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FEKETE-SZEGÖ INEQUALITIES FOR CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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ABSTRACT. In the present investigation, we obtain the Fekete-Szegö inequalities for the classes $S_{\Sigma}(\lambda,\phi)$ and $B(\lambda,\mu,\phi)$ of bi-univalent functions defined by subordination.

1. INTRODUCTION AND DEFINITIONS

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [4] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus, every such univalent function has an inverse f^{-1} which satisfies:

$$f^{-1}(f(z)) = z \quad (z \in U),$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$.

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Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ refer to [11].

Lewin in [7] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu in [9] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha in [2] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0 < \alpha \leq 1)$, respectively, (see [9]). Thus, following Brannan and Taha [2], a function $f(z) \in A$ is in the class $S^{\star}_{\Sigma}(\alpha)$ of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U),$$

where g is the extension of f^{-1} to U. Similarly, a function $f(z) \in A$ is in the class $K_{\Sigma}(\alpha)$ of strongly bi-convex functions of order $\alpha\,(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{z^2 f''(z) + zf'(z)}{zf'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{w^2 g''(w) + wg'(w)}{wg'(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U),$$

where g is the extension of f^{-1} to U. The classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the classes $S^{\star}(\alpha)$ and $K(\alpha)$, were introduced analogously. For each of the classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [5], [11], [12]. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n = 3, 4, \ldots$ is presumably still an open problem.

In this paper we obtain the Fekete Szegö inequalities for the classes $S_{\Sigma}(\lambda, \phi)$ and $B(\lambda, \mu, \phi)$. These inequalities will result in bounds of the third coefficient which are, in some cases, better than the ones obtained in [1], [6] and [8].

In order to derive our main results, we require the following lemma.

Lemma 1.1. *(Pommerenke* [10]*) If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ *is an analytic function in* U *with positive real part, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_2|^2}{2}. \tag{1.2}$$

2. Fekete-Szegö Inequalities for the Class $S_\Sigma(\lambda,\phi)$

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0)=1$ and $\phi'(0)>0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots\quad(B_1>0). \tag{2.1}$$

Definition 2.1. *A function $f\in\Sigma$ is said to be in $S_\Sigma(\lambda,\phi)$, $0<\alpha\le1$ and $0\le\lambda\le1$, if the following subordinations hold*

$$\frac{zf'(z)+\left(2\lambda^2-\lambda\right)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}\prec\phi(z)$$

and

$$\frac{wg'(w)+\left(2\lambda^2-\lambda\right)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)}\prec\phi(w),$$

where $g(w)=f^{-1}(w)$.

Theorem 2.1. *Let f given by (1.1) be in the class $S_\Sigma(\lambda,\phi)$ and $\mu\in\mathbb{R}$. Then*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{B_1}{2(1+2\lambda^2)}\ \ for\ \ |\mu-1|\le\frac{1}{2(2\lambda^2+1)}\cdot\\ \qquad\left|1+2\lambda+15\lambda^2-28\lambda^3+12\lambda^4+\left(1+3\lambda-2\lambda^2\right)^2\frac{(B_1-B_2)}{B_1^2}\right|\\ \\ \frac{B_1^3|\mu-1|}{\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2+(1+3\lambda-2\lambda^2)^2(B_1-B_2)\right|}\ \ for\ \ |\mu-1|\ge\\ \qquad\frac{1}{2(2\lambda^2+1)}\cdot\left|1+2\lambda+15\lambda^2-28\lambda^3+12\lambda^4+\left(1+3\lambda-2\lambda^2\right)^2\frac{(B_1-B_2)}{B_1^2}\right|.\end{cases}$$

Proof. Let $f\in S_\Sigma(\lambda,\phi)$ and g be the analytic extension of f^{-1} to U. Then, there exist two functions u and v, analytic in U with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$, such that

$$\frac{zf'(z)+\left(2\lambda^2-\lambda\right)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}=\phi(u(z))\quad(z\in U) \tag{2.2}$$

and

$$\frac{wg'(w)+\left(2\lambda^2-\lambda\right)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)}=\phi(v(w))\quad(w\in U). \tag{2.3}$$

Next, let define functions p and q by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots \tag{2.4}$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+q_1w+q_2w^2+\cdots. \tag{2.5}$$

Clearly, $\operatorname{Re}p(z)>0$ and $\operatorname{Re}q(w)>0$. From (2.4) and (2.5) one can derive

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac12p_1z+\frac12\left(p_2-\frac12p_1^2\right)z^2+\cdots \tag{2.6}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac12q_1w+\frac12\left(q_2-\frac12q_1^2\right)w^2+\cdots. \tag{2.7}$$

Combining (2.1), (2.2), (2.3), (2.6) and (2.7),

$$\frac{zf'(z)+\left(2\lambda^2-\lambda\right)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} \tag{2.8}$$
$$= 1+\tfrac{1}{2}B_1p_1z+\left(\tfrac{1}{4}B_2p_1^2+\tfrac{1}{2}B_1\left(p_2-\tfrac{1}{2}p_1^2\right)\right)z^2+\cdots$$

and

$$\frac{wg'(w)+\left(2\lambda^2-\lambda\right)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)} \tag{2.9}$$
$$= 1+\tfrac{1}{2}B_1q_1w+\left(\tfrac{1}{4}B_2q_1^2+\tfrac{1}{2}B_1\left(q_2-\tfrac{1}{2}q_1^2\right)\right)w^2+\cdots.$$

From (2.8) and (2.9), we deduce

$$\left(1+3\lambda-2\lambda^2\right)a_2=\frac{1}{2}B_1p_1, \tag{2.10}$$

$$\left(12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1\right)a_2^2+\left(4\lambda^2+2\right)a_3=\frac{1}{4}B_2p_1^2+\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right), \tag{2.11}$$

and

$$-\left(1+3\lambda-2\lambda^2\right)a_2=\frac{1}{2}B_1q_1, \tag{2.12}$$

$$\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)a_2^2-\left(4\lambda^2+2\right)a_3=\frac{1}{4}B_2q_1^2+\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right). \tag{2.13}$$

From (2.10) and (2.12) we obtain

$$p_1=-q_1. \tag{2.14}$$

Subtracting (2.11) from (2.13) and applying (2.14) we have

$$a_3=a_2^2+\frac{1}{8\left(1+2\lambda^2\right)}B_1\left(p_2-q_2\right). \tag{2.15}$$

By adding (2.11) to (2.13), we get

$$2\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)a_2^2=\frac{1}{2}B_1\left(p_2+q_2\right)-\frac{1}{4}\left(B_1-B_2\right)\left(p_1^2+q_1^2\right).$$

Combining this with (2.10) and (2.12) leads to

$$a_2^2=\frac{B_1^3\left(p_2+q_2\right)}{4\left[\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2+\left(1+3\lambda-2\lambda^2\right)^2\left(B_1-B_2\right)\right]}. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$a_3-\mu a_2^2=B_1\left[\left(h\left(\mu\right)+\frac{1}{8\left(1+2\lambda^2\right)}\right)p_2+\left(h\left(\mu\right)-\frac{1}{8\left(1+2\lambda^2\right)}\right)q_2\right],$$

where

$$h\left(\mu\right)=\frac{B_1^2\left(1-\mu\right)}{4\left[\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2+\left(1+3\lambda-2\lambda^2\right)^2\left(B_1-B_2\right)\right]}.$$

Then, in view of (1.2) and (2.1), we conclude that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{B_1}{2(1+2\lambda^2)} & \text{for } 0 \leq |h(\mu)| \leq \frac{1}{8(1+2\lambda^2)} \\ 4B_1\,|h(\mu)| & \text{for } |h(\mu)| \geq \frac{1}{8(1+2\lambda^2)}. \end{cases}$$

□

Taking $\mu=1$ or $\mu=0$ we get

Corollary 2.1. *If* $f \in \mathrm{S}_\Sigma(\lambda,\phi)$ *then*

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(1+2\lambda^2)}. \tag{2.17}$$

Corollary 2.2. *If* $f \in \mathrm{S}_\Sigma(\lambda,\phi)$ *then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2(1+2\lambda^2)} \ \ for \ \frac{B_1-B_2}{B_1^2} \in \\ \qquad \left(-\infty, -\frac{12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3}{(1+3\lambda-2\lambda^2)^2}\right] \cup \left[-\frac{12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1}{(1+3\lambda-2\lambda^2)^2}, \infty\right) \\ \\ \frac{B_1^3}{\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2+(1+3\lambda-2\lambda^2)^2(B_1-B_2)\right|} \ \ for \ \frac{B_1-B_2}{B_1^2} \in \\ \qquad \left[-\frac{12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3}{(1+3\lambda-2\lambda^2)^2}, -\frac{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}{(1+3\lambda-2\lambda^2)^2}\right) \cup \\ \qquad \left(-\frac{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}{(1+3\lambda-2\lambda^2)^2}, -\frac{12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1}{(1+3\lambda-2\lambda^2)^2}\right]. \end{cases} \tag{2.18}$$

Corollary 2.3. *If*

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^\alpha = 1+2\alpha z+2\alpha^2 z^2+\ldots \qquad (0<\alpha\leq 1),$$

then inequalities (2.17) and (2.18) become

$$\left|a_3-a_2^2\right| \leq \frac{\alpha}{1+2\lambda^2}$$

and

$$|a_3| \leq \begin{cases} \frac{\alpha}{1+2\lambda^2} \ \ for \\ \qquad \alpha \in \left(0, \frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-17\lambda^2+2\lambda+3}\right] \cup \left[\frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-33\lambda^2+2\lambda-5}, 1\right] \\ \\ \frac{4\alpha^2}{(20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1)\alpha+(1+3\lambda-2\lambda^2)^2} \ \ for \\ \qquad \alpha \in \left[\frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-17\lambda^2+2\lambda+3}, \frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-25\lambda^2+2\lambda-1}\right) \\ \qquad \cup \left(\frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-25\lambda^2+2\lambda-1}, \frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-33\lambda^2+2\lambda-5}\right] \end{cases}$$

Corollary 2.4. *If*

$$\phi(z)=\frac{1+(1-2\alpha)z}{1-z} = 1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots \qquad (0\leq\alpha<1),$$

then inequalities (2.17) and (2.18) become

$$\left|a_3-a_2^2\right| \leq \frac{1-\alpha}{1+2\lambda^2}$$

and

$$|a_3| \leq \frac{2(1-\alpha)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}.$$

Remark 2.1. *Corollary 2.3 and Corollary 2.4 provide an improvement of the estimate of $|a_3|$ obtained by Magesh and Yamini* [8].

3. Fekete-Szegö Inequalities for the Function Class $B(\lambda,\mu,\phi)$

Definition 3.1. *A function $f\in\Sigma$ is said to be in* $\mathrm{B}(\lambda,\mu,\phi)$, $\lambda\geq 0,\ 0\leq\mu\leq\lambda\leq 1$, *if the following subordinations hold*

$$\frac{\lambda\mu z^3 f'''(z)+(2\lambda\mu+\lambda-\mu)z^2 f''(z)+zf'(z)}{\lambda\mu z^2 f''(z)+(\lambda-\mu)zf'(z)+(1-\lambda+\mu)f(z)}\prec\phi(z)$$

and

$$\frac{\lambda\mu w^3 g'''(w)+(2\lambda\mu+\lambda-\mu)w^2 g''(w)+wg'(w)}{\lambda\mu w^2 g''(w)+(\lambda-\mu)wg'(w)+(1-\lambda+\mu)g(w)}\prec\phi(w),$$

where $g(w)=f^{-1}(w)$ (see [1]*).*

Theorem 3.1. *Let f given by (1.1) be in the class $\mathrm{B}(\lambda,\mu,\phi)$ and $\delta\in\mathbb{R}$. Then*

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}\frac{B_1}{2(1+2\lambda-2\mu+6\lambda\mu)}\ \ for\ \ |\delta-1|\leq\frac{1}{2(1+2\lambda-2\mu+6\lambda\mu)}\cdot\left|2+4\lambda-4\mu+12\lambda\mu-(1+\lambda-\mu+2\lambda\mu)^2\frac{\left(B_1^2-B_1+B_2\right)}{B_1^2}\right|\\[2ex] \frac{B_1^3|\delta-1|}{\left|(2+4\lambda-4\mu+12\lambda\mu)B_1^2-(1+\lambda-\mu+2\lambda\mu)^2\left(B_1^2-B_1+B_2\right)\right|}\ \ for\\ |\delta-1|\geq\frac{1}{2(1+2\lambda-2\mu+6\lambda\mu)}\cdot\left|2+4\lambda-4\mu+12\lambda\mu-(1+\lambda-\mu+2\lambda\mu)^2\frac{\left(B_1^2-B_1+B_2\right)}{B_1^2}\right|\end{cases}.$$

Proof. Let $f\in B(\lambda,\mu,\phi)$ and g be the analytic extension of f^{-1} to U. Then, there exist two functions u and v, analytic in U with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$ $(z,w\in U)$ such that

$$\frac{\lambda\mu z^3 f'''(z)+(2\lambda\mu+\lambda-\mu)z^2 f''(z)+zf'(z)}{\lambda\mu z^2 f''(z)+(\lambda-\mu)zf'(z)+(1-\lambda+\mu)f(z)}=\phi(u(z))\quad(z\in U)\tag{3.1}$$

and

$$\frac{\lambda\mu w^3 g'''(w)+(2\lambda\mu+\lambda-\mu)w^2 g''(w)+wg'(w)}{\lambda\mu w^2 g''(w)+(\lambda-\mu)wg'(w)+(1-\lambda+\mu)g(w)}=\phi(v(w))\quad(w\in U).\tag{3.2}$$

Combining (2.1), (3.1), (3.2), (2.6) and (2.7), we get

$$\begin{aligned}&\tfrac{\lambda\mu z^3 f'''(z)+(2\lambda\mu+\lambda-\mu)z^2 f''(z)+zf'(z)}{\lambda\mu z^2 f''(z)+(\lambda-\mu)zf'(z)+(1-\lambda+\mu)f(z)}\\&=1+\tfrac{1}{2}B_1p_1z+\left(\tfrac{1}{4}B_2p_1^2+\tfrac{1}{2}B_1\left(p_2-\tfrac{1}{2}p_1^2\right)\right)z^2+\cdots\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\tfrac{\lambda\mu w^3 g'''(w)+(2\lambda\mu+\lambda-\mu)w^2 g''(w)+wg'(w)}{\lambda\mu w^2 g''(w)+(\lambda-\mu)wg'(w)+(1-\lambda+\mu)g(w)}\\&=1+\tfrac{1}{2}B_1q_1w+\left(\tfrac{1}{4}B_2q_1^2+\tfrac{1}{2}B_1\left(q_2-\tfrac{1}{2}q_1^2\right)\right)w^2+\cdots.\end{aligned}\tag{3.4}$$

From (3.3) and (3.4), we deduce

$$(1+\lambda-\mu+2\lambda\mu)\,a_2=\frac{1}{2}B_1p_1, \tag{3.5}$$

$$2(1+2\lambda-2\mu+6\lambda\mu)\,a_3-(1+\lambda-\mu+2\lambda\mu)^2\,a_2^2=\frac{1}{4}B_2p_1^2+\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right), \tag{3.6}$$

and

$$-(1+\lambda-\mu+2\lambda\mu)\,a_2=\frac{1}{2}B_1q_1, \tag{3.7}$$

(3.8)

$$2(1+2\lambda-2\mu+6\lambda\mu)\left(2a_2^2-a_3\right)-(1+\lambda-\mu+2\lambda\mu)^2\,a_2^2=\frac{1}{4}B_2q_1^2+\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right).$$

From (3.5) and (3.7) we obtain

$$p_1=-q_1. \tag{3.9}$$

Subtracting (3.6) from (3.8) and applying (3.9) we have

$$a_3=a_2^2+\frac{1}{8(1+2\lambda-2\mu+6\lambda\mu)}B_1(p_2-q_2). \tag{3.10}$$

By adding (3.6) to (3.8), we get

$$4(1+2\lambda-2\mu+6\lambda\mu)\,a_2^2-2(1+\lambda-\mu+2\lambda\mu)^2\,a_2^2=\frac{1}{2}B_1(p_2+q_2)-\frac{1}{4}(B_1-B_2)\left(p_1^2+q_1^2\right).$$

Combining this with (3.5) and (3.7) leads to

$$a_2^2=\frac{B_1^3(p_2+q_2)}{4\left[(2+4\lambda-4\mu+12\lambda\mu)B_1^2-(1+\lambda-\mu+2\lambda\mu)^2\left(B_1^2-B_1+B_2\right)\right]}. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$a_3-\delta a_2^2=B_1\left[\left(h(\delta)+\frac{1}{8(1+2\lambda-2\mu+6\lambda\mu)}\right)p_2+\left(h(\delta)-\frac{1}{8(1+2\lambda-2\mu+6\lambda\mu)}\right)q_2\right],$$

where

$$h(\delta)=\frac{B_1^2(1-\delta)}{4\left[(2+4\lambda-4\mu+12\lambda\mu)B_1^2-(1+\lambda-\mu+2\lambda\mu)^2\left(B_1^2-B_1+B_2\right)\right]}.$$

Then, in view of (1.2) and (2.1), we conclude that

$$|a_3|\leq\begin{cases}\frac{B_1}{2+4\lambda-4\mu+12\lambda\mu} & for\quad 0\leq|h(\delta)|\leq\frac{1}{8(1+2\lambda-2\mu+6\lambda\mu)}\\ 4B_1|h(\delta)| & for\quad |h(\delta)|\geq\frac{1}{8(1+2\lambda-2\mu+6\lambda\mu)}\end{cases}.$$

□

Taking $\delta=1$ or $\delta=0$ we get

Corollary 3.1. *If* $f\in \mathrm{B}(\lambda,\mu,\phi)$ *then*

$$\left|a_3-a_2^2\right|\leq\frac{B_1}{2+4\lambda-4\mu+12\lambda\mu}. \tag{3.12}$$

Corollary 3.2. *If* $f \in \mathrm{B}(\lambda,\mu,\phi)$ *then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2+4\lambda-4\mu+12\lambda\mu} \quad for \quad \frac{B_1^2-B_1+B_2}{B_1^2} \in (-\infty,0] \cup \left[\frac{4(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2},\infty\right) \\ \frac{B_1^3}{\left|(2+4\lambda-4\mu+12\lambda\mu)B_1^2-(1+\lambda-\mu+2\lambda\mu)^2\left(B_1^2-B_1+B_2\right)\right|} \quad for \quad \frac{B_1^2-B_1+B_2}{B_1^2} \in \\ \quad \left[0,\frac{2(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2}\right) \cup \left(\frac{2(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2},\frac{4(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2}\right] \end{cases} \tag{3.13}$$

Corollary 3.3. *If*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots \qquad (0 < \alpha \leq 1),$$

then inequalities (3.12) and (3.13) become

$$\left|a_3 - a_2^2\right| \leq \frac{\alpha}{1+2\lambda-2\mu+6\lambda\mu}$$

and

$$|a_3| \leq \begin{cases} \frac{\alpha}{1+2\lambda-2\mu+6\lambda\mu} \quad for \quad \alpha \in \left(0,\frac{1}{3}\right] \cup \left[\frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-8(1+2\lambda-2\mu+6\lambda\mu)},1\right) \\ \frac{4\alpha^2}{4(1+2\lambda-2\mu+6\lambda\mu)\alpha-(1+\lambda-\mu+2\lambda\mu)^2(3\alpha-1)} \quad for \\ \quad \alpha \in \left[\frac{1}{3},\frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-4(1+2\lambda-2\mu+6\lambda\mu)}\right) \\ \quad \cup \left(\frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-4(1+2\lambda-2\mu+6\lambda\mu)},\frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-8(1+2\lambda-2\mu+6\lambda\mu)}\right] \end{cases}.$$

Corollary 3.4. *If*

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \cdots \qquad (0 \leq \alpha < 1),$$

then inequalities (3.12) and (3.13) become

$$\left|a_3 - a_2^2\right| \leq \frac{1-\alpha}{1+2\lambda-2\mu+6\lambda\mu}$$

and

$$|a_3| \leq \frac{2(1-\alpha)}{2(1+2\lambda-2\mu+6\lambda\mu)-(1+\lambda-\mu+2\lambda\mu)^2}.$$

Remark 3.1. *Corollary 3.3 and Corollary 3.4 provide an improvement of the estimate of* $|a_3|$ *obtained previously by Keerthi and Raja* [6].

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