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AN EXTENDED POISSON-LOMAX DISTRIBUTION

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ABSTRACT. An extension of Poisson-Lomax distribution is introduced and investigated. In particular, the behavior of the density function, the shape of the hazard rate function, a general expansion for moments are derived and studied in detail. The method of maximum likelihood estimators of the involved parameters are obtained. Application of the proposed model to real data is given for illustrative purpose.

1. INTRODUCTION

The study of lifetime of devices, components and systems is of major importance in the field of reliability engineering. The mathematical description of the lifetime can be studied through the failure distribution. Existing distributions may not fit well the actual observations of times to failure. Therefore, many authors have introduced different methods for generalizing or extending the well-known and frequently distributions. Marshall and Olkin [1] first introduced a general method for adding a parameter to the families of exponential and Weibull and then a series of papers adopted the method and proposed new distributions. Another instance of the same generalization, Ghitany et al. [8] introduced Marshal-Olkin extended Lomax distribution. The Lomax distribution has been used in literature in a number of ways see e.g. Bryson [7] and Al-Zahrani and Al-Sobhi [4]. The Poisson distribution has been used extensively for theoretical purposes and in practical applications. The Poisson-exponential distribution is obtained by formally mixing the Poisson distribution with exponential distribution, see Kus [5]. Taking a mixture of the Poisson distribution with a Lindley distribution, Sankaran [9] obtained a discrete Poisson-Lindley distribution. Very recently, Al-Zahrani and Sagor [2] have introduced the Lomax-Logarithmic distribution. Also, Al-Zahrani and Sagor [3] have introduced the Poisson-Lomax distribution (PLD). In this paper we introduce and investigate an extension of Poisson-Lomax distribution.

The article is outlined as follows. In Section 2, the model analysis of the proposed distribution is discussed, particularly, the survival, density, hazard functions and the moments are obtained. Section 3 discusses the estimation of the unknown parameters using the maximum likelihood estimation method. In Section 4, two real examples are presented for illustrative purpose. Finally we conclude in Section 5.

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2. Model Analysis

Suppose that the random variables $Y_1, Y_2, \cdots, Y_z$ are independent and identically distributed, each with distribution function F. Further suppose Z is a random variable distributed according to a shifted Poisson distribution with probability mass function $P_Z(z)$, where

$$P_Z(z) = \frac{\lambda^{z-1} e^{\lambda}}{(z-1)!}, \quad z = 1, 2, \cdots .$$

Define the survival function $G(x)$ as follows:

$$\begin{aligned} \bar{G}(x) &= \sum_{z=1}^{\infty} \bar{F}^z(x) Pr(Z = z) \\ &= \sum_{z=1}^{\infty} \bar{F}^z(x) \, \frac{\lambda^{z-1} e^{\lambda}}{(z-1)!} = \bar{F}(x) \, e^{-\lambda(1-\bar{F}(x))}, \quad x \geq 0, \lambda > 0. \qquad (2.1) \end{aligned}$$

Clearly, the limit of the function $\bar{G}(x)$ is 1 as x tends to 0 and it is 0 as x tends to infinity. The function $\bar{G}$ can be interpreted as an approximation of the life of a component with shortest life among similar other components. The density function corresponding to the survival function $\bar{G}(x)$ given in (2.1) is

$$g(x) = \{1 + \lambda[1 - F(x)]\} f(x) \; e^{-\lambda F(x)}, \quad x \geq 0, \lambda > 0. \qquad (2.2)$$

The hazard rate function, $h(x)$, associated with (2.1) is

$$h_G(x) = \lambda f(x) + h_F(x), \quad x \geq 0, \lambda > 0\,.$$

We start with the survival function of the Lomax distribution, $\bar{F}(x) = (1 + \beta x)^{-\alpha}$, $x > 0$ and define new survival function $\bar{G}(x)$ using (2.1) as follows:

$$\bar{G}(x; \alpha, \beta, \lambda) = (1 + \beta x)^{-\alpha} e^{-\lambda(1-(1+\beta x)^{-\alpha})}, \quad x > 0; \; \lambda \geq 0, \alpha, \beta > 0. \qquad (2.3)$$

2.1. **The density function.** The density function corresponding to (2.3) is given by

$$g(x; \alpha, \beta, \lambda) = \alpha\beta(1 + \beta x)^{-(\alpha+1)} e^{-\lambda(1-(1+\beta x)^{-\alpha})}(1 + \lambda(1 + \beta x)^{-\alpha}), \quad x > 0; \; \lambda \geq 0, \alpha, \beta > 0.$$

Theorem 2.1. *Let X be a random variable with $EPL(\alpha, \beta, \lambda)$ distribution. Then the pdf is a decreasing function in $x > 0$ for all values of α, β and λ.*

Proof. The first derivative of the logarithm of the pdf $g(x)$ is given by

$$(\log g(x))' = -\frac{\beta(\alpha + 1)}{(1 + \beta x)} - \frac{\alpha\beta\lambda}{(1 + \beta x)^{\alpha+1}} \; \xi((1 + \beta x)),$$

where $\xi(y) = 1 + 1/(1 + \lambda y^{-\alpha})$ and $y = (1 + \beta x) > 1$. It is easy to see that the function $(\log g(x))'$ is always negative, which implies that the pdf $g(x)$ is decreasing function with $g(0) = \alpha\beta(1 + \lambda)$ and $g(\infty) = 0$. □

2.2. **The hazard function.** The hazard rate function $h(x)$, $x \geq 0$, of $X \sim EPL(\alpha, \beta, \lambda)$ is defined as follows:

$$h_G(x; \alpha, \beta, \lambda) = \frac{\alpha\beta(1 + \lambda(1 + \beta x)^{-\alpha})}{(1 + \beta x)}.$$

Theorem 2.2. *Let X be a random variable with $EPL(\alpha, \beta, \lambda)$ distribution. Then the hazard function is a decreasing function in $x > 0$ for all values of α, β and λ.*

Proof. The proof is straightforward and therefore omitted. □

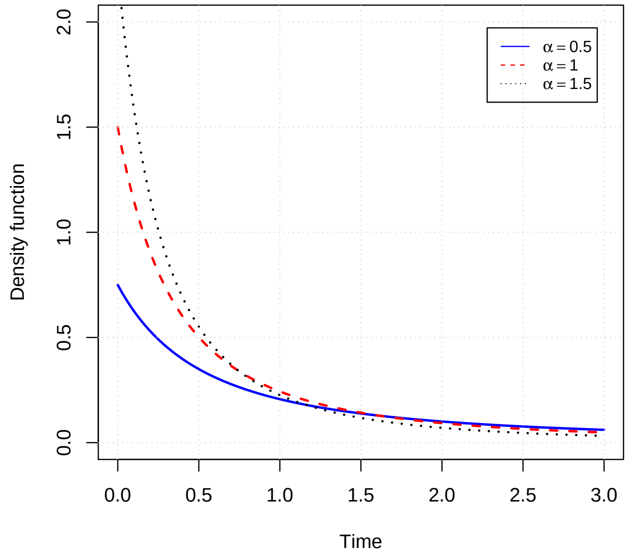

Figure 1. Plot of the density function for $\beta = 1.0$, $\lambda = 0.5$ and different values of the parameters α.

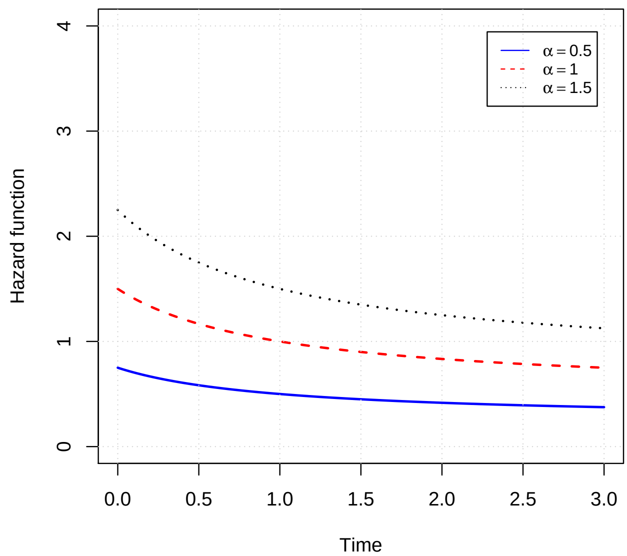

Figure 2. Plot of the hazard function for $\beta = 1.0$, $\lambda = 0.5$ and different values of the parameters α.

2.3. **Moments.** Now we present an infinite sum representation for the rth moment $\mu_r' = E[X^r]$, and consequently the first four moments and variance for the EPL distribution.

Theorem 2.3. *The rth moment about the origin of a random variable X, where $X \sim EPL(\alpha, \beta, \lambda)$, and $\alpha, \beta > 0$, $\lambda \geq 0$, is given by the following:*

(2.4)
$$\mu_r' = E[X^r] = \frac{re^{-\lambda}}{\beta^r} \sum_{n=0}^{\infty} \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{(-1)^{j+r} \lambda^n}{n!(j+1-\alpha(n+1))}, \quad \alpha \neq \frac{i}{n+1}, \quad i = 1, 2, \cdots, r.$$

Proof. The rth moment of X can be determined by direct integration using the formula

$$\begin{aligned} \mu_r' = E[X^r] &= r \int_0^{\infty} X^{r-1} \bar{G}(x) dx \\ &= r \int_0^{\infty} x^{r-1} (1+\beta x)^{-\alpha} e^{-\lambda(1-(1+\beta x)^{-\alpha})} dx \end{aligned}$$

Set $y = 1 + \beta x$ and integrate with respect to y. We use the series representation

$$(1-w)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \, w^j, \quad \text{where } k \text{ is an integer},$$

and the series expansion of the exponential function, $e^x = \sum_{i=0}^{\infty} x^i / i!$. Therefore, after some transformations and integrations we have

$$\begin{aligned} \mu_r' &= \frac{re^{-\lambda}}{\beta^r} \int_1^{\infty} (y-1)^{r-1} y^{-\alpha} e^{\lambda y^{-\alpha}} dy \\ &= \frac{re^{-\lambda}}{\beta^r} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \int_1^{\infty} y^{j-\alpha} e^{\lambda y^{-\alpha}} dy \\ &= \frac{re^{-\lambda}}{\beta^r} \sum_{n=0}^{\infty} \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{(-1)^{r-j} \lambda^n}{n!(j+1-\alpha(n+1))} \end{aligned}$$

Specially, the mean of $X \sim EPL(\alpha, \beta, \lambda)$ is

$$\mu = E(X) = \frac{-e^{-\lambda}}{\beta} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(1-\alpha(n+1))}, \quad \alpha \neq \frac{1}{n+1}.$$

□

Numerical computations for the theoretical mean μ_T and empirical mean μ_E, based on some values of the parameters, are given in Table 1. The empirical mean μ_E has been obtained after generating random numbers from EPL distribution using relation (2.5) for sample size $n = 100$ and values of the parameters. Some little differences between μ_T and μ_E are noted. The second moment of $X \sim EPL(\alpha, \beta, \lambda)$ can be obtained readily from (2.4) and hence the variance of X.

2.4. **Quantile function.**

Lemma 2.1. *Let $X \sim EPL(\alpha, \beta, \lambda)$. Then, the quantile function, say $Q(u)$, has the form*

$$Q(u) = \frac{1}{\beta} \left\{ \left[\frac{1}{\lambda} W(\lambda \, e^{\lambda} (1-u)) \right]^{-1/\alpha} - 1 \right\}, \tag{2.5}$$

where $W(\cdot)$ is the Lambert W-function.

			$\lambda = 0.5$		$\lambda = 1.0$		$\lambda = 1.5$		$\lambda = 2.0$	
n	α	β	μ_T	μ_E	μ_T	μ_E	μ_T	μ_E	μ_T	μ_E
100	2	0.5	1.4496	1.0363	1.0762	1.6019	0.8193	0.6744	0.6400	0.7054
		1.0	0.7248	0.5182	0.5381	0.8009	0.4097	0.3372	0.3200	0.3527
		1.5	0.4832	0.3454	0.3587	0.5340	0.2731	0.2248	0.2133	0.2351
		2.0	0.3624	0.2591	0.2690	0.4005	0.2048	0.1686	0.1600	0.1764
		2.5	0.2899	0.2073	0.2152	0.3204	0.1639	0.1349	0.1280	0.1411
	10	0.5	0.1727	0.1729	0.1371	0.1372	0.1111	0.1120	0.0919	0.0923
		1.0	0.0863	0.0865	0.0685	0.0686	0.0555	0.0560	0.0459	0.0462
		1.5	0.0576	0.0576	0.0457	0.0457	0.0370	0.0373	0.0306	0.0308
		2.0	0.0432	0.0432	0.0343	0.0343	0.0278	0.0280	0.0230	0.0231
		2.5	0.0345	0.0346	0.0274	0.0274	0.0222	0.0224	0.0184	0.0185

Table 1. The theoretical and empirical mean values of the EPL distribution for some values of α, β and λ

Note that the Lambert W-function is the inverse function of $z = f(W) = W \exp\{W\}$. That is, W is the function such that $W(z)\exp\{W(z)\} = z$. We use the R command `lambert_W0` to compute $W(z)$ for each entry in the argument z. Random numbers from the EPL distribution can be generated using (2.5). The quartiles of the EPL distribution for specified values of α, β and λ does not have closed form representation and has to be solved by applying numerical methods through R−program. The quartiles; Q_1 first quartile, Q_2 second quartile, or the median, and Q_3 third quartile are obtained in Table 2.

		λ				
α	Quartiles	0.5	1.0	1.5	2.0	2.5
0.5	Q_1	0.4695	0.3365	0.2608	0.4695	0.1791
	Q_2	1.8142	1.1980	0.8758	1.8142	0.5585
	Q_3	7.3898	4.2369	2.7695	7.3898	1.5271
1.0	Q_1	0.2122	0.1561	0.1229	0.2122	0.0858
	Q_2	0.6776	0.4826	0.3696	0.6776	0.2484
	Q_3	1.8965	1.2884	0.9415	1.8965	0.5897
1.5	Q_1	0.1369	0.1015	0.0803	0.1369	0.0564
	Q_2	0.4118	0.3002	0.2333	0.4118	0.1594
	Q_3	1.0320	0.7366	0.5563	1.0320	0.3621
2.0	Q_1	0.1010	0.0752	0.0596	0.1010	0.0420
	Q_2	0.2952	0.2176	0.1703	0.2952	0.1173
	Q_3	0.7019	0.5128	0.3934	0.7019	0.2608
2.5	Q_1	0.0800	0.0597	0.0474	0.0800	0.0335
	Q_2	0.2299	0.1706	0.1341	0.2299	0.0928
	Q_3	0.5302	0.3926	0.3039	0.5302	0.2037

Table 2. The quartile values of the EPL distribution for $\beta = 1$ and different values of α and λ

3. Statistical inferences

In this section, we consider the statistical inferences of the unknown parameters. First we calculate the MLEs of the unknown parameters and then we consider their asymptotic distribution.

3.1. **Maximum-likelihood estimation.** We consider estimation by the method of maximum likelihood. The log likelihood function for a random sample $X_1, X_2 \ldots, X_n$ from (2.2) is given by

$$\begin{aligned} \ln L(x|\alpha,\beta,\lambda) &= n\ln(\alpha) + n\ln(\beta) - (\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x) \\ &\quad -\lambda\sum_{i=1}^{n}(1-(1+\beta x)^{-\alpha}) + \sum_{i=1}^{n}\ln(1+\lambda(1+\beta x)^{-\alpha}), \end{aligned} \tag{3.1}$$

and by differentiating (3.1) with respect to the parameters α, β and λ, respectively, the gradients are

$$\begin{aligned} \frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n}\ln(1+\beta x) + \lambda\sum_{i=1}^{n}(1+\beta x)^{-\alpha}\ln(1+\beta x) \\ &\quad -\lambda\sum_{i=1}^{n}\frac{(1+\beta x)^{-\alpha}\ln(1+\beta x)}{1+\lambda(1+\beta x)^{-\alpha}}. \\ \frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{(1+\beta x_i)} - \lambda\alpha\sum_{i=1}^{n}x_i(1+\beta x_i)^{-(\alpha+1)} \\ &\quad -\lambda\alpha\sum_{i=1}^{n}\frac{x_i(1+\beta x_i)^{-(\alpha+1)}}{1+\lambda(1+\beta x_i)^{-\alpha}}. \\ \frac{\partial \ln L}{\partial \lambda} &= -\sum_{i=1}^{n}(1-(1+\beta x)^{-\alpha}) + \frac{n}{\lambda} - \frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{(1+\lambda(1+\beta x)^{-\alpha})}. \end{aligned}$$

Note that the last term of (3.1), $\sum\ln(1+\lambda(1+\beta x)^{-\alpha})$, can be rewritten in a convenient way that enables us to deal with it easily. The MLEs of α, β and λ, say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ respectively, are obtained by solving simultaneously the nonlinear normal equations $(\partial\ln L/\partial\alpha) = 0$, $(\partial\ln L/\partial\beta) = 0$ and $(\partial\ln L/\partial\lambda) = 0$. The MLEs cannot be obtained in closed form. Thus, we use any iterative numerical method such as the Newton-Raphson.

Define $A_i = (1+\beta x_i) > 1$. Then we can write the MLEs in terms of A_i as follows.

$$\hat{\alpha} = n\left[\sum_{i=1}^{n}\ln A_i - \lambda\sum_{i=1}^{n}A_i^{-\alpha}\ln A_i\left(1-\frac{1}{1+\lambda A_i^{-\alpha}}\right)\right]^{-1}.$$

$$\hat{\beta} = n\left[(\alpha+1)\sum_{i=1}^{n}\frac{x_i}{A_i} + \alpha\lambda\sum_{i=1}^{n}x_i\ A_i^{-(\alpha+1)}\left(1+\frac{1}{1+\lambda A_i^{-\alpha}}\right)\right]^{-1}.$$

$$\hat{\lambda} = n\left[\sum_{i=1}^{n}(1-A_i^{-\alpha}) + \frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{1+\lambda A_i^{-\alpha}}\right]^{-1}.$$

3.2. **Asymptotic distributions.** The Fisher information matrix of $\theta = (\alpha, \beta, \lambda)$, denoted by $\mathbf{J}(\theta) = \mathbf{E}(\mathbf{I}, \theta)$, where $\mathbf{I} = [I_{i,j}]_{i,j=1,2,3}$ is the observed information matrix, is given by

$$I = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ln L}{\partial \lambda \partial \beta} & \frac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

where the second partial derivatives of the maximum likelihood function, $(\partial^2 \ln L/\partial\theta^2)$, are given as the following:

$$\begin{aligned} I_{11} &= -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \lambda(1+\beta x_i)^{-\alpha} \left(\ln(1+\beta x_i)\right)^2 + \sum_{i=1}^{n} \frac{\lambda(1+\beta x_i)^{-\alpha}\ln(1+\beta x_i)}{(1+\lambda(1+\beta x_i)^{-\alpha})} \\ &\quad -\sum_{i=1}^{n}\left[\frac{\lambda(1+\beta x_i)^{-\alpha}\ln(1+\beta x_i)}{(1+\lambda(1+\beta x_i)^{-\alpha})}\right]^2 \\ &= -\frac{n}{\alpha^2} + \sum_{i=1}^{n}\lambda A_i^{-\alpha}(\ln A_i)^2 + \sum_{i=1}^{n}\frac{\lambda A_i^{-\alpha}\ln A_i}{1+\lambda A_i^{-\alpha}} - \sum_{i=1}^{n}\left[\frac{\lambda A_i^{-\alpha}\ln A_i}{1+\lambda A_i^{-\alpha}}\right]^2. \end{aligned}$$

$$\begin{aligned} I_{12} = I_{21} &= -\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} + \sum_{i=1}^{n}\lambda\alpha x_i(1+\beta x_i)^{-\alpha}\ln(1+\beta x_i) - \sum_{i=1}^{n}\lambda x_i(1+\beta x_i)^{-(\alpha+1)} \\ &\quad +\sum_{i=1}^{n}\frac{\lambda\alpha(1+\beta x_i)^{-(\alpha+1)}\ln(1+\beta x_i)}{1+\lambda(1+\beta x_i)^{-\alpha}} - \sum_{i=1}^{n}\frac{x_i(1+\beta x_i)^{-(\alpha+1)}}{1+\lambda(1+\beta x_i)^{-\alpha}} \\ &\quad -\sum_{i=1}^{n}\frac{\alpha\beta\ln(1+\beta x_i)}{1+\beta x_i}\left[\frac{\lambda(1+\beta x_i)^{-\alpha}}{1+\lambda(1+\beta x_i)^{-\alpha}}\right]^2 \\ &= -\sum_{i=1}^{n}\frac{x_i}{A_i} + \sum_{i=1}^{n}\lambda\alpha x_i A_i^{-\alpha}\ln(A_i) - \sum_{i=1}^{n}\lambda x_i A_i^{-(\alpha+1)} + \sum_{i=1}^{n}\frac{\lambda\alpha A_i^{-(\alpha+1)}\ln(A_i)}{1+\lambda A_i^{-\alpha}} \\ &\quad -\sum_{i=1}^{n}\frac{x_i A_i^{-(\alpha+1)}}{1+\lambda A_i^{-\alpha}} - \sum_{i=1}^{n}\frac{\alpha\beta\ln(A_i)}{1+\beta x_i}\left[\frac{\lambda A_i^{-\alpha}}{1+\lambda A_i^{-\alpha}}\right]^2 \end{aligned}$$

$$\begin{aligned} I_{22} &= -\frac{n}{\beta^2} + (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{(1+\beta x_i)^2} + \lambda\alpha(\alpha+1)\sum_{i=1}^{n}x_i^2(1+\beta x_i)^{-(\alpha+2)} \\ &\quad +\sum_{i=1}^{n}\frac{\lambda\alpha(\alpha+1)x_i^2(1+\beta x_i)^{-(\alpha+2)}}{1+\lambda(1+\beta x_i)^{-\alpha}} + \lambda\sum_{i=1}^{n}\left(\frac{\alpha x_i(1+\beta x_i)^{-(\alpha+1)}}{1+\lambda(1+\beta x_i)^{-\alpha}}\right)^2. \\ &= -\frac{n}{\beta^2} + (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{A_i^2} + \lambda\alpha(\alpha+1)\sum_{i=1}^{n}x_i^2 A_i^{-(\alpha+2)} + \sum_{i=1}^{n}\frac{\lambda\alpha(\alpha+1)x_i^2 A_i^{-(\alpha+2)}}{1+\lambda A_i^{-\alpha}} \\ &\quad +\lambda\sum_{i=1}^{n}\left(\frac{\alpha x_i A_i^{-(\alpha+1)}}{1+\lambda A_i^{-\alpha}}\right)^2. \end{aligned}$$

$$I_{23} = I_{32} = -\sum_{i=1}^{n} \alpha x_i (1+\beta x_i)^{-(\alpha+1)} + \sum_{i=1}^{n} \frac{\alpha x_i (1+\beta x_i)^{-(\alpha+1)}}{(1+\lambda(1+\beta x_i)^{-\alpha})^2}$$
$$= -\sum_{i=1}^{n} \alpha x_i A_i^{-(\alpha+1)} + \sum_{i=1}^{n} \frac{\alpha x_i A_i^{-(\alpha+1)}}{(1+\lambda A_i^{-\alpha})^2}$$

$$I_{13} = I_{31} = -\sum_{i=1}^{n} (1+\beta x_i)^{-\alpha} \ln(1+\beta x_i) - \sum_{i=1}^{n} \frac{(1+\beta x_i)^{-\alpha} \ln(1+\beta x_i)}{1+\lambda(1+\beta x_i)^{-\alpha}}$$
$$+ \sum_{i=1}^{n} \lambda \ln(1+\beta x_i) \left(\frac{(1+\beta x_i)^{-\alpha}}{1+\lambda(1+\beta x_i)^{-\alpha}} \right)^2$$
$$= -\sum_{i=1}^{n} A_i^{-\alpha} \ln(A_i) - \sum_{i=1}^{n} \frac{A_i^{-\alpha} \ln(A_i)}{1+\lambda A_i^{-\alpha}} + \sum_{i=1}^{n} \lambda \ln(A_i) \left(\frac{A_i^{-\alpha}}{1+\lambda A_i^{-\alpha}} \right)^2$$

$$I_{33} = -\sum_{i=1}^{n} \left(\frac{(1+\beta x)^{-\alpha}}{1+\lambda(1+\beta x)^{-\alpha}} \right)^2 = \sum_{i=1}^{n} \left(\frac{A_i^{-\alpha}}{1+\lambda A_i^{-\alpha}} \right)^2$$

The exact mathematical expressions for $\mathbf{J}(\theta) = \mathbf{E}(\mathbf{I}, \theta)$ are very complicated to obtain. Therefore, the observed Fisher information matrix can be used instead of the Fisher information matrix. The variance-covariance matrix may be approximated as $[\mathbf{V_{ij}}] = [\mathbf{I_{ij}}]^{-\mathbf{1}}$. The asymptotic distribution of the maximum likelihood can be written as follows.

$$\left[(\hat{\alpha} - \alpha), (\hat{\beta} - \beta), (\hat{\lambda} - \lambda) \right] \sim N_3 (0, \mathbf{V}) . \tag{3.2}$$

Since $\mathbf{V}$ involves the parameters α, β and λ, we replace the parameters by the corresponding MLEs in order to obtain an estimate of $\mathbf{V}$, which is denoted by $\hat{\mathbf{V}}$. By using (3.2), approximate $100(1-\vartheta)\%$ confidence intervals for α, β and λ are determined, respectively, as

$$\hat{\alpha} \pm Z_{\vartheta/2} \sqrt{\hat{\mathbf{V}}_{\mathbf{11}}}, \ \hat{\beta} \pm Z_{\vartheta/2} \sqrt{\hat{\mathbf{V}}_{\mathbf{22}}}, \ \hat{\lambda} \pm Z_{\vartheta/2} \sqrt{\hat{\mathbf{V}}_{\mathbf{33}}},$$

where Z_{ϑ} is the upper 100ϑ-th percentile of the standard normal distribution.

4. Examples

The fit of the EPL distribution to two sets of real data is examined using maximum likelihood estimates. The first set is corresponding to remission times (in months) of a random sample of 128 bladder cancer patients. The bladder cancer data were analyzed by Lee and Wang [6] and further discussed in Al-Zahrani and Sagor [2]. In the second set the data are corresponding to computer file sizes (in bytes) for 269 files. These data were analyzed by Holland et al. [10] and found to be adequately fitted by the Lomax distribution. The proposed EPL distribution is compared with Lomax, extended Lomax and Poisson-Lomax distributions.

The model selection is carried out using the Akaike information criterion (AIC), the Bayesian information criterion, the consistent Akaike information criteria (CAIC) and the Hannan-Quinn information criterion (HQIC).

$$\begin{aligned}\text{AIC} &= -2l(\hat{\theta}) + 2q \\ \text{BIC} &= -2l(\hat{\theta}) + q\log(n) \\ \text{HQIC} &= -2l(\hat{\theta}) + 2q\log(\log(n)) \\ \text{CAIC} &= -2l(\hat{\theta}) + \frac{2qn}{n-q-1},\end{aligned}$$

where $l(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters, and n is the sample size. The model with minimum AIC or (BIC, CAIC and HQIC) value is chosen as the best model to fit the data. From Table 4 and Table 4, we conclude that the PLD is comparable to the Lomax extended Lomax and Poisson-Lomax models.

	Estimates				Measures			
Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	AIC	BIC	HQIC	CAIC
Lomax	13.9384	121.0221			831.67	837.37	833.98	831.76
	(15.3837)	(142.6940)						
MOEL	23.7437	2.0487	2.2818		825.08	833.64	828.56	825.27
	(35.8106)	(2.5891)	(0.5551)					
PLD	2.8739	8.2719		3.3513	824.77	833.33	828.25	824.96
	(0.8870)	(4.8804)		(1.0303)				
EPLD	0.2387	124.3010		59.8378	833.67	842.22	837.14	833.86
	(1.1424)	(155.3534)		(242.1564)				

Table 3. MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC and CAIC for bladder cancer data.

	Estimates				Measures			
Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	AIC	BIC	HQIC	CAIC
Lomax	0.4978	128.3094			4609.77	4616.96	4612.65	4609.81
	(0.0429)	(23.4788)						
MOEL	1.0513	123.1170	0.5032		4611.76	4622.54	4616.09	4611.85
	(0.5371)	(56.7192)	(0.0699)					
PLD	0.5851	4.5282		9.4728	4590.10	4600.90	4594.40	4590.20
	(0.0323)	(2.4068)		(2.5082)				
EPLD	0.0139	128.3909		35.666	4612.58	4623.37	4616.91	4612.67
	(0.0083)	(23.0281)		(21.2362)				

Table 4. MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC and CAIC for computer file sizes data.

From Table 3 and Table 4, we could conclude that the four models are appropriate to represent this kind of data as assessed by the AIC, BIC, CAIC and HQIC. There are hardly any difference among them, but the Lomax seems to be the best model amongst

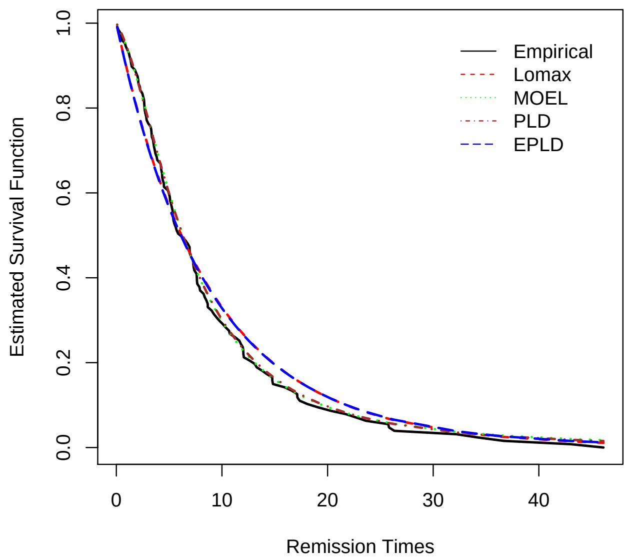

Figure 3. Estimated survivals for bladder cancer data.

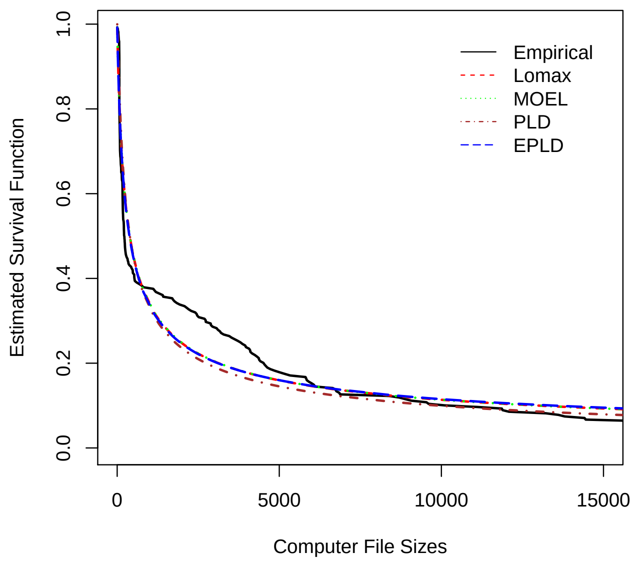

Figure 4. Estimated survivals for computer file sizes data.

the other. This is due to the fact that Lomax is heavy-tailed distribution and this fits the data in our disposal.

5. Concluding Remarks

A new three-parameter distribution, known as extended Poisson-Lomax (EPL) distribution has been proposed. The mathematical properties of the proposed distribution have been provided. The moments of the distribution and the quantile function have been obtained. The estimation of the parameters has been approached by maximum likelihood and the asymptotic variance-covariance matrix have been obtained. Finally, two real data sets were analyzed to show the potential of the distribution. The result indicates that the EPL distribution behaves very similarly, and even in some situations provide a better fit than other well-known distributions.

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