Extremes and ruin of Gaussian processes

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Abstract

For certain Gaussian processes \(X(t)\) with trend \(\text{−}ct^\beta\) and variance \(V^2(t)\) we discuss maxima and ruin probabilities as well as the ruin time. The ruin time is defined as the first time point \(t\) such that \(X(t)\) \text{−}ct^\beta \geq u\) where \(u\) stands typically for the initial capital. The ruin time is of interest in finance and actuarial subjects. But the ruin time is also of interest in other applications e.g. in telecommunications or storage models where it indicates the first time of an overflow. We deal with some asymptotic distributions of maxima, of ruin probability and of the ruin time as \(u \to \infty\). The limiting distributions are dependent on the parameters \(\beta\), \(V(t)\) and the correlation function of \(X(t)\).

Key Words: Maxima, ruin, time to ruin, fractional Brownian motion, Gaussian portfolio processes, locally stationary Gaussian processes, insurance, telecommunications, storage.

1 Introduction

In risk analysis of insurance or finance, ruin is the most important event because it should be avoided. Typically the model starts with a random process which consists of the claim sizes \(Y_k\) occurring at random times \(T_k\). One classically assumes in the Cramér-Lundberg model that the claim sizes are iid positive random variables with a finite mean \(\mu\), and that the inter-arrival times \(T_k - T_{k-1}\) are iid random variables, being independent of the claim sizes. The claim times \(T_k\) define a homogeneous Poisson process with a positive intensity \(\lambda\). Starting with an initial capital \(u\), the corresponding risk process is then defined by \(U(t) = u + ct - \sum_{k:T_k \leq t} Y_k\). The sum in the risk process represents the total claim amount until the time \(t\) and the positive \(c\) is the premium income rate. Now ruin occurs if at some finite time \(t\) we observe \(U(t) < 0\). For large capitals \(u\) one can find asymptotically the probability of ruin \(P\{U(t) < 0\text{ for some }t < \infty\}\), (see e.g. Embrechts et al. (1997)). Classically one assumes that the events of claims follows a

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homogeneous Poisson process and the claim sizes are general iid random variables, which gives the compound Poisson model. For such a risk process the probability of ruin, the ruin time and some related quantities were investigated e.g. by Dufresne and Gerber (1988), Dickson and Egídio dos Reis (1994) and Schmidli (1999). In the non-classical model, adding a Brownian motion \( B(t) \) in the risk process, i.e. considering \( U(t) = u + ct + \sigma B(t) - \sum_{k:T_k \leq t} Y_k, \sigma > 0 \), ruin probability can implicitly be given as solution of an integro-differential equation (Dufresne and Gerber (1991), Chiu and Yin (2003)). In some of these cases the behaviour of the ruin probability can be derived if \( u \) tends to \( \infty \) (e.g. Schmidli (1999)). Also moments of the ruin times were derived for the classical risk process, see Egídio dos Reis (2000).

If we consider a large company with many customers or similar in other applications, the random walk \( \sum_{k:T_k \leq t} Y_k \) with random times is replaced by another random process \( X(t) \), as e.g. a Brownian motion, a Lévy process or a renewal process. The same problem of risk or ruin occurs in other applications also. For instance in storage models, the ruin event indicates the overflow of storage. So other processes \( X(t) \) are of interest in such models, e.g. again the Brownian motion, but also fractional Brownian motions, integrated Gaussian processes (see e.g. Hüsler and Piterbarg (1999), Dębicki (2002), Dieker (2005), Hüsler and Piterbarg (2004), (2005)). Considering a portfolio, the portfolio process \( X(t) \) is a weighted sum of the processes \( X_i(t) \) modelling the individual underlying risk processes. Of interest is not only the probability of ruin or overflow, but again the time of (first) ruin or overflow of such a risk model.

We are going to discuss some of these results related to Gaussian processes in this paper. We consider the ruin event and the time of ruin for different classes of Gaussian processes, as e.g. fractional Brownian motion, integrated fractional Brownian motion or more general locally stationary Gaussian processes and the portfolio process mentioned above.

2 Ruin based on particular Gaussian processes

2.1 Ruin based on fractional Brownian motions

The fractional Brownian motions \( X_H(t) \) with \( 0 < H < 1 \) extend the Brownian motion. They are also Gaussian processes with mean 0, but with
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variance $V^2(t) = t^{2H}$ and covariance function

$$E(X_H(t)X_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

If $H = 1/2$ this process is the well-known Brownian motion with independent increments, a property which is not shared by the fractional Brownian motions with $H \neq 1/2$. The ruin problem for such processes was dealt with by Narayan (1998). By a different method Hüsler and Piterbarg (1999) derived for more general Gaussian processes the asymptotic behaviour of the ruin probability as $u \to \infty$. They showed (Corollary 2 in Hüsler and Piterbarg (1999)) for the fractional Brownian motion that for $\beta > H$ and $H < 1$

$$P\{X(t) > u + ct^{\beta} \text{ for some } t > 0\} \sim Cu^{(1-H/\beta)(1-H)/H} \tilde{\Phi}(Au^{-H/\beta})$$

with $\tilde{\Phi}(x) = 1 - \Phi(x)$ where $\Phi(x)$ denotes the normal distribution function, and some constant $C > 0$ which depends on $H, c$ and $\beta$:

$$C = \frac{H^2 \sqrt{\pi} A^{(2-H)/2H}}{\sqrt{B} 2^{(1-H)/2H} s_0}.$$

$H_\alpha$ denotes Pickands constant: $H_\alpha = \lim_{T \to \infty} E \exp\{\max_{0 \leq t \leq T} U(t)\}/T$ where $U(t)$ is a transformed fractional Brownian motion with drift: $U(t) = \sqrt{2}X_H(t) - t^{\alpha}$. The constants $A, B$ and $s_0$ are defined by

$$A = \frac{\beta}{\beta - H} \left( \frac{H}{c(\beta - H)} \right)^{-H/\beta}, \quad B = H \beta \left( \frac{H}{c(\beta - H)} \right)^{-(H+2)/\beta}$$

and

$$s_0 = \left( \frac{H}{c(\beta - H)} \right)^{1/\beta}.$$ 

Here $s_0$ denotes the point of minimal boundary value of the transformed ruin event.

$$\{\exists t > 0 : X(t) > u + ct^{\beta}\} = \{\exists s > 0 : X(u^{1/\beta}s)/(u^{H/\beta}(1 + cs^{\beta})) > u^{1-H/\beta}\}$$

$$= \{\exists s > 0 : X(u^{1/\beta}s)/(u^{H/\beta} s^H) > u^{1-H/\beta}(1 + cs^{\beta})/s^H\}$$

Note that the Gaussian process $X(u^{1/\beta}s)/(u^{H/\beta} s^H)$ has a constant variance 1. The boundary function in this last event is $u^{1-H/\beta}v(s)$ with
for any $u > 0$ motion in the neighborhood of $s$. Important for this result is the behavior of the fractional Gaussian process. The derivation of the result (1) depends on extreme value theory of Gaussian processes. Hüsler and Piterbarg (1999) analyzed the same results for more general Gaussian processes with mean 0 and variance $\variance{V}$ of this process is given by

$$v(s) = s^{-H} + cs^{3-H}.$$  
Here $s_0$ denotes the minimum point of $v(s) : s_0 = \arg\min s v(s)$ which has the strongest influence on this crossing probability. We may argue also in the other way, by noting that the process $X(u^{1/\beta}/s)/(u^{H/\beta}(1 + cs^3))$ has largest variance at $s_0$, since the variance of this process is given by $v^{-2}(s)$. In the neighborhood of this point an exceedance of the constant level $u^{1-H/\beta}$ is most probable where $A = v(s_0)$ and $B = v''(s_0)$.

The case $H = 1$ defines a degenerated Gaussian process $X(t) = tZ$ with $Z \sim N(0, 1)$. Then for $\beta > 1$ we get immediately that

$$P\{\exists t > 0 : X_H(t) > u + ct^\beta\} = P\{Z > (u + ct_0^\beta)/t_0\} = \Phi(Au^{1-1/\beta})$$

for any $u > 0$ with $t_0 = s_0u^{1/\beta} = \left[\frac{u}{c(\beta - 1)}\right]^{1/\beta}$ and $A = c^{1/\beta}(\beta - 1)^{-1+1/\beta}$.

### 2.2 Ruin based on a Gaussian process similar to a fractional Brownian motion

The derivation of the result (1) depends on extreme value theory of Gaussian processes. Important for this result is the behavior of the fractional motion in the neighborhood of $s_0u^{1/\beta}$. Hüsler and Piterbarg (1999) analyzed the same results for more general Gaussian processes with mean 0 and variance $V^2(t) = t^{2H}$, which are locally stationary in the following way. We consider as in Section 2.1 the transformed Gaussian process

$$X^{(u)}(s) = X(su^{1/\beta})u^{-H/\beta}(1 + cs^3)^{-1}.$$  

Its variance is $v^{-2}(s)$ with $v(s) = s^{-H} + cs^{3-H}$. Hence, $s_0$ denotes the minimum point of $v$ with $A = v(s_0)$ and $B = v''(s_0)$. We consider the class of standardized Gaussian processes $X^{(u)}(s)v(s)$ which are locally stationary in the interval $I_u = (s_0 - \delta(u), s_0 + \delta(u))$ for some $\delta(u)$, e.g. $\delta(u) = u^{H/\beta - 1} \log u$ is suitable for our considerations. A standardized Gaussian process $Y(s)$ is called locally stationary with index $\alpha$ in an interval $I$ if there exist a monotone continuous function $K^2(\cdot)$, regular varying (at 0) with parameter $\alpha \in (0, 2)$, and a (bounded) continuous function $D(t)$, such that

$$\lim_{s \to 0} \frac{E[Y(t + s) - Y(t)]^2}{K^2(|s|)} = D(t) > 0$$

uniformly for $t, t + s \in I$. Since the interval $I = I_u$ shrinks to $s_0$, $D(t) \to D(s_0) = D$ as $u \to \infty$. Hence the inverse function $K^{-1}(y) = \inf\{s : K(s) \geq y\}$ is also regularly varying at 0, but with index $2/\alpha$.  

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Outside of this interval we restrict the covariance by the weak assumption
\[
\limsup_{u \to \infty} E(X^{(u)}(s) - X^{(u)}(t))^2 \leq G|s-t|^\gamma
\]  
(2.4) for all \(s, t > 0\) and for some \(G, \gamma > 0\).

**Theorem 2.1.** (Hüsler and Piterbarg (1999)) Let \(X(t), t \geq 0\), be a Gaussian process with mean zero and variance \(t^{2H}\) and \(c, \beta > 0\) with \(H < \beta\). Assume the conditions (2.3) with \(\alpha < 2\), and (2.4), then as \(u \to \infty\)
\[
P\{X(t) > u + ct^\beta\ \text{ for some } t > 0\} \sim \frac{(D/2)^{2/\alpha} A^{2/\alpha - 3/2} H_c e^{-\frac{1}{2} A^2 u^{2-2H/\beta}}}{\sqrt{B} K^{-1}(u^{-1+H/\beta}) u^{2-2H/\beta}}.
\]  
(2.5)

The proof considers the probabilities of a crossing in the interval \(I_u\) and outside of this interval. It can be shown that the probability of a crossing outside is much smaller than inside \(I_u\). The derivation of the probability of a crossing in the interval \(I_u\) depends on the approximation of the standardized locally stationary process \(X^{(u)}(s)\) by locally stationary (or even stationary) Gaussian processes, being not dependent on \(u\). This is possible by Slepian’s lemma. Furthermore, the crossing probability is then determined by the general result of Bräker (1993, 1995) for locally stationary Gaussian processes which extends the similar result for stationary Gaussian processes by Cuzick (1981). Since the interval is rather small, shrinking to \(s_0\), we can apply Cuzick’s result also, approximating \(X^{(u)}(s)\) by stationary Gaussian processes in \(I_u\).

If one investigates this proof carefully, one notes that we may restrict the interval \(I_u\) further. This allows even to find a limiting distribution for the first time of ruin \(\tau_u\) which is defined by
\[
\tau_u = \min\{t : X(t) > u + ct^\beta\}.
\]
If the set is empty, let \(\tau_u = \infty\). The above result (2.5) gives the asymptotic probability that \(\tau_u < \infty\). Now to deal with the event
\[
\tau_u < u^{1/\beta}(s_0 + xb(u))
\]
for some normalization \(b(u)\), we have to consider crossings in the interval
\[
I_u(x) = (s_0 - \delta(u), s_0 + xb(u)).
\]
The boundary function of the crossing event of the standardized Gaussian process $X^{(u)}(s)v(s)$ is locally a smooth quadratic function

$$u^{1-H/\beta}v(s) = u^{1-H/\beta}\{A^2 + (AB + o(1))(s - s_0)^2\}.$$ 

This implies in the derivation that we get the following asymptotic conditional distribution for $\tau_u$ by taking $b(u) = u^{H/\beta - 1}/\sqrt{AB}$.

**Theorem 2.2.** Let $X(t)$, $t \geq 0$, be a Gaussian process with mean zero and variance $t^{2H}$ and $c, \beta > 0$ with $H < \beta$. Assume the conditions (2.3) with $\alpha < 2$, and (2.4), then as $u \to \infty$

$$P\{X(t) > u + ct^{\beta} \text{ for some } t \in (0, u^{1/\beta}(s_0 + b(u)x))\} \sim \frac{(D/2)^{1/\alpha}A^{2/\alpha-3/2}H_\alpha e^{-\frac{1}{2}A^2u^{2-2H/\beta}}}{\sqrt{B}K^{-1}(u^{1-H/\beta}) u^{2-2H/\beta}} \Phi(x),$$

and thus

$$P\{\tau_u < u^{1/\beta}(s_0 + b(u)x)|\tau_u < \infty\} \to \Phi(x).$$

### 2.3 Ruin based on a portfolio of Gaussian processes

In finance a portfolio consists of several assets or in insurance several insurance contract forms or subjects (e.g. age and gender groups, life and non-life) are combined. These partial processes follows different risk models. Relevant for the investment is the sum of these processes and the possibility of a ruin of the whole. The partial processes do not have all the same weight on the portfolio, so a weighted sum process should be analyzed.

Therefore, we consider with some (positive) weights $w_i$ a finite number $k$ of independent Gaussian processes $X_i(t)$ with mean 0, and their weighted sum process

$$X(t) = \sum_{i=1}^{k} w_i X_i(t),$$

being also a Gaussian process with mean 0. We might consider for simplicity that each $X_i(t)$ is a fractional Brownian motion with variance $\text{Var}X_i(t) =$
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\[ d_i t^{2H_i} \text{ with } H_i \in (0, 1) \text{ and } d_i > 0. \] But as shown in Theorem 2.1 this is not a necessary restriction. Thus we assume that the processes \( X_i(t) \) have mean 0 and variances \( d_i t^{2H_i} \) as in Theorem 2.1, and that \( \beta > H_i \) for all \( i \). It is convenient to order the \( H_i \) values. So by renumbering, let the values \( H_i \) be such that \( H_i \geq H_{i+1} \geq \ldots \geq H_k \). Define \( m \geq 1 \) to be the largest index such that \( H_m = H \).

It is convenient to order the \( H_i \) values. So by renumbering, let the values \( H_i \) be such that \( H_1 \geq H_2 \geq \ldots \geq H_k \). Define \( m \geq 1 \) to be the largest index such that \( H_m = H \). Then we use the standardized process \( \tilde{X}(t) \),

\[
\tilde{X}(t) = \frac{X(t)}{\sqrt{\text{Var}(X(t))}} = \frac{\sum_{i=1}^{k} w_i X_i(t)}{\sqrt{\sum_{i=1}^{k} W_i t^{2H_i}}}
\]

with \( W_i = w_i d_i \) for all \( i \leq k \). Define \( W = \sum_{i=m}^{k} W_i \). Again, we transform the time domain as for the other processes above. Defining \( \tilde{X}^{(u)}(s) = \tilde{X}(u^{1/\beta} s/W^{1/2H}) \), the events of a ruin and for the time of first ruin are transformed in the same way using \( t = u^{1/\beta} s/W^{1/2H} \).

\[
\{ \sup_{0 < t \leq W^{-1/2H} u^{1/\beta} (s_u^* + xb(u))} (X(t) - ct^\beta) > u \}
\]

\[
= \left\{ \exists t \leq W^{-1/2H} u^{1/\beta} (s_u^* + xb(u)) : \tilde{X}(t) > (u + ct^\beta)/\sqrt{\sum_{i=1}^{k} W_i t^{2H_i}} \right\}
\]

\[
= \{ \exists s < s_u^* + xb(u) : \tilde{X}^{(u)}(s) > f_u(s) \}
\]

for some \( x \leq \infty \) where \( f_u(s) = u^{1-H/\beta} v(s) (1 + \delta_u(s)) \) with

\[
\delta_u(s) = \left( 1 + \sum_{j > m} W_j W^{-H_j/H} u^{2(H_j-H)} s^{2(H_j-H)} \right)^{-1/2} - 1
\]

and

\[
v(s) = \frac{1 + \tilde{c}s^\beta}{s^{H}}, \text{ with } \tilde{c} = cW^{-\beta/2H}.
\]

Note that \( \delta(s) \in [-1, 0] \) is monotone increasing in \( s \), for fixed \( u \), and that \( \delta_u(s) \to 0 \) as \( u \to \infty \), for any \( s > 0 \). The boundary function \( f_u(s) \) may have several minimum points, the smallest of these points is denoted by \( s_u^* \).

It can be shown that \( s_u^* \leq s_0 \) and that \( s_u^* \to s_0 \) as \( u \to \infty \).

We assume that each of the standardized Gaussian processes \( \tilde{X}_i^{(u)}(s) = X_i^{(u)}(s)/\sqrt{\text{Var}(X_i^{(u)}(s))} \) is locally stationary in a small neighborhood \( S_u = \)}
\((s_u^* - \epsilon(u), s_u^* + \epsilon(u))\) with \(\epsilon(u) = u^{H/\beta - 1} \log u\). The regular varying function \(K^2\) and the function \(D\) in the definition of the locally stationarity depend on the Gaussian process \(X_i\), so denoted by \(K^2_i(\cdot)\) with index \(\alpha_i \in (0, 2)\) and \(D_i(\cdot)\) (see condition (2.3)). Note that the interval \(S_u\) is shrinking to \(s_0\), so we have again by the continuity of \(D_i(\cdot)\) to consider only the value \(D_i = D_i(s_0)\). This assumption implies that the Gaussian process \(\tilde{X}^{(u)}(t)\) after standardization is also locally stationary with a regularly varying function \(K^2(\cdot)\) with index \(\alpha = \min \alpha_i\). It means we have for some \(K^2\) which is one of the \(K^2_i(\cdot)\)'s

\[
\lim_{s \to s'} \sum_{i \leq m} W_i D_i K^2_i(|s - s'|) = \tilde{W}.
\]

Outside the interval \(S_u\) we assume again that each process \(X^{(u)}_i\) of the portfolio satisfies condition (2.4) with some constants \(G_i\) and \(\gamma_i\). So we can derive the following results where we reuse the above defined constants \(A\) and \(B\) by replacing only \(c\) by \(\tilde{c}\), and \(b(u)\).

**Theorem 2.3.** Let \(X_i(t), t > 0, i = 1, \ldots, k,\) be independent continuous Gaussian processes with mean 0 and variance \(d_i t^{2H_i}\), and \(-ct^\beta\) a trend where \(\beta, c, d_i > 0\) and \(0 < H_k \leq \cdots \leq H_1 = H < \min(1, \beta)\). Let \(w_i \in \mathbb{R}\) be some constant weights. Assume for each process \(X_i\) the conditions (2.3) with \(0 < \alpha_i < 2\), and (2.4). Then, the tail behavior is given by

\[
P\left\{ \sup_{0 < t \leq W^{-1/2} u^{1/\beta} (s_u^* + b(u)x)} \left( \sum_{i=1}^{k} w_i X_i(t) - ct^\beta \right) > u \right\} \sim \frac{\left(\frac{W}{\tilde{W}}\right)^{\frac{2}{\beta}}} {A^{\frac{1}{2}} K^{2 - \frac{2}{\beta} - \frac{1}{\alpha}} \Phi(x)},
\]

as \(u \to \infty\), where \(\alpha = \min \alpha_i \leq m\), \(m\) the number of \(H_i = H\), \(f_u(s)\), \(\tilde{W}\), \(b(u)\) are defined above and \(s_u^* = \inf\{\arg\min f_u(s)\}\).

This implies for the conditional distribution of the first ruin time that

\[
P\{\tau_u \leq W^{-1/2} u^{1/\beta} (s_u^* + b(u)x) | \tau_u < \infty\} \to \Phi(x)
\]

as \(u \to \infty\) for any \(x\).
For the ruin probability we set $x = \infty$ in (2.8) and for the derivation of the conditional distribution of the (first) ruin time we use some finite $x$. The ruin probability ($x = \infty$) was derived by Hüsler and Schmid (2006). Its proof follows similarly to the proof of Theorem 2.1, with adjustment for the less smooth boundary function $f_u(s)$. Since this boundary function can be approximated by quadratic functions with smallest value at $s_u^*$, the result is similar again. By variation of the arguments in Hüsler and Schmid (2006) we get the result (2.8) for $x$ finite which implies immediately the asymptotic ruin time distribution.

We note that this result includes the result of Theorem 2.1 and 2.2 with $m = k = 1$, since the fractional Brownian motions satisfies the assumptions of Theorem 2.3. Note that we have for fractional Brownian motions $\alpha_i = 2H_i$ and $K_i^2(h) = |h|^{2H_i}$.

### 2.4 Ruin based on a physical fractional Brownian motion

Let $\xi(t)$, $t \geq 0$, be a stationary, a.s. continuous Gaussian random process with mean zero, variance one and covariance function $r(t)$. We assume that $r(t)$ is regularly varying at infinity with index $-a$, $0 < a < 1$, and write $r(t) \in RV_{-a}$. It means that for any positive $t$,

$$
\lim_{s \to \infty} \frac{r(ts)}{r(s)} = t^{-a}.
$$

(2.9)

The integrated Gaussian process with stationary increments (called Physical Fractional Brownian Motion (PFBM) with linear drift is defined as

$$
X(t) = \int_0^t \xi(s) \, ds - ct,
$$

(2.10)

for a positive $c$. Here we apply a linear drift, using $\beta = 1$. A time transformation helps to change the drift $ct^\beta$ to a linear one. The ruin probability $P(\exists t \geq 0 : X(t) > u)$ is again analyzed as the level $u \to \infty$.

This process $X(t)$ has differentiable sample paths, its smoothness is depending on $a$. For its variance function and the variance of increments we have as $t \to \infty$

$$
\text{Var}(X(t+v) - X(v)) = \text{Var}(X(t)) \sim \frac{2t^2r(t)}{(1-a)(2-a)}.
$$
The simplest case of regularly varying function is \( r(t) \sim t^{-a} \) as \( t \to \infty \), then \( \text{Var}(X(t) - X(s))^2 \sim C(t - s)^{2-a} \) (for some constant \( C > 0 \)), that is the variance of increments behaves at infinity like a fractional Brownian motion with the Hurst parameter \( H = 1 - a/2 \). Therefore we call it a physical fractional Brownian motion. Other integrated Gaussian processes are dealt with by Debicki (2002) and Dieker (2005) for the analysis of ruin probabilities.

Now let \( g = g(x) \) be the minimal root of the equation
\[
g^2 r(gx) = r^2(x),
\]
that is, with \( R(t) := t^2r(t), g(t) = t^{-1}R^{-1}(t^2r(t)^2) \). It can be shown that \( g(t) \in RV_{-a/(2-a)} \) and \( 0 < g < 1 \) for all sufficiently large \( x \). For example, if \( r(x) \sim Cx^{-a} \) as \( x \to \infty \) we have \( g(x) \sim x^{-\frac{a}{2-a}}C^\frac{1}{2-a} \), and if \( r(x) \sim (\log x)^Ax^{-a} \) with \( A \) some constant, then \( g(x) \sim Cx^{-\frac{a}{2-a}} \), where \( C = C(a, A) > 0 \).

In the proof we consider the transformed process
\[
Y_u(s) = \int_0^u \xi(v)dv/(u\sqrt{r(u)(1 + cs)}).
\]
Its variance is \( \sigma^2_u(s) = [2/(1 + cs)^2] \int_0^s (s - v)r(uv)/r(v)dv \) with supremum \( \sigma^2_u \).

**Theorem 2.4.** (Hüsler and Piterbarg (2004)) Let \( X(t) \) be the physical fractional Brownian motion with drift defined by (2.10). Let (2.9) be fulfilled. Then for the ruin probability we have
\[
P(\exists t \geq 0 : X(t) > u) \sim (\sigma^4(1-a)(2-a)A^4/a^2)^{-1/(2-a)} \frac{\sqrt{\pi H_{2-a}\sigma \sqrt{r(u)}}}{\sqrt{Bg(u)}} \times \left( 1 - \Phi \left( \frac{1}{\sqrt{r(u)^2\sigma_u}} \right) \right)
\]
as \( u \to \infty \), where \( g \) is defined by (2.11) and
\[
\sigma^2_u = \lim_{u \to \infty} \sigma^2_u = \frac{(2-a)^{1-a}a^a}{2a^{2-a}(1-a)^a} \quad B = \frac{c^2a^3}{4(2-a)}.
\]

If one analyzes this derivation also more carefully we can get an asymptotic result for the ruin time also. But we are going to present a general result for locally stationary Gaussian processes in the next section.
3 Locally stationary Gaussian processes

The above mentioned examples have certain properties in common which are necessary for deriving the asymptotic results of the ruin probability and the conditional ruin time distribution.

Thus the results can be extended for a more general class of locally stationary Gaussian processes which satisfies rather weak conditions on the correlation function. This result was derived by Hüsler and Piterbarg (2005). Let \( X(t) \) be a Gaussian process with variance \( V^2(t) \), regularly varying at \( \infty \) with index \( 2H \), \( 0 < H < 1 \). Assume \( X(t) \) has a.s. continuous paths. Let \( \beta > H \) and \( c > 0 \). As above we study the time transformed process \( X(u^1/\beta) = X(su^1/\beta)/V(su^1/\beta) \) for \( s > 0 \), with \( t = su^1/\beta \).

Its variance is \( 1/v^2_u(s) \) with \( v_u(s) = v(s)s^H V(su^1/\beta)/V(su^1/\beta) \) which tends uniformly to \( v(s) \) as \( u \to \infty \) for any \( s > \delta > 0 \) with any \( \delta \) small. The dominating time domain is an interval around \( s_u = \arg \min_s v_u(s) \). Again, let \( A(u) = \min v_u(s) = v_u(s_u) \). The function \( v(s) \) has quadratic behaviour in \( s_0 = \lim u s_u = \arg \min_u v_u(s) \).

Theorem 3.1. Let \( X(t), t \geq 0 \), be a Gaussian process with mean 0 and variance \( V^2(t) \), being regularly varying at infinity with index \( 2H \), \( 0 < H < 1 \). Let \( \beta > H \) and \( c > 0 \). Assume the conditions (2.3), (2.4) and (3.1), then

\[
\frac{v_u(s) - A(u)}{(s - s_u)^2} \to B/2
\]

as \( u \to \infty \), uniformly for \( s \) in a neighborhood of \( s_0 \).

We assume the local stationarity (2.3) of the process in a neighborhood of \( s_u \), i.e. in the interval: \( (s_u - \delta(u), s_u + \delta(u)) \) with \( \delta(u) = u^{-1}V(u^1/\beta) \log u \).

\[
P((\tau_u - s_u u^{1/\beta})/\sigma(u) < x \mid \tau_u < \infty) \to \Phi(x)
\]

as \( u \to \infty \), for all \( x \) where \( \sigma(u) := (AB)^{-1/2}u^{-1/\beta}V(u^1/\beta) \).

Obviously also the ruin probability can be derived for this class (for details see Hüsler and Piterbarg (2005)) in an analogous way as the results in Section 2.
References


