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INEQUALITIES ON CONVEX COMBINATIONS WITH THE COMMON CENTER

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ABSTRACT. The article offers Jensen type inequalities for a class of functions generalizing convex functions. The results are obtained by applying such functions to convex combinations with the common center. Functions of one and several variables are used in these considerations.

1. INTRODUCTION

Recall the concept of convexity and affinity. Let $\mathcal{X}$ be a real linear space. Let $x_1,\dots,x_n \in \mathcal{X}$ be points (vectors), and let $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ be coefficients (scalars). Their linear combination

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

is convex if all coefficients α_i are nonnegative and if their sum is equal to 1. The above combination is affine if only the coefficient sum is equal to 1. The point x itself is called the combination center.

A set $\mathcal{S} \subseteq \mathcal{X}$ is convex (respectively affine) if it contains all convex (respectively affine) combinations of its points. The convex (respectively affine) hull of the set $\mathcal{S}$ is the smallest convex (respectively affine) set containing $\mathcal{S}$, and it consists of all convex (respectively affine) combinations of points of $\mathcal{S}$. The convex (respectively affine) hull of the set $\mathcal{S}$ is usually denoted with $\operatorname{conv}\mathcal{S}$ (respectively $\operatorname{aff}\mathcal{S}$).

Let $\mathcal{C} \subseteq \mathcal{X}$ be a convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is convex if the inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \le \sum_{i=1}^{n} \alpha_i f(x_i) \tag{1.1}$$

holds for all convex combinations of points $x_i \in \mathcal{C}$. Let $\mathcal{A} \subseteq \mathcal{X}$ be an affine set. A function $f : \mathcal{A} \to \mathbb{R}$ is affine if the equality holds in equation (1.1) for all affine combinations of points $x_i \in \mathcal{A}$. If we have two affine combinations with the common center,

$$\sum_{i=1}^{n} \alpha_i x_i = \sum_{j=1}^{m} \beta_j y_j,$$

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then every affine function f satisfies the equality

$$\sum_{i=1}^{n} \alpha_i f(x_i) = \sum_{j=1}^{m} \beta_j f(y_j).$$

To define a convex function we usually take $n = 2$ in equation (1.1). Applying the mathematical induction Jensen has attained the famous inequality in equation (1.1), see [2]. Numerous books and papers have been written on Jensen's inequality. Different approaches can be seen in books [6] and [9], and papers [10], [3], [7] and [5]. An overview of different forms of Jensen's inequality can be found in [4].

2. Main Results

2.1. **Functions of One Variable.** The main result of this subsection is Theorem 2.1 relying on the idea of a convex function graph and its chord line. Using a function that is more general than convex function, and two convex combinations with the common center, we obtained the Jensen type inequality.

Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval with endpoints $a < b$. Then every number $x \in \mathbb{R}$ can be uniquely presented as the binomial affine combination:

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b\,, \tag{2.1}$$

which is convex if, and only if, the number x belongs to the interval $[a, b]$. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval containing $[a, b]$, let $f : \mathcal{I} \to \mathbb{R}$ be a function, and let $f^{\mathrm{line}}_{\{a,b\}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. Applying the affinity of the function $f^{\mathrm{line}}_{\{a,b\}}$ to the combination in (2.1), we obtain its equation

$$f^{\mathrm{line}}_{\{a,b\}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)\,. \tag{2.2}$$

The consequence of the representations in equations (2.1) and (2.2) is the fact that every convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies the inequality

$$f(x) \le f^{\mathrm{line}}_{\{a,b\}}(x) \ \text{ for } x \in [a, b]\,, \tag{2.3}$$

and the reverse inequality

$$f(x) \ge f^{\mathrm{line}}_{\{a,b\}}(x) \ \text{ for } x \in \mathcal{I} \setminus (a, b)\,. \tag{2.4}$$

In the following theorem, we use the functions satisfying the inequalities in equations (2.3) and (2.4).

Theorem 2.1. *Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, let $[a, b] \subseteq \mathcal{I}$ be a closed interval with endpoints $a < b$, and let $f : \mathcal{I} \to \mathbb{R}$ be a function satisfying equations* (2.3) *and* (2.4). *Let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of points $a_i \in [a, b]$, and let $\sum_{j=1}^{m} \beta_j b_j$ be a convex combination of points $b_j \in \mathcal{I} \setminus (a, b)$.*

If the above convex combinations have the common center

$$\sum_{i=1}^{n} \alpha_i a_i = \sum_{j=1}^{m} \beta_j b_j\,, \tag{2.5}$$

then we have the inequality

$$\sum_{i=1}^{n} \alpha_i f(a_i) \le \sum_{j=1}^{m} \beta_j f(b_j). \tag{2.6}$$

Proof. Using the properties of the function f, and applying the affinity of the function $f^{\text{line}}_{\{a,b\}}$, we get

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i f(a_i) &\le \sum_{i=1}^{n} \alpha_i f^{\text{line}}_{\{a,b\}}(a_i) = f^{\text{line}}_{\{a,b\}}\left(\sum_{i=1}^{n} \alpha_i a_i\right) \\ &= f^{\text{line}}_{\{a,b\}}\left(\sum_{j=1}^{m} \beta_j b_j\right) = \sum_{j=1}^{m} \beta_j f^{\text{line}}_{\{a,b\}}(b_j) \\ &\le \sum_{j=1}^{m} \beta_j f(b_j)\,,\end{aligned}$$

finishing the derivation of the inequality in equation (2.6). ☐

The function used in Theorem 2.1 is shown in Figure 1. Although the figure represents a continuous function, it may also be discontinuous.

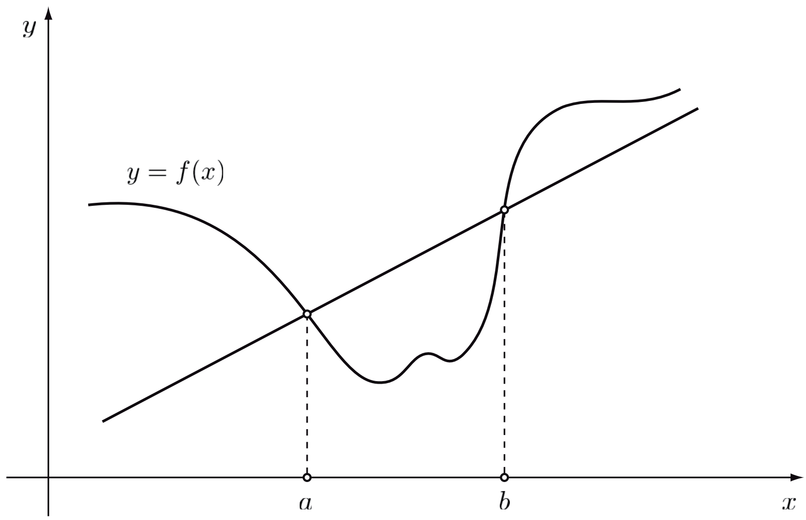

FIGURE 1. A function satisfying equations (2.3) and (2.4)

Involving the binomial convex combination $\alpha a + \beta b$ to the equality in equation (2.5) by assuming that

$$\sum_{i=1}^{n} \alpha_i a_i = \alpha a + \beta b = \sum_{j=1}^{m} \beta_j b_j,$$

and following the proof of Theorem 2.1, we achieve the double inequality

$$\sum_{i=1}^{n} \alpha_i f(a_i) \le \alpha f(a) + \beta f(b) \le \sum_{j=1}^{m} \beta_j f(b_j).$$

The functions used in Theorem 2.1 satisfy Jensen's inequality for convex combinations whose points are outside the open interval (a, b), and whose center is in the closed interval $[a, b]$.

Corollary 2.1. *Let $f : \mathcal{I} \to \mathbb{R}$ be a function satisfying equations* (2.3)-(2.4), *and let $c = \sum_{j=1}^{m} \beta_j b_j$ be a convex combination of points $b_j \in \mathcal{I} \setminus (a, b)$.*

If the center c belongs to $[a, b]$, then

$$f\left(\sum_{j=1}^{m} \beta_j b_j\right) \leq \sum_{j=1}^{m} \beta_j f(b_j). \tag{2.7}$$

Corollary 2.2. *A convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies Theorem 2.1 for every closed interval $[a, b] \subseteq \mathcal{I}$ with endpoints $a < b$.*

Corollary 2.3. *Let $f : \mathcal{I} \to \mathbb{R}$ be a function such that it satisfies Theorem 2.1 for every closed interval $[a, b] \subseteq \mathcal{I}$ with endpoints $a < b$. Then f is convex.*

Proof. Let

$$c = \beta_1 b_1 + \beta_2 b_2 \tag{2.8}$$

be a binomial convex combination of points $b_1, b_2 \in \mathcal{I}$ such that $b_1 < b_2$. Taking $a = b_1$ and $b = b_2$, equation (2.8) becomes the common center of the trivial convex combination $c \in [a, b]$ and the binomial convex combination $\beta_1 b_1 + \beta_2 b_2$ of points $b_1, b_2 \in \mathcal{I} \setminus (a, b)$. We can apply the corollary assumption, and get the inequality

$$f(\beta_1 b_1 + \beta_2 b_2) = f(c) \leq \beta_1 b_1 + \beta_2 b_2 \tag{2.9}$$

which proves the convexity of the function f. □

Bringing together the two previous corollaries, we obtain a characterization of convex functions as follows.

Proposition 2.1. *A function $f : \mathcal{I} \to \mathbb{R}$ is convex if, and only if, it satisfies Theorem 2.1 for every closed interval $[a, b] \subseteq \mathcal{I}$ with endpoints $a < b$.*

2.2. **Generalizations to More Dimensions.** It can be said that the main result in this subsection is Theorem 2.2 generalizing Theorem 2.1 to more dimensions. Example 2.1 shows that we can not transfer all results of the previous subsection to more dimensions.

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a convex set, and let $\triangle = \operatorname{conv}\{A, B, C\} \subseteq \mathcal{C}$ be a triangle with vertices A, B and C. In what follows we use a function $f : \mathcal{C} \to \mathbb{R}$ that satisfies the inequality

$$f(P) \leq f^{\text{plane}}_{\{A,B,C\}}(P) \text{ for } P \in \triangle, \tag{2.10}$$

and the reverse inequality

$$f(P) \geq f^{\text{plane}}_{\{A,B,C\}}(P) \text{ for } P \in \mathcal{C} \setminus \triangle^o, \tag{2.11}$$

where $f^{\text{plane}}_{\{A,B,C\}}$ is the function of the plane passing through the corresponding points of the graph of f, and $\triangle^o$ is the interior of $\triangle$.

The generalization of Theorem 2.1 to two dimensions is as follows.

Lemma 2.1. *Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a convex set, let $\triangle = \operatorname{conv}\{A, B, C\} \subseteq \mathcal{C}$ be a triangle with vertices A, B, C, and let $f : \mathcal{C} \to \mathbb{R}$ be a function satisfying equations* (2.10)-(2.11). *Let $\sum_{i=1}^{n} \alpha_i A_i$ be a convex combination of points $A_i \in \triangle$, and let $\sum_{j=1}^{m} \beta_j B_j$ be a convex combination of points $B_j \in \mathcal{C} \setminus \triangle^o$.*

If the above convex combinations have the common center

$$\sum_{i=1}^{n} \alpha_i A_i = \sum_{j=1}^{m} \beta_j B_j, \tag{2.12}$$

then we have the inequality

$$\sum_{i=1}^{n} \alpha_i f(A_i) \leq \sum_{j=1}^{m} \beta_j f(B_j).$$

Proof. We can implement the same analytic procedure as in Theorem 2.1 using the affinity of the plane function $f^{\text{plane}}_{\{A,B,C\}}$. □

Including the trinomial convex combination $\alpha A + \beta B + \gamma C$ to the equality in equation (2.12) by assuming that

$$\sum_{i=1}^{n} \alpha_i A_i = \alpha A + \beta B + \gamma C = \sum_{j=1}^{m} \beta_j B_j,$$

and applying Lemma 2.1 to the left and right equality of the above equation, we obtain the double inequality

$$\sum_{i=1}^{n} \alpha_i f(A_i) \leq \alpha f(A) + \beta f(B) + \gamma f(C) \leq \sum_{j=1}^{m} \beta_j f(B_j).$$

Corollary 2.4. *Let $f : \mathcal{C} \to \mathbb{R}$ be a function satisfying equations* (2.10)-(2.11), *and let $D = \sum_{j=1}^{m} \beta_j B_j$ be a convex combination of points $B_j \in \mathcal{C} \setminus \triangle^o$.*

If the center D belongs to $\triangle$, then

$$f\left(\sum_{j=1}^{m} \alpha_i B_j\right) \leq \sum_{j=1}^{m} \beta_j f(B_j).$$

The next example demonstrates that a generalization of Corollary 2.2 to convex functions of two variables is not possible.

Example 2.1. *Take the convex function $f(x, y) = x^2 + y^2$, the triangle with vertices $A(0, 0)$, $B(3, 0)$ and $C(0, 3)$, and the outside points $B_1(1, -1)$, $B_2(2, 2)$ and $B_3(-1, 1)$.*

Then we have

$$\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C = \frac{1}{4}B_1 + \frac{1}{2}B_2 + \frac{1}{4}B_3$$

and

$$6 = \frac{1}{3}f(A) + \frac{1}{3}f(B) + \frac{1}{3}f(C) > \frac{1}{4}f(B_1) + \frac{1}{2}f(B_2) + \frac{1}{4}f(B_3) = 5.$$

Lemma 2.1 which refers to the triangles can be generally extended to simplexes. In short, the convex hull of points $C_1, \ldots, C_{p+1} \in \mathbb{R}^p$ is a p-simplex in $\mathbb{R}^p$ if the points $C_1 - C_{p+1}, \ldots, C_p - C_{p+1}$ are linearly independent.

Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a convex set, and let $\mathcal{S} = \mathrm{conv}\{C_1, \dots, C_{p+1}\} \subseteq \mathcal{C}$ be a p-simplex with vertices $C_1, \dots, C_{p+1}$. In this observation we use a function $f : \mathcal{C} \to \mathbb{R}$ that satisfies the inequality

$$f(P) \le f^{\text{hyperplane}}_{\{C_1,\dots,C_{p+1}\}}(P) \text{ for } P \in \mathcal{S}, \tag{2.13}$$

and the reverse inequality

$$f(P) \ge f^{\text{hyperplane}}_{\{C_1,\dots,C_{p+1}\}}(P) \text{ for } P \in \mathcal{C} \setminus \mathcal{S}^o, \tag{2.14}$$

where $f^{\text{hyperplane}}_{\{C_1,\dots,C_{p+1}\}}$ is the function of the hyperplane passing through the corresponding points of the graph of f.

Theorem 2.2. *Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a convex set, let $\mathcal{S} = \mathrm{conv}\{C_1, \dots, C_{p+1}\} \subseteq \mathcal{C}$ be a p-simplex with vertices $C_1, \dots, C_{p+1}$, and let $f : \mathcal{C} \to \mathbb{R}$ be a function satisfying equations* (2.13)-(2.14). *Let $\sum_{i=1}^n \alpha_i A_i$ be a convex combination of points $A_i \in \mathcal{S}$, and let $\sum_{j=1}^m \beta_j B_j$ be a convex combination of points $B_j \in \mathcal{C} \setminus \mathcal{S}^o$.*

If the above convex combinations have the common center

$$\sum_{i=1}^n \alpha_i A_i = \sum_{j=1}^m \beta_j B_j, \tag{2.15}$$

then we have the inequality

$$\sum_{i=1}^n \alpha_i f(A_i) \le \sum_{j=1}^m \beta_j f(B_j). \tag{2.16}$$

Proof. To derive the inequality in equation (2.16), we may also apply the proof of Theorem 2.1 using the affinity of the hyperplane function $f^{\text{hyperplane}}_{\{C_1,\dots,C_{p+1}\}}$. □

Including the $(p + 1)$-membered convex combination $\sum_{k=1}^{p+1} \gamma_k C_k$ to the equality in equation (2.15) in a way that

$$\sum_{i=1}^n \alpha_i A_i = \sum_{k=1}^{p+1} \gamma_k C_k = \sum_{j=1}^m \beta_j B_j,$$

and respecting Theorem 2.2, we get the double inequality

$$\sum_{i=1}^n \alpha_i f(A_i) \le \sum_{k=1}^{p+1} \gamma_k f(C_k) \le \sum_{j=1}^m \beta_j f(B_j).$$

Corollary 2.5. *Let $f : \mathcal{C} \to \mathbb{R}$ be a function satisfying equations* (2.13)-(2.14), *and let $D = \sum_{j=1}^m \beta_j B_j$ be a convex combination of points $B_j \in \mathcal{C} \setminus \mathcal{S}^o$.*

If the center D belongs to $\mathcal{S}$, then

$$f\left(\sum_{j=1}^m \alpha_i B_j\right) \le \sum_{j=1}^m \beta_j f(B_j).$$

3. Applications to Quasi-Arithmetic Means

Functions investigated in Subsection 2.1 can be included to quasi-arithmetic means by applying methods such as those for convex functions. More details on quasi-arithmetic and power means can be found in [1] and [4].

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, and let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in \mathcal{I}$. The discrete φ-quasi-arithmetic mean of the points x_i with the coefficients λ_i is the point

$$M_\varphi(x_i, \lambda_i) = \varphi^{-1}\left(\sum_{i=1}^{n} \lambda_i \varphi(x_i)\right) \tag{3.1}$$

which belongs to $\mathcal{I}$ because the convex combination $\sum_{i=1}^{n} \lambda_i \, \varphi(x_i)$ belongs to $\varphi(\mathcal{I})$.

In order to apply the convexity, we use strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that ψ is convex with respect to φ (ψ is φ-convex), which is to say that the function $f = \psi \circ \varphi^{-1}$ is convex on the interval $\varphi(\mathcal{I})$. A similar notation is used for the concavity.

We want to apply Theorem 2.1 to quasi-arithmetic means.

Theorem 3.1. *Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, and let $[a, b] \subseteq \mathcal{I}$ be a closed interval with endpoints $a < b$. Let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of points $a_i \in [a, b]$, and let $\sum_{j=1}^{m} \beta_j b_j$ be a convex combination of points $b_j \in \mathcal{I} \setminus (a, b)$. Let $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ be strictly monotone continuous functions, and let $f = \psi \circ \varphi^{-1}$ be the composite function.*

If f satisfies equations (2.3)-(2.4) and ψ is increasing, and if the equality

$$M_\varphi(a_i, \alpha_i) = M_\varphi(b_j, \beta_j) \tag{3.2}$$

is valid, then we have the inequality

$$M_\psi(a_i, \alpha_i) \le M_\psi(b_j, \beta_j). \tag{3.3}$$

Proof. Take $\mathcal{J} = \varphi(\mathcal{I})$, and $[c, d] = \varphi([a, b])$ where $c < d$. We will apply Theorem 2.1 to the function $f : \mathcal{J} \to \mathbb{R}$, the points $u_i = \varphi(a_i) \in [c, d]$, and the points $v_j = \varphi(b_j) \in \mathcal{J} \setminus (c, d)$.

Starting with the equality $\varphi\big(M_\varphi(a_i, \alpha_i)\big) = \varphi\big(M_\varphi(b_j, \beta_j)\big)$, that is,

$$\sum_{i=1}^{n} \alpha_i u_i = \sum_{j=1}^{m} \beta_j v_j,$$

and relying on Theorem 2.1, we get

$$\sum_{i=1}^{n} \alpha_i f(u_i) \le \sum_{j=1}^{m} \beta_j f(v_j).$$

Applying the increasing function ψ^{-1} to the above inequality, it follows that

$$\psi^{-1}\left(\sum_{i=1}^{n} \alpha_i f(u_i)\right) \le \psi^{-1}\left(\sum_{j=1}^{m} \beta_j f(v_j)\right),$$

which is actually

$$M_\psi(a_i, \alpha_i) \le M_\psi(b_j, \beta_j)$$

because $f(u_i) = \psi(a_i)$ and $f(v_j) = \psi(b_j)$. $\square$

All the possibilities in the above theorem are as follows.

Corollary 3.1. *Let $f = \psi \circ \varphi^{-1}$ be the composition of functions ψ and φ^{-1} satisfying the conditions of Theorem 3.1.*

If either f satisfies equations (2.3)-(2.4) and ψ is increasing or $-f$ satisfies equations (2.3)-(2.4) and ψ is decreasing, and if the equality in equation (3.2) is valid, then the inequality in equation (3.3) holds.

If either f satisfies equations (2.3)-(2.4) and ψ is decreasing or $-f$ satisfies equations (2.3)-(2.4) and ψ is increasing, and if the equality in equation (3.2) is valid, then the reverse inequality in equation (3.3) holds.

Let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in \mathcal{I}$. As a special case of the quasi-arithmetic means in equation (3.1) using the functions $\varphi_r(x) = x^r$ for $r \neq 0$ and $\varphi_0(x) = \ln x$ on the interval $\mathcal{I} = (0, +\infty)$, we can observe the discrete power means

$$M_r(x_i, \alpha_i) = \begin{cases} \left(\sum_{i=1}^{n} \lambda_i \, x_i^r \right)^{\frac{1}{r}} & \text{for } r \neq 0 \\ \exp\left(\sum_{i=1}^{n} \lambda_i \ln x_i \right) & \text{for } r = 0 \end{cases}.$$

Note that

$$M_1(x_i, \lambda_i) = \sum_{i=1}^{n} \lambda_i x_i.$$

Corollary 3.2. *Let $[a, b] \subset (0, +\infty)$ be a closed interval with endpoints $a < b$. Let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of points $a_i \in [a, b]$, and let $\sum_{j=1}^{m} \beta_j b_j$ be a convex combination of points $b_j \in (0, +\infty) \setminus (a, b)$.*

If the above convex combinations have the common center

$$M_1(a_i, \alpha_i) = M_1(b_j, \beta_j),$$

then

$$M_r(a_i, \alpha_i) \leq M_r(b_j, \beta_j) \ \text{ for } r \geq 1,$$

and

$$M_r(a_i, \alpha_i) \geq M_r(b_j, \beta_j) \ \text{ for } r \leq 1.$$

Proof. The proof follows from Theorem 3.1 and Corollary 3.1 by using convex and concave functions such as $\varphi(x) = x$, $\psi(x) = x^r$ for $r \neq 0$, and $\psi(x) = \ln x$ for $r = 0$. □

4. Applications to Integrals

The aim of the final section is just to indicate how the above results can be transferred to integrals. Something similar was done in [8], but in the reverse direction.

The integral analogy of the concept of convex combination is the concept of barycenter. Let μ be a positive measure on $\mathbb{R}^p$, and let $\mathcal{A} \subseteq \mathbb{R}^p$ be a μ-measurable set with $\mu(\mathcal{A}) > 0$.

Given the positive integer n, let $\mathcal{A} = \cup_{i=1}^{n}\mathcal{A}_{ni}$ be the partition of pairwise disjoint μ-measurable sets $\mathcal{A}_{ni}$. Taking points $A_{ni} \in \mathcal{A}_{ni}$ we determine the convex combination

$$A_n = \sum_{i=1}^{n} \frac{\mu(\mathcal{A}_{ni})}{\mu(\mathcal{A})} A_{ni}$$

whose center A_n belongs to conv$\mathcal{A}$. If the sequence $(A_n)_n$ converges, then the μ-barycenter of the set $\mathcal{A}$ can be defined by the point

$$M(\mathcal{A},\mu) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\mu(\mathcal{A}_{ni})}{\mu(\mathcal{A})} A_{ni} \right) = \frac{1}{\mu(\mathcal{A})} \left(\int_{\mathcal{A}} x_1 \, d\mu, \ldots, \int_{\mathcal{A}} x_r \, d\mu \right).$$

Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a convex set, and let $\mathcal{S} \subseteq \mathcal{C}$ be a p-simplex. To apply the results of Section 2 to integrals, one can use μ-measurable sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{S}$ and $\mathcal{B} \subseteq \mathcal{C} \setminus \mathcal{S}^o$, and that have the common μ-barycenter

$$M(\mathcal{A},\mu) = M(\mathcal{B},\mu).$$

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