A NEW REPRESENTATION OF THE FIELD EQUATIONS OF QUADRATIC METRIC–AFFINE GRAVITY

VEDAD PASIC[1], ELVIS BARAKOVIC AND NERMIN OKICIC

ABSTRACT. We deal with quadratic metric–affine gravity (QMAG), which is an alternative theory of gravity and present a new explicit representation of the field equations of this theory. In our previous work we found new explicit vacuum solutions of QMAG, namely generalised pp-waves of parallel Ricci curvature with purely tensor torsion. Here we do not make any assumptions on the properties of torsion and write down our field equations accordingly. We present a review of research done thus far by several authors in finding new solutions of QMAG and different approaches in generalising pp-waves. We present two conjectures on the new types of solutions of QMAG which the ansatz presented in this paper will hopefully enable us to prove.

1. INTRODUCTION

In 1905, *Albert Einstein* published his work on the theory of *special relativity*. Classical mechanics and classical electromagnetism provide models that are good representations of two sets of actual experiences. As Einstein noted in [5], it is not possible to combine these into a single self–consistent model. The construction of the simplest possible self–consistent model by Einstein is the achievement of Einstein's theory of special relativity. Special relativity gave a very satisfactory representation of the electromagnetic interaction between charged particles, but the theory itself does not deal with gravitational interaction.

General relativity is a theory of gravitation that was developed by Einstein between 1907 and 1915. Hermann Minkowski put Einstein's special relativity model into geometrical terms, and it is widely believed that Einstein constructed his theory of general relativity by experimenting with the generalisation of the geometric model.

Two problems with general relativity became apparent quite quickly. Einstein considered that what are recognised locally as inertial properties of local matter must be determined by the properties of the rest of the universe. To what extent general relativity manages to do this is still unclear to this day, although Einstein's efforts to discover this extent founded the modern study of *cosmology*. The second problem of general relativity was that, although electromagnetism pointed the way to general relativity, it

[1]*corresponding author*

is not included in the theory itself. As is evident from his remarks in [6], Einstein expected much more from general relativity than 'just' the amalgamation of gravitation and electromagnetism at the macroscopic level. He thought the theory should explain the existence of elementary particles and should provide a treatment for nuclear forces. He spent most of the second part of his life in pursuit of this aim, but with no real success.

There are a number of different *alternative* theories of gravity that try to further the completion of Einstein's theory of gravity. One such theory, propagated by Einstein himself for some time, is *metric–affine gravity*, which is the theory employed by this paper. Metric–affine gravity is a natural generalization of Einstein's general relativity, which is based on a spacetime with a Riemannian metric g of Lorentzian signature. In metric–affine gravity, we consider spacetime to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ. The 10 independent components of the symmetric metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ are the unknowns of our theory, see [10] for more details.

Definition 1.1. *We call a spacetime $\{M, g, \Gamma\}$ Riemannian if the connection is Levi–Civita (i.e. $\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$), and non–Riemannian otherwise.*

The spacetime of metric–affine gravity reduces to that of general relativity provided that the torsion (2.1) of the connection Γ vanishes and that the connection is metric compatible (i.e. the covariant derivative of the metric g vanishes, $\nabla g \equiv 0$). In this case the connection is uniquely determined by the metric (Levi–Civita connection) and the same is true for the curvature. Consequently, the metric g is the only unknown quantity of Einstein's equation. In contrast, the metric–affine approach does not involve any a priori assumptions about the connection Γ and thus the metric g and the connection Γ are viewed as two totally independent unknown quantities.

In *quadratic* metric–affine gravity (QMAG), we define our action as

$$(1.1) \qquad S := \int q(R)$$

where q is an $O(1,3)$–invariant quadratic form on curvature R. Independent variation of the metric g and the connection Γ produces Euler–Lagrange equations which we will write symbolically as

$$(1.2) \qquad \partial S / \partial g := 0$$
$$(1.3) \qquad \partial S / \partial \Gamma := 0.$$

Our objective is the study of the combined system of field equations (1.2), (1.3). This is a system of $10 + 64$ real nonlinear partial differential equations with $10 + 64$ real unknowns. The quadratic form $q(R)$ has 16 R^2 terms with 16 real coupling constants, and it can be represented as

$$(1.4) \quad q(R) = b_1 \mathcal{R}^2 + b_1^* \mathcal{R}_*^2$$
$$+ \sum_{l,m=1}^{3} b_{6lm}(\mathcal{A}^{(l)}, \mathcal{A}^{(m)}) + \sum_{l,m=1}^{2} b_{9lm}(\mathcal{S}^{(l)}, \mathcal{S}^{(m)}) + \sum_{l,m=1}^{2} b_{9lm}^*(\mathcal{S}_*^{(l)}, \mathcal{S}_*^{(m)})$$
$$+ b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + b_{30}(R^{(30)}, R^{(30)})_{\text{YM}}$$

with some real constants b_1, b_1^*, $b_{6lm} = b_{6ml}$, $b_{9lm} = b_{9ml}$, $b_{9lm}^* = b_{9ml}^*$, b_{10}, b_{30}. Here $\mathcal{R}$, $\mathcal{R}_*$, $\mathcal{A}^{(l)}$, $\mathcal{S}^{(l)}$, $\mathcal{S}_*^{(l)}$, $R^{(10)}$, $R^{(30)}$ are tensors representing the irreducible pieces of curvature and the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{YM}$ are defined by

$$(K, L) := K_{\mu\nu} L^{\mu\nu}, \qquad (R, Q)_{\mathrm{YM}} := R^\kappa{}_{\lambda\mu\nu} Q^\lambda{}_\kappa{}^{\mu\nu}.$$

Detailed description of the irreducible pieces of curvature and quadratic forms on curvature can be found in [20, 32, 35]. Our motivation comes from the Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of (1.1) with

$$(1.5) \qquad q(R) := R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_\kappa{}^{\mu\nu}.$$

The motivation for choosing a model of gravity which is purely quadratic in curvature is explained in detail in Section 1 of [35], Chapter 1 of [20] and Section 1 of [22]. The idea of using a purely quadratic action in General Relativity goes back to Hermann Weyl [37], where he argued that the most natural gravitational action should be quadratic in curvature and involve all possible invariant quadratic combinations of curvature. In short, by choosing a purely quadratic curvature Lagrangian we are hoping to describe phenomena whose characteristic wavelength is sufficiently small and curvature sufficiently large. One can get more information and form an idea on the historical development of the quadratic metric–affine theory of gravity in e.g. [4, 7, 8, 12, 14, 19, 20, 22, 24, 29, 30, 31, 35, 36, 38].

2. Notation

Our notation follows [13, 21, 22, 23, 32, 35]. We denote local coordinates by x^μ where $\mu = 0, 1, 2, 3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define the covariant derivative of a vector field as $\nabla_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu$. The Christoffel symbol is $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$. We define torsion as

$$(2.1) \qquad T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$$

and contortion as

$$(2.2) \qquad K^\lambda{}_{\mu\nu} = \frac{1}{2} \left(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu \right).$$

Torsion and contortion are also related as

$$(2.3) \qquad T^\lambda{}_{\mu\nu} = K^\lambda{}_{\mu\nu} - K^\lambda{}_{\nu\mu}.$$

The irreducible pieces of torsion are, following [32],

$$(2.4) \qquad T^{(1)} = T - T^{(2)} - T^{(3)}, \quad T^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu, \quad T^{(3)} = *w,$$

where

$$(2.5) \qquad v_\nu = \frac{1}{3} T^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{6} \sqrt{|\det g|} T^{\kappa\lambda\mu} \varepsilon_{\kappa\lambda\mu\nu}.$$

The pieces $T^{(1)}, T^{(2)}$ i $T^{(3)}$ are called *tensor torsion*, *trace torsion*, and *axial torsion* respectively. Substituting formulae (2.4) into formula (2.2), and formula (2.3) into formulae

(2.5) we obtain the irreducible decomposition of contortion:

$$(2.6) \qquad K^{(1)} = K - K^{(2)} - K^{(3)}, \quad K^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\nu\mu}v_\lambda, \quad K^{(3)} = \frac{1}{2} * w,$$

where $v_\nu = \frac{1}{3} K^\lambda{}_{\lambda\nu}$, $w_\nu = \frac{1}{3} \sqrt{|\det g|} K^{\kappa\lambda\mu} \varepsilon_{\kappa\lambda\mu\nu}$. The irreducible pieces of torsion (2.4) and contortion (2.6) are related as $T^{(i)}{}_{\kappa\lambda\mu} = K^{(i)}{}_{\lambda\kappa\mu}$ $(i = 1, 2)$, $T^{(3)}{}_{\kappa\lambda\mu} = 2K^{(3)}{}_{\kappa\lambda\mu}$. We define curvature as $R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta}\Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta}\Gamma^\eta{}_{\mu\lambda}$, Ricci curvature as $Ric_{\lambda\nu} = R^\kappa{}_{\lambda\kappa\nu}$, scalar curvature as $\mathcal{R} = Ric^\kappa{}_\kappa$ and trace-free Ricci curvature as $\mathcal{R}ic = Ric - \frac{1}{4}\mathcal{R}g$. We denote Weyl curvature by $\mathcal{W}$ which is understood as the irreducible piece of curvature defined by conditions $R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}$, $\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0$ and $Ric = 0$. We employ the standard convention of raising and lowering tensor indices by means of the metric tensor. We define the action of the Hodge star on a rank q antisymmetric tensor as $(*Q)_{\mu_{q+1}\ldots\mu_4} := (q!)^{-1} \sqrt{|\det g|}\, Q^{\mu_1\ldots\mu_q} \varepsilon_{\mu_1\ldots\mu_4}$, where ε is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$. When we apply the Hodge star to curvature we have a choice between acting either on the first or the second pair of indices, so we introduce two different Hodge stars: the left Hodge star $({}^*R)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|}\, R^{\kappa'\lambda'}{}_{\mu\nu}\, \varepsilon_{\kappa'\lambda'\kappa\lambda}$ and the right Hodge star $(R^*)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|}\, R_{\kappa\lambda}{}^{\mu'\nu'}\, \varepsilon_{\mu'\nu'\mu\nu}$. Given a scalar function $f : M \to R$ we write for brevity

$$\int f := \int f \sqrt{|\det g|} \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \quad \det g := \det(g_{\mu\nu}).$$

3. Explicit representation of the field equations

We write down explicitly our field equations (1.2), (1.3) under the following assumptions:

(i) our spacetime is metric compatible;
(ii) curvature has symmetries $R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}$ and $\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0$;
(iii) scalar curvature is zero.

The main result of this paper is the following.

Theorem 3.1. *Under the assumptions (i)-(iii) the field equations (1.2) and (1.3) become*

$$(3.1) \qquad d_1 \mathcal{W}^{\kappa\lambda\mu\nu} Ric_{\kappa\mu} + d_3 \left(Ric^{\lambda\kappa} Ric_\kappa{}^\nu - \frac{1}{4} g^{\lambda\nu} Ric_{\kappa\mu} Ric^{\kappa\mu} \right) = 0$$

$$(3.2) \quad d_6 \nabla_\lambda Ric_{\kappa\mu} - d_7 \nabla_\kappa Ric_{\lambda\mu}$$

$$+ d_6 \left(Ric_\kappa{}^\eta (K_{\mu\eta\lambda} - K_{\mu\lambda\eta}) + \frac{1}{2} g_{\lambda\mu} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} (K^\xi{}_{\eta\zeta} - K^\xi{}_{\zeta\eta}) + \frac{1}{2} g_{\mu\lambda} Ric_\xi{}^\eta K^\xi{}_{\eta\kappa} \right.$$

$$\left. + g_{\mu\lambda} Ric_\kappa{}^\eta K^\xi{}_{\xi\eta} - K^\xi{}_{\xi\lambda} Ric_{\kappa\mu} + \frac{1}{2} g_{\mu\lambda} Ric_\kappa{}^\xi (K^\eta{}_{\xi\eta} - K^\eta{}_{\eta\xi}) \right)$$

$$- d_7 \left(Ric_\lambda{}^\eta (K_{\mu\eta\kappa} - K_{\mu\kappa\eta}) + \frac{1}{2} g_{\kappa\mu} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} (K^\xi{}_{\eta\zeta} - K^\xi{}_{\zeta\eta}) + \frac{1}{2} g_{\mu\kappa} Ric_\xi{}^\eta K^\xi{}_{\eta\lambda} \right.$$

$$\left. - g_{\kappa\mu} Ric_\lambda{}^\eta K^\xi{}_{\xi\eta} - K^\xi{}_{\xi\kappa} Ric_{\lambda\mu} + \frac{1}{2} g_{\mu\kappa} Ric_\lambda{}^\xi (K^\eta{}_{\xi\eta} - K^\eta{}_{\eta\xi}) \right)$$

$$+ b_{10} \left(g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} (K^\xi{}_{\zeta\eta} - K^\xi{}_{\eta\zeta}) + g_{\mu\kappa} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} (K^\xi{}_{\eta\zeta} - K^\xi{}_{\zeta\eta}) \right.$$

$$+ g_{\mu\lambda} Ric_\kappa{}^\xi (K^\eta{}_{\eta\xi} - K^\eta{}_{\xi\eta}) + g_{\mu\kappa} Ric_\lambda{}^\xi (K^\eta{}_{\xi\eta} - K^\eta{}_{\eta\xi})$$

$$\left. + g_{\kappa\mu} Ric_\lambda{}^\eta K^\xi{}_{\xi\eta} - g_{\lambda\mu} Ric_\kappa{}^\eta K^\xi{}_{\xi\eta} + Ric_{\mu\kappa} K^\eta{}_{\lambda\eta} - Ric_{\mu\lambda} K^\eta{}_{\kappa\eta} \right)$$

$$+ 2b_{10} \left(\mathcal{W}^\eta{}_{\mu\kappa\xi} (K^\xi{}_{\eta\lambda} - K^\xi{}_{\lambda\eta}) + \mathcal{W}^\eta{}_{\mu\lambda\xi} (K^\xi{}_{\kappa\eta} - K^\xi{}_{\eta\kappa}) \right.$$

$$\left. - \mathcal{W}^{\eta\xi}{}_{\kappa\lambda} K_{\mu\xi\eta} - \mathcal{W}^\eta{}_{\mu\lambda\kappa} K^\xi{}_{\xi\eta} \right) = 0,$$

where

$$d_1 = b_{912} - b_{922} + b_{10}, \quad d_3 = b_{922} - b_{911},$$

$$d_6 = b_{912} - b_{911} + b_{10}, \quad d_7 = b_{912} - b_{922} + b_{10},$$

the b's being coefficients defined in [20, 23].

Remark 3.1. *Note that, by definition, we have the curvature symmetry* $R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}$, *and the symmetry* $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$ *is a consequence of metric compatibility.*

Proof. The LHS of equations (3.1) and (3.2) are respectively the components of tensors A and B from the formula

$$\delta S = \int (2A^{\lambda\nu} \delta g_{\lambda\nu} + 2B^{\kappa\mu}{}_\lambda \delta \Gamma^\lambda{}_{\mu\kappa}).$$

Here δg and $\delta \Gamma$ are the independent variations of the metric and the connection, and δS is the resulting variation of the action. In deriving explicit formulae for tensors A and B we simplified our calculations by adopting the following argument. Formula for the quadratic form (1.4) can be, under the assumptions (i)– (iii), rewritten as

$$q(R) = \sum_{l,m=1}^2 b_{9lm}(\mathcal{S}^{(l)}, \mathcal{S}^{(m)}) + b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + \cdots$$

$$= \sum_{l,m=1}^2 b_{9lm}(\mathcal{R}ic^{(l)}, \mathcal{R}ic^{(m)}) + b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + \cdots,$$

where by $\cdots$ we denote terms which do not contribute to δS when we start our variation using the assumptions (i)–(iii). In accordance with the convention of [35], put

$$P_- := \frac{1}{2}(\mathcal{R}ic^{(1)} - \mathcal{R}ic^{(2)}),$$
$$P_+ := \frac{1}{2}(\mathcal{R}ic^{(1)} + \mathcal{R}ic^{(2)}) = \frac{1}{2}(Ric^{(1)} + Ric^{(2)}).$$

Note that in a metric compatible spacetime $\mathcal{R}ic^{(2)} = -\mathcal{R}ic^{(1)}$, hence $P_+ = 0$ and $P_- = \mathcal{R}ic$. Our quadratic form can now be rewritten as

$$(3.3) \qquad q(R) = b_{10}(R^{(10)}, R^{(10)})_{\text{YM}} + (b_{911} - 2b_{912} + b_{922})(P_-, P_-)$$
$$+ 2(b_{911} - b_{922})(P_-, P_+) + \dots.$$

We also provide another version of this formula which is in accordance with the notation of [32], where most of these terms were studied in detail. The equation (3.3) can be rewritten as

$$(3.4) \qquad q(R) = c_1(R^{(1)}, R^{(1)})_{\text{YM}} + c_3(R^{(3)}, R^{(3)})_{\text{YM}} + 2(b_{911} - b_{922})(P_-, P_+) + \dots$$

where

$$(3.5) \qquad c_1 = -\frac{1}{2}(b_{911} - 2b_{912} + b_{922}), \qquad c_3 = b_{10},$$

and the $R^{(j)}$s are the irreducible pieces of curvature labeled in accordance with [32].

Variation with respect to the connection. The variation of $\int (R^{(j)}, R^{(j)})_{YM}$ was computed in [32]:

$$(3.6) \qquad \int (R^{(j)}, R^{(j)})_{YM} = 4 \int \left((\delta_{YM} R^{(j)})^\mu (\delta\Gamma)_\mu \right)$$

where

$$(\delta_{\text{YM}} R)^\mu := \frac{1}{\sqrt{|\det g|}} \left(\partial_\nu + [\Gamma_\nu, \cdot] \right) \left(\sqrt{|\det g|}\, R^{\mu\nu} \right)$$

is the Yang–Mills divergence where we hide the Lie algebra indices of curvature by using matrix notation

$$(3.7) \qquad [\Gamma_\xi, R_{\mu\nu}]^\kappa{}_\lambda = \Gamma^\kappa{}_{\xi\eta} R^\eta{}_{\lambda\mu\nu} - R^\kappa{}_{\eta\mu\nu} \Gamma^\eta{}_{\xi\lambda}.$$

In our case, because of assumptions (i)–(iii), the curvature has only two irreducible pieces, namely $R^{(1)}$ i $R^{(3)} = \mathcal{W}$, which can be written as

$$R^{(1)}{}_{\kappa\lambda\mu\nu} = \frac{1}{2}(g_{\kappa\mu} Ric_{\lambda\nu} - g_{\lambda\mu} Ric_{\kappa\nu} - g_{\kappa\nu} Ric_{\lambda\mu} + g_{\lambda\nu} Ric_{\kappa\mu}),$$
$$R^{(3)}{}_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} - R^{(1)}{}_{\kappa\lambda\mu\nu}.$$

with the other $R^{(j)}$s being zero. Substituting these expressions into (3.6) we get

$$(3.8) \qquad \delta \int (R^{(1)}, R^{(1)})_{YM} = 2 \int (\delta\Gamma^{\lambda\mu\kappa}) \Big[\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + g_{\kappa\mu} \nabla_\xi Ric_\lambda{}^\xi$$
$$- g_{\mu\lambda} \nabla_\xi Ric_\kappa{}^\xi + Ric_\kappa{}^\eta (K_{\mu\eta\lambda} - K_{\mu\lambda\eta}) + Ric_\lambda{}^\eta (K_{\mu\kappa\eta} - K_{\mu\eta\kappa})$$
$$+ g_{\mu\lambda} K^\xi{}_{\xi\eta} Ric_\kappa{}^\eta - K^\eta{}_{\eta\lambda} Ric_{\kappa\mu} + K^\xi{}_{\xi\kappa} Ric_{\lambda\mu} - g_{\kappa\mu} K^\xi{}_{\xi\eta} Ric_\lambda{}^\eta \Big]$$

and

$$(3.9) \qquad \delta \int (R^{(3)}, R^{(3)})_{YM} = 4 \int (\delta \Gamma^{\lambda \mu \kappa})(\nabla_\nu \mathcal{W}_{\kappa \lambda \mu}{}^\nu - K_{\mu \nu \eta} \mathcal{W}_{\kappa \lambda}{}^{\eta \nu} - K^\xi{}_{\xi \nu} \mathcal{W}_{\kappa \lambda \mu}{}^\nu).$$

Further,

$$(3.10) \qquad \delta \int (P_-, P_+) = -\frac{1}{2} \int (\delta \Gamma^{\lambda \mu \kappa}) \left[\nabla_\lambda Ric_{\kappa \mu} + \nabla_\kappa Ric_{\lambda \mu} - g_{\mu \lambda} \nabla_\xi Ric_\kappa{}^\xi \right.$$
$$- g_{\kappa \mu} \nabla_\xi Ric_\lambda{}^\xi + Ric_\kappa{}^\eta (K_{\mu \eta \lambda} - K_{\mu \lambda \eta}) + Ric_\lambda{}^\eta (K_{\mu \eta \kappa} - K_{\mu \kappa \eta})$$
$$\left. + K^\xi{}_{\xi \eta}(g_{\mu \lambda} Ric_\kappa{}^\eta + g_{\kappa \mu} Ric_\lambda{}^\eta) - K^\xi{}_{\xi \lambda} Ric_{\kappa \mu} - K^\xi{}_{\xi \kappa} Ric_{\lambda \mu} \right].$$

Combining formulae (3.4), (3.5), (3.8)–(3.10), we arrive at the explicit form of the field equation (3.2):

$$(3.11) \qquad d_6' \nabla_\lambda Ric_{\kappa \mu} - d_7' \nabla_\kappa Ric_{\lambda \mu} +$$
$$d_6' \left(-g_{\mu \lambda} \nabla_\eta Ric_\kappa{}^\eta + Ric_\kappa{}^\eta (K_{\mu \eta \lambda} - K_{\mu \lambda \eta}) + g_{\mu \lambda} K^\xi{}_{\xi \eta} Ric_\kappa{}^\eta - K^\xi{}_{\xi \lambda} Ric_{\kappa \mu} \right) -$$
$$d_7' \left(-g_{\kappa \mu} \nabla_\eta Ric_\lambda{}^\eta + Ric_\lambda{}^\eta (K_{\mu \eta \kappa} - K_{\mu \kappa \eta}) + g_{\mu \kappa} K^\xi{}_{\xi \eta} Ric_\lambda{}^\eta - K^\xi{}_{\xi \kappa} Ric_{\lambda \mu} \right) +$$
$$2 b_{10} \left(\nabla_\eta \mathcal{W}^\eta{}_{\mu \lambda \kappa} - K_{\mu \xi \eta} \mathcal{W}^{\eta \xi}{}_{\kappa \lambda} + K^\xi{}_{\xi \eta} \mathcal{W}^\eta{}_{\mu \kappa \lambda} \right) = 0,$$

where

$$d_6' = b_{912} - b_{911}, \quad d_7' = b_{912} - b_{922}.$$

Let us use the Bianchi identity for curvature

$$(\partial_\xi + [\Gamma_\xi, \cdot]) R_{\mu \nu} + (\partial_\nu + [\Gamma_\nu, \cdot]) R_{\xi \mu} + (\partial_\mu + [\Gamma_\mu, \cdot]) R_{\nu \xi} = 0,$$

where we hide the Lie algebra indices of curvature by using formula (3.7). Using our assumptions (i)–(iii) and making one contraction of indices, we get

$$(3.12) \qquad \nabla_\xi Ric_{\lambda \nu} - \nabla_\nu Ric_{\lambda \xi} + g_{\lambda \xi} \nabla_\mu Ric^\mu{}_\nu - g_{\lambda \nu} \nabla_\mu Ric^\mu{}_\xi +$$
$$Ric^\mu{}_\eta (g_{\lambda \xi} K^\eta{}_{\mu \nu} - g_{\lambda \nu} K^\eta{}_{\mu \xi}) + Ric^\mu{}_\xi (K_{\nu \mu \lambda} - K_{\mu \nu \lambda}) +$$
$$Ric^\mu{}_\nu (K_{\mu \xi \lambda} - K_{\xi \mu \lambda}) + Ric_{\lambda \nu}(K^\mu{}_{\xi \mu} - K^\mu{}_{\mu \xi}) + Ric_{\lambda \xi}(K^\mu{}_{\mu \nu} - K^\mu{}_{\nu \mu}) +$$
$$2 \left(\nabla_\mu \mathcal{W}^\mu{}_{\lambda \nu \xi} + \mathcal{W}^\mu{}_{\lambda \xi \eta}(K^\eta{}_{\nu \mu} - K^\eta{}_{\mu \nu}) + \mathcal{W}^\mu{}_{\lambda \nu \eta}(K^\eta{}_{\mu \xi} - K^\eta{}_{\xi \mu}) \right) = 0.$$

Another contraction in (3.12) yields

$$(3.13) \qquad \nabla_\eta Ric^\eta{}_\kappa = -\frac{1}{2} Ric^\eta{}_\xi K^\xi{}_{\eta \kappa} - \frac{1}{2} Ric^\xi{}_\kappa (K^\eta{}_{\xi \eta} - K^\eta{}_{\eta \xi}) - \frac{1}{2} \mathcal{W}^{\eta \zeta}{}_{\kappa \xi}(K^\xi{}_{\eta \zeta} - K^\xi{}_{\zeta \eta}).$$

Substitution of (3.13) into (3.12) gives

$$(3.14) \qquad \nabla_\eta \mathcal{W}^\eta{}_{\mu \lambda \kappa} = \mathcal{W}^\eta{}_{\mu \kappa \xi}(K^\xi{}_{\eta \lambda} - K^\xi{}_{\lambda \eta}) + \mathcal{W}^\eta{}_{\mu \lambda \xi}(K^\xi{}_{\kappa \eta} - K^\xi{}_{\eta \kappa})$$
$$+ \frac{1}{4}(K^\xi{}_{\zeta \eta} - K^\xi{}_{\eta \zeta})(g_{\mu \lambda} \mathcal{W}^{\eta \zeta}{}_{\kappa \xi} - g_{\mu \kappa} \mathcal{W}^{\eta \zeta}{}_{\lambda \xi})$$
$$+ \frac{1}{2} \left[\nabla_\lambda Ric_{\mu \kappa} - \nabla_\kappa Ric_{\mu \lambda} + Ric^\eta{}_\kappa (K_{\eta \lambda \mu} - K_{\lambda \eta \mu}) \right.$$
$$\left. + Ric^\eta{}_\lambda (K_{\kappa \eta \mu} - K_{\eta \kappa \mu}) + Ric_{\mu \lambda}(K^\eta{}_{\eta \kappa} - K^\eta{}_{\kappa \eta}) + Ric_{\mu \kappa}(K^\eta{}_{\lambda \eta} - K^\eta{}_{\eta \lambda}) \right]$$
$$+ \frac{1}{4} Ric^\eta{}_\xi (g_{\mu \lambda} K^\xi{}_{\eta \kappa} - g_{\mu \kappa} K^\xi{}_{\eta \lambda}) + \frac{1}{4}(K^\eta{}_{\eta \xi} - K^\eta{}_{\xi \eta})(g_{\mu \lambda} Ric^\xi{}_\kappa - g_{\mu \kappa} Ric^\xi{}_\lambda).$$

Formulae (3.13) and (3.14) allow us to exclude the terms with $\nabla_\eta Ric_\kappa{}^\eta$, $\nabla_\eta Ric_\lambda{}^\eta$ i $\nabla_\eta \mathcal{W}^\eta{}_{\mu\lambda\kappa}$ from equation (3.11), reducing the latter to (3.2).

Variation with respect to the metric. The field equation (3.1) is identical to the one in the Riemannian case as given in [35], only with the scalar curvature being zero, which is not surprising as the assumptions on the torsion do not influence the form of the equation. Here we present briefly the derivation of equation (3.1). A lengthy but straightforward calculation shows that

$$(3.15) \qquad \delta \int \left(R^{(1)}, R^{(1)} \right)_{YM} = -2 \int \mathcal{W}^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} \delta g_{\alpha\beta}.$$

$$(3.16) \qquad \delta \int \left(R^{(3)}, R^{(3)} \right)_{YM} = -2 \int \mathcal{W}^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} \delta g_{\alpha\beta}.$$

Further,

$$(3.17) \quad \delta \int (P_-, P_+) = \int (Ric, \delta P_+) = \frac{1}{2} \int (Ric, \delta Ric) + \frac{1}{2} \int (Ric, \delta Ric^{(2)}) =$$
$$-\frac{1}{4} \int \left(4 Ric^{\kappa\alpha} Ric_\kappa{}^\beta + 2\mathcal{W}^{\kappa\alpha\beta\nu} Ric_{\kappa\nu} - g^{\alpha\beta} Ric_{\mu\nu} Ric^{\mu\nu} \right) \delta g_{\alpha\beta}.$$

Combining formulae (3.4) - (3.17) we arrive at the explicit form of the field equation (3.1).
This ends the proof of Theorem 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 3.2. *If we assume that torsion is purely tensor, in addition to our assumptions (i)-(iii), the field equations (3.1), (3.2) reduce to those presented in [20, 23], with the correction presented in Appendix C of [22].*

Remark 3.3. *If we assume that the torsion is purely axial, in addition to our assumptions (i)-(iii), the field equations (3.1), (3.2) reduce to*

$$(3.18) \qquad d_1 \mathcal{W}^{\kappa\lambda\mu\nu} Ric_{\kappa\mu} + d_3 \left(Ric^{\lambda\kappa} Ric_\kappa{}^\nu - \frac{1}{4} g^{\lambda\nu} Ric_{\kappa\mu} Ric^{\kappa\mu} \right) = 0,$$

$$(3.19) \qquad d_6 \left(\nabla_\lambda Ric_{\kappa\mu} + Ric_\kappa{}^\eta T_{\mu\eta\lambda} \right) - d_7 \left(\nabla_\kappa Ric_{\lambda\mu} + Ric_\lambda{}^\eta T_{\mu\eta\kappa} \right)$$
$$+ 2b_{10} \left(\mathcal{W}^\eta{}_{\mu\kappa\xi} T^\xi{}_{\eta\lambda} + \mathcal{W}^\eta{}_{\mu\lambda\xi} T^\xi{}_{\kappa\eta} - \frac{1}{2} \mathcal{W}^{\eta\xi}{}_{\kappa\lambda} T_{\mu\xi\eta} \right) = 0.$$

where the d's are the same as given in Theorem 3.1 and the b's are the same as given in [20, 23].

Remark 3.4. *An effective technique for writing down the field equations explicitly can be found in [9, 10]. Namely, according to formulae (142), (143) of [9], our system of field equations reads*

$$(3.20) \qquad e_\alpha \rfloor V - (e_\alpha \rfloor R_\beta{}^\gamma \wedge \frac{\partial V}{\partial R_\beta{}^\gamma}) \quad = \quad 0,$$

$$(3.21) \qquad D \frac{\partial V}{\partial R_\alpha{}^\beta} \quad = \quad 0.$$

Here the notation is anholonomic, $V := *q(R)$ is the Lagrangian, e_α is the frame and D is the covariant exterior differential, $\rfloor$ is the interior product and the exterior product is $\wedge$. Equation (3.21) is the explicit form of equation (1.3), but equations

(3.20) and (1.2) are somewhat different: the difference is that (3.20) is the result of variation with respect to the frame rather than the metric. It is known, however, that the systems (1.2), (1.3) and (3.20), (3.21) are equivalent.

4. Discussion

A comprehensive study of equations (1.2), (1.3) was done only relatively recently. Vassiliev [35] solved the problem of existence and uniqueness for Riemannian solutions (see Definition 1.1). He showed that the Riemannian solutions of the equations (1.2), (1.3) are Einstein spaces, pp-waves with parallel Ricci curvature and Riemannian spacetimes which have zero scalar curvature and are locally a product of Einstein 2–manifolds. Furthermore, in the same paper [35] Vassiliev showed that the above spacetimes are the *only* Riemannian solutions of the system of field equations (1.2), (1.3). It is also interesting that before [35] it had not been noticed that pp-waves were solutions of the problem, although they were well known spacetimes in theoretical physics. Because of the uniqueness result we can now only establish new non-Riemannian solutions of the system (1.2), (1.3).

In [35] Vassiliev also presented one non-Riemannian solution of the system (1.2), (1.3) and it was a *torsion wave* solution with explicitly given torsion. For the Yang–Mills case (1.5) this torsion wave solution was first obtained by Singh and Griffiths: see last paragraph of Section 5 in [28] and the same solution was later independently rediscovered by King and Vassiliev in [13]. It should be pointed out that the torsion wave solution of King and Vassiliev is a highly specialised version of the solution obtained by Singh and Griffiths [28], which is a solution of algebraic type III, where the Riemannian spacetime is a Kundt plane-fronted gravitational wave and the torsion is purely tensor. Vassiliev's contribution in [35] was to show that these spacetimes satisfy equations (1.2), (1.3) in the most general case of the purely quadratic action (1.1). This work of Vassiliev went on to conclude that this torsion wave was a non-Riemannian analogue of a *pp-wave*, whence came the motivation for generalising the notion a classical Riemannian pp-wave to spacetimes with torsion in such a way as to incorporate the non-Riemannian torsion-wave solution into the construction.

PP-waves are well known spacetimes in general relativity, first discovered by Brinkmann [3] in 1923, and subsequently rediscovered by several authors, for example Peres [25] in 1959. We define a *pp-wave* as a Riemannian spacetime which admits a nonvanishing parallel spinor field, or equivalently as a Riemannian spacetime whose metric can be written locally in the form $ds^2 = 2 \, dx^0 \, dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3) \, (dx^3)^2$ in some local coordinates (x^0, x^1, x^2, x^3). In our previous work [20, 21, 22, 23], where a detailed description of pp-waves can be found, we presented results which were new (non-Riemannian) explicit vacuum solutions of the system of our field equations (1.2), (1.3), namely generalised pp-waves with torsion. This generalisation was done by employing the pp-metric and giving an explicit torsion, identical to the torsion-wave obtained by Vassiliev in [35]. The fact that the two solutions, one Riemannian and the other non-Riemannian, 'add up' is extremely non-trivial, as the system we are observing is highly non-linear. We further explored the properties and characteristics of these generalised pp-waves, showing that

they are indeed solutions of the system of our field equations (1.2), (1.3), by writing the field equations explicitly, like in this paper, but with the additional assumption of torsion being purely tensor, which simplifies matters substantially. Our analysis of vacuum solutions of QMAG showed that classical pp-spaces of parallel Ricci curvature should not be viewed on their own, but that they are in fact a particular (degenerate) representative of a wider class of solutions, namely, generalised pp-spaces of parallel Ricci curvature. The latter appear to admit a sensible physical interpretation, which we explored in detail in [22] where we gave a comparison with the classical model describing the interaction of gravitational and massless neutrino fields, namely Einstein–Weyl theory, constructed pp-wave type solutions of this theory and pointed out that generalised pp-waves of parallel Ricci curvature are very similar to pp-wave type solutions of the Einstein–Weyl model. Therefore we proposed that our generalised pp-waves of parallel Ricci curvature represent a metric-affine (i.e. conformally invariant) model for a massless particle, namely the *massless neutrino*. The main difference in using our metric–afine model is that Einstein–Maxwell and Einstein–Weyl theories contain the gravitational constant which dictates a particular relationship between the strengths of the fields in question, whereas our model is conformally invariant and the amplitudes of the two curvatures (i.e. torsion generated and metric generated curvatures) are totally independent.

The main idea of the current paper is to empower us to find new pp-wave type non-Riemannian solutions with *arbitrary* (as opposed to purely tensor) torsion. Note that the assumptions (i)-(iii) used to derive our equations are automatically satisfied by pp-waves and their generalisation, so we are justified in using them.

The observation that one can construct vacuum solutions of QMAG in terms of pp-waves is a recent development. The fact that classical pp-waves of parallel Ricci curvature are solutions was first pointed out in [33, 34, 35]. There are a number of publications in which authors suggested various generalisations of the concept of a classical pp-wave. These generalisations were performed within the Riemannian setting and usually involved the incorporation of a constant non-zero scalar curvature; see [17] and extensive further references therein. Our construction in [20, 21, 22, 23] generalised the concept of a classical pp-wave in a different direction: we added torsion while retaining zero scalar curvature. Note that we keep this assumption in the current paper.

A powerful method which in the past has been used for the construction of vacuum solutions of QMAG is the so-called *double duality ansatz* [1, 2, 14, 15, 32, 35]. For certain types of quadratic actions the following is known to be true: if the spacetime is metric compatible and curvature is irreducible (i.e. all irreducible pieces except one are identically zero) then this spacetime is a solution of (1.2), (1.3). This fact is referred to as the double duality ansatz because the proof is based on the use of the double duality transform $R \mapsto {}^*R^*$ (this idea is due to Mielke [14]) and because the above conditions imply ${}^*R^* = \pm R$. However, solutions presented in [20, 21, 22, 23] and the ones we hope arise from our current work do not fit into the double duality scheme. This is due to reasons that can be found explained in detail in [20, 23]. These solutions are similar to those of Singh and Griffiths [28]. The main differences are as follows:

- The solutions in [28] satisfy the condition $\{Ric\} = 0$ whereas our solutions satisfy the weaker condition $\{\nabla\}\{Ric\} = 0$.

- The solutions in [28] were obtained for the Yang–Mills case (1.5) whereas we deal with a general O(1, 3)-invariant quadratic form q with 16 coupling constants.

One interesting generalisation of the concept of a pp-wave was presented by Obukhov in [18]. Obukhov's motivation comes from his previous work [17] which is the Riemannian case. In fact, the ansatz for the metric and the coframe of [18] is exactly the same as in the Riemannian case. However, the connection extends the Levi-Civita connection is such a way that torsion and *nonmetricity* ($\nabla g \neq 0$) are present, and are determined by this extension of the connection. Obukhov studies the same general quadratic Lagrangian with 16 terms, and the result of [18] does not belong to the triplet ansatz, see [11, 16]. Obukhov's gravitational wave solutions have only two non-trivial pieces of curvature. However, unlike in our setting, the two non-zero pieces of curvature in [18] are equivalent to the pieces of curvature coming from the 10-dimensional $\mathbf{R}^{(10)}$ and the 30-dimensional $\mathbf{R}^{(30)}$ irreducible curvature subspaces. Hence the main differences between our work and Obukhov's generalisation of [18] are the following:

- In Obukhov's plane-fronted waves not only are the torsion waves present, but the non-metricity has a non-trivial wave behaviour as well. As we are only looking at metric–compatible spacetimes, nonmetricity cannot appear in our construction.
- The second ($\mathbf{R}^{(30)}$) irreducible piece of curvature cannot appear in our ansatz, as this piece of curvature is zero for metric-compatible spacetimes.
- Obukhov's gravitational wave solutions provide a minimal generalisation of the pseudoinstanton, see [32], in the sense that nonmetricity does not vanish and that curvature has two non-zero pieces.

In relation to our goal of finding new solutions of QMAG, the two papers of Singh [26, 27] are of special interest to us. Singh constructs solutions for the Yang–Mills case (1.5) with purely axial and purely trace torsion respectively and unlike the solution of [28], $\{Ric\}$ is not assumed to be zero. It is obvious that these solutions differ from the ones presented in [23], as the torsion there is assumed to be purely tensor and the torsion wave produced curvature is purely Weyl, i.e. $\{Ric\} = 0$. It would however be of interest to us to see whether this construction of Singh's can be expanded to our most general O(1, 3)-invariant quadratic form q with 16 coupling constants. In [26] Singh presents solutions of the field equations (1.2), (1.3) for the Yang–Mills case (1.5) for a purely *axial* torsion. The make a class of solutions that cannot be obtained using the double duality ansatz, see [1, 2, 14, 15, 32, 35]. In fact, Singh uses the 'spin coefficient technique' from his previous work with Griffiths [28] in constructing the new solutions. In view of the fact that the previous purely tensor solutions in [28] were shown to also be the solutions in the most general case (1.4), we expect that this is also true in the purely axial case. Therefore, similarly to our previously found purely tensor torsion waves, we suggest the following

Conjecture 4.1. *There exist purely axial torsion waves which are solutions of the field equations (1.2), (1.3).*

We should point out that the explicit forms (3.1), (3.2) of our fields equations (1.2), (1.3) given in Section 3 were obtained without any a priori assumptions on torsion. Hence, under the assumption of purely axial torsion, the field equations would be substantially simplified, as given in equations (3.18), (3.19) in Remark 3.3.

Following the reasoning behind the generalised pp-waves of [23] that were shown to be solutions of the field equations (1.2), (1.3), where we 'combined' the pp-metric and the purely tensor torsion waves to obtain a new class of solutions for QMAG, we hope to be able to do the same with purely axial torsion waves. Therefore, we suggest the following

Conjecture 4.2. *There exists a class of spacetimes equipped with the pp-metric and explicitly given purely axial torsion of parallel Ricci curvature that satisfies the field equations (1.2), (1.3).*

Similarly, in [27] Singh presents solutions of the field equations (1.2), (1.3) for the Yang–Mills case (1.5) for a purely *trace* torsion. At this point we are still not sure whether these torsion waves can also be used to create new generalised pp-wave solutions of QMAG, but we hope to be able to answer this question as well, together with proving the two conjectures above, which would accomplish the main purpose of the current article – to make it simpler for researchers to find and confirm new solutions of metric–affine gravity. The next step would then be to give a physical interpretation of these new solutions by comparing them to existing Riemannian solutions, like it was done in [22] for purely tensor torsion generalised pp-waves, which would represent a very valuable scientific contribution in the field of alternative theories of gravity.

References

[1] P. Baekler, F. W. Hehl, E. W. Mielke: *Vacuum solutions with double duality properties of a quadratic Poincaré gauge field theory*, Proceedings of the Second Marcel Grossmann Meeting on General Relativity edited by Ruffini R (Amsterdam: North–Holland Publishing Company) (1982), 413–453.

[2] P. Baekler, F. W. Hehl, E. W. Mielke: *Nonmetricity and torsion: facts and fancies in gauge approaches to gravity*, Proceedings of the Fourth Marcel Grossmann Meeting on General Relativity edited by Ruffini R (Amsterdam: Elsevier Science Publishers B.V.) (1986), 277–316.

[3] M. W. Brinkmann: *On Riemann spaces conformal to Euclidean space*, Proceedings of the National Academy of Sciences of USA **9** (1923), 1-3.

[4] H. A. Buchdahl: Mathematical Reviews **20** (1959), 1238.

[5] A. Einstein: *Zur Elektrodynamik bewegter Körper*, Annalen der Physik **17** (1905), 891–921.

[6] A. Einstein: *The Meaning of Relativity : Including the Relativistic Theory of the Non–Symmetric Field - With a new Foreword by W.H. McCrea, F.R.S.* Chapman and Hall Science Paperbacks, 1967.

[7] E. E. Fairchild Jr.: *Gauge theory of gravitation*, Phys. Rev. D **14** (1976), 384–391.

[8] E. E. Fairchild Jr.: *Erratum: Gauge theory of gravitation*, Phys. Rev. D **14** (1976), 2833.

[9] F. Gronwald, F. W. Hehl: *On the gauge aspects of gravity*, In: Proc. of the 14th Course of the School of Cosmology and Gravitation on 'Quantum gravity' (Erice, Italy 1995), World Scientific, Singapore 1996, pp.148-198. [arXiv:gr-qc/9602013]

[10] F. W. Hehl, J. D. McCrea, E. W. Mielke, Y. Ne'eman: *Metric–affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance*, Phys. Rep. **258** (1995), 1–171. [arXiv:gr-qc/9402012]

[11] F. W. Hehl, A. Macías: *Metric–affine gauge theory of gravity II. Exact solutions*, Int. J. Mod. Phys. **D8** (1999), 399–416.

[12] P. W. Higgs: *Quadratic Lagrangians and general relativity*, Nuovo Cimento **11** (1959), 816–820.

[13] A. D. King, D. Vassiliev: *Torsion waves in metric–affine field theory*, Class. Quantum Grav. **18** (2001), 2317–2329. [arXiv:gr-qc/0012046]

[14] E. W. Mielke: *On pseudoparticle solutions in Yang's theory of gravity*, Gen. Rel. Grav. **13** (1981), 175–187.

[15] E. W. Mielke: *Duality in Yang's theory of gravity*, Gen. Rel. Grav. **37** (2005), 997–1007.

[16] Yu. N. Obukhov, E. J. Vlachynsky, W. Esser, F. W. Hehl: *Effective Einstein theory from metric–affine gravity models via irreducible decompositions*, Phys. Rev. D **56** (1997), 7769–7778.

[17] Yu. N. Obukhov: *Generalized plane–fronted gravitational waves in any dimension*, Phys. Rev. D **69** (2004), 024013 [7 pages].

[18] Yu. N. Obukhov: *Plane waves in metric–affine gravity*, Phys. Rev. D **73** (2006), 024025 [6 pages].

[19] P. Olesen: *A relation between the Einstein and the Yang–Mills field equations*, Phys. Lett. B **71** (1977), 189–190.

[20] V. Pasic: *New Vacuum Solutions for Quadratic Metric–Affine Gravity*, PhD Doctoral thesis, University of Bath (2009).

[21] V. Pasic: *New Vacuum Solutions for Quadratic Metric–Affine Gravity – a Metric Affine Model for the Massless Neutrino?*, Mathematica Balkanica Vol. 24 (2010), 329–340.

[22] V. Pasic, E. Barakovic: *PP–waves with Torsion - a Metric-affine Model for the Massless Neutrino*, submitted to Gen. Rel. Grav. (2014) [arXiv:gr-qc/1406.1930]

[23] V. Pasic, D. Vassiliev: *PP–waves with torsion and metric– affine gravity*, Class. Quantum Grav. 22 (2005), 3961–3975. [arXiv:gr-qc/0505157]

[24] R. Pavelle: *Unphysical solutions of Yang's gravitational–field equations*, Phys. Rev. Lett. **34** (1975), 1114.

[25] A. Peres: *Some gravitational waves*, Phys. Rev. Lett. **3** (1959), 571–572.

[26] P. Singh: *On axial vector torsion in vacuum quadratic Poincaré gauge field theory*, Phys. Let. **145A** (1990), 7–10.

[27] P. Singh: *On null tratorial torsion in vacuum quadratic Poincaré gauge field theory*, Class. Quantum Grav. **7** (1990), 2125–2130.

[28] P. Singh, J. B. Griffiths: *A new class of exact solutions of the vacuum quadratic Poincaré gauge field theory*, Gen. Rel. Grav. **22** (1990), 947–956.

[29] G. Stephenson: *Quadratic Lagrangians and general relativity*, Nuovo Cimento **9** (1958), 263–269.

[30] A. H. Thompson: *Yang's gravitational field equations*, Phys. Rev. Lett. **34** (1975), 507–508.

[31] A. H. Thompson: *Geometrically degenerate solutions of the Kilmister–Yang equations*, Phys. Rev. Lett. **35** (1975), 320–322.

[32] D. Vassiliev: *Pseudoinstantons in metric-affine field theory*, Gen. Rel. Grav 34 (2002), 1239–1265. [arXiv:gr-qc/0108028]

[33] D. Vassiliev: *Pseudoinstantons in metric–affine field theory*, Quark Confinement and the Hadron Spectrum V, edited by Brambilla, N. and Prosperi, G. M. (Singapore: World Scientific) (2003), 273–275.

[34] D. Vassiliev: *Quadratic non–Riemannian gravity*, Journal of Nonlinear Mathematical Physics **11**, Supplement, (2004) 204–216.

[35] D. Vassiliev: *Quadratic metric-affine gravity*, Ann. Phys. (Lpz.) 14 (2005), 231–252. [arXiv:gr-qc/0304028]

[36] F. Wilczek: *Geometry and Interaction of Instantons. Quark Confinement and Field theory*, eds. D. R. Stump and D. H. Weingarten, Wiley-Interscience, New York (1977), 211–219.

[37] H. Weyl: *Eine neue Erweiterung der Relativitätstheorie*, Ann. Phys. (Lpz.) 59 (1919), 101–133.

[38] C. N. Yang: *Integral Formalism for Gauge Fields*, Phys. Rev. Lett. **33** 59 (1974), 445–447.

Department of Mathematics
University of Tuzla
Bosnia and Herzegovina
E-mail address: vedad.pasic@untz.ba

Department of Mathematics
University of Tuzla
Bosnia and Herzegovina
E-mail address: elvis.barakovic@untz.ba

Department of Mathematics
University of Tuzla
Bosnia and Herzegovina
E-mail address: nermin.okicic@untz.ba