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DIFFERENTIAL MONOMIALS AND THEIR MEASURE OF GROWTHS FROM THE VIEW POINT OF RELATIVE WEAK TYPE

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ABSTRACT. In this paper we explore on some comparative growth properties of composite entire and meromorphic functions on the basis of relative weak type of differential monomials generated by transcendental entire and transcendental meromorphic functions.

1. INTRODUCTION

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ known as maximum modulus function corresponding to f is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)| .$$

When f is meromorphic, $M_f(r)$ can not be defined as f is not analytic. In this situation one may define another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r) \ .$$

If given two meromorphic functions f and g the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's characteristic function. When f is entire function, the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) \ .$$

We called the function $N_f(r, a)$ $\left(\bar{N}_f(r, a)\right)$ as counting function of a-points (distinct a-points) of f. In many occasions $N_f(r, \infty)$ and $\bar{N}_f(r, \infty)$ are denoted by $N_f(r)$ and

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$\bar{N}_f(r)$ respectively. We put

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r \ ,$$

where we denote by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f . Also we denote by $n_{f|=1}(r,a)$,the number of simple zeros of $f-a$ in $|z| \leq r$. Next, $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way and we set, see [9]

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f\ |=1)}{T_f(r)} \quad ,$$

the deficiency of 'a' corresponding to the simple a- points of f i,e. simple zeros of $f-a$. In this connection Yang [8] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which

$$\delta_1(a;f) > 0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \leq 4.$$

On the other hand, $m\left(r, \frac{1}{f-a}\right)$ is denoted by $m_f(r,a)$ and we mean $m_f(r,\infty)$ by $m_f(r)$, which is called the proximity function of f. We also put

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta \,,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$.

Further a meromorphic function $b = b(z)$ is called small with respect to f if it holds $T_b(r) = S_f(r)$ where $S_f(r) = o\{T_f(r)\}$ i.e., $\frac{S_f(r)}{T_f(r)} \to 0$ as $r \to \infty$. Moreover for any transcendental meromorphic function f , we call $P[f] = bf^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$, to be a differential monomial generated by it where $\sum_{i=0}^{k} n_i \geq 1$ (all $n_i \mid i = 0,1,...,k$ are non-negative integers) and the meromorphic function b is small with respect to f. In this connection the numbers $\gamma_{P[f]} = \sum_{i=0}^{k} n_i$ and $\Gamma_{P[f]} = \sum_{i=0}^{k} (i+1)n_i$ are called the degree and weight of $P[f]$ respectively, see [1].

The *order* of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)} \ .$$

Lahiri and Banerjee [6] introduced the definitions of *relative order* and *relative lower order* of a meromorphic function with respect to an entire function to avoid comparing growth just with $\exp z$. To compare the relative growth of two meromorphic functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [3] introduced the notion of *relative weak type* of meromorphic functions with respect to an entire function. Extending these notions of *relative weak type* as cited in the reference, Datta, Biswas and Bhattacharyya [4] gave the definition of *relative weak type* of *differential monomials* generated by entire and meromorphic functions.

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *lower order* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order (* respectively *relative lower order)* and consequently *relative weak type* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysts and they are not aware of the technical advantages of using the relative growth indicators of the functions. Therefore the growth of composite entire and meromorphic functions needs to be modified on the basis of their *relative order (* respectively *relative lower order)* and *relative weak type* some of which has been explored in this paper. Actually in this paper we establish some newly developed results based on the growth properties of *relative weak type* of *monomials* generated by transcendental entire and transcendental meromorphic functions.

2. Notation and preliminary remarks

We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [5] and [7]. Henceforth, we do not explain those in details. Now we just recall some definitions which will be needed in the sequel.

Definition 2.1. *The* order ρ_f *and* lower order λ_f *of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

To determine the relative growth of two meromorphic functions having same non zero finite lower order, Datta and Jha [2] introduced the definition of *weak type* of a meromorphic function of finite positive lower order in the following way:

Definition 2.2. [2] *The* weak type τ_f *of a meromorphic function f of finite positive lower order λ_f is defined by*

$$\tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} .$$

Similarly, one can define the growth indicator $\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} .$$

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$, its Nevanlinna's characteristic function is strictly increasing and continuous. Hence there exists its inverse function $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ with $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [6] introduced the definition of *relative order* of a meromorphic function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [6] if $g(z) = \exp z$. Similarly, one can define the *relative lower order* of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In the case of relative order, it therefore seems reasonable to define suitably the relative weak type of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite relative lower order with respect to an entire function. Datta and Biswas [3] gave such definitions of relative weak type of a meromorphic function f with respect to an entire function g which are as follows:

Definition 2.3. [3] *The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by*

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

In a similar manner, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Definition 2.4. [1] *$P[f]$ is said to be admissible if*
(i) $P[f]$ is homogeneous, or
(ii) $P[f]$ is non homogeneous and $m(r,f) = S(r,f)$.

3. Some Examples

In this section we present some examples in connection with definitions given in the previous section.

Example 3.1 (Order (lower order)). *Given any natural number n, let $f(z) = \exp z^n$. Then $M_f(r) = \exp r^n$. Therefore putting $R = 2$ in the inequality $T_f(r) \le \log M_f(r) \le \frac{R+r}{R-r} T_f(R)$, (see [5]) we get that $T_f(r) \le r^n$ and $T_f(r) \ge \frac{1}{3}\left(\frac{r}{2}\right)^n$. Hence*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} = n \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} = n .$$

Further if we take $g = \exp^{[2]} z$*, then* $T_g(r) \sim \dfrac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ (r \to \infty)$. *Therefore*

$$\rho_f = \lambda_f = \infty \ .$$

Example 3.2 (Weak type). *Let* $f = \exp z$. *Then* $T_f(r) = \dfrac{r}{\pi}$, *and* $\rho_f = 1$. *So*

$$\tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} = \frac{\frac{r}{\pi}}{r} = \frac{1}{\pi} \quad \text{and} \quad \overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} = \frac{\frac{r}{\pi}}{r} = \frac{1}{\pi} \ .$$

Further, if we consider $g = \dfrac{2}{1 - \exp(2z)}$, *then one can easily verify that*

$$\tau_g = \overline{\tau}_g = \frac{2}{\pi} \ .$$

Example 3.3 (Relative order (relative lower order)). *Suppose* $f = g = \exp^{[2]} z$ *then* $T_f(r) = T_g(r) \sim \dfrac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ (r \to \infty)$. *Now we obtain that*

$$\begin{aligned} T_g(r) &\leq \log M_g(r) \leq 3T_g(2r) \quad (\textit{see } [5]) \\ i.e., T_g(r) &\leq \exp r \leq 3T_g(2r) \ . \end{aligned}$$

Therefore

$$\begin{aligned} T_g^{-1} T_f(r) &\geq \log\left(\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right) \\ i.e., \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} &\geq 1 \end{aligned}$$

and

$$\begin{aligned} T_g^{-1} T_f(r) &\leq 2\log\left(\frac{3\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right) \\ i.e., \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} &\leq 1 \ . \end{aligned}$$

Hence

$$\rho_g(f) = \lambda_g(f) = 1 \ .$$

Example 3.4 (Relative weak type). *Suppose* $f = g = \exp z$. *Then* $T_f(r) = T_g(r) = T_{\exp z}(r) = \dfrac{r}{\pi}$ *and* $T_g^{-1} T_f(r) = T_g^{-1}\left(\dfrac{r}{\pi}\right) = r$.
So

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} = 1 \, .$$

Therefore

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}} = 1 \text{ and } \overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}} = 1 \, .$$

4. Lemmas

In this section we present some lemmas which will be needed in the following.

Lemma 4.1. [4] *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. *Also let g be a transcendental entire function of regular growth having non zero finite order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then the relative lower order of $P[f]$ with respect to $P[g]$ are same as those of f with respect to g.*

Lemma 4.2. [4] *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function of regular growth having non zero finite weak type and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$.

Then $\tau_{P[g]}(P[f])$ *and* $\overline{\tau}_{P[g]}(P[f])$ *are* $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g i.e.,*

$$\tau_{P[g]}(P[f])=\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\tau_g(f)\ \ \text{and}$$

$$\overline{\tau}_{P[g]}(P[f])=\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\tau}_g(f)\ .$$

when $\lambda_g(f)$ is positive finite.

5. Main results

In this section we present the main theorems of the paper.

Theorem 5.1. *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. *Also let h be a transcendental entire function of regular growth having non zero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and g be any entire function such that* $0<\tau_h(f\circ g)\le\overline{\tau}_h(f\circ g)<\infty$, $0<\tau_h(f)\le\overline{\tau}_h(f)<\infty$ *and* $\lambda_h(f\circ g)=\lambda_h(f)$. *Then*

$$\frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)}\le\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}$$

$$\le\frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)}$$

$$\le\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\le\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)}\ .$$

Proof. From the definition of $\overline{\tau}_h(f)$, $\tau_h(f\circ g)$ and in view of Lemma 4.1 and Lemma 4.2 we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_h^{-1}T_{f\circ g}(r) \geqslant (\tau_h(f\circ g)-\varepsilon)(r)^{\lambda_h(f\circ g)} \tag{5.1}$$

and

$$T_{P[h]}^{-1}T_{P[f]}(r) \leq \left(\overline{\tau}_{P[h]}(P[f])+\varepsilon\right)(r)^{\lambda_{P[h]}(P[f])}$$

$$i.e.,\ T_{P[h]}^{-1}T_{P[f]}(r)$$

$$\leq \left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)+\varepsilon\right)(r)^{\lambda_h(f)}\ . \tag{5.2}$$

Now from (5.1), (5.2) and using the condition $\lambda_h(f\circ g)=\lambda_h(f)$, it follows for all large values of r that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)} \geqslant \frac{(\tau_h(f\circ g)-\varepsilon)}{\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)+\varepsilon\right)}\ .$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)} \geqslant \frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)}\ . \tag{5.3}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1}T_{f\circ g}(r) \leq (\tau_h(f\circ g)+\varepsilon)(r)^{\lambda_h(f\circ g)} \tag{5.4}$$

and for all sufficiently large values of r,

$$T_{P[h]}^{-1}T_{P[f]}(r) \geqslant \left(\tau_{P[h]}(P[f])-\varepsilon\right)(r)^{\lambda_{P[h]}(P[f])}$$

$$i.e.,\ T_{P[h]}^{-1}T_{P[f]}(r)$$

$$\geq \left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)-\varepsilon\right)(r)^{\lambda_h(f)}\ . \tag{5.5}$$

Combining (5.4) and (5.5) and using the condition $\lambda_h(f\circ g)=\lambda_h(f)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)} \leq \frac{(\tau_h(f\circ g)+\varepsilon)}{\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)-\varepsilon\right)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)} \leq \frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)}\ . \tag{5.6}$$

Also for a sequence of values of r tending to infinity that

$$T_{P[h]}^{-1}T_{P[f]}(r) \leq \left(\tau_{P[h]}(P[f])+\varepsilon\right)(r)^{\lambda_{P[h]}(P[f])}$$

$$i.e.,\ T_{P[h]}^{-1}T_{P[f]}(r)$$

$$\leq \left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)+\varepsilon\right)(r)^{\lambda_h(f)}\ . \tag{5.7}$$

Now from (5.1) and (5.7) and using the condition $\lambda_h(f\circ g)=\lambda_h(f)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\geq\frac{(\tau_h(f\circ g)-\varepsilon)}{\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)+\varepsilon\right)}\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\geq\frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)}\ . \tag{5.8}$$

Also for all sufficiently large values of r,

$$T_h^{-1}T_{f\circ g}(r)\leq(\overline{\tau}_h(f\circ g)+\varepsilon)(r)^{\lambda_h(f\circ g)}\ . \tag{5.9}$$

Since $\lambda_h(f\circ g)=\lambda_h(f)$, it follows from (5.5) and (5.9) for all sufficiently large values of r that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\leq\frac{(\overline{\tau}_h(f\circ g)+\varepsilon)}{\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)-\varepsilon\right)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\leq\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(f)}\ . \tag{5.10}$$

Thus the theorem follows from (5.3), (5.6), (5.8) and (5.10). □

The following theorem can be proved similar as Theorem 5.1 and so its proof is omitted.

Theorem 5.2. *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function with $0<\tau_h(f\circ g)\leq\overline{\tau}_h(f\circ g)<\infty$, $0<\tau_h(g)\leq\overline{\tau}_h(g)<\infty$ and $\lambda_h(f\circ g)=\lambda_h(g)$. Then*

$$\frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(g)}\leq\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[g]}(r)}$$

$$\leq\frac{\tau_h(f\circ g)}{\left(\frac{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(g)}$$

$$\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[g]}(r)}\leq\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_h(g)}\ .$$

Theorem 5.3. *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and g be any entire function such that $0<\overline{\tau}_h(f\circ g)<\infty$, $0<\overline{\tau}_h(f)<\infty$ and $\lambda_h(f\circ g)=\lambda_h(f)$. Then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\le\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)}\le\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}.$$

Proof. From the definition of $\overline{\tau}_{P[h]}(P[f])$ and in view of Lemma 4.1 and Lemma 4.2, we get for a sequence of values of r tending to infinity that

$$T_{P[h]}^{-1}T_{P[f]}(r)\geqslant\left(\overline{\tau}_{P[h]}(P[f])-\varepsilon\right)(r)^{\lambda_{P[h]}(P[f])}$$

$$i.e.,\ T_{P[h]}^{-1}T_{P[f]}(r)$$

$$\geq\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)-\varepsilon\right)(r)^{\lambda_h(f)}\ . \tag{5.11}$$

Now from (5.9), (5.11) and using the condition $\lambda_h(f\circ g)=\lambda_h(f)$, it follows for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\le\frac{(\overline{\tau}_h(f\circ g)+\varepsilon)}{\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)-\varepsilon\right)}\ .$$

As $\varepsilon\,(>0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\le\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)}\ . \tag{5.12}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1}T_{f\circ g}(r)\geqslant(\overline{\tau}_h(f\circ g)-\varepsilon)(r)^{\lambda_h(f\circ g)}\ . \tag{5.13}$$

So combining (5.2) and (5.13) and using the condition $\lambda_h(f\circ g)=\lambda_h(f)$,we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\geqslant\frac{(\overline{\tau}_h(f\circ g)-\varepsilon)}{\left(\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)+\varepsilon\right)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\geqslant\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(f)}\ . \tag{5.14}$$

Thus the theorem follows from (5.12) and (5.14). □

The following theorem can proved similar as Theorem 5.3 and therefore we omit its proof.

Theorem 5.4. *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function with $0<\overline{\tau}_h(f\circ g)<\infty$, $0<\overline{\tau}_h(g)<\infty$ and $\lambda_h(f\circ g)=\lambda_h(g)$. Then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[g]}(r)}\leq\frac{\overline{\tau}_h(f\circ g)}{\left(\frac{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_h(g)}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[g]}(r)}.$$

The following theorem is a natural consequence of Theorem 5.1 and Theorem 5.3.

Theorem 5.5. *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and g be any entire function such that $0<\tau_h(f\circ g)\leq\overline{\tau}_h(f\circ g)<\infty$, $0<\tau_h(f)\leq\overline{\tau}_h(f)<\infty$ and $\lambda_h(f\circ g)=\lambda_h(f)$. Then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}\leq\min\left\{A\cdot\frac{\tau_h(f\circ g)}{\tau_h(f)},A\cdot\frac{\overline{\tau}_h(f\circ g)}{\overline{\tau}_h(f)}\right\}$$
$$\leq\max\left\{A\cdot\frac{\tau_h(f\circ g)}{\tau_h(f)},A\cdot\frac{\overline{\tau}_h(f\circ g)}{\overline{\tau}_h(f)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[f]}(r)}$$

where $A=\frac{1}{\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}}$.

The proof is omitted. Similarly one may state the following theorem, again without its proof.

Theorem 5.6. *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function with $0<\tau_h(f\circ g)\leq\overline{\tau}_h(f\circ g)<\infty$, $0<\tau_h(g)\leq\overline{\tau}_h(g)<\infty$ and $\lambda_h(f\circ g)=\lambda_h(g)$. Then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[g]}(r)}\leq\min\left\{B\cdot\frac{\tau_h(f\circ g)}{\tau_h(g)},B\cdot\frac{\overline{\tau}_h(f\circ g)}{\overline{\tau}_h(g)}\right\}$$
$$\leq\max\left\{B\cdot\frac{\tau_h(f\circ g)}{\tau_h(g)},B\cdot\frac{\overline{\tau}_h(f\circ g)}{\overline{\tau}_h(g)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P[h]}^{-1}T_{P[g]}(r)}$$

where $B=\frac{1}{\left(\frac{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}}$.

6. Conclusion

Actually this paper deals with the extension of the works on the growth properties *differential monomials* generated by transcendental entire and transcendental meromorphic functions on the basis of their *relative weak types*. These theories can also be modified by the treatment of the notions of *generalized relative weak type* and (*(p,q)-th relative weak type*. In addition some extensions of the same may be done in the light of slowly changing functions. Moreover, the notion of *relative weak type* of *differential monomials* generated by transcendental entire and transcendental meromorphic functions may have a wide range of applications in complex dynamics, factorization theory of entire functions of single complex variable, the solution of complex differential equations etc. which should be a vergine area of further research.

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