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Induced 3-Lie superalgebras and their applications in superspace

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Induced 3-Lie superalgebras and their applications in superspace

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Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia.

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ABSTRACT

The aim of the present thesis is to study the properties and characteristics of n-Lie superalgebras that are constructed using an operation from $(n-1)$-Lie superalgebras, especially in the case $n = 3$. A regular Lie algebra can be extended to super- (or $\mathbb{Z}_2$-graded) structures by introducing the notion of Lie superalgebra. Similarly n-Lie algebra, where binary operation is replcaed with n-ary multiplication law, can be extended to superstructures by making use of a graded Filippov–Jacobi identity, giving a notion of n-Lie superalgebra. In the dissertation a classification of low dimensional 3-Lie superalgebras is presented. We show that an n-Lie superalgebra equipped with a supertrace can be used to construct a $(n+1)$-Lie superalgebra, which is referred to as the induced $(n+1)$-Lie superalgebra. It is proved that one can construct induced 3-Lie superalgebras from commutative superalgebras by using involution, even degree derivation, or combination of both of them together. In the thesis a generalization of Nambu–Hamilton equation to a superspace is proposed, and it is shown that it induces a family of ternary Nambu–Poisson brackets of even degree functions on a superspace. Finally a representations of induced 3-Lie algebras and superalgebras are constructed by means of a representation of the initial binary Lie algebra and trace or Lie superalgebra and supertrace, respectively. It is shown that the constructed induced representation of 3-Lie algebra is a representation by traceless matrices, that is, lies in the Lie algebra $\mathfrak{sl}(V)$, where V is a representation space. For 2-dimensional representations the irreduciblility condition of the induced representation of induced 3-Lie algebra is found.

CONTENTS

LIST OF ORIGINAL PUBLICATIONS

Publications included in the thesis

I V. Abramov, P. Lätt, Classification of Low Dimensional 3-Lie Superalgebras. *Engineering Mathematics II, Springer Proceedings in Mathematics & Statistics*, pp. 1–12, 2016. Springer International Publishing.

II V. Abramov, P. Lätt, Ternary Lie Superalgebras and Nambu-Hamilton Equation in Superspace. *Algebraic Structures and Applications, Springer Proceedings in Mathematics & Statistics*, pp. 47–80, 2020. Springer Nature Switzerland.

III V. Abramov, P. Lätt, Induced 3-Lie Algebras, Superalgebras and Induced Representations. *Proceedings of the Estonian Academy of Sciences (69, 2)*, pp. 116–133, 2020. Estonian Academy Publishers.

Other published work of the author

IV V. Abramov, O. Liivapuu, P. Lätt, Preface to the Special Issue of Advances in Applied Clifford Algebras: the 10th Conference on Clifford Algebras and their Applications in Mathematical Physics. *Advances in Applied Clifford Algebras (27)*, pp. 1–7, 2017. Springer Nature Switzerland.

1. INTRODUCTION

1.1. Lie algebras, their generalizations and applications in geometry and physics

Lie algebras play an important role in modern algebra, differential geometry, theoretical physics and theoretical mechanics. Historically, the foundations of the theory of Lie algebras were laid in the works of Marius Sophus Lie (1842–1899), in which the symmetries of differential equations were studied. The importance of Lie algebras and their applications in various fields of mathematics, physics and mechanics is based on the close relation of these algebras with such a fundamental concept of the world around us as symmetry. It is well known that an algebraic structure such as a group is used to describe symmetries in the language of mathematics. In many physical theories, symmetries form a continuous set, and a concept of continuous group is used to describe those symmetries. A continuous or topological group is a topological space in which a group operation (associative, with identity and inverse element) is defined and this operation is continuous in the topology of this space. If we wish to have applications in theories where differential and integral calculus are used, we must impose stronger conditions on the structure of a continuous group, that is, we must assume the existence of a differentiable structure. A topological space on which a differentiable structure is defined is called a differentiable or smooth manifold. Thus, if instead of a topological space we consider a differentiable manifold on which a group operation is defined and this operation is a differentiable map, then we get a concept of a Lie group. A wide class of non-commutative Lie groups gives us the matrix calculus. Matrices with determinant equal to unity, orthogonal, pseudo-orthogonal, unitary matrices give examples of very important Lie groups.

Generally a Lie group is a nonlinear object, that is, if we embed it by means of a differentiable mapping into a space of higher dimension, then its image will be some surface. The study of nonlinear objects is simplified if we consider the tangent space of a nonlinear object (for example, a surface) at some point on the object. A tangent space is linear or vector space and in this case we can apply powerful linear algebra methods. The tangent space to a Lie group at the point that corresponds to the identity element of a Lie group is the Lie algebra, moreover, the Lie bracket (algebraic operation of a Lie algebra) is induced by a group operation of a Lie group. Thus, one can associate to each Lie group its Lie algebra and this algebra contains important information about a Lie group itself. For example, if we consider a group of special (the determinant is equal to one) unitary matrices, then its Lie algebra is also a matrix Lie algebra and this is the vector space of traceless skew-Hermitian matrices on which a Lie bracket is just the ordinary commutator of two matrices. We see that Lie algebras arise naturally as a result of a linearization of Lie groups, and it should be noted that an important structure that connects a Lie group with its Lie algebra is an exponential map.

However, a Lie algebra, as an algebraic structure, can be considered independently, without its connection with a Lie group. In this case, we can define Lie algebra as follows:

Definition 1.1. Let $\mathfrak{g}$ be a vector space and define bilinear operation

$$\mathfrak{g} \times \mathfrak{g} \ni (u,v) \mapsto [u,v] \in \mathfrak{g}, \tag{1.1}$$

such that for all $u,v,w \in \mathfrak{g}$

1. $[u,v] = -[v,u]$, i.e. $[\cdot,\cdot]$ is skew symmetric,
2. $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$, i.e. $[\cdot,\cdot]$ satisfies Jacobi indentity.

Then pair $(\mathfrak{g},[\cdot,\cdot])$ is said to be a *Lie algebra*, and the mapping $[\cdot,\cdot]$ is called *Lie bracket*.

For what follows, it is important that the Jacobi identity can be written in equivalent form

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]], \tag{1.2}$$

which clearly shows that the Lie bracket is a differentiation of itself.

Lie algebras belong to the category of nonassociative algebras and the Jacobi identity is a kind of replacement for the concept of associativity. Any associative algebra $\mathscr{A}$ becomes Lie algebra, if Lie bracket is defined with the help of commutator $[x,y] = x\cdot y - y\cdot x$, where $x,y \in \mathscr{A}$ and $x\cdot y$ is a multiplication in $\mathscr{A}$. For example, if we take an algebra of nth order square matrices over a field $\mathbb{K}$ and equip it with the commutator of two matrices $[A,B] = AB - BA$, then we get the matrix Lie algebra, usually denoted by $\mathfrak{gl}_n(\mathbb{K})$. In connection with this construction, an important question arises: can any Lie algebra be constructed in this way? The answer is contained in the Ado's theorem [9], which states that any finite-dimensional Lie algebra over a field $\mathbb{K}$ of characteristic 0 is isomorphic to a subalgebra of Lie algebra $\mathfrak{gl}_n(\mathbb{K})$.

An emergence and development of supersymmetric field theories in theoretical physics entailed the development of $\mathbb{Z}_2$-graded structures in mathematics, which are also called, taking the terminology of theoretical physics, superstructures. In the framework of this development, a generalization of the concept of Lie algebra was proposed and this generalization was called a Lie superalgebra. In order to give an idea of this generalization of the concept of Lie algebra, we must first introduce the concept of super vector space. One says that a structure of a super vector space is given on a vector space V if it is shown how V is decomposed into a direct sum of subspaces V_0, V_1, i.e. $V = V_0 \oplus V_1$. In a theory of superstructures, the concept of parity of an element plays an important role, that is, a subset of homogeneous elements (each homogeneous element has a certain parity) is distinguished and a homogeneous element can be either even or odd. In the case of a super vector space V a vector v is a homogeneous vector if it belongs either to V_0 or V_1, and v is even vector (its parity is 0) if $v \in V_0$ and v is odd vector (its parity is 1) if $v \in V_1$. The parity function of homogeneous vector v is usually denoted $|v|$ (however occasionally other expressions, such as $\bar{v}$ and $\hat{v}$ are used too). Hence the parity function takes the values $0, 1$ (or, to be more exact, residue classes modulo 2).

Here it should be noted the connection with supersymmetric field theories, which consists in the fact that all elementary particles known at the moment can be divided into two large classes, these are bosons and fermions. Bosons obey Bose-Einstein statistics and fermions follow Fermi-Dirac statistics. This division of all elementary particles into two classes leads to the fact that $\mathbb{Z}_2$-graded structures are suitable for their mathematical description, in terms of which bosons (or their wave functions) refer to even elements and fermions to odd ones. This approach is in a good agreement with the properties of elementary particles, that is, fermions obey the Pauli exclusion principle (the quantum states of two fermions cannot be identical) and this implies the algebraic properties of their wave functions (with two equal arguments, the function vanishes, and its skew-symmetry follows), which are characteristic for odd elements of a superstructure. It should be noted that supersymmetries also arise in the framework of BRST quantization in gauge field theories [31, 41]. However, in this case, the set of fields that are transformed with the help of supersymmetries includes ghost fields.

Now we can give an idea of how a Lie superalgebra is defined. First of all, a Lie superalgebra $\mathfrak{g}$ is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. This super vector space is endowed with a graded Lie bracket that is graded skew-symmetric and satisfies the graded Jacobi identity. As can be seen from this description, that the only difference with the definition of ordinary Lie algebra is that we add everywhere the word "graded". The meaning of this word can be explained as follows. In superstructures, the general rule of signs applies, which is as follows: if the order of two homogeneous elements is interchanged in a formula (we move one element past another), then this operation must be accompanied by multiplying a corresponding term by minus one to the degree of the product of the parities of corresponding elements. Applying this rule of signs, anyone can write the graded skew-symmetry property and the graded Jacobi identity. All things considered, we can formally define Lie superalgebra as follows.

Definition 1.2. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super vector space with bilinear bracket $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We say that $\mathfrak{g}$ is a *Lie superalgebra* if for all homogeneous elements $u, v, w \in \mathfrak{g}$ the bracket is graded skew-symmetric

$$[u, v] = -(-1)^{|u||v|}[v, u], \tag{1.3}$$

and it satisfies graded Jacobi identity

$$(-1)^{|w||u|}[u,[v,w]] + (-1)^{|u||v|}[v,[w,u]] + (-1)^{|v||w|}[w,[u,v]] = 0. \tag{1.4}$$

As in the case of ordinary Lie algebras, we can construct a large class of matrix Lie superalgebras using an associative algebra of Nth order square matrices. The only thing we need to construct a Lie superalgebra is a structure of a super vector space on a vector space of square matrices. This is easily achieved by fixing a block structure of a matrix, that is, we consider a vector space of Nth order square matrices, where a matrix is divided into blocks as follows

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} + \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}, \tag{1.5}$$

where A_{11} is a nth order square matrix, A_{22} is a mth order matrix, A_{12}, A_{21} are $n \times m$ and $m \times n$ rectangular matrices respectively, $n+m=N$. The right hand side of formula (1.5) clearly shows that any matrix A can be represented as a sum of two matrices, where the first is given the parity 0 (even degree matrix) and the second the parity 1 (odd degree matrix). Hence, we have decomposition into direct sum of two subspaces, or, in other words, we defined the structure of super vector space. Finally the graded commutator $[A,B] = AB - (-1)^{|A||B|}BA$, where the parities $|A|, |B|$ of matrices A, B, give us the structure of Lie superalgebra. This matrix Lie superalgebra will be denoted by $\mathfrak{gl}(n,m)$.

An ideal $\mathfrak{I}$ of a Lie superalgebra $\mathfrak{g}$ is defined by $[\mathfrak{g},\mathfrak{I}] \subset \mathfrak{I}$. A Lie superalgebra is said to be a simple Lie superalgebra if it is non-Abelian and has no non-trivial ideals. The classification of simple finite-dimensional Lie superalgebras was given by V. Kac [42].

Obviously, the ordinary Lie algebra can be considered as a particular case of a Lie superalgebra, that is, Lie superalgebra truly is a generalization of the concept of Lie algebra. Indeed, if the odd subspace $\mathfrak{g}_1$ of a Lie superalgebra $\mathfrak{g}$ consists only of the zero element, we have, in fact, a usual Lie algebra. Since we do not have odd elements, the parity function takes only zero value and the graded skew-symmetry becomes just skew-symmetry, and the graded Jacobi identity becomes the usual Jacobi identity.

It is easy to see that the Lie bracket $[u,v]$ of a Lie algebra $\mathfrak{g}$, where $u, v \in \mathfrak{g}$, contains two arguments u, v, that is, it is a binary algebraic operation. In the 60s of the last century, another generalization of the concept of Lie algebra was proposed. This direction is based on the idea to extend a structure of Lie algebra from binary Lie bracket to n-ary Lie bracket, that is, a Lie bracket that contains n arguments. This direction in the development of the theory of Lie algebras was proposed and investigated in the works of Filippov [32] and his collaborators [28–30,61]. The basic concept of this direction is a notion of n-Lie algebra, where n is an integer greater than or equal to two. Then by saying n-Lie algebra we consider a vector space $\mathfrak{g}$ equipped with an n-ary algebraic operation

$$\mathfrak{g} \times \mathfrak{g} \times \ldots \times \mathfrak{g} \ni (u_1, u_2, \ldots, u_n) \mapsto [u_1, u_2, \ldots, u_n] \in \mathfrak{g}. \tag{1.6}$$

The skew-symmetry of an ordinary binary Lie bracket can easily be extended to an n-ary operation (1.6). To do this, we require that an n-ary operation is totally skew-symmetric, that is, any permutation of the arguments either leaves the result of an n-ary algebraic operation unchanged (even permutation), or multiplies it by -1 (odd permutation). A key to how the Jacobi identity can be extended to an n-ary algebraic operation (1.6) is given by formula (1.2). Indeed, as noted above, formula (1.2) means that a Lie bracket is a differentiation of itself, that is, the double-applied Lie bracket must obey the Leibniz rule. In this form, the Jacobi identity can be easily extended to n-ary operation (1.6), i.e. we simply postulate that double-applied n-ary operation (1.6) obeys the Leibniz differentiation rule. The identity obtined like that has been given several names. In papers, where the applications of n-Lie algebras in classical mechanics and theoretical physics are studied, this identity is called Fundamental Identity. In the paper [25] this identity is called the Filippov-Jacobi identity. In this dissertation, we will also call it the Filippov-Jacobi identity. Formally put, this gies us:

Definition 1.3. Vector space $\mathfrak{g}$ together with n-ary bracket $[\cdot,\dots,\cdot]\colon \mathfrak{g}^n \to \mathfrak{g}$ is said to be an *n-Lie algebra* if the bracket is n-linear, skew-symmetric, that is

$$[u_1,\dots,u_i,u_{i+1},\dots,u_n] = -[u_1,\dots,u_{i+1},u_i,\dots,u_n], \tag{1.7}$$

and it satisfies identity

$$[u_1,\dots,u_{n-1},[v_1,\dots,v_n]] = \sum_{i=1}^{n} [v_1,\dots,v_{i-1},[u_1,\dots,u_{n-1},v_i],v_{i+1},\dots,v_n] \tag{1.8}$$

for all $u_1,u_2,\dots u_{n-1},v_1,v_2,\dots,v_n \in \mathfrak{g}$.

An example of n-Lie algebra can be constructed by means of the vector product of vectors. Let $n \geq 3$ and consider an n-dimensional vector space E^n with an Euclidean metric g. Let $\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n$ be a basis for E^n and $g_{\mu\nu} = g(\mathbf{e}_\mu,\mathbf{e}_\nu)$ be the metric tensor in this basis. One can define the vector product of $n-1$ vectors $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{n-1}$ as

$$[\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{n-1}] = g^{\mu\tau}\varepsilon_{\mu_1,\mu_2,\dots,\mu_{n-1},\tau}\, v_1^{\mu_1}\, v_2^{\mu_2} \dots v_{n-1}^{\mu_{n-1}}\, \mathbf{e}_\mu, \tag{1.9}$$

where $\mathbf{v}_i = v^\mu\,\mathbf{e}_\mu$, $(g^{\mu\tau})$ is the inverse matrix of $(g_{\mu\tau})$ and $\varepsilon_{\mu_1,\mu_2,\dots,\mu_{n-1},\tau}$ is the totally antisymmetric tensor in n-dimensional Euclidean space [63]. Euclidean space E^n endowed with the vector product (1.9) is an $(n-1)$-Lie algebra, i.e. the vector product (1.9) satisfies the Filippov-Jacobi identity. This algebra is called the vector product $(n-1)$-Lie algebra. It can be proved that the vector product $(n-1)$-Lie algebra is simple. Moreover, this $(n-1)$-Lie algebra is the only one simple finite-dimensional $(n-1)$-Lie algebra for $n > 3$ [50].

Interest in n-Lie algebras and their active study is connected with promising applications of these algebras in classical mechanics and theoretical physics. In 1973, Nambu proposed a generalization of Hamiltonian mechanics [57], which is based on the notion of n-ary bracket for functions defined on a phase space. Let E^n be an n-dimensional Euclidean space, whose Cartesian coordinates will be denoted x^μ, $\mu = 1,2,\ldots,n$. Let $F_1,F_2,\ldots,F_n$ be smooth functions on Euclidean space E^n. Define the n-ary bracket as follows

$$\{F_1,F_2,\ldots,F_n\} = \frac{\partial(F_1,F_2,\ldots,F_n)}{\partial(x^1,x^2,\ldots,x^n)} = \mathrm{Det}\begin{pmatrix} \partial_{x^1}F_1 & \partial_{x^2}F_1 & \ldots & \partial_{x^n}F_1 \\ \partial_{x^1}F_2 & \partial_{x^2}F_2 & \ldots & \partial_{x^n}F_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial_{x^1}F_n & \partial_{x^2}F_n & \ldots & \partial_{x^n}F_n \end{pmatrix}. \quad (1.10)$$

This n-ary bracket (1.10) of smooth functions is called n-ary Nambu-Poisson bracket. Evidently, the n-ary Nambu-Poisson bracket is skew-symmetric and it can be proved that it satisfies the Filippov-Jacobi identity and, thus, determines the n-Lie algebra structure on the infinite dimensional space of smooth functions of Euclidean space E^n. Moreover, it is easily verified that the n-ary Nambu-Poisson bracket has the derivation property with respect to product of functions. If the algebra of smooth functions on a finite-dimensional smooth manifold is endowed with an n-ary bracket, which is skew-symmetric, has the derivation property with respect to product of functions, and it satisfies the Filippov-Jacobi identity, then this manifold is called Nambu-Poisson manifold [67].

It should be noted that the n-ary Nambu-Poisson bracket retains all its algebraic properties (skew-symmetry, derivation property, Filippov-Jacobi identity), if we replace in (1.10) the partial derivatives with commuting vector fields, that is

$$\partial_{x^\mu} \to X_\mu,$$

where $X_\mu = X_\mu^\nu\,\partial_{x^\nu}$. This suggests an even more general structure. Assume $\mathscr{A}$ is a unital commutative associative algebra and $\delta_1,\delta_2,\ldots,\delta_n$ are derivations of this algebra, which commute, $\delta_i\,\delta_j = \delta_j\,\delta_i$. Then the n-ary bracket

$$\{u_1,u_2,\ldots,u_n\} = \mathrm{Det}\begin{pmatrix} \delta_1(u_1) & \delta_2(u_1) & \ldots & \delta_n(u_1) \\ \delta_1(u_2) & \delta_2(u_2) & \ldots & \delta_n(u_2) \\ \ldots & \ldots & \ldots & \ldots \\ \delta_1(u_n) & \delta_2(u_n) & \ldots & \delta_n(u_n) \end{pmatrix}, \quad (1.11)$$

is an n-ary Nambu-Poisson bracket. Hence, a unital commutative associative algebra $\mathscr{A}$ endowed with the n-ary Nambu-Poisson bracket (1.11) is an n-Lie algebra, which is called a Jacobian algebra defined by $\delta_1,\delta_2,\ldots,\delta_n$ [22].

Another area of theoretical physics, where the n-Lie algebras are applied, is field theory. Particularly the authors of the paper [23] proposed a generalization of the Nahm's equation by means of quantum Nambu 4-bracket and showed that their generalization of the Nahm's equation describes M2 branes ending on M5 branes.

A quantization of Nambu–Hamiltonian mechanics is a problem that already Nambu mentioned and he also outlined possible solutions for this problem in his pioneering work. However, this problem is still unresolved. In [19] the authors set the task to find a quantum version of the ternary Nambu–Poisson bracket using a matrix algebra, where the matrices can be both ordinary (i.e. plane) square matrices and cubic (spatial) matrices. The motivation for this was the possible use of a Nambu–Poisson quantum bracket in M-theory. The authors of [19] proposed several realizations of a quantum Nambu–Poisson bracket using matrices and these realizations are based on a combination of the trace of a matrix and the commutator of two matrices. Let $\mathfrak{gl}(N)$ be the Lie algebra of Nth order square complex matrices. The commutator of two matrices will be denoted by $[A,B] = AB - BA$. The ternary bracket, proposed in [19], has the form

$$[A,B,C] = \mathrm{Tr}(A)[B,C] + \mathrm{Tr}(B)[C,A] + \mathrm{Tr}(C)[A,B], \tag{1.12}$$

where $A,B,C \in \mathfrak{gl}(N)$. It is easy to verify that the ternary bracket (1.12) is totally skew-symmetric. Then, in [19] the authors prove that this ternary bracket satisfies the ternary Filippov-Jacobi identity. Hence, (1.12) is a ternary Lie bracket and the vector space $\mathfrak{gl}(N)$, endowed with this ternary Lie bracket is a 3-Lie algebra. Later, this construction of ternary quantum Nambu–Poisson bracket was extended to n-ary Lie brackets with the help of a generalized trace [13] and to ternary Hom-Nambu-Lie algebras [14].

In [2] it was shown that the quantum Nambu–Poisson bracket (1.12) can be extended to matrix Lie superalgebras. The proposed approach can be described as follows: If $\mathfrak{g}$ is an n-Lie superalgebra, $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ is its representation in a super vector space V, then we define the $(n+1)$-ary bracket for elements $x_1, x_2, \ldots, x_{n+1} \in \mathfrak{g}$ as

$$\begin{aligned}&[x_1,x_2,\ldots,x_{n+1}]_\rho = \\ &\qquad \sum_{k=1}^{n+1} (-1)^{k-1}(-1)^{|x_k||\mathbf{x}|_{\mathbf{k-1}}}\,\mathrm{Str}(\rho(x_k))[x_1,x_2,\ldots,\hat{x}_k,\ldots,x_{n+1}],\end{aligned} \tag{1.13}$$

where $|x_k|$ is the degree of x_k, and hat over an element means that this element is omitted. It was proved that the $(n+1)$-ary bracket (1.13) is graded skew-symmetric and satisfies the Filippov-Jacobi identity. Hence this bracket determines a structure of $(n+1)$-Lie superalgebra on $\mathfrak{g}$. Since Clifford algebra can be considered as superalgebra, given a Clifford algebra one can construct a Lie superalgebra by means of ordinary graded commutator. Clifford algebra with even number of generators has a matrix representation, which in the case of two generators can be constructed by means of Pauli matrices. Hence we can apply an approach based on super analog of quantum Nambu-Poisson bracket (1.13) to Clifford algebra with even number of generators and get a ternary Lie superalgebra. This was done in [6], where the formula for ternary graded Lie bracket of generators was found.

In [1] it was shown that the method of induced 3-Lie algebras based on the ternary commutator (1.12) can be extended to infinite-dimensional Lie algebra of vector fields of a smooth manifold. In the case of the Lie algebra of vector fields of a manifold the author defines a triple commutator of three vector fields by a formula analogous to (1.12), where the trace of a matrix is replaced by a differential 1-form ω. Then it is proved that a triple commutator of vector fields satisfies the Filippov-Jacobi identity if a differential 1-form satisfies the following two conditions

$$\omega(X)\,Y(\omega(Z)) = \omega(Y)\,X(\omega(Z)), \qquad \omega\wedge d\omega = 0, \tag{1.14}$$

where X,Y,Z are vector fields. In the case of a matrix Lie superalgebra $\mathfrak{glt}(n,m)$ the author of [1] defines a graded triple commutator of three matrices by the formula

$$[X,Y,Z] = \mathrm{Str}(X)[Y,Z] + (-1)^{x\overline{yz}}\,\mathrm{Str}(Y)[Z,X] + (-1)^{z\overline{xy}}\,\mathrm{Str}(Z)[X,Y], \tag{1.15}$$

where X,Y,Z are square supermatrices, x,y,z are the degrees of supermatrices, $\overline{xy} = x+y$ and $[\cdot,\cdot]$ is the graded commutator of two supermatrices. Note that (1.15) is a particular case of (1.13). It is proved that a graded triple commutator of matrices satisfies the graded Filippov-Jacobi identity. Hence a matrix Lie superalgebra $\mathfrak{gl}(n,m)$ induces the matrix 3-Lie superalgebra, which is denoted by $\mathfrak{glt}(m,n)$. Then the author considers two particular 3-Lie superalgebras $\mathfrak{glt}(1,1), \mathfrak{glt}(2,2)$. In the case of the first one he takes the Pauli matrices as the generators and finds all non-trivial graded triple commutators of the Pauli matrices. Then he takes the Dirac matrices as the generators of the second 3-Lie superalgebra and also finds all non-trivial graded triple commutators of the Dirac matrices. Finally the author considers a general case of 3-Lie superalgebra $\mathfrak{glt}(n,n)$ and applies a supermodule of spinors over a Clifford algebra [55] in order to find all non-trivial graded triple commutators of $\mathfrak{glt}(n,n)$. The author proposes an analog of supersymmetry transformation δ constructed by means of a graded triple commutator. He shows that the structure of this supersymmetry transformation is similar to usual BRST-supersymmetry operator with additional new term which appears due to a ternary structure of graded commutator. Proposed analog of supersymmetry transformation depends on two "fields", where one is a "bosonic field" and the second is a "fermionic field", which play a role of parameters of supersymmetry transformation. The author shows that a fermionic field can be identified with a ghost field and a bosonic field with an auxiliary field B of BRST-supersymmetry operator. A condition for $\delta^2 = 0$ is found.

In [5] the author develops a calculus of 3-dimensional (spatial) or cubic supermatrices and proposes an extension of (1.15) to cubic supermatrices. For this purpose he defines a $\mathbb{Z}_2$-graded structure of cubic matrix and calls a matrix endowed with this structure a cubic supermatrix. He defines the supertrace of a cubic supermatrix relative to one of three directions of a cubic supermatrix. Then he considers a binary product of two cubic matrices, which is formulated by means of products of sections of certain orientation of cubic matrices. Making use of this product he constructs the triple product of cubic supermatrices with the help of supertrace of a cubic supermatrix, and finds the identities for this triple product. Then he applies this triple product to quantum graded Nambu-Poisson bracket and obtains the graded triple commutator of cubic supermatrices, which he calls quantum super Nambu-Poisson bracket of cubic supermatrices. He proves that quantum super Nambu-Poisson bracket of cubic supermatrices satisfies the graded Filippov-Jacobi identity and thus induces the structure of 3-Lie superalgebra on the super vector space of cubic supermatrices. In this approach he uses the notions and structures of the calculus of cubic matrices developed in [65]. He also generalizes (1.10) by means of cochains of Lie algebra. Given a Lie algebra he constructs the n-ary Lie bracket by means of a $(n-2)$-cochain of given Lie algebra and finds the conditions under which this n-ary bracket satisfies the Filippov-Jacobi identity, thereby inducing the structure of n-Lie algebra. Particular case of this approach applied to matrix Lie algebra of nth order matrices $\mathfrak{gl}(n)$ is the matrix 3-Lie algebra with quantum Nambu-Poisson bracket, which is defined with the help of the trace of a matrix (1.10). Similarly, he proposes a more general approach than (1.15) by making use of cochains of Lie superalgebra, and finds the conditions under which a cochain induces the structure of n-Lie superalgebra.

1.2. Original results presented in dissertation

The main purpose of the present thesis is to study induced ternary Lie superalgebras. By induced ternary Lie superalgebras we mean ternary Lie superalgebras constructed by means of ternary commutator (1.15) or, more generally, the $(n+1)$-ary Lie superalgebra, constructed by means of n-ary commutator (1.13). The present dissertation is based on three scientific publications **I**, **II**, **III** (see section List of original publications). Below I will give a brief description of these publications, where I will shortly describe the main structures that were investigated, the results obtained in this case, and also indicate my contribution to the studies presented in the papers.

In paper **I** we study induced n-Lie superalgebras and propose a classification of ternary Lie superalgebras in low dimensions. In Section 2 of the paper **I** we give basic definitions and notions such as n-Lie algebra, n-Lie superalgebra, ϕ-trace (generalized trace) and explain the method of induced (by means of a generalized trace) n-Lie algebras (Theorem 1). Then we propose a notion of ϕ-supertrace (generalized supertrace) and give statement of the theorem (Theorem 2), which shows that the method of induced n-Lie algebras can be extended to n-Lie superalgebras with the help of a generalized supertrace. It should be noted that this theorem was proved in the paper [2]. In Section 3 we study properties of induced n-Lie superalgebras. First of all we give definitions of ideal of n-Lie superalgebra, the descending central series and derived series. The following four propositions are proved by the author of the present dissertation. Assume that we are given an n-Lie superalgebra $\mathfrak{g}$ equipped with a generalized supertrace. Then the first proposition states that if we have a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then $\mathfrak{h}$ is a subalgebra of the $(n+1)$-Lie superalgebra that is induced by means of generalized supertrace. The second proposition states that if $\mathfrak{h}$ is an ideal in n-Lie superalgebra $\mathfrak{g}$, then $\mathfrak{h}$ is an ideal of induced $(n+1)$-Lie superalgebra if and only if $\mathfrak{h}$ lies in the kernel of a generalized supertrace. Proposition 3 states that $(n+1)$-Lie superalgebra which is induced by means of a generalized supertrace is solvable. In Proposition 4 we study the relations between descending central series of initial n-Lie superalgebra and induced $(n+1)$-Lie superalgebra, and prove that each term of the sequence of the descending central series of induced $(n+1)$-Lie superalgebra is a subspace of corresponding term of the sequence of the descending central series of initial n-Lie superalgebra. In Section 4 we propose a classification of ternary Lie superalgebras in low dimensions. This classification is found by the author of the present thesis. The method, which I use to find a classification of low-dimensional ternary Lie superalgebras, is based on structure constants of these algebras, their transformation rules, when one passes from one set of generators to another, and Filippov-Jacobi identity. I derive a system of quadratic equations, whose solutions are possible structure constants for ternary Lie superalgebras. First of all I prove (Theorem 3) that if dimensions of a 3-Lie superalgebra is either $0|1$ or $1|1$, then this algebra is Abelian. The main results of proposed classification are given in the following two theorems:

Theorem 1.1. *3-Lie superalgebras over $\mathbb{C}$, whose super vector space dimension is $0|2$ or $1|2$, are either Abelian or isomorphic to 3-Lie superalgebra $\mathfrak{h}$ whose non-trivial commutation relations are*

$$\begin{cases} [f_1,f_1,f_1] = -f_1+f_2, \\ [f_1,f_1,f_2] = -f_1+f_2, \\ [f_1,f_2,f_2] = -f_1+f_2, \\ [f_2,f_2,f_2] = -f_1+f_2, \end{cases} \quad or \quad [f_1,f_1,f_1] = f_2,$$

where f_1, f_2 are odd generators of $\mathfrak{h}$.

Theorem 1.2. *3-Lie superalgebras over $\mathbb{C}$, whose super vector space dimension is $2|1$, are either Abelian or isomorphic to 3-Lie superalgebra $\mathfrak{h}$ whose non-trivial commutation relations are*

$$\begin{cases} [e_1,f_1,f_1] = e_1+e_2, \\ [e_2,f_1,f_1] = -e_1-e_2, \end{cases} \quad [e_1,e_1,f_1] = f_1, \quad or \quad [f_1,f_1,f_1] = f_1,$$

where e_1, e_2 are even generators of $\mathfrak{h}$ and f_1 is odd generator of $\mathfrak{h}$.

In the paper **II** we study different methods for constructing ternary Lie algebras and ternary Lie superalgebras. We also propose a generalization of Nambu-Hamilton equation to a superspace and show that this generalization induces a family of ternary Nambu-Poisson brackets of even degree functions determined on a superspace. First of all, in Section 1 we give the definition of n-Lie algebra, its particular case of ternary Lie algebra and a few important well known examples of n-Lie algebras such as vector product and Jacobian n-Lie algebras. Then we show that the method of constructing ternary Lie algebras by means of ternary Lie bracket (1.12) can be generalized by means of 1-cochain of Lie algebra, which satisfies the condition written with the help of differential and wedge product of cochains. Then we describe a method of constructing ternary Lie algebras by means of involution and a derivation of a commutative associative algebra, proposed in [22]. It should be mentioned that we slightly generalize the approach given in [22] by using our approach based on cochains, their wedge products and differential. Section 3 of the paper **II** is devoted to 3-Lie superalgebras. At the beginning of this section we give the definition of n-Lie superalgebra and its particular case of ternary Lie superalgebra. We show that the system of two identities, which are equivalent to ternary Filippov-Jacobi identity [20], can be extended to ternary Lie superalgebras by means of a graded version of these identities. These identities are proved by the author of the present dissertation.

Proposition 1.1. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super vector space endowed with a ternary bracket*

$$\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \ni (x,y,z) \mapsto [x,y,z]_{gr} \in \mathfrak{g}, \tag{1.16}$$

which is trilinear, graded skew-symmetric and satisfies $|[x,y,z]_{gr}| = |x|+|y|+|z|$. *Then the ternary bracket (1.16) is a ternary graded Lie bracket, and a super vector space* $\mathfrak{g}$ *endowed with this ternary bracket is a ternary Lie superalgebra if and only if the ternary bracket (1.16) satisfies two identities*

$$\begin{aligned}[][[x,y,z]_{gr},u,v]_{gr} = (-1)^{\alpha}[[u,y,z]_{gr},x,v]_{gr} + (-1)^{\beta}[[x,u,z]_{gr},y,v]_{gr}\\ + (-1)^{\gamma}[[x,y,u]_{gr},z,v]_{gr},\end{aligned}$$

where

$$\begin{aligned}\alpha &= |u|(|x|+|y|+|z|)+|x|(|y|+|z|),\\ \beta &= |u|(|y|+|z|)+|y||z|,\\ \gamma &= |u||z|,\end{aligned}$$

and

$$\begin{aligned}[][[x,y,z],u,v] + (-1)^{\mu}[[u,v,z],x,y] - (-1)^{\nu}[[x,u,z],y,v]\\ - (-1)^{\lambda}[[v,y,z],u,x] - (-1)^{\rho}[[x,v,z],u,y] - (-1)^{\kappa}[[u,y,z],x,v] = 0,\end{aligned}$$

where

$$\begin{aligned}\mu &= (|x|+|y|)(|u|+|v|),\\ \nu &= |y||u|+|y||z|+|z||u|,\\ \lambda &= |v|(|y|+|z|+|u|)+|x|(|y|+|z|+|u|)+|x||v|,\\ \rho &= (|v|+|y|)(|z|+|u|)+|v||y|,\\ \kappa &= (|x|+|u|)(|y|+|z|)+|x||u|.\end{aligned}$$

Then we shall show that the construction of ternary quantum Nambu-Poisson bracket [19], based on the trace of a matrix, can be extended to matrix Lie superalgebra $\mathfrak{gl}(m,n)$ by means of the supertrace of a matrix. In other words, we show that if we have a matrix Lie superalgebra $\mathfrak{gl}(m,n)$, then we can construct a ternary graded bracket by means of the graded commutator of two supermatrices and the supertrace of a matrix, and prove that this ternary graded bracket is graded skew-symmetric and satisfies the graded Filippov-Jacobi identity. In the last subsection of Section 3 of the paper **II** we extend the method of constructing ternary Lie algebras with the help of a derivation and an involution of commutative associative algebra to commutative superalgebra with superinvolution and even degree derivation. As an example of applications of this method, we construct the ternary Lie superalgebra of functions on a superspace, where the ternary graded Lie bracket is defined by means of superinvolution and even degree vector field. All results of this section are proved by the author of the present thesis and here I will provide a summary of these results.

Let $\mathscr{A}=\mathscr{A}_0\oplus\mathscr{A}_1$ be a superalgebra. As before, the degree of a homogeneous element $u\in\mathscr{A}$ will be denoted by $|u|$. A superalgebra $\mathscr{A}$ is said to be commutative superalgebra if for any two homogeneous elements $u,v\in\mathscr{A}$ it holds

$$uv=(-1)^{|u||v|}vu.$$

A degree m derivation of a superalgebra $\mathscr{A}$, where m is either 0 (even degree derivation) or 1 (odd degree derivation), is a linear mapping $\delta\colon\mathscr{A}\to\mathscr{A}$ such that $|\delta(u)|=|u|+m$, and it satisfies the graded Leibniz rule

$$\delta(uv)=\delta(u)\,v+(-1)^{m|u|}u\,\delta(v). \tag{1.17}$$

The degree of a derivation δ will be denoted by $|\delta|$. Hence if δ is an even degree derivation of a superalgebra $\mathscr{A}$, then $|\delta(u)|=|u|$, i.e. δ does not change the degree of a homogeneous element u, and for any two elements $u,v\in\mathscr{A}$ it satisfies the Leibniz rule

$$\delta(uv)=\delta(u)\,v+u\,\delta(v).$$

A mapping $*:u\in\mathscr{A}\mapsto u^*\in\mathscr{A}$ is said to be a superinvolution of a superalgebra $\mathscr{A}$ if it satisfies the following conditions:

1. a mapping $*\colon u\in\mathscr{A}\mapsto u^*\in\mathscr{A}$ is even degree mapping of a superalgebra $\mathscr{A}$, i.e. $*\colon\mathscr{A}_0\ni u\mapsto u^*\in\mathscr{A}_0$, $*\colon\mathscr{A}_1\ni u\mapsto u^*\in\mathscr{A}_1$, or $|u^*|=|u|$,
2. superinvolution is anti-linear, i.e. $(\lambda\,u+v)^*=\bar{\lambda}\,u^*+v^*$, $\lambda\in\mathbb{C}$, $u,v\in\mathscr{A}$,
3. $(u^*)^*=u$,
4. $(uv)^*=(-1)^{|u||v|}v^*u^*$.

In the case of a commutative superalgebra with superinvolution $*$ the fourth condition takes the form $(uv)^*=u^*v^*$. Any element x of superalgebra $\mathscr{A}$ with superinvolution $\mathscr{A}$ can be written in the form $x=x_1+x_{-1}$, where $x_1^*=x_1$, $x_{-1}^*=-x_{-1}$ and

$$x_1=\frac{1}{2}(x+x^*),\; x_{-1}=\frac{1}{2}(x-x^*).$$

It is worth to mention that the components x_1,x_{-1} have the same degree as x, i.e. $|x_1|=|x_{-1}|=|x|$. A superinvolution and odd degree derivation can be used to construct binary graded Lie brackets on a superalgebra $\mathscr{A}$. Let us define

$$[u,v]_* \;=\; u^*v-(-1)^{|u||v|}v^*u, \tag{1.18}$$

$$[u,v]_\delta \;=\; u\,\delta(v)-(-1)^{|u||v|}v\,\delta(u), \tag{1.19}$$

$$[u,v]_{*,\delta} \;=\; (u-u^*)\,\delta(v)-(-1)^{|u||v|}(v-v^*)\delta(u). \tag{1.20}$$

Then I prove that binary brackets (1.18), (1.19) and (1.20) satisfy the graded Jacobi identity, i.e. we have

$$(-1)^{|w||u|}\big[u,[v,w]_*\big]_*+(-1)^{|u||v|}\big[v,[w,u]_*\big]_*+(-1)^{|v||w|}\big[w,[u,v]_*\big]_*=0,$$
$$(-1)^{|w||u|}\big[u,[v,w]_\delta\big]_\delta+(-1)^{|u||v|}\big[v,[w,u]_\delta\big]_\delta+(-1)^{|v||w|}\big[w,[u,v]_\delta\big]_\delta=0,$$
$$(-1)^{|w||u|}\big[u,[v,w]_{*,\delta}\big]_{*,\delta}+(-1)^{|u||v|}\big[v,[w,u]_{*,\delta}\big]_{*,\delta}+(-1)^{|v||w|}\big[w,[u,v]_{*,\delta}\big]_{*,\delta}=0,$$

and this is stated in the paper **II** in the form of a lemma.

Lemma 1.1. *The binary graded brackets (1.18), (1.19) are graded Lie brackets. The third graded bracket (1.20) is a graded Lie bracket if a superinvolution and an even degree derivation satisfy the condition* $(\delta(u))^* = -\delta(u^*)$.

Now, making use of the binary graded Lie brackets (1.18), (1.19), (1.20) and of a generalized supertrace for corresponding graded Lie brackets, I construct ternary graded brackets and prove that all three satisfy the graded Jacobi identity.

Theorem 1.3. *Let* $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ *be a commutative superalgebra,* $*\colon \mathscr{A} \to \mathscr{A}$ *be its superinvolution and* $\delta\colon \mathscr{A} \to \mathscr{A}$ *be its even degree derivation. If* ξ, η, χ *are generalized supertraces for Lie superalgebras* $\mathscr{A}_*$, $\mathscr{A}_\delta$, $\mathscr{A}_{*,\delta}$ *respectively, then the following ternary graded brackets*

$$\begin{aligned}
[x,y,z]_* &= \xi(x)\,[y,z]_* + (-1)^{|x|(|y|+|z|)}\xi(y)\,[z,x]_* \\
&\qquad + (-1)^{|z|(|x|+|y|)}\xi(z)\,[x,y]_*, \\
[x,y,z]_\delta &= \eta(x)\,[y,z]_\delta + (-1)^{|x|(|y|+|z|)}\eta(y)\,[z,x]_\delta \\
&\qquad + (-1)^{|z|(|x|+|y|)}\eta(z)\,[x,y]_\delta, \\
[x,y,z]_{*,\delta} &= \chi(x)\,[y,z]_{*,\delta} + (-1)^{|x|(|y|+|z|)}\chi(y)\,[z,x]_{*,\delta} \\
&\qquad + (-1)^{|z|(|x|+|y|)}\chi(z)\,[x,y]_{*,\delta}.
\end{aligned}$$

are ternary graded Lie brackets, that is, they satisfy the ternary graded Filippov-Jacobi identity. Hence, the Lie superalgebras $\mathscr{A}_*$, $\mathscr{A}_\delta$, $\mathscr{A}_{*,\delta}$ *equipped with the ternary graded Lie brackets* $[\cdot,\cdot,\cdot]_*$, $[\cdot,\cdot,\cdot]_\delta$, $[\cdot,\cdot,\cdot]_{*,\delta}$ *respectively are ternary Lie superalgebras.*

The proof of the theorem above is rather technical, that is, it checks the ternary graded Filippov-Jacobi identity. It should be noted that the proof requires a lot of computation. In addition, it is easy to notice that the structure of ternary graded Lie brackets in all three cases has the same structure, that is, it is constructed by means of a generalized trace and a graded binary commutator. Thus, a generalized trace is included in the formula of the ternary graded commutator and therefore this construction can be used in such structures where there is an analogue of the trace for a given binary graded Lie bracket. However, it is an interesting observation that one can construct yet another ternary graded bracket that does not contain a generalized trace (which means it has a different structure from the three brackets constructed above), and can be constructed explicitly by a combination of superinvolution and binary graded δ-commutator $[\cdot,\cdot]_\delta$. Making use of this observation I prove the following theorem:

Theorem 1.4. *Let* $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ *be a commutative superalgebra with superinvolution* $*\colon \mathscr{A} \to \mathscr{A}$ *and* $\delta\colon \mathscr{A} \to \mathscr{A}$ *be an even degree derivation of* $\mathscr{A}$*. If for any* $u \in \mathscr{A}$ *a superinvolution and even degree derivation satisfy the condition* $(\delta(u))^* = -\delta(u^*)$*, then the ternary graded bracket*

$$[u,v,w]_{gr} = u^*\,[v,w]_\delta + (-1)^{|u|(|v|+|w|)}v^*\,[w,u]_\delta + (-1)^{|w|(|u|+|v|)}w^*\,[u,v]_\delta, \quad (1.21)$$

is a ternary graded Lie bracket and a superalgebra $\mathscr{A}$ endowed with the ternary graded Lie bracket (1.21) is a ternary Lie superalgebra.

First of all, it should be noted that the proof of this theorem cannot be based on a scheme of the proof for ternary graded brackets constructed by means of a generalized trace. This is because there is a significant difference in the structure of ternary brackets used in these cases. Indeed, the ternary graded Lie bracket in Theorem 1.3 is constructed by means of a generalized supertrace, while the ternary graded bracket (1.21) is constructed by means of superinvolution. The proof of this theorem in the case of commutative algebra, proposed in [22], cannot also be automatically transferred to the case of a superalgebra, since in this case the order of the factors in a product plays a significant role due to the appearance of the factor -1, depending on the degrees of the elements. The difficulty in proving this theorem lies in the fact that the verification of the ternary graded Filippov-Jacobi identity requires a huge amount of computation, and therefore cannot be done manually, at least in a reasonable manner. I checked the ternary graded Filippov-Jacobi identity for the ternary graded bracket (1.21) with the help of a computer program using noncommutative symbolic calculus by utilizing the symbolic capabilities of `Mathematica` language combined with non-commutative algebra package `NCAlgebra` developed in UC San Diego Deparement of Mathematics. For this computer program I derived the formulae, which does not contain the factor -1 to a power of sum of products of degrees of involving elements, but in this case the ordering of elements is essential. Thus, winning in one, we lose in the other. However, using ordered products in noncommutative symbolic algebraic calculus software is much easier than using the factor -1 with complicated expressions in its exponent, which depend on degrees of involved elements. For a computer program to correctly perform calculations, a number of additional relations were necessary. First of all I presented the ternary graded bracket (1.21) in the following form, which does not include explicitly degrees of elements,

$$[u,v,w]_{\text{gr}} = u^*\,v\,\delta(w) - u^*\,\delta(v)\,w + \delta(u)\,v^*\,w - u\,v^*\,\delta(w) + u\,\delta(v)\,w^* - \delta(u)\,v\,w^*,$$

and my computer program expanded ternary graded brackets by means of this formula. One more useful formula, which I used in computer program, is

$$([u,v]_\delta)^* = [v^*,w^*]_\delta.$$

In the paper **II** I propose an application of the proved theorem in the differential geometry of superspace by constructing a ternary Lie superalgebra by means of superalgebra of functions and even degree vector field. Let $\mathbb{R}^{n,2}$ be a superspace with n even degree coordinates x^μ, $\mu = 1,2,\ldots,n$ and two odd degree coordinates $\theta, \bar{\theta}$. Let us denote the superalgebra of smooth complex-valued functions on a superspace $\mathbb{R}^{n,2}$ by $C^\infty(\mathbb{R}^{n,2})$. This superalgebra is commutative superalgebra. A function $F(x,\theta,\bar{\theta})$ can be then expanded in odd degree coordinates as

$$F(x,\theta,\bar{\theta}) = F_0(x) + F_{10}(x)\,\theta + F_{01}(x)\,\bar{\theta} + F_{11}(x)\,\theta\bar{\theta}.$$

The degree of a homogeneous function F will be denoted by $|F|$. Let us endow this commutative superalgebra with superinvolution $F \mapsto F^*$, which is defined as follows:

$$F^*(x,\theta,\bar{\theta}) = \bar{F}_0(x) + \bar{F}_{10}(x)\,\bar{\theta} + \bar{F}_{01}(x)\,\theta + \bar{F}_{11}(x)\,\bar{\theta}\theta,$$

where bar over $F_0, F_{10}, F_{01}, F_{11}$ stands for complex conjugation. Let X be an even degree vector field

$$X = X^\mu\,\frac{\partial}{\partial x^\mu} + \phi\frac{\partial}{\partial\theta} + \psi\frac{\partial}{\partial\bar{\theta}},$$

where every X^μ is an even degree function and ϕ,ψ are odd degree functions. Define the ternary graded bracket of three functions by

$$[F,G,H]_{\mathrm{gr}} = F^*\,[G,H]_X + (-1)G^*\,[H,F]_X + (-1)H^*\,[F,G]_X, \qquad (1.22)$$

where $[F,G]_X = F\,X(G) - (-1)^{|F||G|}G\,X(F)$.

Proposition 1.2. *The ternary graded bracket (1.22) for functions on a superspace $\mathbb{R}^{n,2}$ is a ternary graded Lie bracket if a vector field X has the form*

$$X = X^\mu\,\frac{\partial}{\partial x^\mu} + \phi\frac{\partial}{\partial\theta} - \phi^*\frac{\partial}{\partial\bar{\theta}},$$

where every function X^μ satisfies the condition $(X^\mu)^* = -X^\mu$.

Hence, the superalgebra of smooth functions $C^\infty(\mathbb{R}^{n,2})$, endowed with the ternary graded Lie bracket (1.22), where an even degree vector field X satisfies the condition of Proposition 1.2, is an infinite dimensional ternary Lie superalgebra.

In Section 4 of the paper **II** we develop a generalization of Nambu approach to superspace proposed in [3, 4]. Nambu's approach to a generalization of the Poisson bracket and the Hamilton equations is based on the use of the Jacobian of a mapping defined by a set of functions. Nambu proposed to interpret such a Jacobian as a ternary bracket of three functions (in the case of three-dimensional space) or, in a more general case of n-dimensional space, as the n-ary bracket. It is well known that the Poisson bracket is defined on an even-dimensional space, the phase space of a dynamical system with generalized coordinates and momenta. The ternary bracket proposed by Nambu is defined on three-dimensional space, that is, the Nambu approach makes it possible to extend the Poisson bracket to odd-dimensional spaces. It was later shown that the ternary Nambu bracket, whose arguments are functions defined on three-dimensional space, satisfies the Filippov-Jacobi identity. Thus, it turned out that the concepts of Lie n-algebra and the Nambu n-ary bracket, proposed independently of one another, have a common structure. It should be noted here that the Nambu bracket has one more property, which is called a derivation property. If one of arguments of Nambu bracket is a product of two functions, then Nambu bracket can be expanded with the help of Leibniz rule. An n-Lie algebra with such an additional property is called the Nambu-Leibniz algebra.

We extend the Nambu approach to superspace. In the case of superspace, an analogue of Jacobian is the Berezinian (also called superdeterminant). The Berezinian appears in the theory of the Berezin integral when one performs a change of variables under the sign of the integral. Thus, our approach is based on the Berezinian. However, the Berezinian makes it possible to construct an n-ary bracket only for even degree functions. Recall that in the case of superspace the algebra of functions is a superalgebra, that is, we have even and odd degree functions. Even degree functions form the subalgebra of the algebra of functions on a superspace algebra, while odd degree functions do not. However, the Berezinian, according to its structure, contains, in addition to a set of even degree functions, a set of odd degree functions (their number is determined by the dimension of the odd sector of superspace). Therefore, we interpret the n-ary bracket we have proposed as an n-ary bracket on the algebra of even degree functions, which depends on a set of parameters (odd degree functions), that is, in other words, we have a family of n-ary brackets.

We consider the superspace $\mathbb{R}^{3|2}$ with real coordinates x, y, z and two Grassmann coordinates $\theta, \bar{\theta}$. We propose the following analog of Nambu–Hamilton equation in the superspace $\mathbb{R}^{3|2}$

$$\frac{dF}{dt} = \mathrm{Ber}\,\frac{(F, H, G, \phi, \psi)}{(x, y, z, \theta, \bar{\theta})} = \mathrm{Sdet}\left(\begin{array}{ccc|cc} F'_x & F'_y & F'_z & F'_\theta & F'_{\bar{\theta}} \\ H'_x & H'_y & H'_z & H'_\theta & H'_{\bar{\theta}} \\ G'_x & G'_y & G'_z & G'_\theta & G'_{\bar{\theta}} \\ \hline \phi'_x & \phi'_y & \phi'_z & \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_x & \psi'_y & \psi'_z & \psi'_\theta & \psi'_{\bar{\theta}} \end{array}\right),$$

where F, H, G are even degree functions, ϕ, ψ are odd degree functions, and Ber stands for Berezinian. This equation suggests that it is natural to introduce a new ternary bracket, which can be considered as an analogue of the Nambu-Poisson ternary bracket in the superspace $\mathbb{R}^{3|2}$. We consider even degree functions F, H, G as arguments of this new ternary bracket and two odd degree functions ϕ, ψ as parameters of this new ternary bracket, and denote $\Psi = (\phi, \psi)$. We denote this new ternary bracket by bold curly brackets and define it by

$$\{F, H, G\}_\Psi = \mathrm{Ber}\,\frac{(F, H, G, \phi, \psi)}{(x, y, z, \theta, \bar{\theta})}. \tag{1.23}$$

Then we prove that this ternary bracket can be expanded as the sum of the usual Nambu-Poisson bracket (defined by means of Jacobian and extended in obvious way to even degree functions in superspace) and the ternary bracket

$$\begin{aligned}\{F,H,G\}_\Psi \;=\; &\left|\frac{\partial(F,H,G)}{\partial(x,y,\theta)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)}\right|+\left|\frac{\partial(F,H,G)}{\partial(x,\theta,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)}\right|\\ &+\left|\frac{\partial(F,H,G)}{\partial(\theta,y,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)}\right|-\left|\frac{\partial(F,H,G)}{\partial(x,y,\bar\theta)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\right|\\ &-\left|\frac{\partial(F,H,G)}{\partial(x,\bar\theta,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\right|-\left|\frac{\partial(F,H,G)}{\partial(\bar\theta,y,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\right|,\end{aligned}$$

i.e.

$$\{F,H,G\}_\Psi=\frac{1}{\Delta}\{F,H,G\}-\frac{1}{\Delta^2}\{F,H,G\}_\Psi,$$

where

$$\Delta=\left|\frac{\partial(\phi,\psi)}{\partial(\theta,\bar\theta)}\right|=\left|\begin{array}{cc}\phi'_\theta & \phi'_{\bar\theta}\\ \psi'_\theta & \psi'_{\bar\theta}\end{array}\right|=\phi'_\theta\,\psi'_{\bar\theta}-\phi'_{\bar\theta}\,\psi'_\theta .$$

This formula gives grounds to consider the ternary bracket (1.23) introduced by means of superdeterminant as an extension of usual Nambu-Poisson bracket to the superspace $\mathbb{R}^{3|2}$. It can be proved that this extension preserves all the algebraic properties of the Nambu-Poisson bracket such as skew-symmetry, the derivation property and the Filippov-Jacobi identity (Fundamental Identity).

The purpose of the paper **III** is to study representations of induced ternary Lie superalgebras. The theory of representations of groups and Lie algebras is an important part of the theory of groups and Lie algebras. The theory of representations is especially important in applications in differential geometry and theoretical physics. It is well known that many properties of elementary particles, their internal structure, classification can be explained using the theory of representations of Lie groups and Lie algebras. A representation (finite-dimensional) of a Lie group G is its homomorphism $\varphi\colon G\to \mathrm{GL}(V)$ into a group of invertible linear transformations of a finite-dimensional vector space V. Of particular importance is the class of irreducible representations. A representation of a Lie group is called irreducible if there is no subspace in a representation space V, which is invariant with respect to the representation φ , i.e. there is no subspace $W\subset V$ such that $\varphi(W)\subset W$. The importance of irreducible representations is that they are fundamental so-called building blocks of the theory of representations, from which all other representations are composed, or, in other words, a reducible representation is a direct sum of irreducible ones. If we have a representation of a Lie group $\varphi\colon G\to \mathrm{GL}(V)$, then we can consider its tangent map (or differential $D\varphi$) in the unit of the group, and this tangent map $D\varphi$ is, firstly, linear, and, secondly, it maps the Lie algebra of the Lie group $\mathfrak{g}$, to the Lie algebra of linear transformations $\mathfrak{gl}(V)$ of a space of a representation V. It can be shown that $D\varphi$ is a homomorphism of Lie algebras, i.e.

$$D\varphi([x,y])=[D\varphi(x),D\varphi(y)],\;\; x,y\in\mathfrak{g},$$

and it is called a representation of a Lie algebra $\mathfrak{g}$. It turns out that often it is enough to study the representations of the Lie algebra of a certain Lie group to draw conclusions about the representations of a group itself. For example, irreducible representations of a compact connected Lie group are in one-to-one correspondence with representations of its Lie algebra, and this provides a powerful method for classifying representations of the Lie group itself.

Each Lie algebra $\mathfrak{g}$ has one canonical representation, the so-called adjoint representation, which is denoted by ad, and is defined by the formula $\mathrm{ad}_x(y) = [x,y]$, where $x,y \in \mathfrak{g}$. It is easy to see that the representation space in this case is a vector space of $\mathfrak{g}$ itself. The adjoint representation gives a key to how to define a notion of a representation of a ternary Lie superalgebra. Suppose now that $\mathfrak{g}$ is a ternary Lie superalgebra. Obviously, an analogue of linear transformation ad_x in the case of a ternary Lie superalgebra should have two arguments $\mathrm{ad}_{x,y}$, and not one ad_x, as in the case of a binary Lie algebra. Indeed, we can define it by the formula $\mathrm{ad}_{x,y}(z) = [x,y,z]$, i.e. $\mathrm{ad}\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. In article **III**, I find the properties of the map $\mathrm{ad}\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, which should be taken as the basis for a definition of a representation of a ternary Lie superalgebra. Note that as in paper **III** we denote the parity of an element x by $\widehat{x}$, and $\widehat{xy}$ is a shorthand notion for $\widehat{x}\, + \widehat{y}$. Keeping that in mind, we can give the definition of a representation as follows:

Definition 1.4. A mapping $\rho\colon \mathfrak{h}\otimes\mathfrak{h} \to \mathfrak{sgl}(V)$ is said to be a representation of a 3-Lie superalgebra $\mathfrak{h}$ if the following conditions are satisfied:

1. ρ, as a mapping between two super vector spaces, has grading zero, i.e. $\rho\colon (\mathfrak{h}\otimes\mathfrak{h})_0 \to V_0$ and $\rho\colon (\mathfrak{h}\otimes\mathfrak{h})_1 \to V_1$, or equivalently, $\widehat{\rho}\,(x,y) = \widehat{x}\, + \widehat{y}$,
2. $\rho(x,y) = -(-1)^{\widehat{x}\,\widehat{y}}\,\rho(y,x)$,
3. $[\rho(x,y),\rho(u,v)] = \rho([x,y,u],v) + (-1)^{\widehat{u}\,\widehat{xy}}\,\rho(u,[x,y,v])$,
4. $\rho([x,y,z],u) = \rho(x,y)\rho(z,u) + (-1)^{\widehat{x}\,\widehat{yz}}\,\rho(y,z)\rho(x,u)$
$$+ (-1)^{\widehat{z}\,\widehat{xy}}\,\rho(z,x)\rho(y,u),$$

where $x,y,z,u,v \in \mathfrak{h}$. This representation of 3-Lie superalgebra $\mathfrak{h}$ in a super vector space V will be denoted by $(\mathfrak{h},\rho,V)$.

Then I extend the result, obtained in the paper [20] for 3-Lie algebras, to 3-Lie superalgebras. Actually, this extension is not complicated and consists of checking the rule for the consistency of signs that arise in the case of graded structures.

Theorem 1.5. *Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ be a 3-Lie superalgebra, $V = V_0 \oplus V_1$ be a super vector space, $\rho\colon \mathfrak{h}\otimes\mathfrak{h} \to \mathfrak{sgl}(V)$ be a graded skew-symmetric bilinear mapping. Then $(\mathfrak{h},\rho,V)$ is a representation of 3-Lie superalgebra $\mathfrak{h}$ in a super vector space V if and only if the direct sum of super vector spaces $\mathfrak{h}\oplus V$ equipped with the graded ternary bracket*

$$\begin{aligned}[x_1+v_1,x_2+v_2,x_3+v_3] = {} & [x_1,x_2,x_3] + \rho(x_1,x_2)\,v_3 \\ & + (-1)^{\widehat{x_1}\,\widehat{x_2x_3}}\,\rho(x_2,x_3)\,v_1 + (-1)^{\widehat{x_3}\,\widehat{x_1x_2}}\,\rho(x_3,x_1)\,v_2.\end{aligned}$$

is a 3*-Lie superalgebra, or, in other words, this graded ternary bracket satisfies the graded Filippov–Jacobi identity.*

Then I show that if we have a Lie superalgebra on which a generalized supertrace is defined, we have a representation of this Lie superalgebra, then we have the induced ternary Lie superalgebra and a representation of a binary Lie superalgebra induces a representation of the induced ternary Lie superalgebra. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $S\tau$ be a generalized supertrace of this Lie superalgebra. It can be proved then [2] that the graded ternary bracket

$$[x,y,z] = S\tau(x)\,[y,z] + (-1)^{\hat{x}\,\widehat{yz}}\,S\tau(y)\,[z,x] + (-1)^{\hat{z}\,\widehat{xy}}\,S\tau(z)\,[x,y],\ \ x,y,z \in \mathfrak{g}, \tag{1.24}$$

determines the 3-Lie superalgebra on the super vector space of a Lie superalgebra $\mathfrak{g}$. Recall that this 3-Lie superalgebra constructed by means of a generalized supertrace is called an induced 3-Lie superalgebra. Particularly, if we have a representation $\pi\colon\, \mathfrak{g} \to \mathfrak{sgl}(V)$ of a Lie superalgebra $\mathfrak{g}$, then we construct the induced 3-Lie superalgebra (1.24) by simply using the supertrace of matrices in $\mathfrak{sgl}(V)$, i.e. we define the ternary bracket as

$$\begin{aligned}[x,y,z] = \mathrm{Str}(\pi(x))\,[y,z] + (-1)^{\hat{x}\,\widehat{yz}}\,\mathrm{Str}(\pi(y))\,[z,x]\\ + (-1)^{\hat{z}\,\widehat{xy}}\,\mathrm{Str}(\pi(z))\,[x,y],\end{aligned} \tag{1.25}$$

where $x,y,z \in \mathfrak{g}$. The induced 3-Lie superalgebra with graded ternary bracket (1.25) will be denoted by $\mathfrak{tg}_\pi$. In the paper **III** I prove that the method of constructing induced representations of induced 3-Lie algebras can be extended to induced 3-Lie superalgebras.

Theorem 1.6. *Let $\mathfrak{g}$ be a Lie superalgebra and $\pi\colon\, \mathfrak{g} \to \mathfrak{sgl}(V)$ be a representation of $\mathfrak{g}$. Then mapping $\rho\colon\, \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{sgl}(V)$, defined by the formula*

$$\rho(x,y) = \mathrm{Str}(\pi(x))\,\pi(y) - (-1)^{\hat{x}\,\hat{y}}\,\mathrm{Str}(\pi(y))\,\pi(x), \tag{1.26}$$

where $x,y \in \mathfrak{g}$, is a representation of induced 3-Lie superalgebra $\mathfrak{tg}_\pi$.

A proof of this theorem presented in the paper **III** is based on the following lemma.

Lemma 1.2. *Let $\mathfrak{g}$ be a Lie superalgebra, $\pi\colon\, \mathfrak{g} \to \mathfrak{sgl}(V)$ be a representation of this Lie superalgebra. If we equip the super vector space $\mathfrak{g}\oplus V$ with the graded skew-symmetric bracket*

$$[\![x+v,\ y+w]\!] = [x,y] + \pi(x)\cdot w - (-1)^{\hat{x}\,\hat{y}}\,\pi(y)\cdot v, \tag{1.27}$$

where $x,y \in \mathfrak{g}$, $v,w \in V$ and $[x,y]$ is a Lie bracket in $\mathfrak{g}$, then the direct sum of two super vector spaces $\mathfrak{g}\oplus V$ becomes a Lie superalgebra, i.e. the graded skew-symmetric bracket (1.27) satisfies the graded Jacobi identity.

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SUMMARY IN ESTONIAN

Indutseeritud 3-Lie superalgebrad ja nende rakendused superruumis

Käesoleva doktoritöö eesmärk on uurida selliste n-Lie superalgerbrate omadusi, mis on konstrueeritud kasutades $(n-1)$-Lie superalgebra aluseks olevat $(n-1)$-aarset tehet, seda eriti juhul $n=3$. Tavalise Lie algebra mõistet on võimalik super- (või $\mathbb{Z}_2$-gradueeritud) struktuuridele üle kanda kui toome sisse Lie superalgebra mõiste. Sarnaselt on võimalik n-Lie algebra, kus binaarne tehe on asendatud n-aarse tehtega, üldistada superstruktuuridele, kui kasutame Filippov-Jacobi samasuse gradueeritud analoogi, saades n-Lie superalgebra. Väitekirjas on esitatud madaladimensionaalsete 3-Lie superalgebrate klassifikatsioon. Lisaks näitame, et n-Lie superalgebra abil, mille tehtele leidub superjälg, saab konstrueerida $(n+1)$-Lie superalgebra, mida me nimetame indutseeritud $(n+1)$-Lie superalgebraks. Enamgi veel, on tõestatud, et kommutatiivse superalgebra korral on võimalik indutseerida erinevad 3-Lie superalgebra struktuurid kasutades involutsiooni, derivatsiooni või neid mõlemad korraga. Dissertatsioonis on toodud Nambu–Hamiltoni võrrandi üldistus superruumis jaoks, ja on näidatud, et selle abil on võimalik indutseerida ternaarsete Nambu–Poissoni sulgude pere superruumi paarisfunktsioonide jaoks. Järgnevalt on konstrueeritud indutseeritud 3-Lie superalgebrate indutseeritud esitused, kasutades selleks vastavalt kas esialgset binaarset Lie algebrat koos jäljega või Lie superalgebrat koos superjäljega. Töös on näidatud, et 3-Lie algebra indutseeritud esitus on sisestatav jäljeta maatriksite Lie algebrasse $\mathfrak{sl}(V)$, kus sümboliga V on tähistatud esituse ruum. Kahedimensionaalse indutseeritud esituse korral on esitatud tingimused, mida vastav esitus peab rahuldama, et ta oleks taandumatu.

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Teadustöö põhisuunad

Peamised uurimisvaldkonnad on mittekommutatiivne geomeetria, n-aarsed algebrad ja Lie algebrad.

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