PLURIPOTENTIAL THEORY AND CONVEX BODIES: LARGE DEVIATION PRINCIPLE

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Abstract. We continue the study in [2] in the setting of weighted pluripotential theory arising from polynomials associated to a convex body P in $(\mathbb{R}^+)^d$. Our goal is to establish a large deviation principle in this setting specifying the rate function in terms of $P-$pluripotential-theoretic notions. As an important preliminary step, we first give an existence proof for the solution of a Monge-Ampère equation in an appropriate finite energy class. This is achieved using a variational approach.

Contents

1. Introduction

As in [2], we fix a convex body $P \subset (\mathbb{R}^+)^d$ and we define the logarithmic indicator function

$$H_P(z) := \sup_{J\in P} \log |z^J| := \sup_{(j_1,\ldots,j_d)\in P} \log[|z_1|^{j_1}\cdots|z_d|^{j_d}]. \tag{1.1}$$

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We assume throughout that

$$\Sigma \subset kP \text{ for some } k \in \mathbb{Z}^+ \tag{1.2}$$

where

$$\Sigma := \{(x_1, ..., x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \ \sum_{j=1}^{d} x_i \leq 1\}.$$

Then

$$H_P(z) \geq \frac{1}{k} \max_{j=1,...,d} \log^+ |z_j|$$

where $\log^+ |z_j| = \max[0, \log |z_j|]$. We define

$$L_P = L_P(\mathbb{C}^d) := \{u \in PSH(\mathbb{C}^d) : u(z) - H_P(z) = O(1), \ |z| \to \infty\},$$

and

$$L_{P,+} = L_{P,+}(\mathbb{C}^d) = \{u \in L_P(\mathbb{C}^d) : u(z) \geq H_P(z) + C_u\}.$$

These are generalizations of the classical Lelong classes when $P = \Sigma$. We define the finite-dimensional polynomial spaces

$$Poly(nP) := \{p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C}\}$$

for $n = 1, 2, ...$ where $z^J = z_1^{j_1} \cdots z_d^{j_d}$ for $J = (j_1, ..., j_d)$. For $p \in Poly(nP)$, $n \geq 1$ we have $\frac{1}{n} \log |p| \in L_P$; also each $u \in L_{P,+}(\mathbb{C}^d)$ is locally bounded in $\mathbb{C}^d$. For $P = \Sigma$, we write $Poly(nP) = \mathcal{P}_n$.

Given a compact set $K \subset \mathbb{C}^d$, one can define various pluripotential-theoretic notions associated to K related to L_P and the polynomial spaces $Poly(nP)$. Our goal in this paper is to prove some probabilistic properties of random point processes on K utilizing these notions and their weighted counterparts. We require an existence proof for the solution of a Monge-Ampère equation in an appropriate finite energy class; this is done in Theorem 2.8 using a variational approach and is of interest on its own. The third section recalls appropriate definitions and properties in $P-$pluripotential theory, mostly following [2]. As in [2], our spaces $Poly(nP)$ do not necessarily arise as holomorphic sections of tensor powers of a line bundle. Subsection 3.3 includes a standard elementary probabilistic result on almost sure convergence of probability measures associated to random arrays on K to a $P-$pluripotential-theoretic equilibrium measure. Section 4 sets up the machinery for the more subtle large deviation principle (LDP), Theorem 5.1, for which we provide two proofs (analogous to those in [9]). As in [9], the first proof was inspired by [6] and the second proof was utilized by Berman in [5]. The reader will find far-reaching applications and interpretations of LDP's in the appropriate settings of holomorphic line bundles over a compact, complex manifold in [5]. In particular, the case where P is a convex integral polytope (vertices in $\mathbb{Z}^d$) which is the moment polytope for a toric manifold (P is Delzant) is covered in [5].

2. Monge-Ampère and $P-$pluripotential theory

2.1. **Monge-Ampère equations with prescribed singularity.**

In this section, (X, ω) is a compact Kähler manifold of dimension d.

2.1.1. *Quasi-plurisubharmonic functions.* A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-plurisubharmonic (quasi-psh) if locally $u = \rho + \varphi$, where φ is plurisubharmonic and ρ is smooth.

We let $PSH(X, \omega)$ denote the set of ω-psh functions, i.e. quasi-psh functions u such that $\omega_u := \omega + dd^c u \geq 0$ in the sense of currents on X.

Given $u, v \in PSH(X, \omega)$ we say that u is more singular than v (and we write $u \prec v$) if $u \leq v + C$ on X, for some constant C. We say that u has the same singularity as v (and we write $u \simeq v$) if $u \prec v$ and $v \prec u$.

Given $\phi \in PSH(X, \omega)$, we let $PSH(X, \omega, \phi)$ denote the set of ω-psh functions u which are more singular than ϕ.

2.1.2. *Nonpluripolar Monge-Ampère measure.* For bounded ω-psh functions $u_1, ..., u_d$, the Monge-Ampère product $(\omega + dd^c u_1) \wedge ... \wedge (\omega + dd^c u_d)$ is well-defined as a positive Radon measure on X (see [14], [3]). For general ω-psh functions $u_1, ..., u_d$, the sequence of positive measures

$$\mathbf{1}_{\cap\{u_j > -k\}}(\omega + dd^c \max(u_1, -k)) \wedge ... \wedge (\omega + dd^c \max(u_d, -k))$$

is non-decreasing in k and the limiting measure, which is called the nonpluripolar product of $\omega_{u_1}, ..., \omega_{u_d}$, is denoted by

$$\omega_{u_1} \wedge ... \wedge \omega_{u_d}.$$

When $u_1 = ... = u_d = u$ we write $\omega_u^d := \omega_u \wedge ... \wedge \omega_u$. Note that by definition $\int_X \omega_{u_1} \wedge ... \wedge \omega_{u_d} \leq \int_X \omega^d$.

It was proved in [20, Theorem 1.2] and [11, Theorem 1.1] that the total mass of nonpluripolar Monge-Ampère products is decreasing with respect to singularity type. More precisely,

Theorem 2.1. *Let $\omega_1, ..., \omega_d$ be Kähler forms on X. If $u_j \leq v_j$, $j = 1, ..., d$, are ω_j-psh functions then*

$$\int_X (\omega_1 + dd^c u_1) \wedge ... \wedge (\omega_d + dd^c u_d) \leq \int_X (\omega_1 + dd^c v_1) \wedge ... \wedge (\omega_d + dd^c v_d).$$

As noted above, for a general ω-psh function u we have the estimate $\int_X \omega_u^d \leq \int_X \omega^d$. Following [15] we let $\mathcal{E}(X, \omega)$ denote the set of all ω-psh functions with maximal total mass, i.e.

$$\mathcal{E}(X, \omega) := \left\{ u \in PSH(X, \omega) : \int_X \omega_u^d = \int_X \omega^d \right\}.$$

Given $\phi \in PSH(X, \omega)$, we define

$$\mathcal{E}(X, \omega, \phi) := \left\{ u \in PSH(X, \omega, \phi) \ : \ \int_X \omega_u^d = \int_X \omega_\phi^d \right\}.$$

Proposition 2.2. *Let $\phi \in PSH(X, \omega)$. The following are equivalent :*

(1) $\mathcal{E}(X, \omega, \phi) \cap \mathcal{E}(X, \omega) \neq \emptyset$;
(2) $\phi \in \mathcal{E}(X, \omega)$;
(3) $\mathcal{E}(X, \omega, \phi) \subset \mathcal{E}(X, \omega)$.

Proof. We first prove (1) $\Longrightarrow$ (2). If $u \in \mathcal{E}(X, \omega, \phi) \cap \mathcal{E}(X, \omega)$ then $\int_X \omega_u^d = \int_X \omega^d$. On the other hand, since u is more singular than ϕ, Theorem 2.1 ensures that

$$\int_X \omega^d = \int_X \omega_u^d \leq \int_X \omega_\phi^d \leq \int_X \omega^d,$$

hence equality holds, proving that $\phi \in \mathcal{E}(X, \omega)$.

Now we prove (2) $\Longrightarrow$ (3). If $\phi \in \mathcal{E}(X, \omega)$ and $u \in \mathcal{E}(X, \omega, \phi)$ then

$$\int_X \omega_u^d = \int_X \omega_\phi^d = \int_X \omega^d,$$

hence $u \in \mathcal{E}(X, \omega)$.

Finally (3) $\Longrightarrow$ (1) is obvious. □

Proposition 2.3. *Assume that $\phi_j \in PSH(X, \omega_j)$, $j = 1, ..., d$ with $\int_X (\omega_j + dd^c \phi_j)^d > 0$. If $u_j \in \mathcal{E}(X, \omega_j, \phi_j)$, $j = 1, ..., d$, then*

$$\int_X (\omega_1 + dd^c u_1) \wedge ... \wedge (\omega_d + dd^c u_d) = \int_X (\omega_1 + dd^c \phi_1) \wedge ... \wedge (\omega_d + dd^c \phi_d).$$

Proof. Theorem 2.1 gives one inequality. The other one follows from [11, Proposition 3.1 and Theorem 3.14]. □

2.1.3. *Model potentials.* For a function $f : X \to \mathbb{R} \cup \{-\infty\}$, we let f^* denote its uppersemicontinuous (usc) regularization, i.e.

$$f^*(x) := \limsup_{X \ni y \to x} f(y).$$

Given $\phi \in PSH(X, \omega)$, following J. Ross and D. Witt Nyström [18], we define

$$P_\omega[\phi] := \left(\lim_{t \to +\infty} P_\omega(\min(\phi + t, 0)) \right)^* .$$

Here, for a function f, $P_\omega(f)$ is defined as

$$P_\omega(f) := (x \mapsto \sup\{u(x) : u \in PSH(X, \omega), u \le f\})^* .$$

It was shown in [11, Theorem 3.8] that the nonpluripolar Monge-Ampère measure of $P_\omega[\phi]$ is dominated by Lebesgue measure:

$$(\omega + dd^c P_\omega[\phi])^d \le \mathbf{1}_{\{P_\omega[\phi]=0\}} \omega^d \le \omega^d. \tag{2.1}$$

This fact plays a crucial role in solving the complex Monge-Ampère equation. For the reader's convenience, we note that in the notation of [11] (on the left)

$$P_{[\omega,\phi]}(0) = P_\omega[\phi].$$

Definition 2.4. A function $\phi \in PSH(X, \omega)$ is called a model potential if $\int_X \omega_\phi^d > 0$ and $P_\omega[\phi] = \phi$. A function $u \in PSH(X, \omega)$ has model type singularity if u has the same singularity as $P_\omega[u]$; i.e., $u - P_\omega[u]$ is bounded on X.

There are plenty of model potentials. If $\varphi \in PSH(X, \omega)$ with $\int_X \omega_\varphi^d > 0$ then, by [11, Theorem 3.12], $P_\omega[\varphi]$ is a model potential. In particular, if $\int_X \omega_\varphi^d = \int_X \omega^d$ (i.e. $\varphi \in \mathcal{E}(X, \omega)$) then $P_\omega[\varphi] = 0$.

We will use the following property of model potentials proved in [11, Theorem 3.12]: if ϕ is a model potential then

$$u \in PSH(X, \omega, \phi) \Longrightarrow u - \sup_X u \le \phi. \tag{2.2}$$

In the sequel we always assume that ϕ has *model type singularity* and *small unbounded locus*; i.e., ϕ is locally bounded outside a closed complete pluripolar set, allowing us to use the variational approach of [7] as explained in [11].

2.1.4. *The variational approach.* We call a measure which puts no mass on pluripolar sets a *nonpluripolar measure*. For a positive nonpluripolar measure μ on X we let L_μ denote the following linear functional on $PSH(X, \omega, \phi)$:

$$L_\mu(u) := \int_X (u - \phi) d\mu.$$

For $u \in PSH(X, \omega)$ with $u \simeq \phi$, we define the Monge-Ampère energy

$$\mathbf{E}_\phi(u) := \frac{1}{(d+1)} \sum_{k=0}^{d} \int_X (u - \phi) \omega_u^k \wedge \omega_\phi^{d-k}. \tag{2.3}$$

It was shown in [11, Theorem 4.10] (by adapting the arguments of [7]) that $\mathbf{E}_\phi$ is non-decreasing and concave along affine curves, giving rise to its trivial extension to $PSH(X, \omega, \phi)$.

We define

$$\mathcal{E}^1(X, \omega, \phi) := \{u \in PSH(X, \omega, \phi) : \mathbf{E}_\phi(u) > -\infty\}. \tag{2.4}$$

The following criterion was proved in [11, Theorem 4.13]:

Proposition 2.5. *Let $u \in PSH(X, \omega, \phi)$. Then $u \in \mathcal{E}^1(X, \omega, \phi)$ iff $u \in \mathcal{E}(X, \omega, \phi)$ and $\int_X (u - \phi)\omega_u^d > -\infty$.*

Lemma 2.6. *If E is pluripolar then there exists $u \in \mathcal{E}^1(X, \omega, \phi)$ such that $E \subset \{u = -\infty\}$.*

Proof. Without loss of generality we can assume that ϕ is a model potential. Then (2.1) gives $\int_X |\phi|\omega_\phi^d = 0$. It follows from [7, Corollary 2.11] that there exists $v \in \mathcal{E}^1(X, \omega, 0)$, $v \leq 0$, such that $E \subset \{v = -\infty\}$. Set $u := P_\omega(\min(v, \phi))$. Then $E \subset \{u = -\infty\}$ and we claim that $u \in \mathcal{E}^1(X, \omega, \phi)$. For each $j \in \mathbb{N}$ we set $v_j := \max(v, -j)$ and $u_j := P_\omega(\min(v_j, \phi))$. Then u_j decreases to u and $u_j \simeq \phi$. Using [11, Theorem 4.10 and Lemma 4.15] it suffices to check that $\{\int_X |u_j - \phi|\omega_{u_j}^d\}$ is uniformly bounded. It follows from [11, Lemma 3.7] that

$$\begin{aligned}\int_X |u_j - \phi|\omega_{u_j}^d \leq \int_X |u_j|\omega_{u_j}^d &\leq \int_X |v_j|\omega_{v_j}^d + \int_X |\phi|\omega_\phi^d \\ &= \int_X |v_j|\omega_{v_j}^d.\end{aligned}$$

The fact that $\int_X |v_j|\omega_{v_j}^d$ is uniformly bounded follows from [15, Corollary 2.4] since $v \in \mathcal{E}^1(X, \omega, 0)$. This concludes the proof. □

Lemma 2.7. *Assume that* $\mathcal{E}^1(X,\omega,\phi) \subset L^1(X,\mu)$*. Then, for each* $C > 0$*,* L_μ *is bounded on*

$$E_C := \{u \in PSH(X,\omega,\phi) : \sup_X u \leq 0 \text{ and } \mathbf{E}_\phi(u) \geq -C\}.$$

Proof. By concavity of $\mathbf{E}_\phi$ the set E_C is convex. We now show that E_C is compact in the $L^1(X,\omega^d)$ topology. Let $\{u_j\}$ be a sequence in E_C. We claim that $\{\sup_X u_j\}$ is bounded. Indeed, by [11, Theorem 4.10]

$$\mathbf{E}_\phi(u_j) \leq \int_X (u_j - \phi)\omega_\phi^d$$
$$\leq (\sup_X u_j) \int_X \omega_\phi^d + \int_X (u_j - \sup_X u_j - \phi)\omega_\phi^d.$$

It follows from (2.2) that $u_j - \sup_X u_j \leq P_\omega[\phi] \leq \phi + C_0$, where C_0 is a constant. The boundedness of $\{\sup_X u_j\}$ then follows from that of $\{\mathbf{E}_\phi(u_j)\}$ and the above estimate. This proves the claim.

A subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, converges in $L^1(X,\omega^d)$ to $u \in PSH(X,\omega)$ with $\sup_X u \leq 0$. Since $u_j - \sup_X u_j \leq \phi + C_0$, we have $u - \sup_X u \leq \phi + C_0$. This proves that $u \in PSH(X,\omega,\phi)$. The upper semicontinuity of $\mathbf{E}_\phi$ (see [11, Proposition 4.19]) ensures that $\mathbf{E}_\phi(u) \geq -C$, hence $u \in E_C$. This proves that E_C is compact in the $L^1(X,\omega^d)$ topology.

The result then follows from [7, Proposition 3.4]. □

The goal of this section is to prove the following result:

Theorem 2.8. *Assume that* μ *is a nonpluripolar positive measure on* X *such that* $\mu(X) = \int_X \omega_\phi^d$*. The following are equivalent*

1. *μ has finite energy, i.e.,* L_μ *is finite on* $\mathcal{E}^1(X,\omega,\phi)$*;*
2. *there exists* $u \in \mathcal{E}^1(X,\omega,\phi)$ *such that* $\omega_u^d = \mu$*;*
3. *there exists a unique* $u \in \mathcal{E}^1(X,\omega,\phi)$ *such that*

$$F_\mu(u) = \max_{v \in \mathcal{E}^1(X,\omega,\phi)} F_\mu(v) < +\infty$$

where $F_\mu = \mathbf{E}_\phi - L_\mu$*.*

Remark 2.9. It was shown in [11, Theorem 4.28] that a unique (normalized) solution u in $\mathcal{E}(X,\omega,\phi)$ always exists (without the finite energy assumption on μ). But that proof does not give a solution in $\mathcal{E}^1(X,\omega,\phi)$. Below, we will follow the proof of [11, Theorem 4.28] and use the finite energy condition, $\mathcal{E}^1(X,\omega,\phi) \subset L^1(X,\mu)$, to prove that u belongs to $\mathcal{E}^1(X,\omega,\phi)$.

Lemma 2.10. *Assume that $\mathcal{E}^1(X,\omega,\phi) \subset L^1(X,\mu)$. Then there exists a positive constant C such that, for all $u \in \mathcal{E}^1(X,\omega,\phi)$ with $\sup_X u = 0$,*

$$L_\mu(u) \geq -C(1 + |\mathbf{E}_\phi(u)|^{1/2}). \tag{2.5}$$

The proof below uses ideas in [15, 7].

Proof. Since ϕ has model type singularity, it follows from [11, Theorem 4.10] that $\mathbf{E}_\phi - \mathbf{E}_{P_\omega[\phi]}$ is bounded. Without loss of generality we can assume in this proof that $\phi = P_\omega[\phi]$. Fix $u \in \mathcal{E}^1(X,\omega,\phi)$ such that $\sup_X u = 0$ and $|\mathbf{E}_\phi(u)| > 1$. Then, by [11, Theorem 3.12], $u \leq \phi$. Set $a = |\mathbf{E}_\phi(u)|^{-1/2} \in (0,1)$, and $v := au + (1-a)\phi \in \mathcal{E}^1(X,\omega,\phi)$. We estimate $\mathbf{E}_\phi(v)$ as follows

$$\begin{aligned}
(d+1)\mathbf{E}_\phi(v) &= a \sum_{k=0}^{d} \int_X (u-\phi)\omega_v^k \wedge \omega_\phi^{d-k} \\
&= a \sum_{k=0}^{d} \int_X (u-\phi)(a\omega_u + (1-a)\omega_\phi)^k \wedge \omega_\phi^{d-k} \\
&\geq C(d)a \int_X (u-\phi)\omega_\phi^d + C(d)a^2 \sum_{k=0}^{d} \int_X (u-\phi)\omega_u^k \wedge \omega_\phi^d,
\end{aligned}$$

where $C(d)$ is a positive constant which only depends on d. It follows from $\phi = P_\omega[\phi]$ and [11, Theorem 3.8] that $\omega_\phi^d \leq \omega^d$ (recall (2.1)). This together with [14, Proposition 2.7] give

$$\int_X (u-\phi)\omega_\phi^d \geq -C_1,$$

for a uniform constant C_1. Therefore,

$$(d+1)\mathbf{E}_\phi(v) \geq -C_1 C(d)a + C_2 a^2 \mathbf{E}_\phi(u) \geq -C_3.$$

It thus follows from Lemma 2.7 that $L_\mu(v) \geq -C_4$ for a uniform constant $C_4 > 0$. Thus

$$\int_X (u-\phi)d\mu \geq -C_4/a,$$

which gives (2.5). □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Without loss of generality we can assume that ϕ is a model potential. We first prove (1) $\Longrightarrow$ (2). We write $\mu = f\nu$, where ν is a nonpluripolar positive measure satisfying, for all Borel subsets $B \subset X$,

$$\nu(B) \leq A\mathrm{Cap}_\phi(B),$$

for some positive constant A, and $0 \leq f \in L^1(X, \nu)$ (cf., [11, Lemma 4.26]). Here Cap_ϕ is defined as

$$\mathrm{Cap}_\phi(B) := \sup\left\{ \int_B \omega_u^d : u \in PSH(X, \omega),\ \phi - 1 \leq u \leq \phi \right\}.$$

Set, for $k \in \mathbb{N}$, $\mu_k := c_k \min(f, k)\nu$ where $c_k > 0$ is chosen so that $\mu_k(X) = \int_X \omega_\phi^d$; this is needed in order to solve the Monge-Ampère equation in the class $\mathcal{E}^1(X, \omega, \phi)$. For k large enough, $1 \leq c_k \leq 2$ and $c_k \to 1$ as $k \to +\infty$. It follows from [11, Theorem 4.25] that there exists $u_j \in \mathcal{E}^1(X, \omega, \phi)$, $\sup_X u_j = 0$, such that $\omega_{u_j}^d = \mu_j$; by [11, Theorem 3.12], $u_j \leq \phi$. A subsequence of $\{u_j\}$ which, by abuse of notation, will be denoted by $\{u_j\}$, converges in $L^1(X, \mu)$ to $u \in PSH(X, \omega)$ with $u \leq \phi$. Define $v_k := (\sup_{j \geq k} u_j)^*$. Then $v_k \searrow u$ and $\sup_X v_k = 0$. It follows from (2.5) and [11, Theorem 4.10] that

$$\begin{aligned} |\mathbf{E}_\phi(u_j)| &\leq \int_X |u_j - \phi| \omega_{u_j}^d \leq 2 \int_X |u_j - \phi| d\mu \\ &\leq 2C(1 + |\mathbf{E}_\phi(u_j)|^{1/2}). \end{aligned}$$

Therefore $\{|\mathbf{E}_\phi(u_j)|\}$ is bounded, hence so is $\{|\mathbf{E}_\phi(v_j)|\}$ since $\mathbf{E}_\phi$ is non-decreasing. It then follows from [11, Lemma 4.15] that $u \in \mathcal{E}^1(X, \omega, \phi)$.

Now, repeating the arguments of [11, Theorem 4.28] we can show that $\omega_u^d = \mu$, finishing the proof of (1) $\Longrightarrow$ (2).

We next prove (2) $\Longrightarrow$ (3). Assume that $\mu = \omega_u^d$ for some $u \in \mathcal{E}^1(X, \omega, \phi)$. For all $v \in \mathcal{E}^1(X, \omega, \phi)$, by [11, Theorem 4.10] and Proposition 2.5 we have

$$\begin{aligned} L_\mu(v) &= \int_X (v - \phi)\omega_u^d \\ &= \int_X (v - u)\omega_u^d + \int_X (u - \phi)\omega_u^d \\ &\geq \mathbf{E}_\phi(v) - \mathbf{E}_\phi(u) + \int_X (u - \phi)\omega_u^d > -\infty. \end{aligned}$$

Hence L_μ is finite on $\mathcal{E}^1(X,\omega,\phi)$. Now, for all $v \in \mathcal{E}^1(X,\omega,\phi)$, by [11, Theorem 4.10] we have

$$F_\mu(v) - F_\mu(u) = \mathbf{E}_\phi(v) - \mathbf{E}_\phi(u) - \int_X (v-u)\omega_u^d \le 0.$$

This gives (3). Finally, (3) $\Longrightarrow$ (1) is obvious. □

2.2. **Monge-Ampère equations on $\mathbb{C}^d$ with prescribed growth.** As in the introduction we let P be a convex body contained in $(\mathbb{R}^+)^d$ and fix $r > 0$ such that $P \subset r\Sigma$. We assume (1.2); i.e., $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$. This ensures that H_P in (1.1) is locally bounded on $\mathbb{C}^d$ (and of course $H_P \in L_P^+(\mathbb{C}^d)$). Let $u \in L_P(\mathbb{C}^d)$ and define

$$\tilde{u}(z) := u(z) - \frac{r}{2}\log(1+|z|^2), z \in \mathbb{C}^d. \tag{2.6}$$

Consider the projective space $\mathbb{P}^d$ equipped with the Kähler metric $\omega := r\omega_{FS}$, where

$$\omega_{FS} = dd^c \frac{1}{2}\log(1+|z|^2)$$

on $\mathbb{C}^d$. Then $\tilde{u}$ is bounded from above on $\mathbb{C}^d$. It thus can be extended to $\mathbb{P}^d$ as a function in $PSH(\mathbb{P}^d,\omega)$.

For a plurisubharmonic function u on $\mathbb{C}^d$, we let $(dd^c u)^d$ denotes its nonpluripolar Monge-Ampère measure; i.e., $(dd^c u)^d$ is the increasing limit of the sequence of measures $\mathbf{1}_{\{u>-k\}}(dd^c \max(u,-k))^d$. Then

$$\omega_{\tilde{u}}^d = (\omega + dd^c\tilde{u})^d = (dd^c u)^d \text{ on } \mathbb{C}^d.$$

If $u \in L_P(\mathbb{C}^d)$ then

$$\int_{\mathbb{C}^d} (dd^c u)^d \le \int_{\mathbb{C}^d} (dd^c H_P)^d = d!Vol(P) =: \gamma_d = \gamma_d(P)$$

(cf., equation (2.4) in [2]). We define

$$\mathcal{E}_P(\mathbb{C}^d) := \left\{ u \in L_P(\mathbb{C}^d) : \int_{\mathbb{C}^d} (dd^c u)^d = \gamma_d \right\}.$$

By the construction in (2.6) we have that $\tilde{H}_P \in PSH(\mathbb{P}^d,\omega)$. We define

$$\tilde{\Phi}_P := P_\omega[\tilde{H}_P].$$

The key point here, which follows from [12, Theorem 7.2], is that $\tilde{H}_P$ has model type singularity (recall Definition 2.4) and hence the same singularity as $\tilde{\Phi}_P$. Defining Φ_P on $\mathbb{C}^d$ using (2.6); i.e., for $z \in \mathbb{C}^d$,

$$\Phi_P(z) = \tilde{\Phi}_P(z) + \frac{r}{2}\log(1+|z|^2),$$

we thus have $\Phi_P \in L_{P,+}(\mathbb{C}^d)$. The advantage of using Φ_P is that, by (2.1), $(dd^c\Phi_P)^d \leq \omega^d$ on $\mathbb{C}^d$. Note that $L_{P,+}(\mathbb{C}^d) \subset \mathcal{E}_P(\mathbb{C}^d)$. For $u, v \in L_P^+(\mathbb{C}^d)$ we define

$$E_v(u) := \frac{1}{(d+1)}\sum_{j=0}^{d}\int_{\mathbb{C}^d}(u-v)(dd^cu)^j \wedge (dd^cv)^{d-j}. \tag{2.7}$$

The corresponding global energy (see (2.3)) is defined as

$$\mathbf{E}_{\tilde{v}}(\tilde{u}) := \frac{1}{(d+1)}\sum_{j=0}^{d}\int_{\mathbb{P}^d}(\tilde{u}-\tilde{v})(\omega + dd^c\tilde{u})^j \wedge (\omega + dd^c\tilde{v})^{d-j}.$$

Then E_v is non-decreasing and concave along affine curves in $L_{P,+}(\mathbb{C}^d)$. We extend E_v to $L_P(\mathbb{C}^d)$ in an obvious way. Note that E_v may take the value $-\infty$. We define

$$\mathcal{E}_P^1(\mathbb{C}^d) := \{u \in L_P(\mathbb{C}^d) : E_{H_P}(u) > -\infty\}.$$

We observe that in the above definition we can replace E_{H_P} by E_{Φ_P}, since for $u \in L_{P,+}(\mathbb{C}^d)$, by the cocycle property (cf. Proposition 3.3 [2]),

$$E_{H_P}(u) - E_{H_P}(\Phi_P) = E_{\Phi_P}(u).$$

We thus have the following important identification (see (2.4)):

$$u \in \mathcal{E}_P^1(\mathbb{C}^d) \Longleftrightarrow \tilde{u} \in \mathcal{E}^1(\mathbb{P}^d, \omega, \tilde{\Phi}_P). \tag{2.8}$$

We then have the following local version of Proposition 2.5:

Proposition 2.11. *Let $u \in L_P(\mathbb{C}^d)$. Then $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ iff $u \in \mathcal{E}_P(\mathbb{C}^d)$ and $\int_{\mathbb{C}^d}(u - H_P)(dd^cu)^d > -\infty$. In particular, if supp$(dd^cu)^d$ is compact, $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ iff $\int_{\mathbb{C}^d}(dd^cu)^d = \gamma_d$ and $\int_{\mathbb{C}^d} u(dd^cu)^d > -\infty$.*

Proof. Since $\tilde{H}_P \simeq \tilde{\Phi}_P$,

$$\int_{\mathbb{P}^d}(\tilde{u} - \tilde{H}_P)\omega_{\tilde{u}}^d > -\infty \text{ iff } \int_{\mathbb{P}^d}(\tilde{u} - \tilde{\Phi}_P)\omega_{\tilde{u}}^d > -\infty$$

where $\tilde{u} \in PSH(\mathbb{P}^d, \omega)$ and u are related by (2.6). Moreover, $\Phi_P \in L_{P,+}(\mathbb{C}^d)$ implies $u \leq \Phi_P + c$ so that $\tilde{u} \in PSH(\mathbb{P}^d, \omega, \tilde{\Phi}_P)$. But

$$\int_{\mathbb{P}^d}(\tilde{u} - \tilde{H}_P)\omega_{\tilde{u}}^d = \int_{\mathbb{C}^d}(u - H_P)(dd^cu)^d$$

and the result follows from (2.8) by applying Proposition 2.5 to $\tilde{u}$. For the last statement, note that for general $u \in L_P(\mathbb{C}^d)$ we may have $\int_{\mathbb{C}^d} H_P(dd^c u)^d = +\infty$, but if $(dd^c u)^d$ has compact support then $\int_{\mathbb{C}^d} H_P(dd^c u)^d$ is finite. □

Note that Theorem 2.1 and Proposition 2.3 give the following result:

Theorem 2.12. *Let $u_1, ..., u_d$ be functions in $\mathcal{E}_P(\mathbb{C}^d)$. Then*

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge ... \wedge dd^c u_d = \gamma_d.$$

For $u_1, ..., u_n \in L_{P,+}(\mathbb{C}^d)$ Theorem 2.12 was proved in [1, Proposition 2.7].

Having the correspondence (2.8) we can state a local version of Theorem 2.8; this will be used in the sequel. Let $\mathcal{M}_P(\mathbb{C}^d)$ denote the set of all positive Borel measures μ on $\mathbb{C}^d$ with $\mu(\mathbb{C}^d) = d!Vol(P) = \gamma_d$.

Theorem 2.13. *Assume that $\mu \in \mathcal{M}_P(\mathbb{C}^d)$ is a positive nonpluripolar Borel measure. The following are equivalent*

(1) $\mathcal{E}^1_P(\mathbb{C}^d) \subset L^1(\mathbb{C}^d, \mu)$;
(2) *there exists $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ such that $(dd^c u)^d = \mu$;*
(3) *there exists $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ such that*

$$\mathcal{F}_\mu(u) = \max_{v \in \mathcal{E}^1_P(\mathbb{C}^d)} \mathcal{F}_\mu(v) < +\infty.$$

A priori the functional $\mathcal{F}_\mu$ is defined for $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ by

$$\mathcal{F}_{\mu,\Phi_P}(u) := E_{\Phi_P}(u) - \int_{\mathbb{C}^d} (u - \Phi_P) d\mu.$$

However, using this notation, since

$$\mathcal{F}_{\mu,\Phi_P}(u) - \mathcal{F}_{\mu,H_P}(u) = \mathcal{F}_{\mu,\Phi_P}(H_P),$$

in statement (3) of Theorem 2.13 we can take either of the two definitions $\mathcal{F}_{\mu,\Phi_P}$ or $\mathcal{F}_{\mu,H_P}$ for $\mathcal{F}_\mu$.

Remark 2.14. If μ has compact support in $\mathbb{C}^d$ then $\int_{\mathbb{C}^d} \Phi_P d\mu$ and $\int_{\mathbb{C}^d} H_P d\mu$ are finite. Therefore, the functional $\mathcal{F}_\mu$ can be replaced by

$$u \mapsto E_{H_P}(u) - \int_{\mathbb{C}^d} u d\mu.$$

Using the remark, for $\mu \in \mathcal{M}_P(\mathbb{C}^d)$ with compact support, it is natural to define the Legendre-type transform of E_{H_P}:

$$E^*(\mu) := \sup_{u\in\mathcal{E}_P^1(\mathbb{C}^d)}[E_{H_P}(u) - \int_{\mathbb{C}^d} ud\mu]. \tag{2.9}$$

This functional, which will appear in the rate function for our LDP, will be given a more concrete interpretation using $P-$pluripotential theory in section 4; cf., equation (4.18).

Finally, for future use, we record the following consequence of Lemma 2.6 and the correspondence (2.8).

Lemma 2.15. *If $E \subset \mathbb{C}^d$ is pluripolar then there exists $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ such that $E \subset \{u = -\infty\}$.*

3. $P-$pluripotential theory notions

Given $E \subset \mathbb{C}^d$, the $P-$extremal function of E is

$$V_{P,E}^*(z) := \limsup_{\zeta\to z} V_{P,E}(\zeta)$$

where

$$V_{P,E}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d),\ u \leq 0 \text{ on } E\}.$$

For $K \subset \mathbb{C}^d$ compact, $w : K \to \mathbb{R}^+$ is an admissible weight function on K if $w \geq 0$ is an uppersemicontinuous function with $\{z \in K : w(z) > 0\}$ nonpluripolar. Setting $Q := -\log w$, we write $Q \in \mathcal{A}(K)$ and define the *weighted $P-$extremal function*

$$V_{P,K,Q}^*(z) := \limsup_{\zeta\to z} V_{P,K,Q}(\zeta)$$

where

$$V_{P,K,Q}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d),\ u \leq Q \text{ on } K\}.$$

If $Q = 0$ we write $V_{P,K,Q} = V_{P,K}$, consistent with the previous notation. For $P = \Sigma$,

$$V_{\Sigma,K,Q}(z) = V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}^d),\ u \leq Q \text{ on } K\}$$

is the usual weighed extremal function as in Appendix B of [19].

We write (omitting the dependence on P)

$$\mu_{K,Q} := (dd^cV_{P,K,Q}^*)^d \text{ and } \mu_K := (dd^cV_{P,K}^*)^d$$

for the Monge-Ampère measures of $V^*_{P,K,Q}$ and $V^*_{P,K}$ (the latter if K is not pluripolar). Proposition 2.5 of [2] states that

$$supp(\mu_{K,Q}) \subset \{z \in K : V^*_{P,K,Q}(z) \geq Q(z)\}$$

and $V^*_{P,K,Q} = Q$ q.e. on $supp(\mu_{K,Q})$, i.e., off of a pluripolar set.

3.1. **Energy.** We recall some results and definitions from [2]. For $u, v \in L_{P,+}(\mathbb{C}^d)$, we define the *mutual energy*

$$\mathcal{E}(u,v) := \int_{\mathbb{C}^d} (u-v) \sum_{j=0}^{d} (dd^c u)^j \wedge (dd^c v)^{d-j}.$$

For simplicity, when $v = H_P$, we denote the associated (normalized) energy functional by E:

$$E(u) := E_{H_P}(u) = \frac{1}{d+1} \sum_{j=0}^{d} \int_{\mathbb{C}^d} (u - H_P) dd^c u^j \wedge (dd^c H_P)^{d-j}$$

(recall (2.7)).

For $u, u', v \in L_{P,+}(\mathbb{C}^d)$, and for $0 \leq t \leq 1$, we define

$$f(t) := \mathcal{E}(u + t(u'-u), v),$$

From Proposition 3.1 in [2], $f'(t)$ exists for $0 \leq t \leq 1$ and

$$f'(t) = (d+1) \int_{\mathbb{C}^d} (u'-u)(dd^c(u + t(u'-u)))^d$$

Hence, taking $v = H_P$, we have, for $F(t) := E(u + t(u'-u))$, that

$$F'(t) = \int_{\mathbb{C}^d} (u'-u)(dd^c(u + t(u'-u)))^d.$$

Thus $F'(0) = \int_{\mathbb{C}^d} (u'-u)(dd^c u)^d$ and we write

$$< E'(u), u'-u > := \int (u'-u)(dd^c u)^d. \tag{3.1}$$

We need some applications of a global domination principle. The following version, sufficient for our purposes, follows from [11], Corollary 3.10 (see also Corollary A.2 of [8]).

Proposition 3.1. *Let $u \in L_P(\mathbb{C}^d)$ and $v \in \mathcal{E}_P(\mathbb{C}^d)$ with $u \leq v$ a.e. $(dd^c v)^d$. Then $u \leq v$ in $\mathbb{C}^d$.*

This will be used to prove an approximation result, Proposition 3.3, which itself will be essential in the sequel. First we need a lemma.

Lemma 3.2. *Assume that* $\varphi \leq u, v \leq H_P$ *are functions in* $\mathcal{E}^1_P(\mathbb{C}^d)$. *Then for all* $t > 0$,

$$\int_{\{u \leq H_P - 2t\}} (H_P - u)(dd^c v)^d \leq 2^{d+1} \int_{\{\varphi \leq H_P - t\}} (H_P - \varphi)(dd^c \varphi)^d.$$

In particular, the left hand side converges to 0 *as* $t \to +\infty$ *uniformly in* u, v.

Proof. For $s > 0$, we have the following inclusions of sets:

$$(u \leq H_P - 2s) \subset \left(\varphi \leq \frac{v + H_P}{2} - s\right) \subset (\varphi \leq H_P - s).$$

We first note that the left hand side in the lemma is equal to

$$\int_{\{u \leq H_P - 2t\}} (H_P - u)(dd^c v)^d \tag{3.2}$$

$$= 2t \int_{\{u \leq H_P - 2t\}} (dd^c v)^d + \int_{2t}^{\infty} \left(\int_{\{u \leq H_P - s\}} (dd^c v)^d \right) ds.$$

We claim that, for all $s > 0$,

$$\int_{\{u \leq H_P - 2s\}} (dd^c v)^d \leq 2^d \int_{\{\varphi \leq H_P - s\}} (dd^c \varphi)^d. \tag{3.3}$$

Indeed, the comparison principle ([11, Corollary 3.6]) and the inclusions of sets above give

$$\int_{\{u \leq H_P - 2s\}} (dd^c v)^d \leq \int_{\{\varphi \leq \frac{v+H_P}{2} - s\}} (dd^c v)^d \leq 2^d \int_{\{\varphi \leq \frac{v+H_P}{2} - s\}} \left(dd^c \frac{v + H_P}{2}\right)^d$$

$$\leq 2^d \int_{\{\varphi \leq \frac{v+H_P}{2} - s\}} (dd^c \varphi)^d \leq 2^d \int_{\{\varphi \leq H_P - s\}} (dd^c \varphi)^d.$$

The claim is proved. Using (3.3) and (3.2) we obtain

$$\int_{\{u \leq H_P - 2t\}} (H_P - u)(dd^c v)^d$$

$$\leq 2^{d+1} t \int_{\{\varphi \leq H_P - t\}} (dd^c \varphi)^d + 2^{d+1} \int_t^{+\infty} \left(\int_{\{\varphi \leq H_P - s\}} (dd^c \varphi)^d \right) ds$$

$$= 2^{d+1} \int_{\{\varphi \leq H_P - t\}} (H_P - \varphi)(dd^c \varphi)^d.$$

□

Proposition 3.3. *Let $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ with $(dd^c u)^d = \mu$ having support in a nonpluripolar compact set K so that $\int_K u d\mu > -\infty$ from Proposition 2.11. Let $\{Q_j\}$ be a sequence of continuous functions on K decreasing to u on K. Then $u_j := V^*_{P,K,Q_j} \downarrow u$ on $\mathbb{C}^d$ and $\mu_j := (dd^c u_j)^d$ is supported in K. In particular, $\mu_j \to \mu = (dd^c u)^d$ weak-*. Moreover,*

$$\lim_{j\to\infty} \int_K Q_j d\mu_j = \lim_{j\to\infty} \int_K Q_j d\mu = \int_K u d\mu > -\infty. \tag{3.4}$$

Proof. We can assume $\{Q_j\}$ are defined and decreasing to u on the closure of a bounded open neighborhood Ω of K. By adding a negative constant we can assume that $Q_1 \leq 0$ on Ω. Since $\{Q_j\}$ is decreasing, so is the sequence $\{u_j\}$. Moreover, by [4, Proposition 5.1] $u_j \leq Q_j$ on $K \setminus E_j$ where E_j is pluripolar. But u is a competitor in the definition of V_{P,K,Q_j} so that $u \leq u_j$ on $\mathbb{C}^d$. Thus $\tilde{u} := \lim_{j\to\infty} u_j \geq u$ everywhere and $\tilde{u} \leq u$ on $K \setminus E$, where $E := \cup_j E_j$ is a pluripolar set. Since $(dd^c u)^d$ put no mass on pluripolar sets,

$$\int_{\{u<\tilde{u}\}} (dd^c u)^d \leq \int_{E\cup(\mathbb{C}^d\setminus K)} (dd^c u)^d = 0.$$

It thus follows from Proposition 3.1 that $\tilde{u} \leq u$, hence $\tilde{u} = u$ on $\mathbb{C}^d$.

The second equality in (3.4) follows from the monotone convergence theorem. It remains to prove that

$$\lim_{j\to\infty} \int_K (-Q_j) d\mu_j = \int_K (-u) d\mu.$$

For each k fixed and $j \geq k$ we have

$$\int_K (-Q_j) d\mu_j \geq \int_K (-Q_k) d\mu_j = \int_\Omega (-Q_k) d\mu_j,$$

hence $\liminf_{j\to\infty} \int_K (-Q_j) d\mu_j \geq \int_K (-Q_k) d\mu$ since Ω is open and μ_j, μ are supported on K. Letting $k \to +\infty$ we arrive at

$$\liminf_{j\to\infty} \int_K (-Q_j) d\mu_j \geq \int_K (-u) d\mu.$$

It remains to prove that

$$\limsup_{j\to\infty} \int_K (-Q_j) d\mu_j \leq \int_K (-u) d\mu.$$

The sequence $\{u_j\}$ is not necessarily uniformly bounded below on K. However, using the facts that $Q_j \geq u$ and H_P is continuous in $\mathbb{C}^d$, it suffices to prove that

$$\limsup_{j\to\infty}\int_K (H_P-u)(dd^c u_j)^d \le \int_K (H_P-u)(dd^c u)^d. \tag{3.5}$$

To verify (3.5), we use Lemma 3.2.

By adding a negative constant we can assume that $u_j \le H_P$. For a function v and for $t>0$ we define $v^t := \max(v, H_P - t)$. Note that for each t the sequence $\{u_j^t\}$ is locally uniformly bounded below. Define

$$a(t) := 2^{d+1}\int_{\{u\le H_P - t/2\}} (H_P-u)(dd^c u)^d.$$

Since $u\in\mathcal{E}^1_P(\mathbb{C}^d)$, from Proposition 2.11 we have $a(t)\to 0$ as $t\to+\infty$. By Lemma 3.2 we have

$$\sup_{j\ge 1}\int_{\{u\le H_P-t\}} (H_P-u)(dd^c u_j)^d \le a(t). \tag{3.6}$$

By the plurifine property of non-pluripolar Monge-Ampère measures [10, Proposition 1.4] and (3.6) we have

$$\begin{aligned}\int_K (H_P-u)(dd^c u_j)^d &\le \int_{K\cap\{u>H_P-t\}} (H_P-u)(dd^c u_j)^d + a(t)\\ &= \int_{K\cap\{u>H_P-t\}} (H_P-u^t)(dd^c u_j^t)^d + a(t)\\ &\le \int_K (H_P-u^t)(dd^c u_j^t)^d + a(t).\end{aligned}$$

Since H_P is bounded in Ω, it follows from [16, Theorem 4.26] that the sequence of positive Radon measures $(H_P-u^t)(dd^c u_j^t)^d$ converges weakly on Ω to $(H_P-u^t)(dd^c u^t)^d$. Since K is compact it then follows that

$$\limsup_j \int_K (H_P-u)(dd^c u_j)^d \le \int_K (H_P-u^t)(dd^c u^t)^d + a(t).$$

We finally let $t\to+\infty$ to conclude the proof in the following manner:

$$\begin{aligned}\int_K (H_P-u^t)(dd^c u^t)^d &\le \int_{K\cap\{u>H_P-t\}} (H_P-u^t)(dd^c u^t)^d + a(t)\\ &\le \int_K (H_P-u)(dd^c u)^d + a(t),\end{aligned}$$

where in the first estimate we have used $\{u \leq H_P - t\} = \{u^t \leq H_P - t\}$ and Lemma 3.2 and in the last estimate we use again the plurifine property. $\square$

We now give an alternate description of the Legendre-type transform E^* from (2.9) which will be related to the the rate function in a large deviation principle. Given $K \subset \mathbb{C}^d$ compact, we let $\mathcal{M}_P(K)$ denote the space of positive measures on K of total mass γ_d and we let $C(K)$ denote the set of continuous, real-valued functions on K.

Proposition 3.4. *Let K be a nonpluripolar compact set and $\mu \in \mathcal{M}_P(K)$. Then*

$$E^*(\mu) = \sup_{v \in C(K)} [E(V^*_{P,K,v}) - \int_K v d\mu].$$

Proof. We first treat the case when $E^*(\mu) = +\infty$. By Theorem 2.13 there exists $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ such that $\int_K u d\mu = -\infty$. We take a decreasing sequence $Q_j \in C(K)$ such that $Q_j \downarrow u$ on K and set $u_j := V^*_{P,K,Q_j}$. Then $\{u_j\}$ are decreasing; since $u \in \mathcal{E}^1_P(\mathbb{C}^d)$ and E is non-decreasing, $\{E(u_j)\}$ is uniformly bounded and we obtain

$$E(V^*_{P,K,Q_j}) - \int_K Q_j d\mu \to +\infty,$$

proving the proposition in this case.

Assume now that $E^*(\mu) < +\infty$. Theorem 2.13 ensures that $\int_{\mathbb{C}^d} u d\mu > -\infty$ for all $u \in \mathcal{E}^1_P(\mathbb{C}^d)$. By Lemma 2.15, μ puts no mass on pluripolar sets. From monotonicity of E and the definition of E^* in (2.9) we have

$$E^*(\mu) \geq \sup_{v \in C(K)} [E(V^*_{P,K,v}) - \int_K v d\mu].$$

Here we have used that

$$V^*_{P,K,v} \leq v \text{ q.e. on } K \text{ for } v \in C(K).$$

For the reverse inequality, fix $u \in \mathcal{E}^1_P(\mathbb{C}^d)$. Let $\{Q_j\}$ be a sequence of continuous functions on K decreasing to u on K and set $u_j := V^*_{P,K,Q_j}$. Given $\epsilon > 0$, we can choose j sufficiently large so that, by monotone convergence,

$$\int_K Q_j d\mu \leq \int_K u d\mu + \epsilon;$$

and, by monotonicity of E,

$$E(V^*_{P,K,Q_j}) \geq E(u).$$

Hence

$$E(V^*_{P,K,Q_j}) - \int_K Q_j d\mu \geq E(u) - \int_K u d\mu - \epsilon$$

so that

$$\sup_{v\in C(K)} [E(V^*_{P,K,v}) - \int_K v d\mu] \geq E^*(\mu)$$

and equality holds.

□

3.2. **Transfinite diameter.** Let $d_n = d_n(P)$ denote the dimension of the vector space $Poly(nP)$. We write

$$Poly(nP) = \text{span}\{e_1, ..., e_{d_n}\}$$

where $\{e_j(z) := z^{\alpha(j)}\}_{j=1,...,d_n}$ are the standard basis monomials. Given $\zeta_1, ..., \zeta_{d_n} \in \mathbb{C}^d$, let

$$VDM(\zeta_1, ..., \zeta_{d_n}) := \det[e_i(\zeta_j)]_{i,j=1,...,d_n} \tag{3.7}$$

$$= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix}$$

and for $K \subset \mathbb{C}^d$ compact let

$$V_n = V_n(K) := \max_{\zeta_1,...,\zeta_{d_n} \in K} |VDM(\zeta_1, ..., \zeta_{d_n})|.$$

It was shown in [2] that

$$\delta(K) := \delta(K, P) := \lim_{n\to\infty} V_n^{1/l_n} \tag{3.8}$$

exists where

$$l_n := \sum_{j=1}^{d_n} \deg(e_j) = \sum_{j=1}^{d_n} |\alpha(j)|$$

is the sum of the degrees of the basis monomials for $Poly(nP)$. We call $\delta(K)$ the $P-$*transfinite diameter* of K. More generally, for w an admissible weight function on K and $\zeta_1, ..., \zeta_{d_n} \in K$, let

$$VDM_n^Q(\zeta_1, ..., \zeta_{d_n}) := VDM(\zeta_1, ..., \zeta_{d_n}) w(\zeta_1)^n \cdots w(\zeta_{d_n})^n \tag{3.9}$$

$$= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \cdot w(\zeta_1)^n \cdots w(\zeta_{d_n})^n$$

be a *weighted Vandermonde determinant.* Let

$$W_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n} \in K} |VDM_n^Q(\zeta_1,...,\zeta_{d_n})|.$$

An $n-th$ *weighted* $P-$*Fekete set for* K *and* w is a set of d_n points $\zeta_1,...,\zeta_{d_n} \in K$ with the property that

$$|VDM_n^Q(\zeta_1,...,\zeta_{d_n})| = W_n(K).$$

The limit

$$\delta^Q(K) := \delta^Q(K,P) := \lim_{n\to\infty} W_n(K)^{1/l_n}$$

exists and is called the *weighted* $P-$*transfinite diameter.* The following was proved in [2].

Theorem 3.5. [Asymptotic Weighted $P-$Fekete Measures] *Let* $K \subset \mathbb{C}^d$ *be compact with admissible weight* w. *For each* n, *take points* $z_1^{(n)}, z_2^{(n)}, \cdots, z_{d_n}^{(n)} \in K$ *for which*

$$\lim_{n\to\infty} \left[|VDM_n^Q(z_1^{(n)},\cdots,z_{d_n}^{(n)})|\right]^{\frac{1}{l_n}} = \delta^Q(K) \tag{3.10}$$

(asymptotically weighted $P-$*Fekete arrays) and let* $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$. *Then*

$$\mu_n \to \frac{1}{\gamma_d}\mu_{K,Q} \ \textit{weak} - *.$$

Another ingredient we will use is a Rumely-type relation between transfinite diameter and energy of $V_{P,K,Q}^*$ from [2].

Theorem 3.6. *Let* $K \subset \mathbb{C}^d$ *be compact and* $w = e^{-Q}$ *with* $Q \in C(K)$. *Then*

$$\log \delta^Q(K) = \frac{-1}{\gamma_d dA}\mathcal{E}(V_{P,K,Q}^*, H_P) = \frac{-(d+1)}{\gamma_d dA} E(V_{P,K,Q}^*). \tag{3.11}$$

Here $A = A(P,d)$ was defined in [2]; we recall the definition. For $P = \Sigma$ so that $Poly(n\Sigma) = \mathcal{P}_n$, we have

$$d_n(\Sigma) = \binom{d+n}{d} = 0(n^d/d!) \text{ and } l_n(\Sigma) = \frac{d}{d+1} n d_n(\Sigma).$$

For a convex body $P \subset (\mathbb{R}^+)^d$, define $f_n(d)$ by writing

$$l_n = f_n(d)\frac{nd}{d+1}d_n = f_n(d)\frac{l_n(\Sigma)}{d_n(\Sigma)}d_n.$$

Then the ratio l_n/d_n divided by $l_n(\Sigma)/d_n(\Sigma)$ has a limit; i.e.,

$$\lim_{n\to\infty} f_n(d) =: A = A(P,d). \tag{3.12}$$

3.3. **Bernstein-Markov.** For $K \subset \mathbb{C}^d$ compact, $w = e^{-Q}$ an admissible weight function on K, and ν a finite measure on K, we say that the triple (K, ν, Q) satisfies a weighted Bernstein-Markov property if for all $p_n \in \mathcal{P}_n$,

$$||w^n p_n||_K \leq M_n ||w^n p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1. \tag{3.13}$$

Here, $||w^n p_n||_K := \sup_{z\in K} |w(z)^n p_n(z)|$ and

$$||w^n p_n||^2_{L^2(\nu)} := \int_K |p_n(z)|^2 w(z)^{2n} d\nu(z).$$

Following [1], given $P \subset (\mathbb{R}^+)^d$ a convex body, we say that a finite measure ν with support in a compact set K is a Bernstein-Markov measure for the triple (P, K, Q) if (3.13) holds for all $p_n \in Poly(nP)$.

For any P there exists $A = A(P) > 0$ with $Poly(nP) \subset \mathcal{P}_{An}$ for all n. Thus if (K, ν, Q) satisfies a weighted Bernstein-Markov property, then ν is a Bernstein-Markov measure for $(P, K, \tilde{Q})$ where $\tilde{Q} = AQ$. In particular, if ν is a *strong Bernstein-Markov measure* for K; i.e., if ν is a weighted Bernstein-Markov measure for any $Q \in C(K)$, then for any such Q, ν is a Bernstein-Markov measure for the triple (P, K, Q). Strong Bernstein-Markov measures exist for any nonpluripolar compact set; cf., Corollary 3.8 of [9]. The paragraph following this corollary gives a sufficient mass-density type condition for a measure to be a strong Bernstein-Markov measure.

Given P, for ν a finite measure on K and $Q \in \mathcal{A}(K)$, define

$$Z_n := Z_n(P, K, Q, \nu) := \int_K \cdots \int_K |VDM_n^Q(z_1, ..., z_{d_n})|^2 d\nu(z_1) \cdots d\nu(z_{d_n}). \tag{3.14}$$

The main consequence of using a Bernstein-Markov measure for (P, K, Q) is the following:

Proposition 3.7. *Let $K \subset \mathbb{C}^d$ be a compact set and let $Q \in \mathcal{A}(K)$. If ν is a Bernstein-Markov measure for (P, K, Q) then*

$$\lim_{k\to\infty} Z_n^{\frac{1}{2l_n}} = \delta^Q(K). \tag{3.15}$$

Proof. That $\limsup_{k\to\infty} Z_n^{\frac{1}{2l_n}} \le \delta^Q(K)$ is clear. Observing from (3.7) and (3.9) that, fixing all variables but z_j,

$$z_j \to VDM_n^Q(z_1, ..., z_j, ..., z_{d_n}) = w(z_j)^n p_n(z_j)$$

for some $p_n \in Poly(nP)$, to show $\liminf_{k\to\infty} Z_n^{\frac{1}{2l_n}} \ge \delta^Q(K)$ one starts with an $n-$th weighted $P-$Fekete set for K and w and repeatedly applies the weighted Bernstein-Markov property. □

Recall $\mathcal{M}_P(K)$ is the space of positive measures on K with total mass γ_d. With the weak-* topology, this is a separable, complete metrizable space. A neighborhood basis of $\mu \in \mathcal{M}_P(K)$ can be given by sets

$$G(\mu, k, \epsilon) := \{\sigma \in \mathcal{M}_P(K) : |\int_K (\text{Re}z)^\alpha (\text{Im}z)^\beta (d\mu - d\sigma)| < \epsilon \quad \text{for } 0 \le |\alpha| + |\beta| \le k\} \tag{3.16}$$

where $\text{Re}z = (\text{Re}z_1, ..., \text{Re}z_n)$ and $\text{Im}z = (\text{Im}z_1, ..., \text{Im}z_n)$.

Given ν as in Proposition 3.7, we define a probability measure $Prob_n$ on K^{d_n} via, for a Borel set $A \subset K^{d_n}$,

$$Prob_n(A) := \frac{1}{Z_n} \cdot \int_A |VDM_n^Q(z_1, ..., z_{d_n})|^2 \cdot d\nu(z_1) \cdots d\nu(z_{d_n}). \tag{3.17}$$

We immediately obtain the following:

Corollary 3.8. *Let ν be a Bernstein-Markov measure for (P, K, Q). Given $\eta > 0$, define*

$$A_{n,\eta} := \{(z_1, ..., z_{d_n}) \in K^{d_n} : |VDM_n^Q(z_1, ..., z_{d_n})|^2 \ge (\delta^Q(K) - \eta)^{2l_n}\}. \tag{3.18}$$

Then there exists $n^* = n^*(\eta)$ such that for all $n > n^*$,

$$Prob_n(K^{d_n} \setminus A_{n,\eta}) \le \left(1 - \frac{\eta}{2\delta^Q(K)}\right)^{2l_n}.$$

Remark 3.9. Corollary 3.8 was proved in [9], Corollary 3.2, for ν a probability measure but an obvious modification works for $\nu(K) < \infty$.

Using (3.17), we get an induced probability measure $\mathbf{P}$ on the infinite product space of arrays $\chi := \{X = \{x_j^{(n)}\}_{n=1,2,...;\ j=1,...,d_n} : x_j^{(n)} \in K\}$:

$$(\chi, \mathbf{P}) := \prod_{n=1}^{\infty} (K^{d_n}, Prob_n).$$

Corollary 3.10. *Let ν be a Bernstein-Markov measure for (P, K, Q). For $\mathbf{P}$-a.e. array $X = \{x_j^{(n)}\} \in \chi$,*

$$\nu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{x_j^{(n)}} \to \frac{1}{\gamma_d} \mu_{K,Q} \ \textit{weak-*}.$$

Proof. From Theorem 3.5 it suffices to verify for $\mathbf{P}$-a.e. array $X = \{x_j^{(n)}\}$

$$\liminf_{n\to\infty} \big(|VDM_n^Q(x_1^{(n)}, ..., x_{d_n}^{(n)})|\big)^{\frac{1}{l_n}} = \delta^Q(K). \tag{3.19}$$

Given $\eta > 0$, the condition that for a given array $X = \{x_j^{(n)}\}$ we have

$$\liminf_{n\to\infty} \big(|VDM_n^Q(x_1^{(n)}, ..., x_{d_n}^{(n)})|\big)^{\frac{1}{l_n}} \le \delta^Q(K) - \eta$$

means that $(x_1^{(n)}, ..., x_{d_n}^{(n)}) \in K^{d_n} \setminus A_{n,\eta}$ for infinitely many n. Setting

$$E_n := \{X \in \chi : (x_1^{(n)}, ..., x_{d_n}^{(n)}) \in K^{d_n} \setminus A_{n,\eta}\},$$

we have

$$\mathbf{P}(E_n) \le Prob_n(K^{d_n} \setminus A_{n,\eta}) \le (1 - \frac{\eta}{2\delta^Q(K)})^{2l_n}$$

and $\sum_{n=1}^{\infty} \mathbf{P}(E_n) < +\infty$. By the Borel-Cantelli lemma,

$$\mathbf{P}(\limsup_{n\to\infty} E_n) = \mathbf{P}(\bigcap_{n=1}^{\infty} \bigcup_{k\ge n}^{\infty} E_k) = 0.$$

Thus, with probability one, only finitely many E_n occur, and (3.19) follows. □

The main goal in the rest of the paper is to verify a stronger probabilistic result – a large deviation principle – and to explain this result in $P-$pluripotential-theoretic terms.

4. Relation between E^* and J, J^Q functionals.

We define some functionals on $\mathcal{M}_P(K)$ using L^2-type notions which act as a replacement for an energy functional on measures. Then we show these functionals $\overline{J}(\mu)$ and $\underline{J}(\mu)$ defined using a "lim sup" and a "lim inf" coincide (see Definitions 4.1 and 4.2); this is the essence of our first proof of the large deviation principle, Theorem 5.1. Using Proposition 3.4, we relate this functional with E^* from (2.9).

Fix a nonpluripolar compact set K and a strong Bernstein-Markov measure ν on K. For simplicity, we normalize so that ν is a probability measure. Recall then for any $Q \in C(K)$, ν is a Bernstein-Markov measure for the triple (P, K, Q). Given $G \subset \mathcal{M}_P(K)$ open, for each $s = 1, 2, ...$ we set

$$\tilde{G}_s := \{\mathbf{a} = (a_1, ..., a_s) \in K^s : \frac{\gamma_d}{s}\sum_{j=1}^{s} \delta_{a_j} \in G\}. \tag{4.1}$$

Define, for $n = 1, 2, ...,$

$$J_n(G) := [\int_{\tilde{G}_{d_n}} |VDM_n(\mathbf{a})|^2 d\nu(\mathbf{a})]^{1/2l_n}.$$

Definition 4.1. For $\mu \in \mathcal{M}_P(K)$ we define

$$\overline{J}(\mu) := \inf_{G \ni \mu} \overline{J}(G) \text{ where } \overline{J}(G) := \limsup_{n\to\infty} J_n(G);$$

$$\underline{J}(\mu) := \inf_{G \ni \mu} \underline{J}(G) \text{ where } \underline{J}(G) := \liminf_{n\to\infty} J_n(G).$$

The infima are taken over all neighborhoods G of the measure μ in $\mathcal{M}_P(K)$. A priori, $\overline{J}, \underline{J}$ depend on ν. These functionals are nonnegative but can take the value zero. Intuitively, we are taking a "limit" of $L^2(\nu)$ averages of discrete, equally weighted approximants $\frac{\gamma_d}{s}\sum_{j=1}^{s} \delta_{a_j}$ of μ. An "L^∞" version of $\overline{J}, \underline{J}$ was introduced in [8] where $J_n(G)$ is replaced by

$$W_n(G) := \sup_{\mathbf{a} \in \tilde{G}_{d_n}} |VDM_n(\mathbf{a})|^{1/l_n} \geq J_n(G). \tag{4.2}$$

The weighted versions of these functionals are defined for $Q \in \mathcal{A}(K)$ using

$$J_n^Q(G) := [\int_{\tilde{G}_{d_n}} |VDM_n^Q(\mathbf{a})|^2 d\nu(\mathbf{a})]^{1/2l_n}. \tag{4.3}$$

Definition 4.2. For $\mu \in \mathcal{M}_P(K)$ we define

$$\overline{J}^Q(\mu) := \inf_{G \ni \mu} \overline{J}^Q(G) \text{ where } \overline{J}^Q(G) := \limsup_{n\to\infty} J_n^Q(G);$$

$$\underline{J}^Q(\mu) := \inf_{G \ni \mu} \underline{J}^Q(G) \text{ where } \underline{J}^Q(G) := \liminf_{n\to\infty} J_n^Q(G).$$

The uppersemicontinuity of $\overline{J}, \overline{J}^Q, \underline{J}$ and $\underline{J}^Q$ on $\mathcal{M}_P(K)$ (with the weak-* topology) follows as in Lemma 3.1 of [8]. Set

$$b_d = b_d(P) := \frac{d+1}{Ad\gamma_d}.$$

Proposition 4.3. *Fix $Q \in C(K)$. Then*

1. $\overline{J}^Q(\mu) \le \delta^Q(K)$;
2. $\overline{J}(\mu) = \overline{J}^Q(\mu) \cdot (e^{\int_K Qd\mu})^{b_d}$;
3. $\log \overline{J}(\mu) \le \inf_{v\in C(K)}[\log \delta^v(K) + b_d \int_K v d\mu]$;
4. $\log \overline{J}^Q(\mu) \le \inf_{v\in C(K)}[\log \delta^v(K) + b_d \int_K v d\mu] - b_d \int_K Q d\mu$.

Properties (1)-(4) also hold for the functionals $\underline{J}, \underline{J}^Q$.

Proof. Property (1) follows from

$$J_n^Q(G) \le \sup_{\mathbf{a}\in \tilde{G}_{d_n}} |VDM_n^Q(\mathbf{a})|^{1/l_n} \le \sup_{\mathbf{a}\in K^{d_n}} |VDM_n^Q(\mathbf{a})|^{1/l_n}.$$

The proofs of Corollary 3.4, Proposition 3.5 and Proposition 3.6 of [8] work mutatis mutandis to verify (2), (3) and (4). The relevant estimation, replacing the corresponding one which is two lines above equation (3.2) in [8], is, given $\epsilon > 0$, for $\mathbf{a} \in \tilde{G}_{d_n}$,

$$\begin{aligned} (4.4)\quad |VDM_n^Q(\mathbf{a})|e^{\frac{nd_n}{\gamma_d}(-\epsilon - \int_K Qd\mu)} &\le |VDM_n(\mathbf{a})| \\ &\le |VDM_n^Q(\mathbf{a})|e^{\frac{nd_n}{\gamma_d}(\epsilon + \int_K Qd\mu)}. \end{aligned}$$

To see this, we first recall that

$$|VDM_n(\mathbf{a})| = |VDM_n^Q(\mathbf{a})|e^{n\sum_{j=1}^{d_n} Q(a_j)}.$$

For $\mu \in \mathcal{M}_P(K)$, $Q \in C(K)$, $\epsilon > 0$, there exists a neighborhood G of μ in $\mathcal{M}_P(K)$ with

$$-\epsilon < \int_K Qd\mu - \frac{\gamma_d}{d_n}\sum_{j=1}^{d_n} Q(a_j) < \epsilon$$

for $\mathbf{a} \in \tilde{G}_{d_n}$. Plugging this double inequality into the previous equality we get (4.4). Moreover, from (3.12),

$$\lim_{n\to\infty} \frac{nd_n}{l_n} = \frac{d+1}{Ad} = b_d\gamma_d \tag{4.5}$$

so that $\frac{nd_n}{\gamma_d} \asymp l_n b_d$ as $n \to \infty$. Taking l_n−the roots in (4.4) accounts for the factor of b_d in (2), (3) and (4). □

Remark 4.4. The corresponding $\underline{W}, \underline{W}^Q, \overline{W}, \overline{W}^Q$ functionals, defined using (4.2), clearly dominate their "J" counterparts; e.g., $\overline{W}^Q \geq \overline{J}^Q$.

Note that formula (3.11) can be rewritten:

$$\log \delta^Q(K) = -b_d E(V^*_{P,K,Q}). \tag{4.6}$$

Thus the upper bound in Proposition 4.3 (3) becomes

$$\log \overline{J}(\mu) \leq -b_d \sup_{v\in C(K)} [E(V^*_{P,K,v}) - \int_K v d\mu] = -b_d E^*(\mu). \tag{4.7}$$

For the rest of section 4 and section 5, we will always assume $Q \in C(K)$. Theorem 4.5 shows that the inequalities in (3) and (4) are equalities, and that the $\overline{J}, \overline{J}^Q$ functionals coincide with their $\underline{J}, \underline{J}^Q$ counterparts. The key step in the proof of Theorem 4.5 is to verify this for $\overline{J}^v(\mu_{K,v})$ and $\underline{J}^v(\mu_{K,v})$.

Theorem 4.5. *Let $K \subset \mathbb{C}^d$ be a nonpluripolar compact set and let ν satisfy a strong Bernstein-Markov property. Fix $Q \in C(K)$. Then for any $\mu \in \mathcal{M}_P(K)$,*

$$\log \overline{J}(\mu) = \log \underline{J}(\mu) = \inf_{v\in C(K)} [\log \delta^v(K) + b_d \int_K v d\mu] \tag{4.8}$$

and

(4.9)

$$\log \overline{J}^Q(\mu) = \log \underline{J}^Q(\mu) = \inf_{v\in C(K)} [\log \delta^v(K) + b_d \int_K v d\mu] - b_d \int_K Q d\mu.$$

Proof. It suffices to prove (4.8) since (4.9) follows from (2) of Proposition 4.3. We have the upper bound

$$\log \overline{J}(\mu) \leq \inf_{v\in C(K)} [\log \delta^v(K) + b_d \int_K v d\mu]$$

from (3); for the lower bound, we consider different cases.

Case I: $\mu = \mu_{K,v}$ *for some* $v \in C(K)$.

We verify that

$$\log \overline{J}(\mu_{K,v}) = \log \underline{J}(\mu_{K,v}) = \log \delta^v(K) + b_d \int_K v d\mu_{K,v} \tag{4.10}$$

which proves (4.8) in this case.

To prove (4.10), we use the definition of $\underline{J}(\mu_{K,v})$ and Corollary 3.8. Fix a neighborhood G of $\mu_{K,v}$. For $\eta > 0$, define $A_{n,\eta}$ as in (3.18) with $Q = v$. Set

$$\eta_n := \max\left(\delta^v(K) - \frac{nZ_n^{1/2l_n}}{n+1}, \frac{Z_n^{1/2l_n}}{n+1}\right). \tag{4.11}$$

By Proposition 3.7, $\eta_n \to 0$. We claim that we have the inclusion

$$A_{n,\eta_n} \subset \tilde{G}_{d_n} \text{ for all } n \text{ large enough.} \tag{4.12}$$

We prove (4.12) by contradiction: if false, there is a sequence $\{n_j\}$ with $n_j \uparrow \infty$ and $x^j = (x_1^j, ..., x_{d_{n_j}}^j) \in A_{n_j,\eta_{n_j}} \setminus \tilde{G}_{d_{n_j}}$. However $\mu_j := \frac{\gamma_d}{d_{n_j}} \sum_{i=1}^{d_{n_j}} \delta_{x_i^j} \notin G$ for j sufficiently large contradicts Theorem 3.5 since $x^j \in A_{n_j,\eta_j}$ and $\eta_j \downarrow 0$ imply $\mu_j \to \mu_{K,v}$ weak-*.

Next, a direct computation using (4.11) shows that, for all n large enough,

$$Prob_n(K^{d_n} \setminus A_{n,\eta_n}) \leq \frac{(\delta^v(K) - \eta_n)^{2l_n}}{Z_n} \leq (\frac{n}{n+1})^{2l_n} \leq \frac{n}{n+1} \tag{4.13}$$

(recall ν is a probability measure). Hence

$$\begin{aligned} &\frac{1}{Z_n} \int_{\tilde{G}_{d_n}} |VDM_n^v(z_1, ..., z_{d_n})|^2 \cdot d\nu(z_1) \cdots d\nu(z_{d_n}) \\ &\geq \frac{1}{Z_n} \int_{A_{n,\eta_n}} |VDM_n^v(z_1, ..., z_{d_n})|^2 \cdot d\nu(z_1) \cdots d\nu(z_{d_n}) \\ &\geq \frac{1}{n+1}. \end{aligned}$$

Since $P \subset r\Sigma$ and $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$, $l_n = 0(n^{d+1})$ and we have $\frac{1}{2l_n} \log(n+1) \to 0$. Since ν satisfies a strong Bernstein-Markov property and $v \in C(K)$, using Proposition 3.7 and the above estimate we conclude that

$$\liminf_{n\to\infty}\frac{1}{2l_n}\log\int_{\tilde{G}_{d_n}}|VDM_n^v(z_1,...,z_{d_n})|^2 d\nu(z_1)\cdots d\nu(z_{d_n})$$
$$\geq \log\delta^v(K).$$

Taking the infimum over all neighborhoods G of $\mu_{K,v}$ we obtain

$$\log \underline{J}^v(\mu_{K,v}) \geq \log\delta^v(K).$$

From (1) Proposition 4.3, $\log\overline{J}^v(\mu_{K,v}) \leq \log\delta^v(K)$; thus we have

$$\log \underline{J}^v(\mu_{K,v}) = \log\overline{J}^v(\mu_{K,v}) = \log\delta^v(K). \tag{4.14}$$

Using (2) of Proposition 4.3 with $\mu = \mu_{K,v}$ we obtain (4.10).

Case II: $\mu \in \mathcal{M}_P(K)$ with the property that $E^*(\mu) < \infty$.

From Theorem 2.13 and Proposition 2.11 there exists $u \in L_P(\mathbb{C}^d)$ – indeed, $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ – with $\mu = (dd^c u)^d$ and $\int_K u d\mu > -\infty$. However, since u is only usc on K, μ is not necessarily of the form $\mu_{K,v}$ for some $v \in C(K)$. Taking a sequence of continuous functions $\{Q_j\} \subset C(K)$ with $Q_j \downarrow u$ on K, by Proposition 3.3 the weighted extremal functions V^*_{P,K,Q_j} decrease to u on $\mathbb{C}^d$;

$$\mu_j := (dd^c V^*_{P,K,Q_j})^d \to \mu = (dd^c u)^d \text{ weak-}*;$$

and

$$\lim_{j\to\infty}\int_K Q_j d\mu_j = \lim_{j\to\infty}\int_K Q_j d\mu = \int_K u d\mu. \tag{4.15}$$

From the previous case we have

$$\log\overline{J}(\mu_j) = \log\underline{J}(\mu_j) = \log\delta^{Q_j}(K) + b_d\int_K Q_j d\mu_j.$$

Using uppersemicontinuity of the functional $\mu \to \underline{J}(\mu)$,

$$\limsup_{j\to\infty}\overline{J}(\mu_j) = \limsup_{j\to\infty}\underline{J}(\mu_j) \leq \underline{J}(\mu).$$

Since $Q_j \downarrow u$ on K,

$$\limsup_{j\to\infty}\log\delta^{Q_j}(K) = \lim_{j\to\infty}\log\delta^{Q_j}(K). \tag{4.16}$$

Therefore

$$M := \lim_{j\to\infty}\log\underline{J}(\mu_j) = \lim_{j\to\infty}\big(\log\delta^{Q_j}(K) + b_d\int_K Q_j d\mu_j\big)$$

exists and is less than or equal to $\log \underline{J}(\mu)$. We want to show that

$$\inf_v [\log \delta^v(K) + b_d \int_K v d\mu] \leq M. \tag{4.17}$$

Given $\epsilon > 0$, by (4.15) for $j \geq j_0(\epsilon)$,

$$\int_K Q_j d\mu_j \geq \int_K Q_j d\mu - \epsilon \text{ and } \log \underline{J}(\mu_j) < M + \epsilon.$$

Hence for such j,

$$\inf_v [\log \delta^v(K) + b_d \int_K v d\mu] \leq \log \delta^{Q_j}(K) + b_d \int_K Q_j d\mu$$

$$\leq \log \delta^{Q_j}(K) + b_d \int_K Q_j d\mu_j + b_d\epsilon = \log \underline{J}(\mu_j) + b_d\epsilon < M + (b_d + 1)\epsilon,$$

yielding (4.17). This finishes the proof in Case II.

Case III: $\mu \in \mathcal{M}(K)$ with the property that $E^*(\mu) = +\infty$.

It follows from Proposition 3.4 and Theorem 3.6 that the right-hand side of (4.8) is $-\infty$, finishing the proof.

□

Remark 4.6. From now on, we simply use the notation J, J^Q without the overline or underline. Using Proposition 3.4 and Theorem 3.6, we have

$$\log J(\mu) = \inf_{Q \in C(K)} [\log \delta^Q(K) + b_d \int_K Q d\mu]$$

$$= - \sup_{Q \in C(K)} [-\log \delta^Q(K) - b_d \int_K Q d\mu]$$

$$= - \sup_{Q \in C(K)} [b_d E(V^*_{P,K,Q}) - b_d \int_K Q d\mu] = -b_d \sup_{Q \in C(K)} [E(V^*_{P,K,Q}) - \int_K Q d\mu]$$

(recall (4.6)) which one can compare with

$$E^*(\mu) = \sup_{Q \in C(K)} [E(V^*_{P,K,Q}) - \int_K Q d\mu]$$

from Proposition 3.4 to conclude

$$\log J(\mu) = -b_d E^*(\mu). \tag{4.18}$$

In particular, J, J^Q are independent of the choice of strong Bernstein-Markov measure for K.

Following the idea in Proposition 4.3 of [9], we observe the following:

Proposition 4.7. *Let $K \subset \mathbb{C}^d$ be a nonpluripolar compact set and let ν satisfy a strong Bernstein-Markov property. Fix $Q \in C(K)$. The measure $\mu_{K,Q}$ is the unique maximizer of the functional $\mu \to J^Q(\mu)$ over $\mu \in \mathcal{M}_P(K)$; i.e.,*

$$J^Q(\mu_{K,Q}) = \delta^Q(K) \ (\textit{and } J(\mu_K) = \delta(K)). \tag{4.19}$$

Proof. The fact that $\mu_{K,Q}$ maximizes J^Q (and μ_K maximizes J) follows from (4.10), (4.14) and Proposition 4.3.

Assume now that $\mu \in \mathcal{M}_P(K)$ maximizes J^Q. From Remark 4.4 and the definitions of the functionals, for any neighborhood $G \subset \mathcal{M}_P(K)$ of μ,

$$\overline{J}^Q(\mu) \leq \overline{W}^Q(\mu) \leq \sup\{\limsup_{n\to\infty} |VDM_n^Q(\mathbf{a}^{(n)})|^{1/l_n}\} \leq \delta^Q(K)$$

where the supremum is taken over all arrays $\{\mathbf{a}^{(n)}\}_{n=1,2,...}$ of d_n-tuples $\mathbf{a}^{(n)}$ in K whose normalized counting measures $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n} \delta_{a_j^{(n)}}$ lies in G. Since $\overline{J}^Q(\mu) = \delta^Q(K)$ there is an asymptotic weighted Fekete array $\{\mathbf{a}^{(n)}\}$ as in (3.10). Theorem 3.5 yields that $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n} \delta_{a_j^{(n)}}$ converges weak-* to $\mu_{K,Q}$, hence $\mu_{K,Q} \in \overline{G}$. Since this is true for each neighborhood $G \subset \mathcal{M}_P(K)$ of μ, we must have $\mu = \mu_{K,Q}$. □

5. Large deviation.

As in the previous section, we fix $K \subset \mathbb{C}^d$ a nonpluripolar compact set; $Q \in C(K)$; and a measure ν on K satisfying a strong Bernstein-Markov property. For $x_1, ..., x_{d_n} \in K$, we get a discrete measure $\frac{\gamma_d}{d_n}\sum_{j=1}^{d_n} \delta_{x_j} \in \mathcal{M}_P(K)$. Define $j_n : K^{d_n} \to \mathcal{M}_P(K)$ via

$$j_n(x_1, ..., x_{d_n}) := \frac{\gamma_d}{d_n}\sum_{j=1}^{d_n} \delta_{x_j}.$$

From (3.17), $\sigma_n := (j_n)_*(Prob_n)$ is a probability measure on $\mathcal{M}_P(K)$: for a Borel set $B \subset \mathcal{M}_P(K)$,

$$\sigma_n(B) = \frac{1}{Z_n}\int_{\tilde{B}_{d_n}} |VDM_n^Q(x_1, ..., x_{d_n})|^2 d\nu(x_1)\cdots d\nu(x_{d_n}) \tag{5.1}$$

where $\tilde{B}_{d_n} := \{\mathbf{a} = (a_1, ..., a_{d_n}) \in K^{d_n} : \frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} \delta_{a_j} \in B\}$(recall (4.1)). Here, $Z_n := Z_n(P, K, Q, \nu)$. Note that

$$\sigma_n(B)^{1/2l_n} = \frac{1}{Z_n^{1/2l_n}} \cdot J_n^Q(B). \tag{5.2}$$

For future use, suppose we have a function $F : \mathbb{R} \to \mathbb{R}$ and a function $v \in C(K)$. We write, for $\mu \in \mathcal{M}_P(K)$,

$$< v, \mu >:= \int_K v d\mu$$

and then

$$\int_{\mathcal{M}_P(K)} F(< v, \mu >) d\sigma_n(\mu) := \tag{5.3}$$

$$\frac{1}{Z_n} \int_K \cdots \int_K |VDM_n^Q(x_1, ..., x_{d_n})|^2 F\left(\frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} v(x_j)\right) d\nu(x_1) \cdots d\nu(x_{d_n}).$$

With this notation, we offer two proofs of our LDP, Theorem 5.1. We state the result; define LDP in Definition 5.2; and then proceed with the proofs. This closely follows the exposition in section 5 of [9].

Theorem 5.1. *The sequence $\{\sigma_n = (j_n)_*(Prob_n)\}$ of probability measures on $\mathcal{M}_P(K)$ satisfies a* **large deviation principle** *with speed $2l_n$ and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}_P(K)$,*

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu).$$

This means that $\mathcal{I} : \mathcal{M}_P(K) \to [0, \infty]$ is a lowersemicontinuous mapping such that the sublevel sets $\{\mu \in \mathcal{M}_P(K) : \mathcal{I}(\mu) \leq \alpha\}$ are compact in the weak-* topology on $\mathcal{M}_P(K)$ for all $\alpha \geq 0$ ($\mathcal{I}$ is "good") satisfying (5.4) and (5.5):

Definition 5.2. The sequence $\{\mu_k\}$ of probability measures on $\mathcal{M}_P(K)$ satisfies a **large deviation principle** (LDP) with good rate function $\mathcal{I}$ and speed $2l_n$ if for all measurable sets $\Gamma \subset \mathcal{M}_P(K)$,

$$-\inf_{\mu \in \Gamma^0} \mathcal{I}(\mu) \leq \liminf_{n \to \infty} \frac{1}{2l_n} \log \mu_n(\Gamma) \text{ and} \tag{5.4}$$

$$\limsup_{n \to \infty} \frac{1}{2l_n} \log \mu_n(\Gamma) \leq -\inf_{\mu \in \overline{\Gamma}} \mathcal{I}(\mu). \tag{5.5}$$

In the setting of $\mathcal{M}_P(K)$, to prove a LDP it suffices to work with a base for the weak-* topology. The following is a special case of a basic general existence result for a LDP given in Theorem 4.1.11 in [13].

Proposition 5.3. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)$. Let $\mathcal{B}$ be a base for the topology of $\mathcal{M}_P(K)$. For $\mu \in \mathcal{M}_P(K)$ let*

$$\mathcal{I}(\mu) := -\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\liminf_{\epsilon\to 0}\epsilon\log\sigma_\epsilon(G)\big).$$

Suppose for all $\mu \in \mathcal{M}_P(K)$,

$$\mathcal{I}(\mu) = -\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\limsup_{\epsilon\to 0}\epsilon\log\sigma_\epsilon(G)\big).$$

Then $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$.

There is a converse to Proposition 5.3, Theorem 4.1.18 in [13]. For $\mathcal{M}_P(K)$, it reads as follows:

Proposition 5.4. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)$. Suppose that $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$. Then for any base $\mathcal{B}$ for the topology of $\mathcal{M}_P(K)$ and any $\mu \in \mathcal{M}_P(K)$*

$$\begin{aligned}\mathcal{I}(\mu) &:= -\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\liminf_{\epsilon\to 0}\epsilon\log\sigma_\epsilon(G)\big)\\ &= -\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\limsup_{\epsilon\to 0}\epsilon\log\sigma_\epsilon(G)\big).\end{aligned}$$

Remark 5.5. Assuming Theorem 5.1, this shows that, starting with a strong Bernstein-Markov measure ν and the corresponding sequence of probability measures $\{\sigma_n\}$ on $\mathcal{M}_P(K)$ in (5.1), the existence of an LDP with rate function $\mathcal{I}(\mu)$ and speed $2l_n$ implies that necessarily

$$\mathcal{I}(\mu) = \log J^Q(\mu_{K,Q}) - \log J^Q(\mu). \tag{5.6}$$

Uniqueness of the rate function is basic (cf., Lemma 4.1.4 of [13]).

We turn to the first proof of Theorem 5.1, using Theorem 4.5, which gives a pluripotential theoretic description of the rate functional.

Proof. As a base $\mathcal{B}$ for the topology of $\mathcal{M}_P(K)$, we can take the sets from (3.16) or simply all open sets. For $\{\sigma_\epsilon\}$, we take the sequence of probability measures $\{\sigma_n\}$ on $\mathcal{M}_P(K)$ and we take $\epsilon = \frac{1}{2l_n}$. For $G \in \mathcal{B}$, from (5.2),

$$\frac{1}{2l_n}\log\sigma_n(G) = \log J_n^Q(G) - \frac{1}{2l_n}\log Z_n.$$

From Proposition 3.7, and (4.14) with $v = Q$,

$$\lim_{n\to\infty} \frac{1}{2l_n} \log Z_n = \log \delta^Q(K) = \log J^Q(\mu_{K,Q});$$

and by Theorem 4.5,

$$\inf_{G\ni\mu} \limsup_{n\to\infty} \log J_n^Q(G) = \inf_{G\ni\mu} \liminf_{n\to\infty} \log J_n^Q(G) = \log J^Q(\mu).$$

Thus by Proposition 5.3 $\{\sigma_n\}$ satisfies an LDP with rate function

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu)$$

and speed $2l_n$. This rate function is good since $\mathcal{M}_P(K)$ is compact. □

Remark 5.6. From Proposition 4.7, $\mu_{K,Q}$ is the unique maximizer of the functional

$$\mu \to \log J^Q(\mu)$$

over all $\mu \in \mathcal{M}_P(K)$. Thus

$$\mathcal{I}_{K,Q}(\mu) \geq 0 \text{ with } \mathcal{I}_{K,Q}(\mu) = 0 \iff \mu = \mu_{K,Q}.$$

To summarize, $\mathcal{I}_{K,Q}$ is a good rate function with unique minimizer $\mu_{K,Q}$. Using the relations

$$\log J(\mu) = -b_d \sup_{Q\in C(K)} [E(V^*_{P,K,Q}) - \int_K Q d\mu]$$

$$J(\mu) = J^Q(\mu) \cdot (e^{\int_K Q d\mu})^{b_d}, \text{ and } J^Q(\mu_{K,Q}) = \delta^Q(K)$$

(the latter from (4.19)), we have

$$\begin{aligned}
\mathcal{I}(\mu) &:= \log \delta^Q(K) - \log J^Q(\mu) \\
&= \log \delta^Q(K) - \log J(\mu) + b_d \int_K Q d\mu \\
&= b_d \sup_{Q\in C(K)} [E(V^*_{P,K,Q}) - \int_K Q d\mu] + \log \delta^Q(K) + b_d \int_K Q d\mu \\
&= b_d \sup_{v\in C(K)} [E(V^*_{P,K,v}) - \int_K v d\mu] - b_d[E(V^*_{P,K,Q}) - \int_K Q d\mu]
\end{aligned}$$

from (4.6).

The second proof of our LDP follows from Corollary 4.6.14 in [13], which is a general version of the Gärtner-Ellis theorem. This approach was originally brought to our attention by S. Boucksom and was also utilized by R. Berman in [5]. We state the version of the [13] result for an appropriate family of probability measures.

Proposition 5.7. *Let $C(K)^*$ be the topological dual of $C(K)$, and let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}_P(K) \subset C(K)^*$ (equipped with the weak-* topology). Suppose for each $\lambda \in C(K)$, the limit*

$$\Lambda(\lambda) := \lim_{\epsilon \to 0} \epsilon \log \int_{C(K)^*} e^{\lambda(x)/\epsilon} d\sigma_\epsilon(x)$$

exists as a finite real number and assume Λ is Gâteaux differentiable; i.e., for each $\lambda, \theta \in C(K)$, the function $f(t) := \Lambda(\lambda + t\theta)$ is differentiable at $t = 0$. Then $\{\sigma_\epsilon\}$ satisfies an LDP in $C(K)^*$ with the convex, good rate function Λ^*.

Here

$$\Lambda^*(x) := \sup_{\lambda \in C(K)} \big(< \lambda, x > - \Lambda(\lambda)\big),$$

is the Legendre transform of Λ. The upper bound (5.5) in the LDP holds with rate function Λ^* under the assumption that the limit $\Lambda(\lambda)$ exists and is finite; the Gâteaux differentiability of Λ is needed for the lower bound (5.4). To verify this property in our setting, we must recall a result from [2].

Proposition 5.8. *For $Q \in \mathcal{A}(K)$ and $u \in C(K)$, let*

$$F(t) := E(V^*_{P,K,Q+tu})$$

for $t \in \mathbb{R}$. Then F is differentiable and

$$F'(t) = \int_{\mathbb{C}^d} u (dd^c V^*_{P,K,Q+tu})^d.$$

In [2] it was assumed that $u \in C^2(K)$ but the result is true with the weaker assumption $u \in C(K)$ (cf., Theorem 11.11 in [16] due to Lu and Nguyen [17], see also [11, Proposition 4.20]).

We proceed with the second proof of Theorem 5.1. For simplicity, we normalize so that $\gamma_d = 1$ to fit the setting of Proposition 5.7 (so members of $\mathcal{M}_P(K)$ are probability measures).

Proof. We show that for each $v \in C(K)$,

$$\Lambda(v) := \lim_{n \to \infty} \frac{1}{2l_n} \log \int_{C(K)^*} e^{2l_n < v, \mu >} d\sigma_n(\mu)$$

exists as a finite real number. First, since σ_n is a measure on $\mathcal{M}_P(K)$, the integral can be taken over $\mathcal{M}_P(K)$. Consider

$$\frac{1}{2l_n}\log\int_{\mathcal{M}_P(K)} e^{2l_n<v,\mu>}d\sigma_n(\mu).$$

By (5.3), this is equal to

$$\frac{1}{2l_n}\log\frac{1}{Z_n}\cdot\int_{K^{d_n}}|VDM_n^{Q-\frac{l_n}{nd_n}v}(x_1,...,x_{d_n})|^2d\nu(x_1)\cdots d\nu(x_{d_n}).$$

From (4.5), with $\gamma_d=1$, $\frac{l_n}{nd_n}\to\frac{1}{b_d}$; hence for any $\epsilon>0$,

$$\frac{1}{b_d+\epsilon}v\le\frac{l_n}{nd_n}v\le\frac{1}{b_d-\epsilon}v \text{ on } K$$

for n sufficiently large. Recall that

$$Z_n=\int_{K^{d_n}}|VDM_n^Q(x_1,...,x_{d_n}))|^2d\nu(x_1)\cdots d\nu(x_{d_n}).$$

Define

$$\tilde{Z}_n:=\int_{K^{d_n}}|VDM_n^{Q-v/b_d}(x_1,...,x_{d_n})|^2d\nu(x_1)\cdots d\nu(x_{d_n}).$$

Then we have

$$\lim_{n\to\infty}\tilde{Z}_n^{\frac{1}{2l_n}}=\delta^{Q-v/b_d}(K) \text{ and } \lim_{n\to\infty}Z_n^{\frac{1}{2l_n}}=\delta^Q(K)$$

from (3.15) in Proposition 3.7 and the assumption that $(K,\nu,\tilde{Q})$ satisfies the weighted Bernstein-Markov property for *all* $\tilde{Q}\in C(K)$. Thus

$$\Lambda(v)=\lim_{n\to\infty}\frac{1}{2l_n}\log\frac{\tilde{Z}_n}{Z_n}=\log\frac{\delta^{Q-v/b_d}(K)}{\delta^Q(K)}. \tag{5.7}$$

Define now, for $v,v'\in C(K)$,

$$f(t):=E(V^*_{P,K,Q-(v+tv')}).$$

Proposition 5.8 shows that Λ is Gâteaux differentiable and Proposition 5.7 gives that Λ^* is a rate function on $C(K)^*$.

Since each σ_n has support in $\mathcal{M}_P(K)$, it follows from (5.4) and (5.5) in Definition 5.2 of an LDP with $\Gamma\subset C(K)^*$ that for $\mu\in C(K)^*\setminus\mathcal{M}_P(K)$, $\Lambda^*(\mu)=+\infty$. By Lemma 4.1.5 (b) of [13], the restriction of Λ^* to $\mathcal{M}_P(K)$ is a rate function. Since $\mathcal{M}_P(K)$ is compact, it is a good rate function. Being a Legendre transform, Λ^* is convex.

To compute Λ^*, we have, using (5.7) and (3.11),

$$\Lambda^*(\mu) = \sup_{v\in C(K)} \Big(\int_K v d\mu - \log \frac{\delta^{Q-v/b_d}(K)}{\delta^Q(K)}\Big)$$

$$= \sup_{v\in C(K)} \Big(\int_K v d\mu - b_d[E(V^*_{P,K,Q}) - E(V^*_{P,K,Q-v/b_d}])\Big).$$

Thus

$$\Lambda^*(\mu) + b_d E(V^*_{P,K,Q}) = \sup_{v\in C(K)} \Big(\int_K v d\mu + b_d E(V^*_{P,K,Q-v/b_d})\Big)$$

$$= \sup_{u\in C(K)} \Big(b_d E(V^*_{P,K,Q+u}) - b_d \int_K u d\mu\Big) \text{ (taking } u = -v/b_d).$$

Rearranging and replacing u in the supremum by $v = u + Q$,

$$\Lambda^*(\mu) = \sup_{u\in C(K)} \Big(b_d E(V^*_{P,K,Q+u}) - b_d \int_K u d\mu\Big) - b_d E(V^*_{P,K,Q})$$

$$= b_d\big[\sup_{v\in C(K)} E(V^*_{P,K,v}) - \int_K v d\mu\big] - b_d\big[E(V^*_{P,K,Q}) - \int_K Q d\mu\big]$$

which agrees with the formula in Remark 5.6 (since μ is a probability measure).

□

Remark 5.9. Thus the rate function can be expressed in several equivalent ways:

$$\mathcal{I}(\mu) = \Lambda^*(\mu) = \log J^Q(\mu_{K,Q}) - \log J^Q(\mu)$$

$$= b_d\big[\sup_{v\in C(K)} E(V^*_{P,K,v}) - \int_K v d\mu\big] - b_d\big[E(V^*_{P,K,Q}) - \int_K Q d\mu\big]$$

$$= b_d E^*(\mu) - b_d\big[E(V^*_{P,K,Q}) - \int_K Q d\mu\big]$$

which generalizes the result equating (5.3), (5.10) and (5.11) in [9] for the case $P = \Sigma$ and $b_d = 1$. Note in the last equality we are using the slightly different notion of E^* in (2.9) and Proposition 3.4 than that used in [9].

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