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ON WEIGHTED BANACH SEQUENCE SPACES

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ABSTRACT. We consider Banach sequence spaces $l_{p,\sigma}$ with a weighted sequence σ, which are generalizations of standard sequence spaces. We investigate the relationships between these spaces for a fixed p $(1 \leq p \leq +\infty)$ and for different weighted functions, as well as for fixed σ and various p, q $(1 \leq p < q \leq +\infty)$. We also present the representation of bounded linear functionals on these spaces.

1. INTRODUCTION

In addition to looking at some standard sequence spaces s, c, c_0 and l_p, it is of interest to consider some generalizations of such spaces. One of the ways to make such a generalization is by considering the sequence space $l_p(X)$ such that coordinates of the sequence $x \in l_p(X)$ belong to a metric space X. Particularly, if $X = \mathbb{R}^n$ or $X = \mathbb{C}^n$ $(n \in \mathbb{N})$ we get standard sequence spaces. We can also change norms on standard spaces. Bynum [4] considered spaces $l_{p,q}$, for $1 \leq p, q < +\infty$, of all $x \in l_p$, where the norm is given by

$$||x||_{l_{p,q}} = \left(||x^+||_{l_p}^q + ||x^-||_{l_p}^q \right)^{\frac{1}{q}} .$$

The case when $q = +\infty$ gives us the space $l_{p,\infty}$ where the norm is

$$||x||_{l_{p,\infty}} = \max\{||x^+||_{l_p}, ||x^-||_{l_p}\}$$

and $x_n^+ = \max\{x_n, 0\}$, $x_n^- = \max\{-x_n, 0\}$ for $n \in \mathbb{N}$.

Banas et al. [2] defined Baernstein spaces. A sequence $x = (x_n)_{n\in\mathbb{N}}$ belongs to the Baernstein space if

$$||x||_B = \sup \left\{ \left[\sum_{k=1}^{\infty} \left(\sum_{i\in\gamma_k} |x_i| \right)^2 \right]^{\frac{1}{2}} : (\gamma_n)_{n\in\mathbb{N}} \in A \right\} < +\infty ,$$

where A is the set of all sequences $(\gamma_n)_{n\in\mathbb{N}}$ of finite subsets of natural numbers such that $card(\gamma_n)_{n\in\mathbb{N}} \leq \min_{n\in\mathbb{N}} \gamma_n$ and $\max_{n\in\mathbb{N}} \gamma_n < \min_{n\in\mathbb{N}} \gamma_{n+1}$ $(n \in \mathbb{N})$.

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Shue [15] introduced Cesaro sequence spaces ces_p for $1 < p < +\infty$ as

$$ces_p = \left\{ x = (x_n)_{n\in\mathbb{N}} \ : \ ||x|| = \left(\sum_{n\in\mathbb{N}} \left(\frac{1}{n} \sum_{i=1}^{n} |x_i| \right)^p \right)^{\frac{1}{p}} \right\} .$$

In the spaces mentioned above, the idea was to change the norm with respect to standard spaces in order to obtain some new spaces in such a way.

In this paper we consider one generalization of the Banach sequence spaces that are p-power summable, that is we make the generalization of the space $l_p(\Phi)$ where $\Phi = \mathbb{R}, \mathbb{C}$ and $1 \leq p \leq \infty$, so that we give some weight to every coordinate of the sequence $x = (x_n)_{n\in\mathbb{N}}$. In [5, 6, 12, 13] Lorentz spaces were considered. Let $1 \leq p < \infty$ and let $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary non-increasing sequence of positive numbers. The space of all sequences $x = (x_n)_{n\in\mathbb{N}}$, such that

$$||x|| = \sup_{\pi} \left(\sum_{n\in\mathbb{N}} |x_{\pi(n)}|^p \sigma_n \right)^{\frac{1}{p}} < +\infty ,$$

where π represents an arbitrary permutation of the set of natural numbers is called a Lorentz sequence space, denoted by $d(\sigma, p)$. If by $(x_n^*)_{n\in\mathbb{N}}$ we denote the non-increasing rearrangement of the sequence $x = (x_n)_{n\in\mathbb{N}}$, that is the non-increasing sequence that we get from $(|x_n|)_{n\in\mathbb{N}}$ by using an appropriate permutation of $\mathbb{N}$, then for $x \in d(\sigma, p)$ we have that

$$||x|| = \left(\sum_{n\in\mathbb{N}} x_n^{*p} \sigma_n \right)^{\frac{1}{p}} .$$

Now we can consider another class of spaces that are also the generalization of the space l_p. Moreover, we don't require that the sequence σ is non-increasing.

Definition 1.1. *Let $1 \leq p < +\infty$ and $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of nonnegative numbers. The set of all sequences $x = (x_n)_{n\in\mathbb{N}}$ such that*

$$\sum_{n\in\mathbb{N}} |x_n|^p \sigma_n < +\infty$$

is called a weighted sequence space and denoted by $l_{p,\sigma}$.
For $p = +\infty$ the corresponding space $l_{\infty,\sigma}$ is called a weighted space of bounded sequences. The function σ is the weighted function or the weighted sequence.

If we define the norm on the space $l_{p,\sigma}$ as

$$||x||_{l_{p,\sigma}} = \left(\sum_{n\in\mathbb{N}} |x_n|^p \sigma_n \right)^{\frac{1}{p}} ,$$

for $1 \leq p < \infty$, i.e.

$$||x||_{l_{\infty,\sigma}} = \sup_{n\in\mathbb{N}} |x_n| \sigma_n ,$$

for $p = +\infty$, then $l_{p,\sigma}$ is complete, i.e. a Banach space. Notice that for $0 < p < 1$, $l_{p,\sigma}$ is not a normed space. However, the functional

$$[x] = \sum_{n\in\mathbb{N}} (x_n)^p \sigma_n$$

defines the k-norm on $l_{p,\sigma}$ with $k = 2^{\frac{1-p}{p}}$ and in this case $l_{p,\sigma}$ is complete k-normed space. The inequality $\|a+b\| \leq k(\|a\| + \|b\|)$ is called the k-triangle inequality and the functional $\|\cdot\|$ is called k-norm.

Some important properties of the space $l_{p,\sigma}$ are given with the following lemma.

Lemma 1.1. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a weighted function such that $\sigma_n \geq 1$ for all $n \in \mathbb{N}$ and let $1 \leq p < \infty$. Then*

(1) *the space $l_{p,\sigma}$ is ideal;*
(2) *the space $l_{p,\sigma}$ is regular.*

Proof. (1) Let $y \in l_{p,\sigma}$ and let x be such that $|x| \leq |y|$. Since a weighted function is nonnegative, we have that $|x_n|\sigma_n \leq |y_n|\sigma_n$, for all $n \in \mathbb{N}$ and

$$\left(\sum_{n\in\mathbb{N}} |x_n|^p \sigma_n\right)^{\frac{1}{p}} \leq \left(\sum_{n\in\mathbb{N}} |y_n|^p \sigma_n\right)^{\frac{1}{p}},$$

that is, $x \in l_{p,\sigma}$. Furthermore, $\|x\|_{l_{p,\sigma}} \leq \|y\|_{l_{p,\sigma}}$. Hence $l_{p,\sigma}$ is ideal space.

(2) Let $x \in l_{p,\sigma}$ be arbitrary. It means that $\|x\|^p_{l_{p,\sigma}} = \sum_{n\in\mathbb{N}} |x_n|^p \sigma_n < \infty$. Then

$$R_k = \sum_{n=k+1}^{\infty} |x_n|^p \sigma_n = \sum_{n\in\mathbb{N}} |P_k x_n|^p \sigma_n \to 0, \ k \to \infty,$$

where P_k is a projection operator. This is equivalent to $\lim_{n\to\infty} \|P_n x\|_{l_{p,\sigma}} = 0$, so we can conclude that $x \in l_{p,\sigma}$ has an absolutely continuous norm. Thus, $l_{p,\sigma}$ is a regular space.

□

Since every regular space is almost perfect and because it is closed, we can conclude that $l_{p,\sigma}$ is a perfect space. Thus $l_{p,\sigma}$ is a completely regular space.

Now let X and Y be ideal spaces. The set Y/X^n ($n \in \mathbb{N}$) of all z such that $zx^n \in Y$, for all $x \in X$, equipped with the norm

$$\|z\|_{Y/X^n} = \sup\{\|zx^n\|_Y : \ \|x\|_X \leq 1\},$$

is called *n-th space of multipliers* of X with respect to Y. Particularly for $n = 1$, the space Y/X is called the space of multipliers of X with respect to Y.
The space of multipliers of $l_{p,\sigma}$ with respect to $l_{q,\tau}$ is given by

$$l_{q,\tau}/l_{p,\sigma} = \begin{cases} l_{\frac{pq}{p-q}, \tau^{\frac{p}{p-q}} \sigma^{-\frac{q}{p-q}}} & ; \ p > q \\ l_{\infty, \tau^{\frac{1}{q}} \sigma^{-\frac{1}{q}}} & ; \ p \leq q \, . \end{cases}$$

Indeed, for arbitrary $x \in l_{p,\sigma}$, let y be arbitrary and let $p > q$. By Hölder's inequality (see [9]), we have that

$$\sum_{n\in\mathbb{N}} |x_n y_n|^q \tau_n \leq \|x\|_{l_{p,\sigma}} \left[\sum_{n\in\mathbb{N}} \left(|y_n|^{\frac{pq}{p-q}} \tau_n^{\frac{p}{p-q}} \sigma_n^{-\frac{q}{p-q}}\right)\right]^{\frac{p-q}{p}}. \tag{1.1}$$

The right hand side of (1.1) is finite if $y \in l_{\frac{pq}{p-q}, \tau^{\frac{p}{p-q}} \sigma^{-\frac{q}{p-q}}}$. Hence, the left hand side of (1.1) is also finite, i.e. $y \in l_{q,\tau}/l_{p,\sigma}$. On the other hand, if $p \leq q$, the finiteness of the right hand side of (1.1) is achieved by an arbitrary bounded function y. Now, we will consider the relationships between $l_{p,\sigma}$ spaces for different values of p and different weighted functions σ, as well as relationships between $l_{p,\sigma}$ and standard l_p spaces.

2. Relationships between $l_{p,\sigma}$ and $l_{q,\tau}$ spaces for $p = q$ and $\sigma \neq \tau$

First let us consider some relationships between $l_{p,\sigma}$ and $l_{p,\tau}$ spaces, where $1 \leq p \leq +\infty$, and σ, τ are arbitrary sequences of nonnegative numbers.

Theorem 2.1. *If* $\limsup\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} \in (0, +\infty)$, *then* $l_{p,\sigma} \subseteq l_{p,\tau}$.

Proof. Let $\limsup\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} = \dfrac{\sigma_0}{2}$, for some $\sigma_0 \in \mathbb{R}^+$. Then there exists an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ we have that $\dfrac{\tau_n}{\sigma_n} < \sigma_0$. Let $1 \leq p < +\infty$ and $x \in l_{p,\sigma}$ be arbitrary. Then:

$$\sum_{n\geq n_0} |x_n|^p \tau_n \leq \sum_{n\geq n_0} |x_n|^p \sigma_0 \sigma_n \leq \sigma_0 \sum_{n\in\mathbb{N}} |x_n|^p \sigma_n < +\infty \ . \tag{2.1}$$

Adding a finite number of summands to the left hand side of (2.1), we can conclude that $||x||_{l_{p,\tau}} < +\infty$, that is $x \in l_{p,\tau}$.

For $p = +\infty$ and $x \in l_{\infty,\sigma}$ arbitrary, we have that $|x_n|\tau_n < |x_n|\sigma_0\sigma_n$, for $n \in \mathbb{N}$. Accordingly,

$$\sup_{n\in\mathbb{N}} |x_n|\tau_n \leq \sigma_0 \sup_{n\in\mathbb{N}} |x_n|\sigma_n = \sigma_0 ||x||_{l_{\infty,\sigma}} \ .$$

Thus, $x \in l_{\infty,\tau}$. □

Theorem 2.2. *If* $\liminf\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} \in (0, +\infty)$, *then* $l_{p,\tau} \subseteq l_{p,\sigma}$.

Proof. Let $\liminf\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} \in (0, +\infty)$. Then

$$\frac{1}{\liminf\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n}} = \limsup_{n\to\infty} \frac{\sigma_n}{\tau_n} \in (0, +\infty) \ ,$$

and by Theorem 2.1 we get that $l_{p,\tau} \subseteq l_{p,\sigma}$. □

Theorems 2.1 and 2.2 imply that if $\lim\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} = \sigma_0 \in (0, +\infty)$ then $l_{p,\sigma}$ and $l_{p,\tau}$ are equal as spaces, i.e. they are isomorphic, taking the identity as the isomorphism between them. We can now state this in a more generalized form.

Corollary 2.1. *If* $\limsup\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n}$, $\liminf\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} \in (0, +\infty)$, *then* $l_{p,\sigma} = l_{p,\tau}$.

If $\limsup\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} = +\infty$ then $l_{p,\sigma} \nsubseteq l_{p,\tau}$. Without loss of generality we can assume that $\lim\limits_{n\to\infty} \dfrac{\tau_n}{\sigma_n} = +\infty$. This means that $\dfrac{\tau}{\sigma} \sim \alpha$, that is

$$\frac{\tau_n}{\sigma_n} = \alpha_n \quad , \quad n \in \mathbb{N} \ ,$$

and $\lim_{n\to\infty} \alpha_n = +\infty$. If $\alpha \sim \ln n$ $(n \to \infty)$, then $l_{p,\tau} \subset l_{p,\sigma}$.
Hence, let $\alpha \not\sim \ln n$, $(n \to \infty)$. Let $k \in \mathbb{R}^+$ be such that

$$\sum_{n\in\mathbb{N}} \frac{1}{\alpha_n^k} < +\infty \text{ i } (\forall \varepsilon > 0) \sum_{n\in\mathbb{N}} \frac{1}{\alpha_n^{k-\varepsilon}} = +\infty \ . \tag{2.2}$$

If we define the sequence $x^* = (x_n^*)_{n\in\mathbb{N}}$ in such a way so that

$$x_n^* = \frac{1}{\alpha_n^{\frac{k}{p}}} \frac{1}{\sigma_n^{\frac{1}{p}}} \ ,$$

then $\sum_{n\in\mathbb{N}} |x_n^*|^p \sigma_n = \sum_{n\in\mathbb{N}} \frac{1}{\alpha_n^k} < +\infty$, i. e. $x^* \in l_{p,\sigma}$. We also have that

$$\sum_{n\in\mathbb{N}} |x_n^*|^p \tau_n = \sum_{n\in\mathbb{N}} \frac{1}{\alpha_n^k} \frac{\tau_n}{\sigma_n} = \sum_{n\in\mathbb{N}} \frac{1}{\alpha_n^{k-1}} \ . \tag{2.3}$$

The last series in (2.3) is divergent because of the value of k chosen in (2.2), so we can conclude that $x^* \notin l_{p,\sigma_2}$.

Lemma 2.1. *If* $\limsup_{n\to\infty} \frac{\tau_n}{\sigma_n} = +\infty$, *then* $l_{p,\tau} \subsetneq l_{p,\sigma}$.

Proof. Without lost of generality we can assume that $\lim_{n\to\infty} \frac{\tau_n}{\sigma_n} = +\infty$. This means that $\frac{\tau_n}{\sigma_n} = \alpha_n$ for $n \in \mathbb{N}$, where $\lim_{n\to\infty} \alpha_n = +\infty$. We can also assume that $\alpha_n \geq 1$ for all $n \in \mathbb{N}$.
Let $1 \leq p < +\infty$ and $x \in l_{p,\tau}$ be arbitrary. Because of our previous assumptions, we can conclude that

$$\sum_{n\in\mathbb{N}} |x_n|^p \sigma_n \quad = \quad \sum_{n\in\mathbb{N}} |x_n|^p \frac{\tau_n}{\alpha_n} \leq \sum_{n\in\mathbb{N}} |x_n|^p \tau_n < +\infty \ .$$

Hence $x \in l_{p,\sigma}$, that is $l_{p,\tau} \subseteq l_{p,\sigma}$. Since there exists an element in $l_{p,\sigma}$ that is not in $l_{p,\tau}$, we get that $l_{p,\tau} \subsetneq l_{p,\sigma}$.

Let $p = +\infty$ and $x \in l_{\infty,\tau}$. For an arbitrary $n \in \mathbb{N}$ we have that $|x_n|\sigma_n = |x_n|\frac{\tau_n}{\alpha_n} \leq |x_n|\tau_n$, so if we take the supremum over $n \in \mathbb{N}$ we have that $||x||_{l_{\infty,\sigma}} \leq ||x||_{l\infty,\tau}$. Since $x \in l_{\infty,\tau}$ is arbitrary, we can conclude that $l_{\infty,\tau} \subseteq l_{\infty,\sigma}$. Particularly, if we choose $\sigma = (\frac{1}{n})_{n\in\mathbb{N}}$, $\tau = (n^2)_{n\in\mathbb{N}}$ and $x = (\frac{1}{n})_{n\in\mathbb{N}}$, we can conclude that $x \in l_{\infty,\sigma}$ and $x \notin l_{\infty,\tau}$. Hence, in this case we have that $l_{\infty,\tau} \subsetneq l_{\infty,\sigma}$. □

Lemma 2.2. *If* $\liminf_{n\to\infty} \frac{\tau_n}{\sigma_n} = 0$, *then* $l_{p,\sigma} \subsetneq l_{p,\tau}$.

Proof. From the fact that $\liminf_{n\to\infty} \frac{\tau_n}{\sigma_n} = 0$, and using some known properties we get that

$$\liminf_{n\to\infty} \frac{\tau_n}{\sigma_n} = 0 \Leftrightarrow \limsup_{n\to\infty} \frac{\sigma_n}{\tau_n} = +\infty \ ,$$

and by Lemma 2.1 we can conclude that $l_{p,\sigma} \subsetneq l_{p,\tau}$. □

3. Relationships between spaces $l_{p,\sigma}$ and $l_{q,\tau}$ for $p \neq q$ and $\sigma = \tau$

In the Section 2 we established some relationships between two weighted spaces of sequences for the same value of p and different weighted functions. Now let $1 \leq p < q \leq +\infty$ and let σ be an arbitrary sequence of positive numbers. It is well known that for standard l_p spaces in the case $1 \leq p < q \leq +\infty$, we have that $l_p \subset l_q$. In this section we want to see under which conditions we will have the same relationships between $l_{p,\sigma}$ and $l_{q,\sigma}$ spaces. We show that the weighted function plays an important role in the ordering weighted spaces. First, we give one sufficient condition.

Theorem 3.1. *Let $1 \leq p < q < +\infty$. If*

$$\liminf_{n\to\infty} \sigma_n > 0 \ , \tag{3.1}$$

then $l_{p,\sigma} \subseteq l_{q,\sigma}$.

Proof. Let $1 \leq p < q < +\infty$ and $\liminf_{n\to\infty} \sigma_n > 0$. Let $\liminf_{n\to\infty} \sigma_n = 2\rho_0 > 0$. Then, there exists $n_0 \in \mathbb{N}$, such that $\sigma_n \geq \rho_0$, for all $n \geq n_0$. Now, let $x \in l_{p,\sigma}$ be arbitrary. This means that $\sum_{n\in\mathbb{N}} |x_n|^p \sigma_n < +\infty$. However, in that case we have that

$$\sum_{n\geq n_0} |x_n|^p \sigma_n \geq \rho_0 \sum_{n\geq n_0} |x_n|^p \ .$$

Because of the fact that $x \in l_{p,\sigma}$ we can conclude that $\sum_{n\in\mathbb{N}} |x_n|^p < +\infty$, i. e. $\lim_{n\to\infty} |x_n| = 0$, which means that there exists an $n_1 \in \mathbb{N}$, such that $|x_n| \leq 1$, for all $n \geq n_1$. Now, since $p < q$ we have $|x_n|^p \geq |x_n|^q$, for $n \geq n_1$, i.e. we have that

$$\sum_{n\geq n_1} |x_n|^q \sigma_n \leq \sum_{n\geq n_1} |x_n|^p \sigma_n \leq \sum_{n\in\mathbb{N}} |x_n|^p \sigma_n < +\infty \ ,$$

whence we get that $x \in l_{q,\sigma}$. $\square$

We can't weaken the condition (3.1), i.e. we can't demand the condition $\liminf_{n\to\infty} \sigma_n \geq 0$. Namely, if we consider the spaces $l_{1,\frac{1}{n^3}}$ i $l_{2,\frac{1}{n^3}}$ and the weighted function $\sigma_n = \frac{1}{n^3}$, such that $\lim_{n\to\infty} \sigma_n = 0$, we realize that for a sequence $x = (n)_{n\in\mathbb{N}}$ it is clear that $x \in l_{1,\frac{1}{n^3}}$ but $x \notin l_{2,\frac{1}{n^3}}$, This means that $l_{1,\frac{1}{n^3}} \nsubseteq l_{2,\frac{1}{n^3}}$.

The following theorem gives us the necessary and sufficient condition for the expected order of weighted spaces in the sense of inclusion.

Theorem 3.2. *Let $1 \leq p < q < +\infty$. Then $l_{p,\sigma} \subseteq l_{q,\sigma}$ if and only if the weighted function σ satisfies the condition*

$$(\forall n \in \mathbb{N}) \ \ \sigma_n \geq 1 \ . \tag{3.2}$$

Proof. Let $1 \leq p < q < \infty$ and $l_{p,\sigma} \subseteq l_{q,\sigma}$. Let k be an arbitrary fixed natural number. Let us choose the sequence $x = (x_n)_{n\in\mathbb{N}}$ such that $x_k = 1$ and $x_n = 0$ for $n \neq k$. Then, since $l_{p,\sigma} \subseteq l_{q,\sigma}$, we have that $||x||_{l_{p,\sigma}} \geq ||x||_{l_{q,\sigma}}$, and because of the choice of the sequence x, it would mean that $(\sigma_k)^{\frac{1}{p}} \geq (\sigma_k)^{\frac{1}{q}}$. Since $p < q$, we can conclude that $\sigma_k \geq 1$, for arbitrary k.

Now suppose that condition (3.2) is satisfied, and let $\beta > 1$ be such that $q = p\beta$. Without loss of generality assume that $x \in l_{p,\sigma}$ is such that $\sum_{s\in\mathbb{N}} |x_s|^p \sigma_s = 1$. Then, for all $s \in \mathbb{N}$, we have that $|x_s|\sigma_s \leq 1$, hence

$$\sum_{s\in\mathbb{N}} |x_s|^q \sigma_s = \sum_{s\in\mathbb{N}} (|x_s|^p \sigma_s)^\beta (\sigma_s)^{1-\beta} \leq \sum_{s\in\mathbb{N}} (|x_s|^p \sigma_s)^\beta \leq \sum_{s\in\mathbb{N}} |x_s|^p \sigma_s \ .$$

Thus, $||x||_{l_{q,\sigma}} \leq ||x||_{l_{p,\sigma}}$, i.e. $l_{p,\sigma} \subseteq {}_{q,\sigma}$. □

Theorem 3.3. *Let $1 \leq p < +\infty$. If $\sup\limits_{n\in\mathbb{N}} \sigma_n < +\infty$ then $l_{p,\sigma} \subseteq l_{\infty,\sigma}$.*

Proof. Let $\sup\limits_{n\in\mathbb{N}} \sigma_n < +\infty$ and $x \in l_{p,\sigma}$ be arbitrary. Then $||x||_{l_{p,\sigma}} < +\infty$, so we can conclude that $\lim\limits_{n\to\infty} |x_n|^p \sigma_n = 0$. Since the weighted sequence is bounded, it follows that $\lim\limits_{n\to\infty} x_n = 0$, our convergent sequence x is bounded and there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. We have that $|x_n|\sigma_n \leq M\sigma_n$ i.e., $\sup\limits_{n\in\mathbb{N}} |x_n|\sigma_n \leq M \sup\limits_{n\in\mathbb{N}} \sigma_n$. Thus,

$$||x||_{l_{\infty,\sigma}} \leq M \sup_{n\in\mathbb{N}} \sigma_n < \infty \ ,$$

i.e. $x \in l_{\infty,\sigma}$. □

The following example shows that in Theorem 3.3 the strict inclusion can hold. Let $x = (n)_{n\in\mathbb{N}}$ and $\sigma = \left(\frac{1}{n^p}\right)_{n\in\mathbb{N}}$. Since $\sup\limits_{n\in\mathbb{N}} |x_n|\sigma_n = \sup\limits_{n\in\mathbb{N}} \dfrac{1}{n^{p-1}} = 1$, we have that $x \in l_{\infty,\frac{1}{n^p}}$. However, the series $\sum\limits_{n=1}^{\infty} |x_n|^p \sigma_n = \sum\limits_{n=1}^{\infty} 1$ is divergent, i.e. $x \notin l_{p,\frac{1}{n^p}}$.

4. Relationships between $l_{p,\sigma}$ and l_p spaces

From the condition (3.2) of Theorem 3.2 we see that for the special choice of the weighted function, for which $\sigma_n = 1$ for $n \in \mathbb{N}$, we obtain the standard l_p space as the special case of the weighted $l_{p,\sigma}$ space. In this section we consider the relationships between l_p and $l_{p,\sigma}$ spaces.

Theorem 4.1. *Let $1 \leq p \leq +\infty$ and let σ be the sequence of nonnegative numbers. If $\sup\limits_{n\in\mathbb{N}} \sigma_n < +\infty$, then $l_p \subseteq l_{p,\sigma}$.*

Proof. Let $1 \leq p < +\infty$, $\sigma_n \leq M \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $x \in l_p$ be arbitrary. We have that

$$\sum_{n\in\mathbb{N}} |x_n|^p \sigma_n \leq M \sum_{n\in\mathbb{N}} |x_n|^p < +\infty \ .$$

Thus, $x \in l_{p,\sigma}$.

If $p = +\infty$ and $\sigma_n \leq M \in \mathbb{R}$ for all $n \in \mathbb{N}$, then for arbitrary $x \in {}_\infty$ we have that $|x_n|\sigma_n \leq |x_n|M$, i.e. $\sup\limits_{n\in\mathbb{N}} |x_n|\sigma_n \leq M \sup\limits_{n\in\mathbb{N}} |x_n|$. Therefore,

$$||x||_{l_{\infty,\sigma}} \leq M||x||_{l_\infty} < +\infty \ ,$$

and $x \in l_{\infty,\sigma}$. □

Considering the l_p space as a weighted $l_{p,1}$ space and using Lemma 2.2 and Theorem 4.1 we get the following corollary.

Corollary 4.1. *If σ is the sequence bounded from above and such that $\liminf_{n\to\infty} \sigma_n = 0$, then $l_p \subsetneq l_{p,\sigma}$.*

For example, for $p = +\infty$ we consider the sequence $x = (n)_{n\in\mathbb{N}}$ and weighted function $\sigma_n = \frac{1}{n}$ $(n \in \mathbb{N})$. Clearly $x \in l_{\infty,\frac{1}{n}}$ and $x \notin l_\infty$. Hence, we have a strict inclusion in the relation from the Theorem 4.1 and in this case we have that $l_\infty \subset l_{\infty,\frac{1}{n}}$.

Theorem 4.2. *Let $1 \le p \le +\infty$ and let σ be the sequence of nonnegative numbers. If $\inf_{n\in\mathbb{N}} \sigma_n > 0$ then $l_{p,\sigma} \subseteq l_p$.*

Proof. Let $1 \le p < +\infty$ and $\sigma_n \ge m > 0$ for all $n \in \mathbb{N}$. For an arbitrary $x \in l_{p,\sigma}$ we have that

$$m \sum_{n\in\mathbb{N}} |x_n|^p \le \sum_{n\in\mathbb{N}} |x_n|^p \sigma_n < +\infty \ ,$$

i.e. $x \in l_p$.

Let $p = +\infty$ and $\sigma_n \ge m > 0$ for all $n \in \mathbb{N}$. Let $x \in l_{\infty,\sigma}$ be arbitrary. For arbitrary $n \in \mathbb{N}$ we have that $m|x_n| \le |x_n|\sigma_n$ and hence $m \sup_{n\in\mathbb{N}} |x_n| \le \sup_{n\in\mathbb{N}} |x_n|\sigma_n$, i.e.

$$m||x||_{l_\infty} \le ||x||_{l_{\infty,\sigma}} \ .$$

Hence $x \in l_\infty$. □

Considering the l_p space as a weighted $l_{p,1}$ space and using Lemma 2.1 and Theorem 4.2 we obtain the following result.

Corollary 4.2. *If $\inf_{n\in\mathbb{N}} \sigma_n > 0$ and $\limsup_{n\to\infty} \sigma_n = +\infty$, then $l_{p,\sigma} \subsetneq l_p$.*

Let us consider $x = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ and weighted sequence $\sigma = (n^2)_{n\in\mathbb{N}}$. It is clear that $\sup_{n\in\mathbb{N}} |x_n|\sigma_n = \sup_{n\in\mathbb{N}} n = +\infty$, and $x \notin l_{\infty,n^2}$. But it is also clear that $x \in l_\infty$. This example justifies the previous corollary, i.e. in this case we have that $l_{\infty,n^2} \subset l_\infty$. Now we can give a more general statement.

Corollary 4.3. *Let $1 \le p \le +\infty$ and let σ be the sequence of nonnegative numbers. If there exist $m, M \in \mathbb{R}$, such that $0 < m \le \sigma_n \le M < +\infty$ holds for all $n \in \mathbb{N}$, then $l_p = l_{p,\sigma}$.*

5. The reflexivity of the space $l_{p,\sigma}$

Theorem 5.1. *Any bounded linear functional x^* on the space $l_{p,\sigma}$, $1 < p < +\infty$, has the following representation*

$$x^*(x) = \sum_{i\in\mathbb{N}} \eta_i \xi_i, \ \ y = (\eta_i)_{i\in\mathbb{N}} \in l_{p,\tau} \ , \tag{5.1}$$

where $\tau = \sigma^{-\frac{1}{p-1}}$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then the functional x^* on $l_{p,\sigma}$ defines a unique point $y \in l_{q,\tau}$ and $\|x^*\| = \|y\|_{l_{p,\tau}}$.

Proof. Let $x = (\xi_i)_{i\in\mathbb{N}} \in l_{p,\sigma}$ and $y = (\eta_i)_{i\in\mathbb{N}} \in l_{p,\tau}$, where $\tau_i = \sigma_i^{-\frac{1}{p-1}}$ i $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Hölder's inequality implies

$$\sum_{i\in\mathbb{N}} |\eta_i\xi_i| = \sum_{i\in\mathbb{N}} |\eta_i\sigma_i^{-\frac{1}{p}}\xi_i\sigma_i^{\frac{1}{p}}| \le \left(\sum_{i\in\mathbb{N}} |\eta_i\sigma_i^{-\frac{1}{p}}|^q\right)^{\frac{1}{q}} \left(\sum_{i\in\mathbb{N}} |\xi_i\sigma_i^{\frac{1}{p}}|^p\right)^{\frac{1}{p}} .$$

Since

$$\left(\sum_{i\in\mathbb{N}} |\eta_i|^q\sigma_i^{-\frac{q}{p}}\right)^{\frac{1}{q}} = \left(\sum_{i\in\mathbb{N}} |\eta_i|^q\sigma_i^{-\frac{1}{p-1}}\right)^{\frac{1}{q}} = \left(\sum_{i\in\mathbb{N}} |\eta_i|^q\tau_i\right)^{\frac{1}{q}} = \|y\|_{l_{q,\tau}} ,$$

we have that

$$\sum_{i\in\mathbb{N}} |\eta_i\xi_i| \le \|y\|_{l_{q,\tau}}\|x\|_{l_{p,\sigma}} < \infty ,$$

hence the expression $\sum_{i\in\mathbb{N}} |\eta_i\xi_i|$ always make sense. If we consider this expression as the function of $x \in l_{p,\sigma}$ then it defines a functional x^* on the space $l_{p,\sigma}$. The linearity of the functional x^* is clear, while its boundedness follows from the inequality $|x^*(x)| \le \|y\|_{l_{q,\tau}}\|x\|_{l_{p,\sigma}}$. Thus

$$\|x^*\| \le \|y\|_{l_{q,\tau}}. \tag{5.2}$$

Conversely, let x^* be a bounded linear functional on $l_{p,\sigma}$, $1 < p < +\infty$. Then for all $x \in l_{p,\sigma}$ we have that

$$\left\| x - \sum_{i\in\mathbb{N}} \xi_i e_i \right\|_{l_{p,\sigma}} = \left\| \sum_{i=n+1}^{\infty} \xi_i e_i \right\|_{l_{p,\sigma}} = \left(\sum_{i=n+1}^{\infty} |\xi_i|^p\sigma_i \right)^{\frac{1}{p}} \to 0,\ (n\to\infty) ,$$

hence we can write $x = \sum_{i\in\mathbb{N}} \xi_i e_i$. By the linearity and boundedness of the functional x^* we conclude that

$$x^*(x) = \sum_{i\in\mathbb{N}} \xi_i x^*(e_i) .$$

If we take $y = (\eta_i)_{i\in\mathbb{N}} = (x^*(e_i))_{i\in\mathbb{N}}$, we can see that the functional x^* has the form (5.1).

Now consider the sequence $x_n = (\xi_i^n)_{i\in\mathbb{N}}$, $n = 1, 2, ..$ where

$$\xi_i^n = \begin{cases} \operatorname{sgn} \eta_i|\eta_i|^{q-1}\sigma_i^{-\frac{1}{p-1}} & for \quad i \le n, \\ 0 & for \quad i > n. \end{cases}$$

Then

$$\|x_n\|_{l_{p,\sigma}} = \left(\sum_{i=1}^{n} |\eta_i|^{pq-p} \left(\sigma_i^{-\frac{1}{p-1}}\right)^p \sigma_i \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |\eta_i|^q\sigma_i^{-\frac{1}{p-1}} \right)^{\frac{1}{p}},$$

and $x^*(x_n) = \sum_{i\in\mathbb{N}} \xi_i^n x^*(e_i) = \sum_{i=1}^{n} |\eta_i|^q\tau_i$. Since the functional x^* is bounded we have that $|x^*(x_n)| \le \|x^*\|\|x_n\|$, i.e.

$$\sum_{i=1}^{n} |\eta_i|^q\tau_i \le \|x^*\| \left(\sum_{i=1}^{n} |\eta_i|^q\tau_i \right)^{\frac{1}{p}} .$$

Thus $\left(\sum_{i=1}^{n}|\eta_i|^q\tau_i\right)^{\frac{1}{q}} \leq \|x^*\|$ and

$$\|y\|_{l_{q,\tau}} = \left(\sum_{i\in\mathbb{N}}|\eta_i|^q\tau_i\right)^{\frac{1}{q}} \leq \|x^*\|. \tag{5.3}$$

We conclude that $y \in l_{q,\tau}$, and by using (5.2) and (5.3) we get that $\|y\|_{l_{q,\tau}} = \|x^*\|$.

Now we wish to show that the representation (5.1) is unique. Suppose that there exist two different points $y_1 = (\eta_i^{(1)})_{i\in\mathbb{N}}$ and $y_2 = (\eta_i^{(2)})_{i\in\mathbb{N}}$ such that

$$x^*(x) = \sum_{i\in\mathbb{N}}\eta_i^{(1)}\xi_i = \sum_{i\in\mathbb{N}}\eta_i^{(2)}\xi_i \ .$$

This would mean that for $x = (e_i)_{i\in\mathbb{N}}$ we have that $x^*(e_i) = \eta_i^{(1)} = \eta_i^{(2)}$, for all $i = 1, 2, ...,$ which contradicts the assumption that $y_1 \neq y_2$. □

Therefore $l_{p,\sigma}^* = l_{q,\tau}$, where $\tau = \sigma^{-\frac{1}{p-1}}$ i $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For the second dual of the space $l_{p,\sigma}$, we have that

$$l_{p,\sigma}^{**} = l_{q,\tau}^* = l_{p,\rho} \ ,$$

where $\rho = \tau^{-\frac{1}{q-1}}$. Since

$$\rho = \tau^{-\frac{1}{q-1}} = \left(\sigma^{-\frac{1}{p-1}}\right)^{-\frac{1}{q-1}} = \sigma^{\frac{1}{pq-q-p+1}} = \sigma \ ,$$

we have that $l_{p,\sigma}^{**} = l_{p,\sigma}$. Thus, $l_{p,\sigma}$ $(1 < p < +\infty)$ is a reflexive space.

Theorem 5.2. *Any bounded linear functional x^* on the space $l_{1,\sigma}$, has the following representation*

$$x^*(x) = \sum_{i\in\mathbb{N}}\eta_i\xi_i, \ \ y = (\eta_i)_{i\in\mathbb{N}} \in l_{\infty,\frac{1}{\sigma}} \ . \tag{5.4}$$

Then the functional x^* on $l_{1,\sigma}$ defines a unique point $y \in l_{\infty,\frac{1}{\sigma}}$ and $\|x^*\| = \|y\|_{l_{\infty,\frac{1}{\tau}}}$.

Proof. Let $y = (\eta_i)_{i\in\mathbb{N}} \in l_{\infty,\frac{1}{\sigma}}$. Equality (5.4) defines a bounded linear functional on the space $l_{1,\sigma}$. Indeed,

$$|x^*(x)| \leq \sum_{i\in\mathbb{N}}|\eta_i\xi_i| \leq \sup_{i\in\mathbb{N}}|\eta_i|\frac{1}{\sigma_i}\sum_{i\in\mathbb{N}}|\xi_i|\sigma_i = \|y\|_{l_{\infty,\frac{1}{\sigma}}}\|x\|_{l_{1,\sigma}}.$$

Thus,

$$\|x^*\| \leq \|y\|_{l_{\infty,\frac{1}{\sigma}}}. \tag{5.5}$$

Let us prove that for every bounded linear functional x^* on $l_{1,\sigma}$ there exists a unique point $y \in l_{\infty,\frac{1}{\sigma}}$ such that (5.4) holds. Notice that for all $x = (\xi_i)_{i\in\mathbb{N}} \in l_{1,\sigma}$ we have that

$$\left\|x - \sum_{i=1}^{n}\xi_i e_i\right\|_{l_{1,\sigma}} = \sum_{i=n+1}^{\infty}|\xi_i|\sigma_i \to 0, \ (n\to\infty) \ ,$$

hence we can write $x = \sum_{i\in\mathbb{N}} \xi_i e_i$. Consequently $x^*(x) = \sum_{i\in\mathbb{N}} \xi_i x^*(e_i)$. If we put $\eta_i = x^*(e_i)$, we see that the functional x^* has the form (5.4). Now we will prove that $y = (\eta_i)_{i\in\mathbb{N}}$ belongs to the space $l_{\infty,\frac{1}{\sigma}}$. Consider the sequence $x_n = (\xi_i^n)_{i\in\mathbb{N}}$, $n = 1, 2, ..,$ where

$$\xi_i^n = \begin{cases} \operatorname{sgn} \eta_n & , \quad i = n, \\ 0 & , \quad i \neq n. \end{cases}$$

For $n \in \mathbb{N}$ we have that $\|x_n\|_{l_{1,\sigma}} = \sigma_n$ and $x^*(x_n) = |\eta_n|$. Since x^* is a bounded linear functional, i.e. $|x^*(x_n)| \leq \|x^*\| \|x_n\|_{l_{1,\sigma}}$, we have t the inequality $|\eta_n|\frac{1}{\sigma_n} \leq \|x^*\|$. Taking the supremum over all $n \in \mathbb{N}$ yields

$$\|y\|_{l_{\infty,\frac{1}{\sigma}}} \leq \|x^*\|. \tag{5.6}$$

Thus $y \in l_{\infty,\frac{1}{\sigma}}$ and the equality $\|y\|_{l_{\infty,\frac{1}{\sigma}}} = \|x^*\|$ holds due to (5.5) and (5.6).
The uniqueness of the representation (5.1) can be shown in the same way as was done in Theorem 5.1. □

6. Discussion

Borwein and Gao [3] gave some necessary and sufficient conditions for a nonnegative matrix to bounded operator from l_p to l_q, $1 < p \leq q < \infty$. We could consider acting of the mentioned matrix from the weighted sequence space $l_{p,\sigma}$ to $l_{q,\tau}$.

Let $l_{p,\sigma}$ and $l_{q,\tau}$ be weighted spaces such that $\lim\limits_{n\to\infty} \frac{\tau_n}{\sigma_n} \in (0, \infty)$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a bounded weighted sequence, such that $0 < m \leq \sigma_n \leq M < +\infty$. If we assume that there exists a real positive number C and a positive sequence $(u_j)_{j\in\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} a_{ij} \left(\sum_{k=1}^{\infty} a_{ik} u_k \right)^{p-1} \leq C u_j^{p-1}, \quad j = 1, 2, ...,$$

where $A = (a_{ij})$ is a real nonnegative matrix, then using Theorem A from [3] and Corollaries 2.1 and 4.3 we conclude that the matrix operator A acts from $l_{p,\tau}$ to $l_{p,\sigma}$ and that $\|A\|_{l_{p,\sigma}} \leq C^{\frac{1}{p}}$ holds.

There are many results about the acting conditions of different classes of operators (matrix, integral, etc.) which are defined on sequence spaces as well as on weighted sequence spaces ([1, 10, 14]). Furthermore, some attention was given to the calculating of the norm of such operators. For example, Jameson and Laksharipour [7] determined the norm of some operators on weighted l_p spaces and corresponding Lorentz sequence spaces $d(w, p)$ with the power weighting sequence $w_n = n^{-\alpha}$ or the variant defined by $w_1 + ... + w_n = n^{1-\alpha}$. The problem of finding a lower bound of such operators was also considered by Jameson and Laksharipour [8]. The norm of arbitrary weighted mean matrix acting on an arbitrary weighted space $l_{1,\sigma}$ where σ is a decreasing, nonnegative sequence such that $\lim\limits_{n\to\infty} \sigma_n = 0$ and $\sum\limits_{n=1}^{\infty} \sigma_n$ is divergent, was presented by Lashkaripour in [11].

In view of the relationships between weighted sequence spaces presented in this paper, we could then consider the action of different operators on weighted sequence spaces under some other assumptions about the weighted sequence.

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