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Colloquia: VILASIFEST

Superintegrable deformations of the KC and HO potentials on curved spaces

D. Latini and O. Ragnisco

Dipartimento di Matematica e Fisica, Università Roma Tre and
INFN, Sezione di Roma 3 - Rome, Italy

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Summary. — This is a paper written to celebrate the 70th birthday of our dear colleague Gaetano Vilasi where we collect some recent results about a couple of maximally superintegrable systems. Both the classical and the quantum version will be considered, and the corresponding solution techniques will be illustrated: namely, the spectrum generating algebra (SGA) for the classical systems and the shape invariance potentials approach (SIP) for the quantum case.

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PACS `02.40.Ky` – Riemannian geometries.

1. – Description

As is well known, maximally superintegrable Hamiltonian systems play a distinguished role among the larger family of integrable systems. Indeed, the existence of a dynamical symmetry group, associated to the specific features of the potential, implies that no matter what might be the number of dimensions of the ambient space, those systems are characterized by only one degree of freedom. As a consequence their dynamics is described by the solution of an ordinary differential equation. In the case of a Euclidean metric, the Bertrand Theorem holds [1]: *any bounded orbit is periodic and among those periodic orbits there exist stable circular ones*. Moreover, and most important, only two potentials enjoy the maximal superintegrability property, *i.e.* the *Kepler-Coulomb* (KC) and the *Harmonic Oscillator* (HO) systems. Bertrand's Theorem dates back to 1873.

Since then, there have been several generalisations enabling to include for instance metric spaces with nonzero but constant curvature. Already in those mild generalisations, it was apparent that the form of the potential and that of the metric were functionally related. However, one had to wait until 1992, when there was a crucial turning point, due the results obtained by the theoretical astrophysicist V. Perlick [2]. Restricting attention to the case of conformally flat metrics with (hyper-)spherical symmetry, he proved that there exist only two multi-parametric families of Bertrand systems, each of them identified by a pair $\big(\mathcal{U}_\alpha(r), ds^2_\alpha(r)\big)$, where (α) is just a shorthand notation for the parameter set. In the following they will be denoted as Perlick I (PI) and Perlick II (PII). Those two families are natural generalisations of the KC and the HO systems, that are their limit in the Euclidean case. More than ten years later it has been explicitly proven by Ballesteros *et al.* [3,4] that for bounded motion all solutions of PI and PII correspond to periodic orbits.

In this paper, we revise in the framework of *spectrum generating algebras* (SGA) [5,6] and *supersymmetric quantum mechanics* (SUSYQM) [7,8], the results holding for two prototype examples of maximally superintegrable systems on manifolds with non constant curvature, both pertaining to the family PII. They are summarized in the formulas yielding the solution of the equations of motion for the classical Taub-NUT (TN) and Darboux III (DIII) systems, as well as by the explicit expression for the spectrum of the corresponding quantum problems for a standard choice of the parameters. In this respect, the present paper includes, in a somehow simplified form, both the content of [9,10], focussed on the classical case, as well as the results for the quantum problems, that have been published, for instance, in [11,12].

2. – The Classical Case: SGA and the solution of the motion

2˙1. *Defining Taub-NUT*. – The TN system is defined by the Hamiltonian function:

$$\mathcal{H}_\eta(\mathbf{q},\mathbf{p}) = \mathcal{T}_\eta(\mathbf{q},\mathbf{p}) + \mathcal{U}_\eta(\mathbf{q}) = \frac{|\mathbf{q}|\mathbf{p}^2}{2m(\eta+|\mathbf{q}|)} - \frac{k}{\eta+|\mathbf{q}|}, \tag{1}$$

- $m>0,\ k>0,\ \eta>0$;
- $\mathbf{q},\mathbf{p}\in\mathbb{R}^n$;
- $\mathrm{d}\mathbf{q}\wedge\mathrm{d}\mathbf{p}$: standard symplectic form on $\mathbb{R}^{2n}$;
- $|\mathbf{u}|^2 := \mathbf{u}\cdot\mathbf{u} = \sum_i u_i^2,\ \forall u\in\mathbb{R}^n$.

The Hamiltonian (1) generates a motion on the conformally flat Riemannian manifold $\mathcal{M}^n := \big(\mathbb{R}^n, ds^2_\eta\big)$ with metric

$$\mathrm{d}s^2_\eta = \left(1+\frac{\eta}{|\mathbf{q}|}\right)\mathrm{d}\mathbf{q}^2, \tag{2}$$

under the potential

$$\mathcal{U}_\eta(\mathbf{q}) = -\frac{k}{\eta+|\mathbf{q}|}. \tag{3}$$

It provides an example of a Hamiltonian on a Riemannian space of nonconstant curvature which is *Maximally Superintegrable* (MS) in any dimension [13]. Actually, according to Perlick's classification [2,14], it is a special case of Bertrand system of type II.

Moreover, it can be viewed as an *η-deformation* of the Euclidean KC, with coupling constant k, which is recovered in the smooth limit $\eta\to 0$.

The system (1) can be expressed in terms of hyperspherical coordinates r, φ_j and canonical momenta p_r, p_{φ_j} $(j = 1, \ldots, N-1)$, defined by

$$q_j = r\cos\varphi_j \prod_{k=1}^{j-1} \sin\varphi_k \quad (1 \le j \le N-1), \qquad q_N = r \prod_{k=1}^{N-1} \sin\varphi_k, \tag{4}$$

such that

$$r = |\mathbf{q}|, \qquad \mathbf{p}^2 = p_r^2 + \frac{\mathbf{L}^2}{r^2}, \quad \text{with} \quad \mathbf{L}^2 \doteq \sum_{j=1}^{N-1} p_{\varphi_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2\varphi_k}. \tag{5}$$

Then, for a fixed value of the total angular momentum $\mathbf{L}^2 = l^2$, it can be written as a one-degree of freedom radial system.

2˙2. *Spectrum generating algebra for the classical TN system.* – In what follow we will use the spectrum generating algebra approach [5, 6] to investigate the Hamiltonian:

$$\mathcal{H}_\eta(r,p) = \frac{rp^2}{2m(r+\eta)} + \frac{l^2}{2mr(r+\eta)} - \frac{k}{r+\eta} \doteq \mathcal{C}_\eta(r)\mathcal{H}_0(r,p), \tag{6}$$

with $\{r, p\} = 1$. Here $\mathcal{H}_0(r,p)$ is the "undeformed" KC Hamiltonian and $\mathcal{C}_\eta(r) \doteq \frac{r}{r+\eta}$ is a conformal factor. For the sake of simplicity we shall set $2m = k = 1$. Rewriting for $r \neq 0$ the Hamiltonian (6) in the form

$$r(r+\eta)\mathcal{H}_\eta = r^2\left(p^2 + \frac{l^2}{r^2} - \frac{1}{r}\right) = r^2p^2 + l^2 - r, \tag{7}$$

we propose a variant of the *Euclidean* factorization [6], namely

$$r^2p^2 - r(1+\eta\mathcal{H}_\eta) - r^2\mathcal{H}_\eta = \mathcal{A}_\eta^+\mathcal{A}_\eta^- + \gamma(\mathcal{H}_\eta) = -l^2, \tag{8}$$

where $\mathcal{A}_\eta^+$, $\mathcal{A}_\eta^-$ are unknown functions of r, p. Recalling the Euclidean case, we take:

$$\mathcal{A}_\eta^\pm = \left(\mp irp + a\sqrt{-\mathcal{H}_\eta}\,r + \frac{b(\mathcal{H}_\eta)}{\sqrt{-\mathcal{H}_\eta}}\right) e^{\pm f_\eta(r,p)}. \tag{9}$$

The "arbitrary function" $f_\eta(r,p)$, which depends crucially on the metric, will be determined by requiring the closure of the Poisson algebra generated by $\mathcal{H}_\eta$ and $\mathcal{A}^\pm$:

$$\{\mathcal{H}_\eta, \mathcal{A}_\eta^\pm\} = \mp i\alpha(\mathcal{H}_\eta)\mathcal{A}_\eta^\pm \tag{10}$$

$$\{\mathcal{A}_\eta^+, \mathcal{A}_\eta^-\} = i\beta(\mathcal{H}_\eta), \tag{11}$$

where $\alpha(\mathcal{H}_\eta)$, $\beta(\mathcal{H}_\eta)$ are still to be determined. Inserting $\mathcal{A}_\eta^\pm$ in (8) we get

$$a = 1, \qquad b(\mathcal{H}_\eta) = -\frac{1}{2}(1+\eta\mathcal{H}_\eta), \qquad \gamma(\mathcal{H}_\eta) = \frac{(1+\eta\mathcal{H}_\eta)^2}{4\mathcal{H}_\eta}, \tag{12}$$

and by requiring that $\mathcal{A}_\eta^\pm$ obey (11) we arrive at

$$f_\eta(r,p) = -\frac{2irp\sqrt{-\mathcal{H}_\eta}}{1-\eta\mathcal{H}_\eta} \qquad \alpha(\mathcal{H}_\eta) = -\frac{4\mathcal{H}_\eta\sqrt{-\mathcal{H}_\eta}}{1-\eta\mathcal{H}_\eta} \qquad \beta(\mathcal{H}_\eta) = \frac{1+\eta\mathcal{H}_\eta}{\sqrt{-\mathcal{H}_\eta}}, \tag{13}$$

and finally

$$\mathcal{A}_\eta^\pm = \left(\mp irp + \sqrt{-\mathcal{H}_\eta}r - \frac{1+\eta\mathcal{H}_\eta}{2\sqrt{-\mathcal{H}_\eta}}\right) e^{\mp\frac{2irp\sqrt{-\mathcal{H}_\eta}}{1-\eta\mathcal{H}_\eta}}, \tag{14}$$

$$\{\mathcal{H}_\eta, \mathcal{A}_\eta^\pm\} = \pm i\frac{4\mathcal{H}_\eta\sqrt{-\mathcal{H}_\eta}}{1-\eta\mathcal{H}_\eta}\mathcal{A}_\eta^\pm, \qquad \{\mathcal{A}_\eta^+, \mathcal{A}_\eta^-\} = i\frac{1+\eta\mathcal{H}_\eta}{\sqrt{-\mathcal{H}_\eta}}. \tag{15}$$

In the limit $\eta \to 0$ one gets back the undeformed Poisson algebra presented in [6].

To make the identification even more perspicuous we can introduce the new generator $\tilde{\mathcal{A}}_\eta \doteq \frac{1+\eta\mathcal{H}_\eta}{2\sqrt{-\mathcal{H}_\eta}}$ entailing the following $\mathfrak{su}(1,1)$ algebra:

$$\{\tilde{\mathcal{A}}_\eta, \mathcal{A}_\eta^\pm\} = \mp i\mathcal{A}_\eta^\pm \qquad \{\mathcal{A}_\eta^+, \mathcal{A}_\eta^-\} = 2i\tilde{\mathcal{A}}_\eta. \tag{16}$$

Now, we can define the "time-dependent constants of motion":

$$\mathcal{Q}_\eta^\pm = \mathcal{A}_\eta^\pm e^{\mp i\alpha(\mathcal{H}_\eta)t}, \tag{17}$$

such that $\frac{\mathrm{d}\mathcal{Q}_\eta^\pm}{\mathrm{d}t} = \{\mathcal{Q}_\eta^\pm, \mathcal{H}_\eta\} + \partial_t\mathcal{Q}_\eta^\pm = 0$. Those dynamical variables take complex values admitting the polar decomposition $\mathcal{Q}_\eta^\pm = q_\eta\, e^{\pm i\varphi_0}$ and allowing in fact to determine the motion, which turns out to be bounded for $\mathcal{H}_\eta = \mathcal{E} < 0$. Indeed we have

$$\left(\mp irp + \sqrt{-\mathcal{E}}r - \frac{1+\eta\mathcal{E}}{2\sqrt{-\mathcal{E}}}\right) e^{\mp i\left(\frac{2rp\sqrt{-\mathcal{E}} - 4\mathcal{E}\sqrt{-\mathcal{E}}t}{1-\eta\mathcal{E}}\right)} = q_\eta\, e^{\pm i\varphi_0}, \tag{18}$$

or else

$$\begin{cases} -irp + \sqrt{-\mathcal{E}}r - \dfrac{1+\eta\mathcal{E}}{2\sqrt{-\mathcal{E}}} &= q_\eta\, e^{i\left(\frac{2rp\sqrt{-\mathcal{E}} - 4\mathcal{E}\sqrt{-\mathcal{E}}t}{1-\eta\mathcal{E}} + \varphi_0\right)}, \\ irp + \sqrt{-\mathcal{E}}r - \dfrac{1+\eta\mathcal{E}}{2\sqrt{-\mathcal{E}}} &= q_\eta\, e^{-i\left(\frac{2rp\sqrt{-\mathcal{E}} - 4E\sqrt{-\mathcal{E}}t}{1-\eta\mathcal{E}} + \varphi_0\right)}, \end{cases} \tag{19}$$

where $q_\eta = \sqrt{-l^2 - \frac{(1+\eta\mathcal{E})^2}{4\mathcal{E}}}$ (derived from the equality $\mathcal{A}_\eta^+\mathcal{A}_\eta^- = q_\eta^2$).

Thanks to the above relations, we can obtain t as a function of the radial coordinate r:

$$\Omega_\eta(\mathcal{E})t(r) + \varphi_0 = \arccos\left[-\frac{1}{\epsilon_\eta}\left(1 - \frac{r}{a_\eta}\right)\right] - \frac{a_\eta}{\eta + a_\eta}\sqrt{\epsilon_\eta^2 - \left(1 - \frac{r}{a_\eta}\right)^2}, \tag{20}$$

where we have defined $\Omega_\eta(\mathcal{E}) \doteq -\frac{4\mathcal{E}\sqrt{-\mathcal{E}}}{1-\eta\mathcal{E}} \equiv \alpha(\mathcal{H}_\eta)$, $a_\eta \doteq -\frac{1+\eta\mathcal{E}}{2\mathcal{E}}$ and $\epsilon_\eta \doteq \sqrt{1 + \frac{4l^2\mathcal{E}}{(1+\eta\mathcal{E})^2}}$, representing respectively the frequency of the motion, the major semi-axes of the ellipse and its eccentricity. Obviously the motion is not isochronous (the frequency depends on the initial conditions). Moreover, it is easy to check that in the limit $\eta \to 0$, the results for the flat KC are recovered [6]. For a more general discussion on the classical Taub-NUT system we refer the reader to the paper [9].

2˙3. *Defining Darboux III.* – Let us consider now the DIII system (see ref. [10] and references therein), another system belonging to the Perlick family of type II [2, 14], representing a one parameter deformation of the Harmonic Oscillator. This system is characterized by the Hamiltonian function:

$$\mathcal{H}_\lambda(\mathbf{q},\mathbf{p}) = \mathcal{T}_\lambda(\mathbf{q},\mathbf{p}) + \mathcal{U}_\lambda(\mathbf{q}) = \frac{\mathbf{p}^2}{2m(1+\lambda\mathbf{q}^2)} + \frac{m\omega^2\mathbf{q}^2}{2(1+\lambda\mathbf{q}^2)}, \tag{21}$$

- $m > 0,\ \omega > 0,\ \lambda > 0$;
- $\mathbf{q}, \mathbf{p} \in \mathbb{R}^n$;
- $\mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{p}$: standard symplectic form on $\mathbb{R}^{2n}$;
- $|\mathbf{u}|^2 := \mathbf{u}\cdot\mathbf{u} = \sum_i u_i^2,\ \forall u \in \mathbb{R}^n$.

The Hamiltonian (21) generates a motion on the conformally flat Riemannian manifold $\mathcal{M}^n := \left(\mathbb{R}^n, ds_\lambda^2\right)$ with metric

$$\mathrm{d}s_\lambda^2 = \left(1+\lambda\mathbf{q}^2\right)\mathrm{d}\mathbf{q}^2, \tag{22}$$

under the deformed harmonic potential

$$\mathcal{U}_\lambda(\mathbf{q}) = \frac{m}{2}\frac{\omega^2\mathbf{q}^2}{1+\lambda\mathbf{q}^2}. \tag{23}$$

It provides another example of a Hamiltonian system on a Riemannian space of non-constant curvature which is MS in any dimension [15, 16]. Moreover, it represents a λ-*deformation* of the Euclidean HO, with frequency ω, which is recovered for $\lambda \to 0$.

2˙4. *Spectrum generating algebra for the classical DIII system.* – In what follow we solve the classical motion through the same technique that has been used for the Taub-NUT. The DIII system, using hyperspherical coordinate, is described by the Hamiltonian:

$$\mathcal{H}_\lambda(r,p) = \frac{p^2}{2m(1+\lambda r^2)} + \frac{l^2}{2mr^2(1+\lambda r^2)} + \frac{m\,\omega^2 r^2}{2(1+\lambda r^2)} \doteq \mathcal{C}_\lambda(r)\mathcal{H}_0(r,p), \tag{24}$$

$\mathcal{H}_0(r,p)$ being the undeformed Hamiltonian and $\mathcal{C}_\lambda(r) \doteq \frac{1}{1+\lambda r^2}$ a conformal factor. With the previous cautions we again take $2m = 1$. Before tackling this system, let us study the undeformed problem (with $\lambda = 0$), *i.e.*:

$$\mathcal{H}_0 = p^2 + \frac{l^2}{r^2} + \frac{\omega^2 r^2}{4}, \tag{25}$$

for which we try the following factorization:

$$p^2r^2 - r^2\mathcal{H}_0 + \frac{\omega^2 r^4}{4} = \mathcal{A}_0^+\mathcal{A}_0^- + \gamma(\mathcal{H}_0) = -l^2. \tag{26}$$

As before the functions $\mathcal{A}_0^\pm$ will be determined on the basis of algebraic considerations. After not difficult calculations, we get

$$\mathcal{A}_0^\pm = \left(\mp irp + \frac{\omega r^2}{2} - \frac{\mathcal{H}_0}{\omega}\right) e^{\pm f_0(r,p)}, \tag{27}$$

where the auxiliary function $f_0(r,p)$ will be identified by requiring the closure of the Poisson algebra generated by the generators $\mathcal{A}_0^\pm, \mathcal{H}_0$. As a matter of fact, one has

$$\{\mathcal{H}_0, \mathcal{A}_0^\pm\} = \mp i\, 2\,\omega\, \mathcal{A}_0^\pm, \tag{28}$$

$$\{\mathcal{A}_0^+, \mathcal{A}_0^-\} = i\,\frac{4\mathcal{H}_0}{\omega}, \tag{29}$$

with $\gamma(\mathcal{H}_0) = -\frac{\mathcal{H}_0^2}{\omega^2}$ and $f_0(r,p) = 0$.

Paraphrasing what has been done for the Taub-NUT system, we go over to the time-dependent constants of motion $\mathcal{Q}_0^\pm = \mathcal{A}_0^\pm e^{\mp i\alpha(\mathcal{H}_0)t}$, that allow us to write

$$\left(\mp irp + \frac{\omega r^2}{2} - \frac{\mathcal{H}_0}{\omega}\right) e^{\mp 2i\,\omega t} = q_0\, e^{\pm i\varphi_0}, \tag{30}$$

that is:

$$\begin{cases} -irp + \dfrac{\omega r^2}{2} - \dfrac{\mathcal{H}_0}{\omega} = q_0\, e^{i(2\,\omega t + \varphi_0)} \\ irp + \dfrac{\omega r^2}{2} - \dfrac{\mathcal{H}_0}{\omega} = q_0\, e^{-i(2\,\omega t + \varphi_0)}, \end{cases} \tag{31}$$

with $q_0 = \sqrt{-l^2 + \frac{\mathcal{H}_0^2}{\omega^2}}$. Again, by easy algebraic manipulations we find (for $\mathcal{H}_0 = \mathcal{E}_0$):

$$\begin{cases} r(t) = \sqrt{\dfrac{2\mathcal{E}_0}{\omega^2} + \dfrac{2q_0}{\omega}\cos(2\omega t + \varphi_0)}, \\ p(t) = -\dfrac{q_0 \sin(2\omega t + \varphi_0)}{\sqrt{\frac{2\mathcal{E}_0}{\omega^2} + \frac{2q_0}{\omega}\cos(2\omega t + \varphi_0)}}, \end{cases} \tag{32}$$

which represents the trajectory in the phase-space of the system we are dealing with.

So far, we have restricted our considerations to the undeformed case. In the deformed case we can proceed on the same way. We start from the Hamiltonian function:

$$\mathcal{H}_\lambda = \frac{p^2}{(1+\lambda r^2)} + \frac{l^2}{r^2(1+\lambda r^2)} + \frac{\omega^2 r^2}{4(1+\lambda r^2)} = \frac{\mathcal{H}_0}{1+\lambda r^2}, \tag{33}$$

and impose the following factorization:

$$p^2r^2 - r^2\mathcal{H}_\lambda + \left(\frac{\omega^2}{4} - \lambda\mathcal{H}_\lambda\right) r^4 = \mathcal{A}_\lambda^+\mathcal{A}_\lambda^- + \gamma(\mathcal{H}_\lambda) = -l^2. \tag{34}$$

Under the (*essential*) constraint

$$\omega^2 - 4\lambda\mathcal{H}_\lambda > 0, \tag{35}$$

we get

$$\mathcal{A}_\lambda^\pm = \left(\mp irp + \frac{r^2}{2}\sqrt{\omega^2 - 4\lambda\mathcal{H}_\lambda} - \frac{\mathcal{H}_\lambda}{\sqrt{\omega^2 - 4\lambda\mathcal{H}_\lambda}}\right) e^{\pm f_\lambda(r,p)}, \tag{36}$$

in terms of a so far arbitrary function $f_\lambda(r,p)$, with $\gamma(\mathcal{H}_\lambda)$ reading: $\gamma(\mathcal{H}_\lambda) = -\frac{\mathcal{H}_\lambda^2}{\omega^2 - 4\lambda\mathcal{H}_\lambda}$.

As usual we require a "deformed Poisson algebra" to be satisfied by our generators:

$$\{\mathcal{H}_\lambda, \mathcal{A}_\lambda^\pm\} = \mp i\alpha(\mathcal{H}_\lambda)\mathcal{A}_\lambda^\pm \tag{37}$$

$$\{\mathcal{A}_\lambda^+, \mathcal{A}_\lambda^-\} = i\beta(\mathcal{H}_\lambda). \tag{38}$$

Formulas (38) enable to determine explicitly the function $f_\lambda(r,p)$, which turns out to be $f_\lambda(r,p) = -\frac{2i\lambda rp\sqrt{\omega^2-4\lambda\mathcal{H}_\lambda}}{\omega^2-2\lambda\mathcal{H}_\lambda}$, moreover:

$$\alpha(\mathcal{H}_\lambda) = \frac{2(\omega^2 - 4\lambda\mathcal{H}_\lambda)^{\frac{3}{2}}}{\omega^2 - 2\lambda\mathcal{H}_\lambda} \qquad \beta(\mathcal{H}_\lambda) = \frac{4\mathcal{H}_\lambda}{\sqrt{\omega^2 - 4\lambda\mathcal{H}_\lambda}}. \tag{39}$$

So that, in conclusion, we find the following relations:

$$\mathcal{A}_\lambda^\pm = \left(\mp irp + \frac{r^2}{2}\sqrt{\omega^2 - 4\lambda\mathcal{H}_\lambda} - \frac{\mathcal{H}_\lambda}{\sqrt{\omega^2 - 4\lambda\mathcal{H}_\lambda}}\right) e^{\mp\frac{2i\lambda rp\sqrt{\omega^2-4\lambda\mathcal{H}_\lambda}}{\omega^2-2\lambda\mathcal{H}_\lambda}}, \tag{40}$$

$$\{\mathcal{H}_\lambda, \mathcal{A}_\lambda^\pm\} = \mp i\frac{2(\omega^2 - 4\lambda\mathcal{H}_\lambda)^{\frac{3}{2}}}{\omega^2 - 2\lambda\mathcal{H}_\lambda}\mathcal{A}_\lambda^\pm, \qquad \{\mathcal{A}_\lambda^+, \mathcal{A}_\lambda^-\} = i\frac{4\mathcal{H}_\lambda}{\sqrt{\omega^2 - 4\lambda\mathcal{H}_\lambda}}. \tag{41}$$

It is easy to check that in the $\lambda \to 0$ limit we recover the Poisson algebra of the Euclidean harmonic oscillator (28), (29) previously discussed.

Furthermore, we can define even in the deformed case the time-dependent constants of the motion, going over to the polar decomposition. In this way we get two coupled equations, namely:

$$\begin{cases} \dfrac{r^2}{2}\sqrt{\omega^2-4\lambda\mathcal{E}} - \dfrac{\mathcal{E}}{\sqrt{\omega^2-4\lambda\mathcal{E}}} = q_\lambda \cos\left(\dfrac{2\lambda rp\sqrt{\omega^2-4\lambda\mathcal{E}}+2(\omega^2-4\lambda\mathcal{E})^{\frac{3}{2}}t}{\omega^2-2\lambda\mathcal{E}} + \varphi_0\right) \\ rp = -q_\lambda \sin\left(\dfrac{2\lambda rp\sqrt{\omega^2 - 4\lambda\mathcal{E}} + 2(\omega^2 - 4\lambda\mathcal{E})^{\frac{3}{2}}t}{\omega^2 - 2\lambda\mathcal{E}} + \varphi_0\right). \end{cases} \tag{42}$$

As we did for the Taub-NUT system, but in contrast with the Euclidean case where it is well known that one can get explicitly r and p as a function of t (32), here the best we can do is to write t as an explicit function of the radial coordinate r:
(43)

$$\Omega_\lambda(\mathcal{E})t(r)+\varphi_0=\arccos\left[-\frac{1}{\epsilon_\lambda}\left(1-\left(\frac{r}{a_\lambda}\right)^2\right)\right]-\frac{\lambda a_\lambda^2}{1+\lambda a_\lambda^2}\sqrt{\epsilon_\lambda^2-\left[1-\left(\frac{r}{a_\lambda}\right)^2\right]^2}.$$

Here $\Omega_\lambda(\mathcal{E}) \doteq \frac{2(\omega^2-4\lambda\mathcal{E})^{\frac{3}{2}}}{\omega^2-2\lambda\mathcal{E}} \equiv \alpha(\mathcal{H}_\lambda)$, $q_\lambda \doteq \sqrt{-l^2+\frac{\mathcal{E}^2}{\omega^2-4\lambda\mathcal{E}}}$, $a_\lambda^2 \doteq \frac{2\mathcal{E}}{\left(\omega^2-4\lambda\mathcal{E}\right)}$ and $\epsilon_\lambda \doteq \sqrt{1-\frac{(\omega^2-4\lambda\mathcal{E})l^2}{\mathcal{E}^2}}$, a_λ^2 being the square of the major semi-axes of the ellipse and ϵ_λ a parameter directly related to its eccentricity. For a more general discussion regarding the classical DIII system we refer the reader to [10].

In the next section we will see that the existence of a region where the orbits of the classical motion are bounded has an obvious quantum counterpart in the existence of a discrete spectrum.

3. – Shape-invariance, SUSY partners and quantisation

3·1. *The quantum Taub-NUT system.* – The present subsection is devoted to deriving the spectrum of the quantum Hamiltonian (from now on we will fix $m=1$):

$$\hat{\mathcal{H}}_\eta(\mathbf{q},\mathbf{p})=-\frac{\hbar^2}{2}\frac{|\mathbf{q}|}{\eta+|\mathbf{q}|}\nabla^2-\frac{k}{\eta+|\mathbf{q}|}, \tag{44}$$

where $\mathbf{q}=(q_1,\ldots,q_N)\in\mathbb{R}^N$ and $k,\eta\in\mathbb{R}^+$, using the SUSYQM formalism (factorization method and shape invariance condition (SIC)) [7, 8]. We remind that in hyperspherical coordinates the radial Schrödinger equation reads (see for instance ref. [12]):

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\hbar^2\,(N-1)}{2r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\hbar^2 l(l+N-2)}{2r^2}-\frac{\mathcal{K}}{r}\right]\Psi(r)=\mathcal{E}\Psi(r), \tag{45}$$

where $\mathcal{K}\doteq k+\eta\mathcal{E}$. To solve for the spectrum using this approach we have to get rid of the term containing the first r-derivative, and to this aim we gauge-transform the wave function, *i.e.*:

$$\Psi(r)=e^{f(r)}\,\Phi(r), \tag{46}$$

where $f(r)$ has to be determined. We easily calculate

$$\begin{cases}\Psi(r) &= e^{f(r)}\Phi(r),\\ \Psi'(r) &= e^{f(r)}[\Phi'(r)+f'(r)\Phi(r)],\\ \Psi''(r) &= e^{f(r)}[\Phi''(r)+2\,f'(r)\Phi'(r)+[f'(r)]^2\Phi(r)+f''(r)\Phi(r)].\end{cases} \tag{47}$$

By replacing the above expressions in the radial Schrödinger equation we get

$$
\begin{aligned}
(48)\qquad & -\frac{\hbar^2}{2}\left[\Phi''(r)+\Phi'(r)\left[2\,f'(r)+\frac{N-1}{r}\right]\right.\\
&\left.+\Phi(r)\left[[f'(r)]^2+f''(r)+\frac{N-1}{r}f'(r)\right]\right]+\left[\frac{\hbar^2 l(l+N-2)}{2r^2}-\frac{\mathcal{K}}{r}\right]\Phi(r)=\mathcal{E}\Phi(r).
\end{aligned}
$$

Thus, the term in the first r-derivative will cancel out iff the following first order differential equation is satisfied:

$$
(49)\qquad 2\,f'(r)+\frac{N-1}{r}=0 \quad\Rightarrow\quad f(r)=\ln r^{-\frac{N-1}{2}}.
$$

Then, plugging $f(r)$ in the Schrödinger equation we obtain:

$$
(50)\qquad \left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\hbar^2\left[l(l+N-2)+\frac{(N-1)(N-3)}{4}\right]}{2r^2}-\frac{\mathcal{K}}{r}\right]\Phi(r)=\mathcal{E}\,\Phi(r),
$$

with $\Phi(r)=r^{\frac{N-1}{2}}\Psi(r)$. The above equation has the form

$$
(51)\qquad \left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V_1^{(l,N,\eta)}(r)\right]\Phi(r)=\mathcal{E}\,\Phi(r),
$$

it is then possible to use SUSYQM in order to solve the eigenvalues problem. As the constant $\mathcal{K}$ is itself energy-dependent, we expect an algebraic equation to be solved.

As a starting point, we have to shift the potential $V_1^{(l,N,\eta)}(r)$ subtracting the ground state eigenvalue $\mathcal{E}_0$ in order to define the first SUSY potential:

$$
(52)\qquad \tilde{V}_1^{(l,N,\eta)}(r)\doteq V_1^{(l,N,\eta)}(r)-\mathcal{E}_0.
$$

From SUSYQM theory it is known that this shifted potential can be in turn re-expressed through a function $\mathcal{W}(r)$, the so-called Super-potential, which solves a Riccati-type differential equation:

$$
\begin{aligned}
(53)\qquad \tilde{V}_1^{(l,N,\eta)}(r)=V_1^{(l,N,\eta)}(r)-\mathcal{E}_0 &=\frac{\hbar^2\left[l(l+N-2)+\frac{(N-1)(N-3)}{4}\right]}{2r^2}-\frac{\mathcal{K}}{r}-\mathcal{E}_0\\
&=\mathcal{W}^2(r)-\frac{\hbar}{\sqrt{2}}\mathcal{W}'(r).
\end{aligned}
$$

Relying on the *known* functional form of the potential we can try an ansatz for the Super-potential. We seek the simplest form:

$$
(54)\qquad \mathcal{W}(r)=\mathcal{A}-\frac{\mathcal{B}}{r},
$$

where the coefficients $\mathcal{A}, \mathcal{B}$ have to be determined. So, we get the equality

$$\frac{\hbar^2\left[l(l+N-2)+\frac{(N-1)(N-3)}{4}\right]}{2r^2}-\frac{\mathcal{K}}{r}-\mathcal{E}_0=\frac{[\mathcal{B}^2-\frac{\hbar}{\sqrt{2}}\mathcal{B}]}{r^2}-\frac{2\mathcal{A}\mathcal{B}}{r}+\mathcal{A}^2, \tag{55}$$

entailing:

$$\begin{cases} \mathcal{B}^2-\dfrac{\hbar}{\sqrt{2}}\mathcal{B}=\dfrac{\hbar^2}{2}\left[l(l+N-2)+\dfrac{(N-1)(N-3)}{4}\right], \\ 2\mathcal{A}\mathcal{B}=\mathcal{K}, \\ \mathcal{E}_0=-\mathcal{A}^2. \end{cases} \tag{56}$$

Solving the equation for $\mathcal{B}$ (where the positive root has been chosen to keep the analogy with the undeformed case) and plugging the result in the remaining two equations we immediately get:

$$\begin{cases} \mathcal{B}=\dfrac{\hbar}{2\sqrt{2}}\left[1+\sqrt{1+4l(l+N-2)+(N-1)(N-3)}\right]=\dfrac{\hbar}{2\sqrt{2}}[N+2l-1], \\ \mathcal{A}=\dfrac{\mathcal{K}}{2B}=\dfrac{\sqrt{2}\mathcal{K}}{\hbar\left[1+\sqrt{1+4l(l+N-2)+(N-1)(N-3)}\right]}=\dfrac{\sqrt{2}\mathcal{K}}{\hbar[N+2l-1]}, \\ \mathcal{E}_0=-\mathcal{A}^2=-\dfrac{2\mathcal{K}^2}{\hbar^2[N+2l-1]^2}. \end{cases} \tag{57}$$

Clearly, in the $\eta \to 0$ limit and for $N = 3$ we have to recover the well-known result holding for Hydrogen-like atom [7, 8]. In fact we get

$$\begin{cases} \mathcal{B}=\dfrac{\hbar}{\sqrt{2}}(l+1), \\ \mathcal{A}=\dfrac{k}{\sqrt{2}\hbar(l+1)}, \\ \mathcal{E}_0=-\dfrac{k^2}{2\hbar^2(l+1)^2}, \end{cases} \tag{58}$$

namely the values holding in the flat case (for $m = 1$ and $k = \frac{e^2}{4\pi\epsilon_0}$). Now, given the form of the Super-potential $\mathcal{W}(r)$, we can construct the SUSY partner potential to $\tilde{V}_1^{(l,N,\eta)}$. In particular, as the Super-potential reads

$$\mathcal{W}(r)=\frac{\sqrt{2}\mathcal{K}}{\hbar[N+2l-1]}-\frac{\hbar}{2\sqrt{2}r}[N+2l-1], \tag{59}$$

using the Riccati equation (with a change of sign in the second term), we obtain the expression:

$$\begin{aligned} \tilde{V}_2^{(l,N,\eta)}(r) &= \mathcal{W}^2(r)+\frac{\hbar}{\sqrt{2}}\mathcal{W}'(r) \\ &= \frac{\hbar^2\left[l(l+N-2)+\frac{(N-1)(N-3)}{4}+[N+2l-1]\right]}{2r^2}-\frac{\mathcal{K}}{r}+\frac{2\mathcal{K}^2}{\hbar^2[N+2l-1]^2}. \end{aligned}$$

Hence the SUSY partner potentials read:

$$\begin{cases} \tilde{V}_1^{(l,N,\eta)}(r) = \dfrac{\hbar^2\left[l(l+N-2)+\frac{(N-1)(N-3)}{4}\right]}{2r^2} - \dfrac{\mathcal{K}}{r} + \dfrac{2\mathcal{K}^2}{\hbar^2[N+2l-1]^2}, \\ \tilde{V}_2^{(l,N,\eta)}(r) = \dfrac{\hbar^2\left[l(l+N)+\frac{(N-1)(N+1)}{4}\right]}{2r^2} - \dfrac{\mathcal{K}}{r} + \dfrac{2\mathcal{K}^2}{\hbar^2[N+2l-1]^2}. \end{cases} \tag{60}$$

Even in that case, in the limit $\eta \to 0$ we recover the SUSY partner potentials of the Euclidean case. Moreover we can check explicitly whether the shape-invariance condition is fullfilled. Should this be the case the potential $\tilde{V}_2^{(l,N,\eta)}(r)$ would be related to its super partner through the formula $\tilde{V}_2^{(l,N,\eta)} = \tilde{V}_1^{(f(l),g(N),\eta)}(r) + \mathcal{R}(l,N)$. It is indeed clear that *SIC* holds in our case. To elucidate this fact, let us take $f(l) = l$ and $g(N) = N+2$, which entail:

$$\begin{aligned} \tilde{V}_2^{(l,N,\eta)} &= \tilde{V}_1^{(l,N+2,\eta)}(r) + \mathcal{R}(l,N) \\ &= \tilde{V}_1^{(l,N+2,\eta)}(r) + \frac{8\mathcal{K}^2(2l+N)}{\hbar^2(N+2l-1)^2(N+2l+1)^2}. \end{aligned} \tag{61}$$

Once again, in the limit $\eta \to 0$ the above equation collapses to the ordinary one holding for the hydrogen-like atom. Then, in full analogy with the ordinary case where, however, we remind that k is replaced by $\mathcal{K}$, we can write the (discrete) energy spectrum, namely

$$\mathcal{E}_n^{(l,N,\eta)} = \mathcal{E}_0 + \sum_{j=1}^{n} \frac{(2l+b_j)8\mathcal{K}^2}{\hbar^2(b_j+2l-1)^2(b_j+2l+1)^2}, \tag{62}$$

with $b_j = N + 2(j-1)$. We stress that the above formula, in contrast with the flat case, leads to an algebraic equation for the energy levels. In fact

$$\begin{aligned} \mathcal{E}_n^{(l,N,\eta)} &= -\frac{\mathcal{K}^2}{2\hbar^2}\left[\frac{4}{[N+2l-1]^2} - 16\sum_{j=1}^{n}\frac{(2l+b_j)}{(b_j+2l-1)^2(b_j+2l+1)^2}\right] \\ &= -\frac{\left(k+\eta\mathcal{E}_n^{(l,N,\eta)}\right)^2}{2\hbar^2}\,\Sigma(n,l,N), \end{aligned}$$

where we have defined the quantity:

$$\Sigma(n,l,N) \doteq \frac{4}{[N+2l-1]^2} - 16\sum_{j=1}^{n}\frac{2(l+j)+N-2}{[2(l+j)+N-3]^2[2(l+j)+N-1]^2}, \tag{63}$$

which can be evaluated in closed form:

$$\Sigma(n,l,N) = \frac{1}{(l+n+\frac{N-1}{2})^2}. \tag{64}$$

Therefore, the equation for the spectrum can be written as

$$\mathcal{E}^{(l+n,N,\eta)} = -\frac{\left(k + \eta \mathcal{E}^{(l+n,N,\eta)}\right)^2}{2\hbar^2 (l+n+\frac{N-1}{2})^2}, \tag{65}$$

that is nothing but the algebraic equation derived in [17] (see also [18]), which can be easily solved for the *perturbed* energy:

$$\mathcal{E}^{(l+n,N,\eta)} = \frac{-\hbar^2\left(n+l+\frac{N-1}{2}\right)^2 - \eta k + \sqrt{[\hbar^2\left(n+l+\frac{N-1}{2}\right)^2 + \eta k]^2 - \eta^2 k^2}}{\eta^2}. \tag{66}$$

Actually, the closed formula (66), as well as its inverse, can be cast in a much simpler and illuminating form. Indeed, omitting for simplicity the dependence upon N, first of all we notice that, introducing the parameter $\epsilon \doteq \frac{\eta}{k} > 0$ and defining

$$X \doteq \mathcal{E}^{(l+n,0)} = -\frac{k^2}{2\hbar^2(l+n+\frac{N-1}{2})^2}, \qquad Y \doteq \mathcal{E}^{(l+n,\epsilon)}, \tag{67}$$

we can rewrite (66) as

$$X = \frac{Y}{1+(\epsilon Y)^2}. \tag{68}$$

Moreover, the same parameter ϵ can be reabsorbed through a trivial rescaling ($\epsilon X = \mathcal{X}$, $\epsilon Y = \mathcal{Y}$), yielding

$$\mathcal{X} = \frac{\mathcal{Y}}{1+\mathcal{Y}^2}; \qquad (\mathcal{Y} \le 0,\ \mathcal{X} \le 0), \tag{69}$$

entailing

$$(1+\mathcal{Y}^2) - \mathcal{Y}/\mathcal{X} = 0 \quad \Rightarrow \quad \mathcal{Y} = 1/(2\mathcal{X}) + \sqrt{1/(4\mathcal{X}^2) - 1}. \tag{70}$$

Or, in parametric form,

$$\mathcal{Y} = -\sinh \mathcal{T}, \quad \mathcal{X} = -\frac{\sinh \mathcal{T}}{\cosh^2 \mathcal{T}} \quad (\mathcal{T} > 0). \tag{71}$$

In other words, the (*discrete*) *distributions* of the energy eigenvalues in the undeformed and deformed cases are exactly interpolated by the continuous flows (71).

3˙2. *The quantum Darboux III system.* – In this last subsection our purpose is to derive, using the same technique, the discrete spectrum of the DIII Hamiltonian:

$$\hat{\mathcal{H}}_\lambda(\mathbf{q},\mathbf{p}) = -\frac{\hbar^2}{2(1+\lambda \mathbf{q}^2)}\nabla^2 + \frac{\omega^2 \mathbf{q}^2}{2(1+\lambda \mathbf{q}^2)}, \tag{72}$$

where $\mathbf{q} = (q_1, \ldots, q_N) \in \mathbb{R}^N$ and $\omega, \lambda \in \mathbb{R}^+$. In this case, the radial Schrödinger equation written in hyperspherical coordinates reads [11]:

$$\frac{1}{1+\lambda r^2}\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\hbar^2\,(N-1)}{2r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\hbar^2 l(l+N-2)}{2r^2} + \frac{\Omega^2 r^2}{2}\right]\Psi(r) = \mathcal{E}\Psi(r), \tag{73}$$

that can be rewritten as

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\hbar^2\,(N-1)}{2r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\hbar^2 l(l+N-2)}{2r^2} + \frac{\Omega^2 r^2}{2}\right]\Psi(r) = \mathcal{E}\Psi(r), \tag{74}$$

where $\Omega^2 \doteq \omega^2 - 2\lambda\mathcal{E}$. Now, as in the Taub-NUT case, we get rid of the term containing the first derivative by means of the gauge transformation (46).

Performing the same steps, we can rewrite the last equation as follows:

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_1^{(l,N,\lambda)}(r)\right]\Phi(r) = \mathcal{E}\,\Phi(r), \tag{75}$$

the latter suitable to be cast in the framework of SUSYQM. Accordingly, we assume the first SUSY partner potential to be defined as

$$\tilde{V}_1^{(l,N,\lambda)}(r) \doteq V_1^{(l,N,\lambda)}(r) - \mathcal{E}_0, \tag{76}$$

which can be expressed in terms of the Super-potential $\mathcal{W}(r)$ through the equation:

$$\begin{aligned}\tilde{V}_1^{(l,N,\lambda)}(r) = V_1^{(l,N,\lambda)}(r) - \mathcal{E}_0 &= \frac{\hbar^2\left[l(l+N-2) + \frac{(N-1)(N-3)}{4}\right]}{2r^2} + \frac{\Omega^2 r^2}{2} - \mathcal{E}_0 \\ &= \mathcal{W}^2(r) - \frac{\hbar}{\sqrt{2}}\mathcal{W}'(r).\end{aligned} \tag{77}$$

As an ansatz for the Super-potential we try the natural one:

$$\mathcal{W}(r) = \mathcal{A}\,r - \frac{\mathcal{B}}{r}, \tag{78}$$

where the coefficients $\mathcal{A}, \mathcal{B}$ wait to be determined. Doing this, we get the equality:

$$\begin{aligned}&\frac{\hbar^2\left[l(l+N-2) + \frac{(N-1)(N-3)}{4}\right]}{2r^2} + \frac{\Omega^2 r^2}{2} - \mathcal{E}_0 \\ &= \frac{[\mathcal{B}^2 - \frac{\hbar}{\sqrt{2}}\mathcal{B}]}{r^2} + \mathcal{A}^2 r^2 - 2\mathcal{A}\mathcal{B} - \frac{\hbar}{\sqrt{2}}\mathcal{A},\end{aligned} \tag{79}$$

that implies:

$$\begin{cases} \mathcal{B}^2 - \frac{\hbar}{\sqrt{2}}\mathcal{B} = \frac{\hbar^2}{2}\left[l(l+N-2) + \frac{(N-1)(N-3)}{4}\right] \\ \mathcal{A}^2 = \frac{\Omega^2}{2} \\ 2\mathcal{A}\mathcal{B} + \frac{\hbar}{\sqrt{2}}\mathcal{A} = \mathcal{E}_0. \end{cases} \tag{80}$$

Solving for $\mathcal{B}$ and substituting the result in the remaining two equations we get:

$$\begin{cases} \mathcal{B} = \frac{\hbar}{2\sqrt{2}}\left[1 + \sqrt{1 + 4l(l+N-2) + (N-1)(N-3)}\right] = \frac{\hbar}{2\sqrt{2}}[N+2l-1], \\ \mathcal{A} = \frac{\Omega}{\sqrt{2}}, \\ \mathcal{E}_0 = \frac{\hbar\Omega}{2}[N+2l]. \end{cases} \tag{81}$$

We are now enabled to write down the Super-potential $\mathcal{W}(r)$, and consequently the SUSY partner of $\tilde{V}_1^{(l,N,\lambda)}$. In fact, taking into account that

$$\mathcal{W}(r) = \frac{\Omega}{\sqrt{2}}r - \frac{\hbar}{2\sqrt{2}r}[N+2l-1], \tag{82}$$

by using the second Riccati equation we get the expression:

$$\begin{aligned} \tilde{V}_2^{(l,N,\lambda)}(r) &= \mathcal{W}^2(r) + \frac{\hbar}{\sqrt{2}}\mathcal{W}'(r) \\ &= \frac{\hbar^2\left[l(l+N-2) + \frac{(N-1)(N-3)}{4} + [N+2l-1]\right]}{2r^2} + \frac{\Omega^2 r^2}{2} - \frac{\hbar\Omega}{2}[N+2l-2]. \end{aligned} \tag{83}$$

Finally, the SUSY partner potentials turn out to be:

$$\begin{cases} \tilde{V}_1^{(l,N,\lambda)}(r) = \frac{\hbar^2\left[l(l+N-2) + \frac{(N-1)(N-3)}{4}\right]}{2r^2} + \frac{\Omega^2 r^2}{2} - \frac{\hbar\Omega}{2}[N+2l], \\ \tilde{V}_2^{(l,N,\lambda)}(r) = \frac{\hbar^2\left[l(l+N) + \frac{(N-1)(N+1)}{4}\right]}{2r^2} + \frac{\Omega^2 r^2}{2} - \frac{\hbar\Omega}{2}[N+2l-2]. \end{cases} \tag{84}$$

In this case, if we take $f(l) = l$ and $g(N) = N+2$, we obtain

$$\tilde{V}_2^{(l,N,\lambda)}(r) = \tilde{V}_1^{(l,N+2,\lambda)}(r) + \mathcal{R}(l,N) = \tilde{V}_1^{(l,N+2,\lambda)}(r) + 2\,\hbar\,\Omega. \tag{85}$$

The additional term does not depend on the parameters l, N and the spectrum reads:

$$\mathcal{E}_n^{(l,N,\lambda)} = \mathcal{E}_0 + \sum_{j=1}^{n} 2\hbar\Omega = \frac{\hbar\Omega}{2}[N+2l] + 2n\hbar\Omega = \hbar\Omega\left(2n + l + \frac{N}{2}\right), \tag{86}$$

which corresponds to the solution found in [11]. Therefore, also in the DIII case the spectrum is defined through a quadratic equation:

$$(87)\qquad (\mathcal{E}_n^{(l,N,\lambda)})^2 + 2\lambda\hbar^2\left(2n+l+\frac{N}{2}\right)^2 \mathcal{E}_n^{(l,N,\lambda)} - \hbar^2\omega^2\left(2n+l+\frac{N}{2}\right)^2 = 0.$$

Its *physical solution* (corresponding to the positive branch of the square root) reads

$$(88)\qquad \begin{aligned}\mathcal{E}_n^{(l,N,\lambda)} &= -\lambda\hbar^2\left(2n+l+\frac{N}{2}\right)^2 \\ &\quad +\hbar\left(2n+l+\frac{N}{2}\right)\sqrt{\omega^2+\hbar^2\lambda^2\left(2n+l+\frac{N}{2}\right)}.\end{aligned}$$

As it was the case for the Taub-NUT system, it is convenient to introduce in (87) the natural parameter $\epsilon \doteq \frac{2\lambda}{\omega^2}$: it turns out that maximal degeneracy is present at any order and again, having made the right choice for the physical branch, all the coefficients in perturbative expansion are simple analytic functions of the unperturbed energy levels. Indeed we can write (87) as

$$(89)\qquad X^2(\epsilon) + \epsilon X(\epsilon)X(0)^2 - X(0)^2 = 0,$$

whence

$$(90)\qquad X(0)^2 = \frac{X^2(\epsilon)}{1-\epsilon X(\epsilon)}.$$

A simple but crucial remark is that (90) makes sense iff $\epsilon X(\epsilon)$ is confined to the open interval $(0,1)$. In terms of the original definitions this restriction is equivalent to

$$(91)\qquad 0 < E < \frac{\omega^2}{2\lambda}.$$

Once this point has been clarified, we can assert that even for DIII the parameter ϵ can be reabsorbed by rescaling the variables ($\epsilon X(\epsilon) = \mathcal{Y}$, $\epsilon X(0) = \mathcal{X}$), arriving at the equation:

$$(92)\qquad \mathcal{Y}^2 + (\mathcal{Y}-1)\mathcal{X}^2 = 0; \qquad 0 < \mathcal{Y} < 1,$$

whose "physical" solution is given by

$$(93)\qquad \mathcal{Y} = -\mathcal{X}^2/2 + \mathcal{X}\sqrt{1+\mathcal{X}^2/4},$$

which is consistent with the bounds (91), and is amenable to the parametric form (compare with (71)):

$$(94)\qquad \mathcal{X} = 2\sinh\mathcal{T}, \qquad \mathcal{Y} = 1 - e^{-2\mathcal{T}} \qquad (\mathcal{T} > 0).$$

Hence, unlike the flat case, in the curved case the energy levels are no longer equally spaced, but approach the limit point $\mathcal{E}^{\infty} = \frac{\omega^2}{2\lambda}$ as the quantum number $n + l$ diverges. As it has been already pointed out in [11] this implies that above such limit point the energy spectrum becomes continuous.

4. – Concluding remarks

We end this paper by adding a few remarks, which might be useful to outline some possible future developments. One important question that has to be faced, and hopefully solved, concerns the extension of the approaches used in this paper to the whole classes of systems that have been shown by V. Perlick [2] to be, at a classical level, multiparametric families of maximally superintegrable deformations of HO and KC. At the quantum level, this question got an essentially positive answer in a seminal paper by S. Post and D. Riglioni [19]. We expect that an analogous affirmative answer will apply for the classical systems as well, as it is suggested by the findings contained in [4].

A further issue worthing a deeper analysis, and only partially answered by the authors in two very recent papers [9, 10], has to do with the classical and quantum behaviour of both Taub-NUT and DIII Hamiltonians for negative values of the deformation parameter, such that the potential acquires a confining shape.

Finally, a fairly general and intriguing question concerns the *discretization* of the systems described above. Starting from the remarkable results already available in the literature, mostly due to S. Odake and R. Sasaki [20], one may ask for a finite-difference version of the Schrödinger equation on a curved space preserving (maximal) superintegrability. This should imply in general a nontrivial discretisation both of the potential and of the kinetic energy, entailing then the onset of novel features related with the notion of "discrete curvature".

* * *

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