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# ON TWO FORMULAS CONTIGUOUS TO A TRANSFORMATION DUE TO BAILEY

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ABSTRACT. The aim of this short research note is to provide two interesting formulas contiguous to the following transformation due to Bailey (written here in **corrected** form) viz.

$$(1-x)^{-1}\,{}_2F_1\left[\begin{matrix} 1, & b; \\ & & -\frac{4x}{(1-x)^2} \\ 2b+\frac{1}{2}; & \end{matrix}\right]$$

$$= {}_2F_1\left[\begin{matrix} 1, & \frac{3}{4}-b; \\ & & x^2 \\ \frac{3}{4}+b; & \end{matrix}\right] - \frac{x(4b-1)}{(4b+1)}\,{}_2F_1\left[\begin{matrix} 1, & \frac{5}{4}-b; \\ & & x^2 \\ \frac{5}{4}+b; & \end{matrix}\right]$$

The results are derived with the help of two formulas contiguous to classical Whipple's summation theorem on the sum of a ${}_3F_2$ obtained earlier by Lavoie, et al.

The results established in this short research note are simple, interesting, easily established and may are useful.

## 1. INTRODUCTION

In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role. Several formulae were given by Gauss and Kummer expressing the product of two hypergeometric series, such as $e^{-x}\,{}_1F_1(x)$ as a series of the type ${}_1F_1(-x)$ and

$$(1+x)^{-p}\,{}_2F_1\left(\frac{4x}{(1+x)^2}\right)$$

as a series of the type ${}_2F_1(x)$. By employing the above mentioned classical summation theorems, in 1928, Bailey [1] made a systematic search for several formulae. Evidently if the product of two hypergeometric series can be expressed as a hypergeometric series with argument $x$, the coefficient of $x^n$ in the product must be expressible in terms of the Gamma function.

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Recently good progress has been done in the direction of generalizing and extending the above mentioned classical summation theorems. For this, we refer to the research papers [2] - [5].

In our present investigation, we are interested in the following transformation due to Bailey [1]

$$(1-x)^{-1}\,{}_2F_1\left[\begin{matrix} 1, & b; & \\ & & -\frac{4x}{(1-x)^2} \\ 2b+\frac{1}{2}; & & \end{matrix}\right]$$

$$= {}_2F_1\left[\begin{matrix} 1, & \frac{3}{4}-b; & \\ & & x^2 \\ \frac{3}{4}+b; & & \end{matrix}\right] - \frac{x(4b-1)}{(4b+1)}\,{}_2F_1\left[\begin{matrix} 1, & \frac{5}{4}-b; & \\ & & x^2 \\ \frac{5}{4}+b; & & \end{matrix}\right]. \tag{1.1}$$

Bailey [1] obtained this result by employing the following classical Whipple's summation theorem [8] viz

$${}_3F_2\left[\begin{matrix} a, & b, & c; & \\ & & & 1 \\ & e, & f; & \end{matrix}\right]$$

$$= \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}f\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}f\right)} \tag{1.2}$$

provided $a+b=1$ and $e+f=1+2c$.

In 1996, Lavoie, et al. [5] generalized the above mentioned classical Whipple's summation theorem (1.2) and have obtained explicit expressions of

$${}_3F_2\left[\begin{matrix} a, & b, & c; & \\ & & & 1 \\ & e, & f; & \end{matrix}\right] \tag{1.3}$$

provided $a+b=1+i+j$ and $e+f=2c+i+1$ for $i,j=0,\pm1,\pm2,\pm3$.

In our present investigation, we shall require the following results which can be obtained from (1.3) by taking $i=1, j=-1$ and $i=-1, j=1$ respectively, these are

- For $i=1, j=-1$

$${}_3F_2\left[\begin{matrix} a, & b, & c; & \\ & & & 1 \\ & e, & f; & \end{matrix}\right] = \frac{\Gamma(e)\Gamma(f)\Gamma(e-c-1)}{2^{2a}\Gamma(e-a)\Gamma(f-a)\Gamma(e-c)}$$

$$\times\left\{\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{1}{2}\right)\Gamma\left(\frac{f}{2}+\frac{a}{2}\right)} - \frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}\right)\Gamma\left(\frac{f}{2}+\frac{a}{2}-\frac{1}{2}\right)}\right\} \tag{1.4}$$

provided $a+b=1, e+f=2c+2$.

- For $i=-1, j=1$

$$
{}_3F_2\left[\begin{matrix} a, & b, & c; & \\ & & & 1 \\ & e, & f; & \end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(c-1)}{2^{2a}\Gamma(e-a)\,\Gamma(f-a)\,\Gamma(c)}
$$

$$
\times\left\{\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{1}{2}\right)\Gamma\left(\frac{f}{2}+\frac{a}{2}\right)}+\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}\right)\Gamma\left(\frac{f}{2}+\frac{a}{2}-\frac{1}{2}\right)}\right\} \tag{1.5}
$$

provided $a+b=1, e+f=2c$.

The aim of this short research note is to find the explicit expressions of

$$
(1-x)^{-1}\,{}_2F_1\left[\begin{matrix} 1, & b; & \\ & & -\frac{4x}{(1-x)^2} \\ 2b+\frac{1}{2}+l; & & \end{matrix}\right]
$$

for $l=\pm 1$.

The results are derived with the help of contiguous Whipple's summation theorems (1.4) and (1.5). The results established in this short research note are simple interesting, easily established and may be useful.

## 2. Main Results

The transformation formulas closely related to the Bailey's transformation to be established in this short research note are given by

$$
(1-x)^{-1}\,{}_2F_1\left[\begin{matrix} 1, & b; & \\ & & -\frac{4x}{(1-x)^2} \\ 2b+\frac{3}{2}; & & \end{matrix}\right]
$$

$$
= \frac{1}{2(2b+1)}\,{}_3F_2\left[\begin{matrix} 1, & \frac{5}{4}, & \frac{1}{4}-b; & \\ & & & x^2 \\ \frac{1}{4}, & \frac{5}{4}+b; & & \end{matrix}\right] + \frac{(4b+1)}{2(2b+1)}\,{}_2F_1\left[\begin{matrix} 1, & \frac{3}{4}-b; & \\ & & x^2 \\ \frac{3}{4}+b; & & \end{matrix}\right]
$$

$$
+ \frac{3(4b+1)x}{2(2b+1)(4b+3)}\,{}_3F_2\left[\begin{matrix} 1, & \frac{7}{4}, & \frac{3}{4}-b; & \\ & & & x^2 \\ \frac{3}{4}, & \frac{7}{4}+b; & & \end{matrix}\right] - \frac{(4b-1)x}{2(2b+1)}\,{}_2F_1\left[\begin{matrix} 1, & \frac{5}{4}-b; & \\ & & x^2 \\ \frac{5}{4}+b; & & \end{matrix}\right] \tag{2.1}
$$

and

$$(1-x)^{-1}\,{}_2F_1\left[\begin{array}{cc} 1, & b; \\ & & -\frac{4x}{(1-x)^2} \\ 2b-\frac{1}{2}; & \end{array}\right]$$

$$= \frac{1}{4(1-b)}\,{}_3F_2\left[\begin{array}{ccc} 1, & \frac{5}{4}, & \frac{5}{4}-b; \\ & & & x^2 \\ \frac{1}{4}, & b+\frac{1}{4}; & \end{array}\right] - \frac{(4b+3)}{4(1-b)}\,{}_2F_1\left[\begin{array}{cc} 1, & \frac{7}{4}-b; \\ & & x^2 \\ b-\frac{1}{4}; & \end{array}\right]$$

(2.2)

$$+\frac{3(4b-3)x}{4(1-b)(4b-1)}\,{}_3F_2\left[\begin{array}{ccc} 1, & \frac{7}{4}, & \frac{7}{4}-b; \\ & & & x^2 \\ \frac{3}{4}, & \frac{3}{4}+b; & \end{array}\right] + \frac{(4b-5)x}{4(1-b)}\,{}_2F_1\left[\begin{array}{cc} 1, & \frac{9}{4}-b; \\ & & x^2 \\ b+\frac{1}{4}; & \end{array}\right]$$

*Proof.* In order to derive (2.1), we proceed as follows. Denoting the left-hand side of (2.1) by $S$, we have

$$S = (1-x)^{-1}\,{}_2F_1\left[\begin{array}{cc} 1, & b; \\ & & -\frac{4x}{(1-x)^2} \\ 2b+\frac{3}{2}; & \end{array}\right].$$

Expressing ${}_2F_1$ as a series, we have after some algebra

$$S = \sum_{n=0}^{\infty} \frac{(b)_n(-1)^n 2^{2n}}{\left(2b+\frac{3}{2}\right)_n} x^n (1-x)^{-(2n+1)},$$

using Binomial theorem, we have

$$S = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(b)_n(-1)^n 2^{2n}}{\left(2b+\frac{3}{2}\right)_n} \frac{(2n+1)_m}{m!} x^{n+m}.$$

Now, replacing $m$ by $m-n$ and using the known result [6, Lemma 10, p. 56 Eq. (1)]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

we have

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(b)_n(-1)^n 2^{2n}}{\left(2b+\frac{3}{2}\right)_n} \frac{(2n+1)_{m-n}}{(m-n)!} x^m,$$

using the elementary identities [6]

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n},$$

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n}$$

and

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}},$$

we have, after some simplification

$$S = \sum_{m=0}^{\infty} x^m \sum_{n=0}^{m} \frac{(-m)_n(1+m)_n(b)_n}{\left(\frac{1}{2}\right)_n \left(2b+\frac{3}{2}\right)_n n!}.$$

Summing up the inner series, we have

$$S = \sum_{m=0}^{\infty} x^m \, {}_3F_2 \left[ \begin{array}{cccc} -m, & 1+m, & b; & \\ & & & 1 \\ \frac{1}{2}, & 2b+\frac{3}{2}; & & \end{array} \right].$$

Now, separating into even and odd powers of $x$, we have

$$\begin{aligned} S = &\sum_{m=0}^{\infty} x^{2m} \, {}_3F_2 \left[ \begin{array}{cccc} -2m, & 1+2m, & b; & \\ & & & 1 \\ \frac{1}{2}, & 2b+\frac{3}{2}; & & \end{array} \right] \\ &+ x \sum_{m=0}^{\infty} x^{2m} \, {}_3F_2 \left[ \begin{array}{cccc} -2m-1, & 2m+2, & b; & \\ & & & 1 \\ \frac{1}{2}, & 2b+\frac{3}{2}; & & \end{array} \right]. \end{aligned}$$

Now assume that

$$S = A + B, \tag{2.3}$$

then we have

$$A = \sum_{m=0}^{\infty} x^{2m} \, {}_3F_2 \left[ \begin{array}{cccc} -2m, & 1+2m, & b; & \\ & & & 1 \\ \frac{1}{2}, & 2b+\frac{3}{2}; & & \end{array} \right].$$

It is now, easy to see that, the ${}_3F_2$ can be evaluated with the help of the contiguous Whipple's summation theorem (1.4) by taking $a = -2m, b = 1+2m, c = b, e = \frac{1}{2}$ and $f = 2b + \frac{3}{2}$ and after some simplification, we get

$$\begin{aligned} A = &\frac{1}{2(2b+1)} \sum_{m=0}^{\infty} \frac{(1)_m \left(\frac{5}{4}\right)_m \left(\frac{1}{4}-b\right)_m}{\left(\frac{1}{4}\right)_m \left(\frac{5}{4}+b\right)_m m!} x^{2m} \\ &+ \frac{(4b+1)}{2(2b+1)} \sum_{m=0}^{\infty} \frac{(1)_m \left(\frac{3}{4}-b\right)_m}{\left(\frac{3}{4}+b\right)_m m!} x^{2m} \end{aligned}$$

summing up the series, we have

$$\begin{aligned} A = &\frac{1}{2(2b+1)} \, {}_3F_2 \left[ \begin{array}{cccc} 1, & \frac{5}{4}, & \frac{1}{4}-b; & \\ & & & x^2 \\ \frac{1}{4}, & b+\frac{5}{4}; & & \end{array} \right] \\ &+ \frac{(4b+1)}{2(2b+1)} \, {}_2F_1 \left[ \begin{array}{ccc} 1, & \frac{3}{4}-b; & \\ & & x^2 \\ \frac{3}{4}+b; & & \end{array} \right]. \end{aligned} \tag{2.4}$$

In exactly the same manner, we can show that

$$\begin{aligned} B = &\frac{3(4b+1)x}{2(2b+1)(4b+3)} \, {}_3F_2 \left[ \begin{array}{cccc} 1, & \frac{7}{4}, & \frac{3}{4}-b; & \\ & & & x^2 \\ \frac{3}{4}, & \frac{7}{4}+b; & & \end{array} \right] \\ &- \frac{(4b-1)x}{2(2b+1)} \, {}_2F_1 \left[ \begin{array}{ccc} 1, & \frac{5}{4}-b; & \\ & & x^2 \\ \frac{5}{4}+b; & & \end{array} \right]. \end{aligned} \tag{2.5}$$

Substituting the results of $A$ and $B$ from (2.4) and (2.5) in (2.3), we easily arrive at the right-hand side of our first main result (2.1). This completes the proof of (2.1).

In exactly the same manner, the result (2.2) can be established by employing the contiguous Whipple's summation theorem (1.5), so we prefer to omit the details. □

Clearly the results (2.1) and (2.2) are closely related to the Bailey's transformation (1.1).

## Authors' Contributions

The authors have equal contributions to each part of this paper. All authors have read and approved the final manuscript.

## Competing Interests

The authors declare that they have no competing interests.

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