

**ALMOST EVERYWHERE CONVERGENCE OF FEJÉR MEANS OF
SOME SUBSEQUENCES OF FOURIER SERIES FOR INTEGRABLE
FUNCTIONS WITH RESPECT TO THE KACZMARZ SYSTEM**

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ABSTRACT. The first result of this work is about almost everywhere convergence of Fejér means for some subsequences of partial sums of Fourier series of integrable functions with respect to the Kaczmarz system. The second result establishes almost everywhere convergence of subsequences of Fejér means for some specific families of subsequences.

1. INTRODUCTION

For the Walsh-Paley system it is known that the sequence of Fejér means converges almost everywhere for every integrable function, see [3]. The question on the degree to which elements from the partial sums of Fourier series can be removed so that means of the remaining subsequence still converge almost everywhere was considered in [7].

In [2, Theorem 1] G.Gát proves that the mean sequence of every lacunary sequence of partial sums of Fourier series for integrable functions with respect to the Walsh-Paley system is almost everywhere convergent to the function itself. The main point was about the level of condensation of the elements of the subsequence since this same question was mentioned for the trigonometric system, see [1]. Namely, [2, Theorem 1] gives the convergence result for less dense sequences than the sequence considered in [1].

The methods used in this work provide some answers on a.e. convergence with respect to the Kaczmarz system for some specific subsequences, where the question is more about structure and homogeneity rather than density of elements. In the sequel it would be interesting to study convergence of means for larger families of subsequences of Fourier series.

Let \mathbb{Z}_2 denote the discrete cyclic group $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is addition modulo 2. If $|E|$ denotes the measure of the subset $E \subset \mathbb{Z}_2$, then we have $|\{0\}| = |\{1\}| = \frac{1}{2}$.

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The dyadic group G is obtained from $G = \prod_{i=0}^{\infty} \mathbb{Z}_2$, see[5], where topology and measure are obtained from the product.

In this paper the notation $|E|$ is used for both of the probability measure for subsets of the dyadic group G and for the discrete measure of finite subsets of natural numbers.

Let $x = (x_n)_{n \geq 0} \in G$. The sets $I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$, $n \geq 1$ and $I_0(x) := G$ are dyadic intervals of G . Let $I_n = I_n(0)$, and $e_n := (\delta_{in})_i$. It is easily seen that $(I_n)_n$ is a decreasing sequence of subgroups.

Since every nonnegative integer i can be written in the form $i = \sum_{k=0}^{\infty} i_k 2^k$, we define the sequence $(z_i)_{i \geq 0}$ by

$$z_i = \sum_{k=0}^{\infty} i_k e_k .$$

It is easily seen that for each positive integer n , the set $\{z_i, i < 2^n\}$ is a set of representatives of I_n -cosets.

The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\omega_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k}, \quad n \in \mathbb{N}, \quad x \in G,$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ and $r_k(x) = (-1)^{x_k}$. The n -th Walsh-Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k},$$

for $n \geq 1$, $\kappa_0(x) := 1$ and $|n| = \max_{n_k \neq 0} k$.

For every natural number $n = \sum_{k=0}^{|n|} n_k 2^k$, we define the corresponding numbers

$$n^{(s)} = \sum_{k=s}^{|n|} n_k 2^k, \quad n_{(s)} = \sum_{k=0}^{s-1} n_k 2^k .$$

We denote the Dirichlet kernel function by:

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad D_n^\kappa := \sum_{k=0}^{n-1} \kappa_k, \quad n \geq 1 .$$

For every integrable function f the maximal function f^* is defined by

$$f^*(x) = \sup_{n \geq 0} |D_{2^n} * f(x)|, \quad x \in G .$$

In [4] it was noticed that the Dirichlet kernel function with respect to the Kaczmarz system can be written in the form

$$D_n^\kappa(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}(\tau_{|n|}(x)) .$$

The notation C is used for independent positive constant which may vary in different contexts.

2. MAIN RESULTS

Lemma 2.1. *For every positive integers s, n, k such that $s < 2^k < 2^n$, and every integrable function f we have*

$$\|(\omega_s(D_{2^{n-k}} \circ \tau_n)) * f\|_1 \leq \|f\|_1.$$

Proof. Let s, k and n be fixed. Put $F = (\omega_s(D_{2^{n-k}} \circ \tau_n)) * f$. Since this convolution only depends on the mean values $S_{2^n} f$ of the function f , we get

$$F(x) = \sum_{i=0}^{2^n-1} (\omega_s(D_{2^{n-k}} \circ \tau_n)) * (S_{2^n} f(z_i) 1_{I_n(z_i)})(x).$$

We have

$$\begin{aligned} (\omega_s(D_{2^{n-k}} \circ \tau_n)) * 1_{I_n(z_i)}(x) &= 2^{n-k} \int_{I_n(x+z_i) \cap \{\tau_n(t) \in I_{n-k}\}} \omega_s(t) dt \\ &= 2^{-k} \omega_s(x+z_i) 1_{\{\tau_n(t) \in I_{n-k}\}}(x+z_i). \end{aligned}$$

It follows

$$\|(\omega_s(D_{2^{n-k}} \circ \tau_n)) * 1_{I_n(z_i)}\|_1 = 2^{-k} \int_{\{\tau_n(x+z_i) \in I_{n-k}\}} dx = 2^{-n},$$

and we get

$$\|F\|_1 \leq \sum_{i=0}^{2^n-1} 2^{-n} |S_{2^n} f(z_i)| \leq \|f\|_1,$$

proving the lemma. \square

Lemma 2.2. *Let k and s be positive integers such that $s < 2^k$. Let $(\varepsilon(n))_n$ be a sequence where every term is either 0 or 1. Then, the operator L_k defined on L^1 by*

$$L_k f(x) = \sup_N \frac{1}{|\{n : k+1 \leq n \leq N, \varepsilon(n) = 1\}|} \left| \sum_{n=k+1}^N \varepsilon(n) (r_n \omega_s((D_{2^{n-k}} \circ \tau_n)) * f(x)) \right|,$$

is of weak type (L^1, L^1) .

Proof. Let $\lambda > \|f\|_1$. There exists some N_0 depending on f and λ such that

$$\begin{aligned} |\{L_k f(x) > \lambda\}| &\leq 2 \left| \left\{ \sup_{N \leq N_0} \frac{1}{|\{n : k+1 \leq n \leq N, \varepsilon(n) = 1\}|} \right. \right. \\ &\quad \left. \left. \left| \sum_{n=k+1}^N \varepsilon(n) (r_n \omega_s((D_{2^{n-k}} \circ \tau_n)) * f(x)) \right| > \lambda \right\} \right|. \end{aligned}$$

Then, we choose $F = (\omega_s(D_{2^{N_0-k}} \circ \tau_{N_0})) * f$. Notice that for every $n : k < n < N_0$,

$$\begin{aligned} r_n D_{2^n} * (\omega_s((D_{2^{N_0-k}} \circ \tau_{N_0}))(x) &= 2^{n+N_0-k} r_n(x) \int_{I_n(x) \cap \{\tau_{N_0}(t) \in I_{N_0-k}\}} r_n(t) \omega_s(t) dt \\ &= 2^{n+N_0-k} r_n(x) \omega_s(x) \int_{I_n(x) \cap \{\tau_{N_0}(t) \in I_{N_0-k}\}} dt \\ &= 2^{n-k} r_n(x) \omega_s(x) 1_{\{\tau_n(t) \in I_{n-k}\}}(x) = r_n(x) \omega_s(x) (D_{2^{n-k}} \circ \tau_n)(x). \end{aligned}$$

Since

$$\begin{aligned}
& \frac{1}{|\{n : k+1 \leq n \leq N, \varepsilon(n) = 1\}|} \left| \sum_{n=k+1}^N \varepsilon(n) (r_n D_{2^n} * F)(x) \right| \\
&= \frac{1}{|\{n : k+1 \leq n \leq N, \varepsilon(n) = 1\}|} \left| \sum_{n=k+1}^N \varepsilon(n) (D_{2^{n+1}} - D_{2^n}) * F(x) \right| \\
&\leq \sup_{n \leq N} \varepsilon(n) |(D_{2^{n+1}} - D_{2^n}) * F(x)| \leq 2F^*(x).
\end{aligned}$$

Applying Lemma 2.1 we have

$$\begin{aligned}
& |\{ \sup_{N \leq N_0} \frac{1}{|\{n : k+1 \leq n \leq N, \varepsilon(n) = 1\}|} \left| \sum_{n=k+1}^N \varepsilon(n) (r_n \omega_s((D_{2^{n-k}} \circ \tau_n)) * f)(x) \right| > \lambda \}| \\
&= |\{ \sup_{N \leq N_0} \frac{1}{|\{n : k+1 \leq n \leq N, \varepsilon(n) = 1\}|} \left| \sum_{n=k+1}^N \varepsilon(n) (r_n D_{2^n} * F)(x) \right| > \lambda \}| \\
&\leq |\{2F^*(x) > \lambda\}| \leq C \frac{\|F\|_1}{\lambda} \leq C \frac{\|f\|_1}{\lambda},
\end{aligned}$$

where the constant $C > 0$ is independent on N_0, k, f or λ . \square

Theorem 2.1. *Let $f \in L^1$ and $(k_n)_n$ be a fixed sequence of positive integers. Suppose that $(\alpha(n))_n$ is an increasing sequence of positive integers satisfying $\alpha(n+1) = 2^{k_n} \alpha(n)$, for every $n \geq m$, where m is some fixed positive integer. Then,*

$$\frac{1}{N} \sum_{n=1}^N D_{\alpha(n)}^{\kappa} * f \rightarrow f,$$

almost everywhere on G .

Proof. Following Gát in [2], it suffices to prove that the operator

$$\sup_N \left| \frac{1}{N} \sum_{n=1}^N D_{\alpha(n)}^{\kappa} * f \right|,$$

is of weak type (L^1, L^1) .

Putting $D_{\alpha(n)}^{\kappa} * f$ in the form

$$D_{\alpha(n)}^{\kappa} * f = D_{2^{|\alpha(n)|}} * f + r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f,$$

since

$$\sup_N \left| \frac{1}{N} \sum_{n=1}^N D_{2^{|\alpha(n)|}} * f \right| \leq f^*,$$

then it suffices to estimate

$$\sup_N \left| \frac{1}{N} \sum_{n=1}^N r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right|.$$

We have

$$\begin{aligned}
& \sup_N \left| \frac{1}{N} \sum_{n=1}^N r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| \leq \\
& \leq \sup_{N \leq m} \left| \frac{1}{N} \sum_{n=1}^N r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| + \\
& \quad + \sup_{N > m} \left| \frac{1}{N} \sum_{n=1}^N r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| \\
& \leq \sup_{N \leq m} \left| \frac{1}{N} \sum_{n=1}^N r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| + \\
& \quad + \sup_{N > m} \left| \frac{1}{N} \sum_{n=1}^m r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| + \\
& \quad + \sup_{N > m} \left| \frac{1}{N} \sum_{n=m+1}^N r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right|.
\end{aligned}$$

Since the first two terms are obviously bounded, we only estimate the third term. For every $n > m$ we have

$$\alpha(n) = 2^{k_n + \dots + k_{m+1}} \alpha(m), \quad |\alpha(n)| = k_n + \dots + k_{m+1} + |\alpha(m)|,$$

and

$$\alpha(n) - 2^{|\alpha(n)|} = 2^{k_n + \dots + k_{m+1}} (\alpha(m) - 2^{|\alpha(m)|}).$$

Suppose that $\alpha(m) = \sum_{s=0}^{|\alpha(m)|} m_s 2^s$, where $m_s \in \{0, 1\}$, for every s . Then we have

$$\alpha(n) - 2^{|\alpha(n)|} = \sum_{s=0}^{|\alpha(m)|-1} m_s 2^{s+k_n + \dots + k_{m+1}},$$

which gives from [6, Theorem 1]

$$D_{\alpha(n)-2^{|\alpha(n)|}} = \sum_{s=0}^{|\alpha(m)|-1} m_s D_{2^{s+|\alpha(n)|-|\alpha(m)|}} \prod_{k=s+1}^{|\alpha(m)|-1} r_{k+|\alpha(n)|-|\alpha(m)|}^{m_k}.$$

The result is true if we prove that the operator

$$\sup_{N > m} \left| \frac{1}{N} \sum_{n=m+1}^N (r_{|\alpha(n)|} (D_{2^{s+|\alpha(n)|-|\alpha(m)|}} \prod_{k=s+1}^{|\alpha(m)|-1} r_{k+|\alpha(n)|-|\alpha(m)|}^{m_k} \circ \tau_{|\alpha(n)|})) * f \right|$$

is of weak type (L^1, L^1) for every s . This is true and follows from Lemma 2.2 since for every $k : s+1 \leq k \leq |\alpha(m)|-1$, we have

$$r_{k+|\alpha(n)|-|\alpha(m)|} \circ \tau_{|\alpha(n)|} = r_{|\alpha(m)|-k-1}.$$

□

Lemma 2.3. *Let m, n be natural numbers where $|n| > |m|$. Suppose that $m = 2^{-s} n^{(s)}$, where $1 \leq s \leq |n|$. Then,*

$$\begin{cases} S_{2|m|} D_{n-2|n|}(\tau_{|n|}(x)) = D_{m-2|m|}(\tau_{|m|}(x)), & n = n^{(s)}; \\ S_{2|m|} D_{n-2|n|}(\tau_{|n|}(x)) = D_{m-2|m|}(\tau_{|m|}(x)) + \omega_{m-2|m|}(\tau_{|m|}(x)), & n > n^{(s)}. \end{cases}$$

Proof. Let $n = \sum_{k=0}^{|n|} n_k 2^k$ and $m = \sum_{k=0}^{|m|} m_k 2^k$. Then, for every $k \in \{0, \dots, |m|\}$ we have $m_k = n_{k+s}$.

From the proof of [6, Theorem 1] it can be seen that

$$D_{n-2|n|} = \sum_{k=0}^{|n|-1} n_k D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}.$$

Hence, it suffices to prove the following statements:

(1) For every $k \geq s$, if $n_k \neq 0$ we have

$$S_{2|m|} [(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] = (D_{2^{k-s}} \prod_{i=k-s+1}^{|m|-1} r_i^{m_i}) \circ \tau_{|m|}.$$

(2) For every $k \leq s-1$ satisfying the property $n_i = 0$ for each $i \in \{k+1, \dots, s-1\}$ if this set is not empty, we have

$$S_{2|m|} [(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] = \omega_{m-2|m|} \circ \tau_{|m|}.$$

(3) If $k \leq s-2$ is so that there exists some $i \in \{k+1, \dots, s-1\}$ where $n_i = 1$, then

$$S_{2|m|} [(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] = 0.$$

In this way since if $n_{(s)} \neq 0$ we have:

$$\begin{aligned} S_{2|m|} [D_{n-2|n|} \circ \tau_{|n|}] &= S_{2|m|} [(\sum_{k=0}^{s-1} n_k D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] \\ &\quad + (\sum_{k=s}^{|n|-1} n_k D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}, \\ S_{2|m|} [(\sum_{k=0}^{s-1} n_k D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] &= \omega_{m-2|m|} \circ \tau_{|m|}, \end{aligned}$$

and if $n_{(s)} = 0$:

$$S_{2|m|} [(\sum_{k=0}^{s-1} n_k D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] = 0.$$

Also we have

$$\begin{aligned} S_{2|m|} [(\sum_{k=s}^{|n|-1} n_k D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}] &= (\sum_{k=s}^{|n|-1} n_k D_{2^{k-s}} \prod_{i=k-s+1}^{|m|-1} r_i^{m_i}) \circ \tau_{|m|} \\ &= (\sum_{k=0}^{|m|-1} m_k D_{2^k} \prod_{i=k+1}^{|m|-1} r_i^{m_i}) \circ \tau_{|m|} = D_{m-2|m|} \circ \tau_{|m|}, \end{aligned}$$

and then, the result will follow.

Now we prove the statements (1),(2) and (3). Let $x \in G$ be fixed and arbitrary.

(1) Let $k \geq s$ and $n_k \neq 0$. Then we have

$$\begin{aligned}
S_{2^{|m|}}[(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}](x) &= 2^{|m|+k} \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t)) dt \\
&= 2^{|m|+k-|n|} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(x)) 1_{\{\tau_{|m|}(t) \in I_{k-s}\}}(x) \\
&= 2^{|m|+k-|n|} \prod_{i=k+1}^{|n|-1} r_{i-s}^{n_i}(\tau_{|m|}(x)) 1_{\{\tau_{|m|}(t) \in I_{k-s}\}}(x) \\
&= 2^{|m|+k-|n|} \prod_{i=k+1-s}^{|m|-1} r_i^{m_i}(\tau_{|m|}(x)) 1_{\{\tau_{|m|}(t) \in I_{k-s}\}}(x) \\
&= (D_{2^{k-s}} \prod_{i=k-s+1}^{|m|-1} r_i^{m_i})(\tau_{|m|}(x)).
\end{aligned}$$

(2) Let $k \leq s-1$, and $n_k = 1$, $n_i = 0$ for each $i \in \{k+1, \dots, s-1\}$. In this case

$$r_i^{n_i}(\tau_{|n|}(t)) = r_i^{n_i}(\tau_{|n|}(x)),$$

whenever $i \geq k+1$ and $t \in I_{|m|}(x)$. Therefore,

$$\begin{aligned}
S_{2^{|m|}}[(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}](x) &= 2^{|m|+k} \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t)) dt \\
&= 2^{|m|+k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(x)) \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} dt \\
&= \prod_{i=s}^{|n|-1} r_i^{n_i}(\tau_{|n|}(x)) \\
&= \prod_{i=0}^{|m|-1} r_i^{m_i}(\tau_{|m|}(x)) = \omega_{m-2^{|m|}}(\tau_{|m|}(x)).
\end{aligned}$$

(3) Suppose that $k \leq s - 2$ is so that there exists some $j \in \{k + 1, \dots, s - 1\}$ where $n_j = 1$. Then we have:

$$\begin{aligned}
S_{2^{|m|}}[(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}](x) &= 2^{|m|+k} \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t)) dt \\
&= 2^{|m|+k} \int_{[I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}] + e_{|n|-j-1}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t)) dt \\
&= 2^{|m|+k} \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t + e_{|n|-j-1})) dt \\
&= 2^{|m|+k} \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t)) r_i^{n_i}(e_j) dt \\
&= -2^{|m|+k} \int_{I_{|m|}(x) \cap \{\tau_{|n|}(t) \in I_k\}} \prod_{i=k+1}^{|n|-1} r_i^{n_i}(\tau_{|n|}(t)) dt.
\end{aligned}$$

This gives that

$$S_{2^{|m|}}[(D_{2^k} \prod_{i=k+1}^{|n|-1} r_i^{n_i}) \circ \tau_{|n|}](x) = 0.$$

□

Lemma 2.4. *Let $m, n \in \mathbb{N}$ be such that $|m| < |n|$. Then:*

$$D_{2m} \circ \tau_{|n|+1}(x) - D_m \circ \tau_{|n|}(x) = r_{|n|}(x) D_m \circ \tau_{|n|}(x), \quad \forall x \in G.$$

Proof. Let $m = \sum_{i=0}^{|m|} m_i 2^i$. Then we have

$$2m = \sum_{i=1}^{|m|+1} m_{i-1} 2^i.$$

We get

$$D_m = \sum_{i=0}^{|m|} m_i D_{2^i} \prod_{k=i+1}^{|m|} r_k^{m_k},$$

and

$$D_{2m} = \sum_{i=0}^{|m|} m_i D_{2^{i+1}} \prod_{k=i+1}^{|m|} r_{k+1}^{m_k}.$$

Suppose that $x_{|n|} = 1$ for some $x \in G$. Then, for every $k \geq 0$, $D_{2^{k+1}} \circ \tau_{|n|+1}(x) = 0$. Therefore, $D_{2m} \circ \tau_{|n|+1}(x) = 0$, which gives

$$D_{2m} \circ \tau_{|n|+1}(x) - D_m \circ \tau_{|n|}(x) = -D_m \circ \tau_{|n|}(x) = r_{|n|}(x) D_m \circ \tau_{|n|}(x).$$

For $x_{|n|} = 0$, we have

$$D_{2^{k+1}} \circ \tau_{|n|+1}(x) = 2D_{2^k} \circ \tau_{|n|}(x), \quad \forall k \geq 0.$$

Besides, for every $i \geq 1$ it holds $r_{i+1} \circ \tau_{|n|+1}(x) = r_i \circ \tau_{|n|}(x)$, which implies:

$$D_{2m} \circ \tau_{|n|+1}(x) = 2 \left(\sum_{i=0}^{|m|} m_i D_{2^i} \prod_{k=i+1}^{|m|} r_k^{m_k} \right) \circ \tau_{|n|}(x) = 2 D_m \circ \tau_{|n|}(x).$$

We easily get

$$D_{2m} \circ \tau_{|n|+1}(x) - D_m \circ \tau_{|n|}(x) = D_m \circ \tau_{|n|}(x) = r_{|n|}(x) D_m \circ \tau_{|n|}(x).$$

□

Theorem 2.2. *Let f be an integrable function. Let $(\alpha(n))_n$ be an increasing sequence of numbers satisfying the following conditions :*

(1) *If*

$$\{|\alpha(k)|, k \in \mathbb{N}\} \cap [m, n] = \{m, n\},$$

where m and n are natural numbers then the number $\frac{|\{k: |\alpha(k)|=n\}|}{|\{k: |\alpha(k)|=m\}|}$ denoted by b_m is a natural number. Moreover, if k is so that $|\alpha(k)| = m$, then there are exactly b_m elements from the set $\{k' : |\alpha(k')| = n\}$, satisfying $k = 2^{|m|-|n|} k' (|n|-|m|)$,

(2) *$|\{k : |\alpha(k)| = n\}| \sim n^\beta$, for some fixed $\beta > 0$.*

(3) *For every $k \in \mathbb{N}$,*

$$\sum_{i=0}^k \sum_{n: |\alpha(n)|=k} (\alpha(n))_i = O(k).$$

Then,

$$\frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)|\}|} \sum_{n: |\alpha(n)| \leq |\alpha(N)|} D_n^\kappa * f \rightarrow f, \quad N \rightarrow \infty,$$

almost everywhere.

Proof. Following the first steps in Theorem 2.1 it suffices to prove that the operator

$$\sup_N \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)|\}|} \left| \sum_{n: |\alpha(n)| \leq |\alpha(N)|} r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right|$$

is of weak type (L^1, L^1) .

From the property (2) we can easily deduce that

$$N \sim \sum_{l \leq |\alpha(N)|} l^\beta \sim |\alpha(N)|^{\beta+1},$$

because,

$$|\{n : |\alpha(n)| \leq |\alpha(N)| - 1\}| < N \leq |\{n : |\alpha(n)| \leq |\alpha(N)|\}|.$$

Let $\lambda > \|f\|_1$. There exists a natural number N_1 such that

$$\begin{aligned} & \left| \sup_N \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)|\}|} \sum_{n: |\alpha(n)| \leq |\alpha(N)|} r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| > \lambda \Big\} \\ < 2 \left\{ \sup_{N \leq N_1} \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)|\}|} \sum_{n: |\alpha(n)| \leq |\alpha(N)|} r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \right| > \lambda \Big\}. \end{aligned}$$

Of course, the number N_1 depends on f and λ . For every $s \in \{0, \dots, \lceil \log_2 N_1 \rceil\}$, we define the function

$$F_s = \frac{2^s}{N_1} \sum_{n: |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|} (D_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|}) * f.$$

Writing $D_{\alpha(n)-2|\alpha(n)|}$ in the form

$$D_{\alpha(n)-2|\alpha(n)|} = \sum_{i=0}^{|\alpha(n)|-1} (\alpha(n))_i D_{2^i} \prod_{k=i+1}^{|\alpha(n)|-1} r_k^{(\alpha(n))_k},$$

we obtain that

$$\begin{aligned} \sum_{n: |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|} D_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|} &= \sum_{i=0}^{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|-1} \sum_{n: |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|} (\alpha(n))_i (D_{2^i} \circ \tau_{|\alpha(n)|}) \\ &\quad \prod_{k=i+1}^{|\alpha(n)|-1} (r_k^{(\alpha(n))_k} \circ \tau_{|\alpha(n)|}). \end{aligned}$$

Applying Lemma 2.1 and the property (3) mentioned in this theorem, we get

$$\begin{aligned} \|F_s\|_1 &\leq \frac{2^s}{N_1} \sum_{i=0}^{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|-1} \sum_{n: |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|} (\alpha(n))_i \|f\|_1 \\ &= O\left(2^s \frac{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|}{N_1}\right) \|f\|_1 \sim |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|^{-\beta} \|f\|_1. \end{aligned}$$

Let n, k be so that $|\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|$, and $\alpha(k) = 2^{|\alpha(k)|-|\alpha(n)|} (\alpha(n))^{(|\alpha(n)|-|\alpha(k)|)}$. We have

$$(2.1) \quad r_{|\alpha(k)|} D_{2^{|\alpha(k)|}} * (D_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|}) = S_{2^{|\alpha(k)|+1}} - S_{2^{|\alpha(k)|}} (D_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|}).$$

Then for $(\alpha(n))_{|\alpha(n)|-|\alpha(k)|-1} = 0$, we have

$$(\alpha(n))^{(|\alpha(n)|-|\alpha(k)|-1)} = (\alpha(n))^{(|\alpha(n)|-|\alpha(k)|)} = 2^{|\alpha(n)|-|\alpha(k)|-1} 2\alpha(k),$$

and from Lemma 2.3 we have:

$$(2.2) \quad \begin{aligned} S_{2^{|\alpha(k)|+1}} (D_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|}) &= \\ &= D_{2\alpha(k)-2|\alpha(k)|+1} \circ \tau_{|\alpha(k)|+1} + \epsilon_1 \omega_{2\alpha(k)-2|\alpha(k)|+1} \circ \tau_{|\alpha(k)|+1}, \end{aligned}$$

where

$$\epsilon_1 = \begin{cases} 0, & \alpha(n) = (\alpha(n))^{(|\alpha(n)|-|\alpha(k)|-1)}, \\ 1, & \alpha(n) > (\alpha(n))^{(|\alpha(n)|-|\alpha(k)|-1)}. \end{cases}$$

In a similar way if $(\alpha(n))_{|\alpha(n)|-|\alpha(k)|-1} = 1$, we have:

$$(\alpha(n))^{(|\alpha(n)|-|\alpha(k)|-1)} = (\alpha(n))^{(|\alpha(n)|-|\alpha(k)|)} + 2^{|\alpha(n)|-|\alpha(k)|-1} = 2^{|\alpha(n)|-|\alpha(k)|-1} (2\alpha(k) + 1),$$

and from Lemma 2.3 we have:

$$\begin{aligned}
& S_{2^{|\alpha(k)|+1}}(D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|}) = \\
& = D_{2\alpha(k)-2^{|\alpha(k)|+1}+1} \circ \tau_{|\alpha(k)|+1} + \epsilon_1 \omega_{2\alpha(k)-2^{|\alpha(k)|+1}+1} \circ \tau_{|\alpha(k)|+1} \\
& = D_{2\alpha(k)-2^{|\alpha(k)|+1}} \circ \tau_{|\alpha(k)|+1} + \omega_{2\alpha(k)-2^{|\alpha(k)|+1}} \circ \tau_{|\alpha(k)|+1} + \\
(2.3) \quad & + \epsilon_1 \omega_{2\alpha(k)-2^{|\alpha(k)|+1}+1} \circ \tau_{|\alpha(k)|+1}.
\end{aligned}$$

Then by Lemma 2.4 and equalities (2.1), (2.2) and (2.3) we get

$$\begin{aligned}
& (r_{|\alpha(k)|} D_{2^{|\alpha(k)|}}) * (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|}) = \\
& = (D_{2\alpha(k)-2^{|\alpha(k)|+1}} \circ \tau_{|\alpha(k)|+1}) - (D_{\alpha(k)-2^{|\alpha(k)|}} \circ \tau_{|\alpha(k)|}) \\
& + \epsilon_1 \omega_{2\alpha(k)-2^{|\alpha(k)|+1}+\epsilon_2} \circ \tau_{|\alpha(k)|+1} + \epsilon_2 \omega_{2\alpha(k)-2^{|\alpha(k)|+1}} \circ \tau_{|\alpha(k)|+1} \\
& - \epsilon_3 \omega_{\alpha(k)-2^{|\alpha(k)|}} \circ \tau_{|\alpha(k)|} = r_{|\alpha(k)|} (D_{\alpha(k)-2^{|\alpha(k)|}} \circ \tau_{|\alpha(k)|}) \\
& + \epsilon_1 \omega_{2\alpha(k)-2^{|\alpha(k)|+1}+\epsilon_2} \circ \tau_{|\alpha(k)|+1} + \epsilon_2 \omega_{2\alpha(k)-2^{|\alpha(k)|+1}} \circ \tau_{|\alpha(k)|+1} \\
(2.4) \quad & - \epsilon_3 \omega_{\alpha(k)-2^{|\alpha(k)|}} \circ \tau_{|\alpha(k)|},
\end{aligned}$$

where $\epsilon_2 = (\alpha(n))_{|\alpha(n)-|\alpha(k)|-1}$, and

$$\epsilon_3 = \begin{cases} 0, & \alpha(n) = (\alpha(n))_{(|\alpha(n)|-|\alpha(k)|)}, \\ 1, & \alpha(n) > (\alpha(n))_{(|\alpha(n)|-|\alpha(k)|)}. \end{cases}$$

It follows that for every $l < |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|$, where $s \in \{0, \dots, [\log_2 N_1]\}$ is fixed,

$$\begin{aligned}
r_l D_{2^l} F_s & = \frac{2^s}{N_1} \frac{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|\}|}{|\{n : |\alpha(n)| = l\}|} \sum_{n:|\alpha(n)|=l} r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \\
& + \frac{2^s}{N_1} \frac{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|\}|}{|\{n : |\alpha(n)| = l\}|} \\
& \sum_{n:|\alpha(n)|=l} \epsilon_1 \omega_{2\alpha(n)-2^{|\alpha(n)|+1}+\epsilon_2} \circ \tau_{|\alpha(n)|+1} + \epsilon_2 \omega_{2\alpha(n)-2^{|\alpha(n)|+1}} \circ \tau_{|\alpha(n)|+1} \\
& - \epsilon_3 \omega_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|},
\end{aligned}$$

where ϵ_1 , ϵ_2 and ϵ_3 are defined as previously. We get that for every $N : \frac{N_1}{2^{s+1}} < N \leq \frac{N_1}{2^s}$,

$$\begin{aligned}
& \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{n:|\alpha(n)| \leq |\alpha(N)|} r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \\
& = \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{l=1}^{|\alpha(N)|} \sum_{n:|\alpha(n)|=l} r_{|\alpha(n)|} (D_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f \\
& = \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \frac{N_1}{2^s} \sum_{l=1}^{|\alpha(N)|} \frac{|\{n : |\alpha(n)| = l\}|}{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|\}|} r_l D_{2^l} F_s \\
& - \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{l=1}^{|\alpha(N)|} \sum_{n:|\alpha(n)|=l} (\epsilon_1 \omega_{2\alpha(n)-2^{|\alpha(n)|+1}+\epsilon_2} \circ \tau_{|\alpha(n)|+1} \\
& + \epsilon_2 \omega_{2\alpha(n)-2^{|\alpha(n)|+1}} \circ \tau_{|\alpha(n)|+1} - \epsilon_3 \omega_{\alpha(n)-2^{|\alpha(n)|}} \circ \tau_{|\alpha(n)|}) * f.
\end{aligned}$$

It is easily seen that for every $x \in G$,

$$\begin{aligned} & \sup_{N: \frac{N_1}{2^s+1} < N \leq \frac{N_1}{2^s}} \left| \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{l=1}^{|\alpha(N)|} \sum_{n: |\alpha(n)|=l} (\epsilon_1 \omega_{2\alpha(n)-2|\alpha(n)|+1+\epsilon_2} \circ \tau_{|\alpha(n)|+1} \right. \\ & \left. + \epsilon_2 \omega_{2\alpha(n)-2|\alpha(n)|+1} \circ \tau_{|\alpha(n)|+1} - \epsilon_3 \omega_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|}) * f(x) \right| \leq 3 \|f\|_1, \end{aligned}$$

from which we get:

$$\begin{aligned} & \left\| \sup_{N \leq N_1} \left| \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{l=1}^{|\alpha(N)|} \sum_{n: |\alpha(n)|=l} (\epsilon_1 \omega_{2\alpha(n)-2|\alpha(n)|+1+\epsilon_2} \circ \tau_{|\alpha(n)|+1} \right. \right. \\ & \left. \left. + \epsilon_2 \omega_{2\alpha(n)-2|\alpha(n)|+1} \circ \tau_{|\alpha(n)|+1} - \epsilon_3 \omega_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|}) * f \right\|_1 \leq \\ & \left\| \sup_{N \leq N_1} \left| \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{l=1}^{|\alpha(N)|} \sum_{n: |\alpha(n)|=l} (\epsilon_1 \omega_{2\alpha(n)-2|\alpha(n)|+1+\epsilon_2} \circ \tau_{|\alpha(n)|+1} \right. \right. \\ & \left. \left. + \epsilon_2 \omega_{2\alpha(n)-2|\alpha(n)|+1} \circ \tau_{|\alpha(n)|+1} - \epsilon_3 \omega_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|}) * f \right\|_\infty \leq 3 \|f\|_1. \end{aligned}$$

On the other side the term

$$\left| \sum_{l=1}^{|\alpha(N)|} \frac{|\{n : |\alpha(n)| = l\}|}{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)\}|} r_l D_{2^l} F_s \right|$$

can be written as

$$\begin{aligned} & \left| \frac{1}{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)\}|} \sum_{1 \leq k \leq C|\alpha(N)|^\beta} k \sum_{l: |\{n: |\alpha(n)|=l\}|=k} r_l D_{2^l} F_s \right| \\ & = \left| \frac{1}{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)\}|} \sum_{1 \leq k \leq C|\alpha(N)|^\beta} \sum_{l: |\{n: |\alpha(n)|=l\}| \geq k} r_l D_{2^l} F_s \right| \\ & \leq \frac{1}{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)\}|} C |\alpha(N)|^\beta F_s^* \leq C F_s^*. \end{aligned}$$

Now we have

$$\begin{aligned} & \left| \left\{ \sup_{N \leq N_1} \left| \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{n: |\alpha(n)| \leq |\alpha(N)|} r_{|\alpha(n)|} (D_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|}) * f \right| > \lambda \right\} \right| \\ & \leq \left| \left\{ \sup_{N \leq N_1} \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \sum_{l=1}^{|\alpha(N)|} \sum_{n: |\alpha(n)|=l} (\epsilon_1 \omega_{2\alpha(n)-2|\alpha(n)|+1+\epsilon_2} \circ \tau_{|\alpha(n)|+1} \right. \right. \\ & \left. \left. + \epsilon_2 \omega_{2\alpha(n)-2|\alpha(n)|+1} \circ \tau_{|\alpha(n)|+1} - \epsilon_3 \omega_{\alpha(n)-2|\alpha(n)|} \circ \tau_{|\alpha(n)|}) * f \right| > \frac{\lambda}{2} \right\} \right| \\ & + \sum_{s=0}^{\lfloor \log_2 N_1 \rfloor} \left| \left\{ \sup_{N: \frac{N_1}{2^s+1} < N \leq \frac{N_1}{2^s}} \frac{1}{|\{n : |\alpha(n)| \leq |\alpha(N)\}|} \frac{N_1}{2^s} \right. \right. \\ & \left. \left. \left| \sum_{l=1}^{|\alpha(N)|} \frac{|\{n : |\alpha(n)| = l\}|}{|\{n : |\alpha(n)| = |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)\}|} r_l D_{2^l} F_s \right| > \frac{\lambda}{2} \right\} \right| \end{aligned}$$

$$\begin{aligned} &\leq 6 \frac{\|f\|_1}{\lambda} + C \sum_{s=0}^{\lfloor \log_2 N_1 \rfloor} |\{F_s^* > \frac{\lambda}{2}\}| \leq 6 \frac{\|f\|_1}{\lambda} + C \sum_{s=0}^{\lfloor \log_2 N_1 \rfloor} \frac{\|F_s^*\|_1}{\lambda} \\ &\leq 6 \frac{\|f\|_1}{\lambda} + C \sum_{s=0}^{\lfloor \log_2 N_1 \rfloor} |\alpha(\lfloor \frac{N_1}{2^s} \rfloor)|^{-\beta} \frac{\|f\|_1}{\lambda} \leq C \frac{\|f\|_1}{\lambda} N_1^{-\frac{\beta}{\beta+1}} \sum_{s=0}^{\lfloor \log_2 N_1 \rfloor} 2^{s \frac{\beta}{\beta+1}} \leq C \frac{\|f\|_1}{\lambda}. \end{aligned}$$

□

We provide an example that shows that the set of increasing sequences $(\alpha(n))_n$ mentioned in Theorem 2.2 is not empty.

Example 2.1. Let for example $\beta = \frac{1}{2}$. For every $l : 4 \leq l < 16$, we define the set

$$\{k : |\alpha(k)| = l\} = \{2^l, 2^l + 2^{l-4}\}.$$

Then if l is so that $2^{2n} \leq l < 2^{2n+2}$, we put $|\alpha(k)| = l$ if and only if

$$k \in \{2^{l-2^{2n-2}}s, 2^{l-2^{2n-2}}s + 1\},$$

for some s being such that $|\alpha(s)| = 2^{2n-2}$. The condition (1) in the theorem is obviously satisfied. Recursively on n we get for $2^{2n} \leq l < 2^{2n+2}$,

$$|\{k : |\alpha(k)| = l\}| = 2|\{s : |\alpha(s)| = 2^{2n-2}\}| = 2 \cdot 2^{n-1} \sim l^\beta.$$

We also use induction on n in order to prove the property (3), if we suppose that

$$\sum_{i=0}^{2^{2n-2}} \sum_{k:|\alpha(k)|=2^{2n-2}} (\alpha(k))_i \leq 2^{2n-2}, \text{ we get for } 2^{2n} \leq l < 2^{2n+2}$$

$$\begin{aligned} \sum_{i=0}^l \sum_{k:|\alpha(k)|=l} (\alpha(k))_i &= 2 \sum_{i=0}^{2^{2n-2}} \sum_{k:|\alpha(k)|=2^{2n-2}} (\alpha(k))_i + |\{k : |\alpha(k)| = 2^{2n-2}\}| \\ &\leq 2^{2n-1} + 2^{n-1} < l. \end{aligned}$$

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