum_{k=0}^{\infty} a_k z^k,\ \ g(z)=\sum_{k=0}^{\infty} b_k z^k\ \ \mapsto f*g(z):=\sum_{k=0}^{\infty} a_k b_k z^k. \tag{1.2}$$

In this survey we aim to emphasize the efficient role of the special functions and their properties for obtaining some mapping, distortion and other characterization properties of the operators of the generalized fractional calculus (GFC) in $A(n), S(n), T(n)$ etc. subclasses. We demonstrate an unified approach to attack similar problems for all mentioned particular cases of our operators of generalized fractional integration. The results surveyed here extend the corresponding ones for many known linear integral operators considered in classes of univalent functions by various authors, as well as our previous results.

2. Definitions of some special functions

We need to remind briefly the definitions of some special functions used in this paper. For details and properties, see the basic contemporary handbooks as e.g. [7], [24], [29], [12], etc., also Appendix of [14].

Definition 2.1. *The Wright generalized hypergeometric functions ${}_p\Psi_q(z)$, called also Fox-Wright functions are defined as:*

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\dots,(\alpha_p,A_p)\\(\beta_1,B_1),\dots,(\beta_q,B_q)\end{array}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+kA_1)\dots\Gamma(\alpha_p+kA_p)}{\Gamma(\beta_1+kB_1)\dots\Gamma(\beta_q+kB_q)}\ \frac{z^k}{k!}. \tag{2.1}$$

When all $A_1=\dots=A_p=1, B_1=\dots=B_q=1$, these are reduced to the more popular generalized hypergeometric ${}_pF_q$-functions, namely:

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,1),\dots,(\alpha_p,1)\\(\beta_1,1),\dots,(\beta_q,1)\end{array}\middle| z\right]=\omega^{-1}\ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;\,z)\,,$$

$${}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;\,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\ \frac{z^k}{k!}\,,$$

with the Pochhammer symbol and the constant ω denoted as follows:

$$(\alpha)_k:=\Gamma(\alpha+k)/\Gamma(\alpha),\quad \omega:=\Big[\prod_{j=1}^{q}\Gamma(\beta_j)/\prod_{i=1}^{p}\Gamma(\alpha_i)\Big].$$

The series ${}_p\Psi_q$ are usually considered for parameters $\alpha_i,\beta_j\in\mathbb{C}$ and $A_i>0, B_j>0,\ i=1,\dots,p;\, j=1,\dots,q$, and the numbers $\Delta=\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i,\ r=\prod_{i=1}^{p}A_i^{-A_i}\prod_{j=1}^{q}B_j^{B_j}$, $\mu=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i+(p-q)/2$ play important role for its properties as analytic functions, see e.g. [12, p.56, Th. 1.5]. If $\Delta>-1$, (2.1) is absolutely convergent series for all $z\in\mathbb{C}$, and if $\Delta=-1$, then it is absolutely convergent for $|z|<r$ and $\Re(\mu)>1/2$. For example, the ${}_pF_q$-function, when $p\le q$, $\Delta\ge 0$, is an entire function. But if $p=q+1$, it is absolutely convergent in the unit disk $U=\{z:|z|<1\}$ $(r=1)$, and diverges for all $z\neq 0$ if $p>q+1$. In the case $z=1$, for ${}_{q+1}F_q$ we require the condition (see [7], §4.1)

$$\Re\{\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{q+1}\alpha_i\}>0.$$

For the considered operators of GFC the related generalized hypergeometric functions $({}_{m+1}\Psi_m,\ {}_{m+1}F_m)$ have all real parameters $\alpha_i,\ \beta_j$ (so omit the sign $\Re$), $p=q+1$, $r=1$.

The ${}_p\Psi_q$- and ${}_pF_q$ functions are special cases of the more general special functions known as Fox's H-functions and Meijer's G-functions.

Definition 2.2. *By a Fox's H-function we mean a generalized hypergeometric function defined by means of the Mellin-Barnes type contour integral*

$$(2.2)\quad H^{m,n}_{p,q}\left[\sigma\left|\begin{array}{c}(a_k,A_k)_1^p\\(b_k,B_k)_1^q\end{array}\right.\right]=\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\frac{\prod\limits_{k=1}^{m}\Gamma(b_k-B_ks)\prod\limits_{j=1}^{n}\Gamma(1-a_j+sA_j)}{\prod\limits_{j=n+1}^{p}\Gamma(a_j-sA_j)\prod\limits_{k=m+1}^{q}\Gamma(1-b_k+sB_k)}\sigma^s\,ds,$$

one can see more in [24], [29], [12]. *When* $A_1=\ldots=A_p=1, B_1=\ldots=B_q=1$, (2.2) *turns into the simpler Meijer's G-function* $G^{m,n}_{p,q}$, *see* [7, Vol.1, Ch.5], [14, Appendix].

Here $\mathcal{L}$ is a suitable contour in $\mathbb{C}$, the orders (m,n,p,q) are integers $0\le m\le q,\ 0\le n\le p$ and the parameters $a_j\in\mathbb{R},\ A_j>0\ (j=1,\ldots,p),\ b_k\in\mathbb{R},\ B_k>0\ (k=1,\ldots,q)$ are such that $A_j(b_k+l)\ne B_k(a_j-l'-1)\ (l,l'=0,1,2,\ldots)$. For various type of contours and conditions for existence and analyticity of these special functions inside or outside disks $\subset\mathbb{C}$ with radii $\rho=\prod_{j=1}^{p}A_j^{-A_j}\prod_{k=1}^{q}B_k^{B_k}>0$, see the mentioned handbooks. The H- and G-functions are analytic functions of z with a branch point at the origin. Especially, the kernel functions $H^{m,0}_{m,m}$ and $G^{m,0}_{m,m}$ $(n=0,m=p=q)$ of the operators of generalized fractional calculus that we consider, are analytic functions in the unit disc U and vanish identically outside it (for $|z|>1$).

We like to emphasize that the H- and G-functions encompass almost all the elementary and special functions as particular cases, and thus the knowledge on them is very useful, see some long lists of examples in Kiryakova [14, Appendix], etc. Specially,

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\sigma)=[\prod_{k=1}^{q}\Gamma(b_k)/\prod_{j=1}^{p}\Gamma(a_j)]\,G^{1,p}_{p,q+1}\left[-\sigma\left|\begin{array}{c}1-a_1,\ldots,1-a_p\\0,1-b_1,\ldots,1-b_q\end{array}\right.\right],$$

$$\begin{aligned}{}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\middle|\sigma\right]&=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\ldots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\ldots\Gamma(b_q+kB_q)}\,\frac{\sigma^k}{k!}\\&=H^{1,p}_{p,q+1}\left[-\sigma\left|\begin{array}{c}(1-a_1,A_1),\ldots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)\end{array}\right.\right].\end{aligned}$$

3. Operators of generalized fractional calculus

In the papers on classes of univalent functions, various linear integral or differ-integral operators have been introduced by different authors which are variations or generalizations of the operators of the fractional calculus, even if not announced or observed to be of this kind. On the other hand, several generalizations of the classical fractional calculus have been introduced since the 70's of last century by means of various special functions as kernel-functions, on the place of the elementary functions there, see some details in [16], [14], and other Kiryakova's papers cited here. It was Kalla who in 1970 proposed the most general form of the *operators of generalized fractional integration*, as

$$(3.1)\qquad \mathcal{I}f(z)=z^{-\gamma-1}\int\limits_0^z\Phi(t/z)t^{\gamma}f(t)dt=\int\limits_0^1\Phi(\sigma)\sigma^{\gamma}f(z\sigma)d\sigma,$$

with an arbitrary continuous (or analytic in a disk $\in \mathbb{C}$) kernel-function $\Phi(\sigma)$, instead of the kernels $(1-\sigma)^{\delta}\sigma^{\gamma}/\Gamma(\delta)$ of the *Erdélyi-Kober* (E-K), or resp. of the *Riemann-Liouville* (R-L, $\gamma = 0$) operators of integration of arbitrary (noninteger) order $\delta > 0$.

In Kiryakova [14], see also paper [13] and next ones, we have introduced operators of the form (3.1) but with very suitable choice of the special functions: to be G- and H-functions of peculiar orders $(m, 0, m, m)$ that allow to develop a full theory with applications of the so-called *generalized fractional calculus.* The generalized fractional integrals there are based on commutable compositions of E-K operators (depending each on 3 parameters $\delta \geq 0, \gamma$ and additional one $\beta > 0$) but instead of by repeated integrals, defined by means of equivalent single integral operators involving $H^{m,0}_{m,m}$- and $G^{m,0}_{m,m}$- kernel functions with vector parameters $(\delta_i)_1^m$, $(\gamma_i)_1^m$, $(\beta_i)_1^m$.

Definition 3.1. *Let $m \geq 1$ be an integer; $\delta_i \geq 0, \gamma_i \in \mathbb{R}, \beta_i > 0$ $(i = 1, \ldots, m)$. We consider $\delta = (\delta_1, \ldots, \delta_m)$ as a multi-order of fractional integration, resp., $\gamma = (\gamma_1, \ldots, \gamma_m)$ as multi-weight, $\beta = (\beta_1, \ldots, \beta_m)$ as additional parameter. The integral operators defined by:*

$$(3.2)\quad I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_i + \delta_i + 1 - 1/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + 1 - 1/\beta_i, 1/\beta_i)_1^m \end{matrix} \right. \right] f(z\sigma)d\sigma, \quad \text{if} \quad \sum_{i=1}^m \delta_i > 0,$$

or as the identity $I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = f(z)$, if $\delta_1 = \delta_2 = \cdots = \delta_m = 0$, are said to be multiple (m-tuple) Erdélyi-Kober fractional integration operators. And more generally, all the operators of the form

$$If(z) = z^{\delta_0} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \quad \text{with} \quad \delta_0 \geq 0,$$

are called briefly generalized (m-tuple) fractional integrals.

The corresponding generalized fractional derivatives are denoted by $D^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$ and defined by means of explicit differintegral expressions (see [14]), similarly to the idea for the classical Riemann-Liouville derivative. For $m = 1$ operators (3.2) turn into the *Erdélyi-Kober fractional integrals* $I^{\gamma,\delta}_{\beta}$, widely used in the applied mathematical analysis and to the classical *Riemann-Liouville fractional integrals* I^{δ} (with $\gamma = 0$):

$$(3.3)\qquad I^{\gamma,\delta}_{\beta} f(z) = \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} \sigma^{\gamma} f(z\sigma^{1/\beta})d\sigma, \quad I^{\delta} f(z) = z^{\delta} \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(z\sigma)d\sigma,$$

namely:

$$I^{\gamma,\delta}_{\beta} f(z) = I^{\gamma,\delta}_{1,1} f(z), \quad I^{\delta} f(z) = z^{\delta} I^{0,\delta}_{1,1} f(z);$$

for $m = 2$ – into the *hypergeometric fractional integrals* (Love, Saigo, Hohlov, etc.); and for various other special choices of $m \geq 1$ and of parameters, to many other generalized integration and differentiation operators used in analysis, including in univalent functions theory, integral transforms and special functions, differential and integral equations, etc. The main feature of the generalized (m-tuple) fractional integrals is that single integrals (3.2) involving H-functions (or G-functions in the simpler case of all equal $\beta_i = \beta > 0, i = 1, \ldots, m$) can be equivalently represented by means of *commutative compositions of finite number* (m) *of Erdélyi-Kober integrals* (3.3), namely:

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \left[\prod_{i=1}^m I^{\gamma_k,\delta_k}_{\beta_k}\right] f(z) = \int_0^1 \ldots \int_0^1 \left[\prod_{i=1}^m \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right] f\left(z\sigma_1^{\frac{1}{\beta_1}} \ldots \sigma_m^{\frac{1}{\beta_m}}\right) d\sigma_1 \ldots d\sigma_m.$$

This decomposition formula is the key to numerous applications of (3.2), while the simple but quite effective tools of the G- and H-functions make essentially easier their study.

Using the simple properties of the Fox H-function, evaluating the integral in (3.2) according to our formula (E.21) from [14, App.], see also in [20], Lemma 0, one easily obtains the following.

Lemma 3.1. *For $\delta_i \geq 0, \gamma_i \in \mathbb{R}, \beta_i > 0$ $(i = 1, \dots, m)$, and each $p > \max_i [-\beta_i(\gamma_i + 1)]$,*

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z^p\} = \lambda_p\, z^p \quad \text{with} \quad \lambda_p = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + p/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + p/\beta_i)} > 0. \tag{3.4}$$

Then under the conditions

$$\delta_i \geq 0, \quad \gamma_i \geq -1, \quad \beta_i > 0, \quad i = 1, \dots, m, \tag{3.5}$$

the image (3.4) *holds for each $p \geq 0$, i.e. in the classes A, $A(n)$ and their subclasses.*

In view of formula (3.4), to stay in the classes $A(n)$, $S(n)$, $T(n)$, it is suitable to *normalize* the operators (3.2) by the multiplier constant $N := [\lambda_1]^{-1}$ $(p = 1)$. Therefore, further we consider the generalized fractional integrals (using the *same name* for the normalized version, but stressing this fact by an additional "tilde" in the denotation: $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} := [\lambda_1]^{-1} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$), as

$$\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) := \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + 1 + 1/\beta_i)}\; I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = N\, I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z). \tag{3.6}$$

Before to proceed with next statements for the properties of the operators (3.2)-(3.6), we first provide some auxiliary results for the multiplier sequence $\theta(k)$ that appears in their representation as Hadamard products with the specific special functions. As these are used essentially in the further proofs, we formulate them as a separate proposition.

Proposition 3.1. *Let us introduce and consider the following auxiliary function of the index k, $k = n + 1, n + 2, \cdots$:*

(3.7)

$$\theta(k) = \prod_{i=1}^{m} \left[\frac{\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + 1 + 1/\beta_i)}\; \frac{\Gamma(\gamma_i + 1 + k/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + k/\beta_i)} \right] = N \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + k/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + k/\beta_i)}.$$

Evidently,

$$\theta(1) = 1 \quad \text{and} \quad \theta(k) > 0 \quad \text{for all} \quad k. \tag{3.8}$$

The following properties hold:

a)

$$\lim_{k \to \infty} |\theta(k)|^{1/k} = 1;$$

b) $\theta(k)$ is nonincreasing function of k, therefore

$$0 < \theta(k) \leq \theta(n + 1) \quad \text{for each} \quad k \geq n + 1. \tag{3.9}$$

Proof. For brevity, denote

$$a_i = \gamma_i + \delta_i + 1, \quad b_i = \gamma_i + 1, \quad \kappa_i = k/\beta_i, \quad i = 1, \dots, m; \; k = n + 1, \dots, \tag{3.10}$$

from where and according to (3.5), we have $a_i \geq b_i$, and $\kappa_i \to \infty$ as $k \to \infty$. To prove a), we use the known asymptotic formula for the Γ-function ([7, §1.18, (4)]):

$$\frac{\Gamma(b + \kappa)}{\Gamma(a + \kappa)} \sim \kappa^{b-a} \;\text{ as }\; \kappa \to \infty, \;\text{ which yields }\; \left[\frac{\Gamma(b_i + \kappa)}{\Gamma(a_i + \kappa)}\right]^{1/k} \sim \left(\kappa_i^{-\delta_i}\right)^{1/k} = \left(k^{1/k}\right)^{-\delta_i} \cdot q_i^{1/k}$$

with $q_i := \beta_i^{\delta_i}$, and the limit equalities $\lim\limits_{k\to\infty} k^{1/k} = 1$, $\lim\limits_{k\to\infty} q_i^{1/k} = 1$ for $q_i = \text{const}$, give:

$$\lim_{\kappa_i\to\infty}\left[\frac{\Gamma(b_i+\kappa_i)}{\Gamma(a_i+\kappa_i)}\right]^{1/k} = 1, \quad \text{and also} \quad \lim_{k\to\infty}\left[\frac{\Gamma(a_i+1/\beta_i)}{\Gamma(b_i+1/\beta_i)}\right]^{1/k} = 1, \quad i=1,\dots,m.$$

We have then

$$\lim_{k\to\infty}|\theta(k)|^{1/k} = \lim_{k\to\infty} N^{1/k}\cdot\lim_{k\to\infty}\prod_{i=1}^{m}\left[\frac{\Gamma(b_i+\kappa_i)}{\Gamma(a_i+\kappa_i)}\right]^{1/k} = 1\cdot 1 = 1.$$

To verify b), we start with the fact that the digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing for all $x>0$, since $\Psi'(x) > 0$ for all $x \neq -j$, $j = 0,1,2,\dots$, see the representation of $\Psi^{(n)}(x)$, [7, §1.16,(9)], [24, II, §3/eq.4 on p.723]. Therefore,

$$\Psi(x+\varepsilon) = \frac{\Gamma'(x+\varepsilon)}{\Gamma(x+\varepsilon)} > \frac{\Gamma'(x)}{\Gamma(x)} = \Psi(x) \quad \text{for} \quad \varepsilon > 0,$$

or, the auxiliary function

$$\widetilde{\Gamma}(x) := \frac{\Gamma(x+\varepsilon)}{\Gamma(x)}$$

has a positive derivative

$$\widetilde{\Gamma}'(x) = \frac{\Gamma'(x+\varepsilon)\Gamma(x) - \Gamma(x+\varepsilon)\Gamma'(x)}{\Gamma^2(x)} > 0 \quad \text{for} \quad x>0,\ \varepsilon>0.$$

Then, $\widetilde{\Gamma}(x)$ is also an increasing function, and so,

$$\frac{\Gamma(x+\varepsilon)}{\Gamma(x)} \geq \frac{\Gamma(y+\varepsilon)}{\Gamma(y)} \quad \text{whenever} \quad x \geq y > 0.$$

From this, by the replacement $\varepsilon \mapsto 1/\beta_i, x \mapsto a_i + k/\beta_i, y \mapsto b_i + k/\beta_i$ (according to the notations assumed in beginning of the proof) and by $a_i \geq b_i > 0$, we have for each $i = 1,\dots,m$:

$$\frac{\Gamma(a_i+(k+1)/\beta_i)}{\Gamma(a_i+k/\beta_i)} \geq \frac{\Gamma(b_i+(k+1)/\beta_i)}{\Gamma(b_i+k/\beta_i)}.$$

Thus the required nonincreasing property for $\theta(k)$ follows:

$$\frac{\theta(k)}{\theta(k+1)} = \prod_{i=1}^{m}\frac{\Gamma(b_i+k/\beta_i)}{\Gamma(b_i+(k+1)/\beta_i)}\cdot\frac{\Gamma(a_i+(k+1)/\beta_i)}{\Gamma(a_i+k/\beta_i)} \geq 1,$$

and

$$0 < \theta(k) \leq \theta(n+1) \quad \text{for each} \quad k \geq n+1.$$

□

Then, we continue with *properties of the generalized fractional integrals in the considered classes of analytic functions.*

Theorem 3.1. *Under the parameters' conditions* (3.5), *the generalized fractional integral* $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}$ *maps the class* $A(n)$ *into itself, and the image of a power series* (1.1) *has the form*

$$\widetilde{I}f(z) = \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left\{z + \sum_{k=n+1}^{\infty} a_k z^k\right\} = z + \sum_{k=n+1}^{\infty}\theta(k)\ a_k z^k \in A(n), \tag{3.11}$$

with multipliers' sequence $\theta(k)$ *as defined in* (3.7).

Proof. Under the assumptions of the theorem, Lemma 3.1 guarantees that

$$\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z\} = z \quad \text{and} \quad \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\{z^k\} = \frac{\lambda_k}{\lambda_1} z^k = \theta(k)\, z^k,$$

and term-by-term integration of power series (1.1) gives series (3.11). By virtue of the Cauchy-Hadamard formula, the radius of convergence of the first series, as an analytic function in the unit disk, is $R = \left\{ \overline{\lim_{k\to\infty}} |a_k|^{1/k} \right\}^{-1} \geq 1$, and that of the latter series is calculated, in view of Proposition 3.1, a), as

$$\widetilde{R} = \left\{ \overline{\lim_{k\to\infty}} \left[\, |a_k|^{1/k} \cdot |\theta(k)|^{1/k} \right] \right\}^{-1} = \left\{ \overline{\lim_{k\to\infty}} |a_k|^{1/k} \right\}^{-1} = R \geq 1,$$

Therefore the image $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)$ given by series (3.11) is analytic in the unit disc, too.

Note that due to positiveness of the multipliers $\theta(k)$, see (3.8), series with positive (like in $A(n)$) and negative (like in $T(n)$) coefficients *map into series of the same kind.* $\square$

Theorem 3.2. *In the class $A(n)$ the generalized fractional integral* (3.6) *can be represented by the Hadamard product* (1.2) *in U as*

$$(3.12)\quad \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = (h * f)(z), \quad \textit{with the function} \quad h(z) = z + \sum_{k=n+1}^{\infty} \theta(k) z^k \in A(n),$$

expressed by the Wright generalized hypergeometric function (2.1):

$$(3.13)\qquad h(z) = z + N z^{n+1}\, {}_{m+1}\Psi_m \left[\begin{array}{c} (1,1), (\gamma_i + 1 + (n+1)/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + (n+1)/\beta_i, 1/\beta_i)_1^m z \end{array} \Bigg| z \right],$$

where the normalizing constant $N = 1/\lambda_1$ is as in (3.6).

Proof. In the expression for $h(z)$ we change the index of summation k to j via $k = j + (n+1)$, using the denotations from (3.10) and for briefness, put additionally $c_i = a_i + (n+1)/\beta_i$, $d_i = b_i + (n+1)/\beta_i$, $i = 1, \dots, m$; $k = n+1, \dots$. Thus we get

$$h(z) = z + \sum_{k=n+1}^{\infty} \theta(k) z^k = z + \frac{z^{n+1}}{\lambda_1} \sum_{j=0}^{\infty} \lambda_{j+(n+1)}\, z^j = z + \frac{z^{n+1}}{\lambda_1} \sum_{j=0}^{\infty} \Big\{ \Gamma(1+j)$$

$$\times \prod_{i=1}^{m} \frac{\Gamma(d_i + j/\beta_i)}{\Gamma(c_i + j/\beta_i)} \Big\} \frac{z^j}{j!} = z + \frac{z^{n+1}}{\lambda_1}\, {}_{m+1}\Psi_m \left[\begin{array}{c} (1,1), (d_1, 1/\beta_1), \dots, (d_m, 1/\beta_m) \\ (c_1, 1/\beta_1), \dots, (c_m, 1/\beta_m) \end{array} \Bigg| z \right],$$

according to the definition (2.1), which gives (3.13). $\square$

Remark. If we ignore the requirement for the convolution function $h(z)$ to be in same class $A(n)$, we can look for a function analytic in U of the form $\hat{h}(z) = b_0 + b_1 z + b_2 z^2 + \dots$. with the only condition $b_k = \theta(k)$, i.e. $b_1 = 1$, but $b_k \neq 0$, $k = 0, 1, 2, \dots, n, n+1, \dots$. Thus

$$\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = (\hat{h} * f)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \dots + a_n b_n z^n + a_{n+1} b_{n+1} z^{n+1} + \dots$$

$= z + a_{n+1} b_{n+1} z^{n+1} + \dots$, having in mind that $a_0 = 0, a_1 = 1, a_2 = \dots = a_n = 0$, for $n \geq 2$, since $f \in A(n)$. Then $\hat{h}(z)$ is represented much simpler as *the Wright function*

$$\hat{h}(z) = N\, {}_{m+1}\Psi_m \left[\begin{array}{c} (1,1), (\gamma_i + 1, 1/\beta_1)_1^m \\ (\gamma_i + \delta_i + 1, 1/\beta_1)_1^m \end{array} \Bigg| z \right], \quad \text{analytic in } U.$$

Corollary 3.1. *For $n = 1$ the representation of the "convolution function" $h(z)$ in* (3.12) *in the classes A, S and T simplifies as:*

$$h(z) = z + N z^2\, {}_{m+1}\Psi_m \left[\begin{array}{c} (1,1), (\gamma_i + 1 + 2/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + 2/\beta_i, 1/\beta_i)_1^m \end{array} \Bigg| z \right] \in A.$$

Corollary 3.2. *When all* $\beta_i = \beta > 0$ $(i = 1, \dots, m)$, *and especially for shortness of denotations it is taken* $\beta = 1$, *for the generalized fractional integrals with Meijer's G-function in the kernel,*

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) := \widetilde{I}_{(1,1,\dots,1),m}^{(\gamma_i),(\delta_i)} f(z) = N \int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_i + \delta_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right] f(z\sigma) d\sigma, \tag{3.14}$$

we get the simpler representation of multipliers' sequence $\theta(k)$:

$$\theta(k) = \prod_{i=1}^m \frac{(\gamma_i + 2)_{k-1}}{(\gamma_i + \delta_i + 2)_{k-1}} > 0, \quad k = n+1, n+2, \dots,$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$ *denoting the known Pochhammer symbol, and respectively, for the convolution function* $h(z)$ *as follows*:

$$h(z) = z + \prod_{i=1}^m \frac{(\gamma_i + 2)_n}{(\gamma_i + \delta_i + 2)_n}\, z^{n+1}\ {}_{m+1}F_m\left[\begin{matrix} 1, (\gamma_i + 2 + n)_1^m \\ (\gamma_i + \delta_i + 2 + n)_1^m \end{matrix} \,\middle|\, z\right] \in A(n).$$

For $n = 1$ (*i.e. in the classes* A, S, T), $h(z)$ *simplifies to a* ${}_{m+1}F_m$*-generalized hypergeometric function:*

$$h(z) = z + z^2\ {}_{m+1}F_m\left[\begin{matrix} 1, (\gamma_i + 3)_1^m \\ (\gamma_i + \delta_i + 3)_1^m \end{matrix} \,\middle|\, z\right] \in A.$$

4. Distortion inequalities and some characterization theorems

Here we provide some examples of distortion inequalities in terms of the generalized fractional integration operators (3.6). We use the following auxiliary results.

Lemma 4.1. (Chatterjea [6]) *Let the function* $f(z) \in A(n)$. *Then* $f(z)$ *is in the class* $T_\alpha(n)$, *resp. in* $L_\alpha(n)$, *if and only if*

$$\sum_{k=n+1}^\infty \frac{k-\alpha}{1-\alpha} a_k \le 1, \quad \textit{resp.} \quad \sum_{k=n+1}^\infty \frac{k(k-\alpha)}{1-\alpha} a_k \le 1. \tag{4.1}$$

Applying Lemma 4.1 and Theorem 3.1, we obtain *the following distortion theorems.*

Theorem 4.1. *Let conditions* (3.5) *be satisfied and* $f(z)$ *defined by* (1.1) *belong to the class* $T_\alpha(n)$. *Then the following inequalities hold for each* $n \ge 1$ *and* $z \in U$:

$$\left|\widetilde{I}_{(\beta_i),m}^{(\gamma_i),(\delta_i)} f(z)\right| \ge |z| - \frac{1-\alpha}{n+1-\alpha}\, \theta(n+1)\, |z|^{n+1} \tag{4.2}$$

and

$$\left|\widetilde{I}_{(\beta_i),m}^{(\gamma_i),(\delta_i)} f(z)\right| \le |z| + \frac{1-\alpha}{n+1-\alpha}\, \theta(n+1)\, |z|^{n+1}, \tag{4.3}$$

where the multiplier $\theta(n+1)$ *is defined as in* (3.7), *namely*:

$$\theta(n+1) = \prod_{i=1}^m \frac{\Gamma(\gamma_i + 1 + (n+1)/\beta_i)\, \Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + (n+1)/\beta_i)\, \Gamma(\gamma_i + 1 + 1/\beta_i)} > 0. \tag{4.4}$$

Equalities in (4.2) *and* (4.3) *are attained by the function*

$$f(z) = z - \frac{1-\alpha}{n+1-\alpha}\, z^{n+1}.$$

Theorem 4.2. *Let conditions* (3.5) *be satisfied and the function* $f(z)$ *defined by* (1.1) *belong to the class* $L_\alpha(n)$. *Then the following inequalities hold for* $n \ge 1$ *and* $z \in U$:

$$\left|\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)\right| \geq |z| - \frac{1-\alpha}{n+1-\alpha}\,\frac{\theta(n+1)}{n+1}\,|z|^{n+1} \tag{4.5}$$

and

$$\left|\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)\right| \leq |z| + \frac{1-\alpha}{n+1-\alpha}\,\frac{\theta(n+1)}{n+1}\,|z|^{n+1}, \tag{4.6}$$

where the multiplier $\theta(n+1)$ *is defined as above,* (4.4). *Equalities in* (4.5) *and* (4.6) *are attained by the function*

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)}\, z^{n+1}.$$

Proof of Theorems 4.1 and 4.2. The main point in this proof is that the multiplier function $\theta(k)$ is nonincreasing, see Proposition 3.1, b), that is,

$$0 < \theta(k) \leq \theta(n+1) \quad \text{for each} \quad k \geq n+1.$$

Then, for $f(z) \in T_\alpha(n)$ with image of the form (3.11) with negative coefficients,

$$\begin{aligned}\left|\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)\right| &\geq |z| - \left|\sum_{k=n+1}^{\infty} \theta(k)\, a_k z^k\right| \\ &\geq |z| - \theta(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \geq |z| - \theta(n+1)|z|^{n+1}\,\frac{1-\alpha}{n+1-\alpha},\end{aligned}$$

since in view of Lemma 4.1 (see (4.1)), we have also

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{1-\alpha}{n+1-\alpha}.$$

Thus, inequality (4.2) is obtained. The next inequality (4.3) can be proved similarly, and Theorem 4.2 follows in analogous way by application of the same lemma, but the second inequality in (4.1). Details can be found in Kiryakova et al. [21]. □

Corollary 4.1. *Setting* $n = 1$ *and* $\alpha = 0$, *we obtain for the subclasses of starlike and convex functions in* U *with negative coefficients, respectively*:

$$f \in S^* \cap T(1) \implies |\widetilde{I} f(z)| \geq |z| - \frac{\theta(2)}{2}\, |z|^2, \quad |\widetilde{I} f(z)| \leq |z| + \frac{\theta(2)}{2}\, |z|^2$$

$$f \in K \cap T(1) \implies |\widetilde{I} f(z)| \geq |z| - \frac{\theta(2)}{4}\, |z|^2, \quad |\widetilde{I} f(z)| \leq |z| + \frac{\theta(2)}{4}\, |z|^2$$

with the multiplier

$$\theta(2) = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + 2/\beta_i)\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + 2/\beta_i)\Gamma(\gamma_i + 1 + 1/\beta_i)}.$$

Remark. The case $m = 1$ gives corresponding estimates for the classical Erdélyi-Kober operators (3.3).

As applications of the above general results, we can derive the same kind for the operators of Saigo and Hohlov as well as for the fractional integrals involving the Appell's F_3-function, etc. special cases in Section 5.

Now we consider *some sufficient conditions for the operators of generalized fractional calculus to produce starlike and convex functions.* From Silverman's results [27], one can formulate the following auxiliary lemmas.

Lemma 4.2. ([27]) *If the function $f(z) \in A(n)$ satisfies the condition*

$$\sum_{k=n+1}^{\infty} k|a_k| \le 1, \tag{4.7}$$

then $f(z) \in S^*(n)$. The equality in (4.7) *is attained by the function*

$$g_1(z) = z + \varepsilon(n+1) \sum_{k=n+1}^{\infty} \frac{z^k}{k^2(k+1)} \quad (\varepsilon = \mathrm{const},\ |\varepsilon| = 1,\ z \in U).$$

Lemma 4.3. ([27]) *If the function $f(z) \in A(n)$ satisfies the condition*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \le 1, \tag{4.8}$$

then $f(z) \in K(n)$. The equality in (4.8) *is attained by the function*

$$g_2(z) = z + \varepsilon(n+1) \sum_{k=n+1}^{\infty} \frac{z^k}{k^3(k+1)} \quad (\varepsilon = \mathrm{const},\ |\varepsilon| = 1,\ z \in U).$$

Then, for the generalized fractional integrals (3.6) we obtain *the following sufficient conditions.*

Theorem 4.3. *Under the condition* (3.5), *if the function $f(z) \in A(n)$ defined by* (1.1) *satisfies*

$$\sum_{k=n+1}^{\infty} k|a_k| \le \frac{1}{\theta(n+1)}, \tag{4.9}$$

then $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)$ belongs to the class $S^*(n)$.

Proof. We use again the inequality (3.9) for $\theta(k)$, Proposition 3.1, b). Then, for the function

$$\widetilde{I} f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \quad \text{with coefficients} \quad b_k = \theta(k) a_k,$$

we obtain

$$\sum_{k=n+1}^{\infty} k b_k \le \theta(n+1) \sum_{k=n+1}^{\infty} k a_k \le 1. \qquad \square$$

Analogously, using Lemma 4.3, we obtain

Theorem 4.4. *Under condition* (3.5), *if the function $f(z) \in A(n)$ satisfies*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \le \frac{1}{\theta(n+1)}, \tag{4.10}$$

then $\widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z)$ belongs to the class $K(n)$.

Details can be found in Kiryakova et al. [21].

Remark. Examples of functions satisfying conditions (4.9), (4.10) are the functions

$$g_3(z) = z + \frac{1}{\theta(k_0)} \frac{z^{k_0}}{k_0} \quad \text{and} \quad g_4(z) = z + \frac{1}{\theta(k_0)} \frac{z^{k_0}}{k_0^2}, \quad \text{resp., with some} \ \ k_0 > n+1.$$

Another circle of rather general results are *theorems about preserving the univalency* of functions, under operators of GFC. For simplicity, we consider univalent functions from the class A ($n = 1$).

Theorem 4.5. *Let* $\widetilde{I}f(z) = \widetilde{I}^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = h(z) * f(z)$, *with convolution function* $h(z)$ *being the Wright hypergeometric function* (3.13), *be the normalized generalized fractional integration operator* (3.6) *in the class* A. *Suppose that its parameters* $\gamma_i, \delta_i, \beta_i$, $i = 1, \ldots, m$ *satisfy conditions* (3.5) *and additionally,* $\sum_{i=1}^{m} \delta_i > 3$, *i.e.*

$$\gamma_i > -1, \quad \delta_i \geq 0, \quad \beta_i > 0, \quad i = 1, \ldots, m; \quad \delta_1 + \delta_2 + \ldots + \delta_m > 3,$$

as well as the inequality

$$\begin{aligned} &{}_{m+1}\Psi_m \left[\begin{array}{c} (3,1), (\gamma_i + 1 + 3/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + 3/\beta_i, 1/\beta_i)_1^m \end{array} \middle| 1 \right] + 3\, {}_{m+1}\Psi_m \left[\begin{array}{c} (2,1), (\gamma_i + 1 + 2/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + 2/\beta_i, 1/\beta_i)_1^m \end{array} \middle| 1 \right] \\ &+ {}_{m+1}\Psi_m \left[\begin{array}{c} (1,1), (\gamma_i + 1 + 1/\beta_i, 1/\beta_i)_1^m \\ (\gamma_i + \delta_i + 1 + 1/\beta_i, 1/\beta_i)_1^m \end{array} \middle| 1 \right] < 2\lambda_1 = 2 \prod_{i=1}^{m} u \frac{\Gamma(\gamma_i + 1 + 1/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + 1/\beta_i)} . \end{aligned} \tag{4.11}$$

Then for each univalent function f *in* A, *the image* $\widetilde{I}f$ *is also univalent, that is,* $\widetilde{I} : S \mapsto S$.

Proof. The details of the proof have been given in [20], and for the more general case of the Dziok-Srivastava operator with convolution function ${}_p\Psi_q$ (arbitrary $p \leq q + 1$) in [18], [19]. It is based on a result of Avhadiev and Aksent'ev [2] that for the image-function $\widetilde{I}f(z) = z + \sum_{k=2}^{\infty} \theta(k) a_k z^k$ to be univalent function, it is sufficient to have $\sigma_1 = \sum_{k=2}^{\infty} \theta(k) a_k < 1$, and on using the well-known estimate (de Branges' theorem [5]) for the coefficients of the univalent function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, namely: $|a_k| \leq k$.

The point we like to stress is that all rest of the proof uses algebra with the Γ- and factorial functions, and *properties of the Wright hypergeometric function* ${}_{m+1}\Psi_m$ *and its values at the boundary point* $z = 1$, as these in (4.11). For the mentioned Dziok-Srivastava operator ([8], [9]) the function ${}_p\Psi_q$ is concerned, while in the simpler case of G-function operator (3.14) with all equal $\beta_i = \beta > 0$, the function is ${}_{m+1}F_m$. □

A similar theorem provides conditions, again in terms of the values of ${}_{m+1}\Psi_m(1)$, for $\widetilde{I} : K \mapsto S$, that is for mapping the convex functions in univalent functions.

5. Special cases of the generalized fractional integrals used often in GFT

The results from Sections 3, 4 and the others from our papers [17], [18], [19], [20], [21], etc., can be specialized for a great number of linear integral operators used in Geometric Function Theory, starting from *the classical operators of Biernacki, Libera, Bernardi, Komatu, Rusheweyh, Saigo, Hohlov, Srivastava and Owa*, and going to the more general operators studied recently by different authors, say Dziok-Srivastava operators [8], [9], [28]. To this end it is enough to choose suitable particular parameters $m, \gamma_i, \delta_i, \beta_i$ $(i = 1, ..., m)$ for the operators (3.6) of the generalized fractional calculus (GFC).

We list below examples, and their presentation in the denotations of (3.2)-(3.6).

For $\underline{\mathbf{m = 1}}$, we have the examples (see longer list and more details in [14], [16], [17]):

Biernacki operator: ([4]) $Bf(z) = I_{1,1}^{-1,1} f(z) = \log(\dfrac{1}{1-z}) * f(z) = \displaystyle\int_0^z \frac{f(\sigma)}{\sigma} d\sigma$;

Libera operator: $Lf(z) = 2I_{1,1}^{0,1} f(z) = z\, {}_2F_1(1,2;3;z) * f(z) = \dfrac{2}{z} \int\limits_0^z f(\sigma) d\sigma;$

Generalized Libera operator: ([23]) $B_c f(z) = (c+1) I_{1,1}^{c-1,1} f(z) = \dfrac{c+1}{z^c} \int\limits_0^z \sigma^{c-1} f(\sigma) d\sigma$
$= z^{c+1} {}_2F_1(1, c+1; c+2; z) * f(z);$ For integer $c \in \mathbb{N}$, it is called *Bernardi operator* ([3]);

Carlson-Shaffer operator: $L(a,c)f(z) = \dfrac{\Gamma(c)}{\Gamma(a)} I_{1,1}^{a-2,c-a} f(z) = z\, {}_2F_1(1,a;c;z) * f(z)$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int\limits_0^1 (1-\sigma)^{c-a-1} \sigma^{a-2} f(z\sigma) d\sigma.$$

Examples of GFC operators for $\underline{\mathbf{m=2}}$ are the so-called *hypergeometric integral operators of Hohlov and Saigo*. Let us pay here some more attention on them.

In [10],[11] Hohlov introduced the hypergeometric operator $\mathbf{F}$(a,b,c) (we call it as *Hohlov operator*) defined in the class A by means of the Hadamard product with the *Gauss hypergeometric function* ${}_2F_1(a,b;c;z)$:

$$\mathbf{F}(a,b,c) f(z) = \{z\ {}_2F_1(a,b;c;z)\} * f(z). \tag{5.1}$$

We can write (5.1) in terms of the GFC operators (3.2)-(3.6) with $m=2$, as:

$$\mathbf{F}(a,b,c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} I_{1,2}^{(a-2,b-2),(1-a,c-b)} = \widetilde{I}_{1,2}^{(a-2,b-2),(1-a,c-b)}.$$

Another class of hypergeometric fractional integration operators has been introduced by Saigo [25] (see [15]) for solving the Euler-Darboux equation, and studied from view of univalent functions' theory in series of papers by Srivastava, Saigo and Owa, as for example [30]. This linear integral operator, named as *Saigo operator*, can be represented also as a generalized fractional integral in the sense of (3.2) with $m=2$ (details in [15]):

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int\limits_0^z (z-\zeta)^{\alpha-1} {}_2F_1\left[\begin{array}{c} \alpha+\beta, -\eta \\ \alpha \end{array} \middle| 1-\frac{\zeta}{z}\right] f(\zeta) d\zeta = z^{-\beta} I_{1,2}^{(\eta-\beta,0),(-\eta,\alpha+\eta)} f(z).$$

To preserve the class A, the Saigo operator is normalized by the constant N, in this case as

$$\widetilde{I}_{0,z}^{\alpha,\beta,\eta} f(z) := \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)}\ z^{\beta}\ I_{0,z}^{\alpha,\beta,\eta} f(z).$$

Operators (3.2)-(3.6) with "multiplicity" $\underline{\mathbf{m>2}}$ have been not so popular. Such one is the Saigo operator (see in [14], [15]) with the *Appel F_3-function* in the kernel, that is, an operator (3.2) with $\mathbf{m=3}$:

$$I(\alpha,\alpha',\beta,\beta';\gamma) f(z) = z^{-\alpha} \int\limits_0^z \frac{(z-\xi)^{\gamma-1}}{\Gamma(\gamma)} \xi^{-\alpha'}\ F_3\left(\alpha,\alpha',\beta,\beta';\gamma; 1-\frac{\xi}{z}, 1-\frac{z}{\xi}\right) f(\xi) d\xi$$

$$= z^{-\alpha-\alpha'+\gamma} \int\limits_0^1 G_{3,3}^{3,0}\left[\sigma \middle| \begin{array}{c} \alpha-\alpha'+\beta, \gamma-2\alpha', \gamma-\alpha'-\beta' \\ \alpha-\alpha', \beta-\alpha', \gamma-2\alpha'-\beta' \end{array}\right] f(z\sigma) d\sigma$$

$$= z^{-\alpha-\alpha'+\gamma}\ I_{1,3}^{(\alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'),(\beta,\gamma-\alpha'-\beta,\alpha')} f(z).$$

A typical example of (3.2) with arbitrary $\mathbf{m > 2}$ is given by the *integral operator* $L = z^{\beta} I_{(\beta,\dots,\beta),m}^{(\gamma_1,\dots,\gamma_m),(1,1,\dots 1)}$ which is the linear right inverse to the so-called *hyper-Bessel differential operator, introduced by Dimovski* (see in [14, Ch.3]), having the form

$$B = z^{\alpha_0}\frac{d}{dz}z^{\alpha_1}\frac{d}{dz}\dots\frac{d}{dz}z^{\alpha_m} = z^{-\beta}\prod_{i=1}^{m}\left(z\frac{d}{dz}+\beta\gamma_k\right),\quad \beta>0,\ \alpha_i,\gamma_i\in\mathbb{R}.$$

In this relation, let us mention the *Salagean differential operator* ([26], e.g. [22]), defined in A for functions $f(z)$ of the form (1.1) and for $m = 1, 2, 3, \dots$ by the recurrence relation

$$S_0 f(z) = f(z);\, S_1 f(z) = z f^{'}(z), \dots, S_m f(z) = S_1\left(S_{m-1} f(z)\right) = z + \sum_{k=2}^{\infty} k^m a_k z^k.$$

This operator can be seen as an interesting case of hyper-Bessel differential operator with $\beta = 1$ and all $\gamma_k = -1, \delta_k = 1,\ k = 1, \dots, m$. Its linear right inverse operator is the *integral operator of Alexander* ([1], see e.g. [22]): A_m, $m = 1, 2, 3\dots$,

$$A_0 f(z) = f(z), A_1 f(z) = \int_0^1 \frac{f(\sigma)}{\sigma}d\sigma,\ \dots, A_m f(z) = A_1(A_{m-1} f(z)) = z + \sum_{k=2}^{\infty}\frac{1}{k^m}a_k z^k,$$

which can be written in the form of generalized fractional integral, namely:

$$A_m f(z) = I_{(1,\dots,1),m}^{(-1,-1,\dots,-1),(1,1,\dots,1)} f(z), \quad \text{put} \quad \theta(k) = 1/k^m = [\Gamma(k)/\Gamma(1+k)]^m \ \text{ in } \ (3.7).$$

The above list of examples of operators of classical and generalized fractional calculus, that are special cases also of the more general Dziok-Srivastava operator (we treated similarly in [18]), shows that great amount of results in GFT – some of them mentioned here, in other our papers, as well as from many papers of different authors, can be obtained from the considered general case by suitable choice of parameters for each particular operator. Once again, *we emphasize that these results – in the mentioned general case, are obtained by the efficient use of the special functions as tools, thus providing an unified approach.*

Acknowledgment

A paper on the working program on Bilateral Research Project "Mathematical Modeling by Means of Integral Transform Methods, Partial Differential Equations, Special and Generalized Functions, Numerical Analysis" between Bulgarian and Serbian Academies of Sciences.

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