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## THE FINITE MODEL PROPERTY FOR THE VARIETY OF HEYTING ALGEBRAS WITH SUCCESSOR

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ABSTRACT. The finite model property of the variety of  $S$ -algebras was proved by X. Caicedo using Kripke model techniques of the associated calculus. A more algebraic proof, but still strongly based on Kripke model ideas, was given by Muravitskii. In this article we give a purely algebraic proof for the finite model property which is strongly based on the fact that for every element  $x$  in a  $S$ -algebra the interval  $[x, S(x)]$  is a Boolean lattice.

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### 1. INTRODUCTION

In [4], Kuznetsov introduced an operation on Heyting algebras as an attempt to build an intuitionistic version of the provability logic of Gödel-Löb, which formalizes the concept of provability in Peano arithmetic. This unary operation, which we shall call *successor* [1], was also studied by Caicedo and Cignoli in [1] and by Esakia in [3]. In particular, Caicedo and Cignoli considered it as an example of an implicit compatible operation on Heyting algebras.

The successor,  $S$ , can be defined on the variety of Heyting algebras by the following set of equations:

- (S1):  $x \leq S(x)$ ,
- (S2):  $S(x) \leq y \vee (y \rightarrow x)$ ,
- (S3):  $S(x) \rightarrow x = x$ .

There is at most one operation satisfying the previous equations. We shall call  $S$ -algebra to a Heyting algebra endowed with its successor function, when it exists.

The finite model property of the variety of  $S$ -algebras was proved by X. Caicedo in [2], using Kripke model techniques of the associated calculus. A more algebraic proof, but still strongly based on Kripke model ideas, was given by Muravitskii in [5]. In this article we give a purely algebraic proof for the finite model property which is strongly based on the fact that for every element  $x$  in a  $S$ -algebra the interval  $[x, S(x)]$  is a Boolean lattice.

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2. THE FINITE MODEL PROPERTY

Let  $T$  be the type of Heyting algebras with successor built in the usual way from the operation symbols  $\wedge, \vee, \rightarrow, 0$  and  $S$  corresponding to meet, join, implication, bottom and successor, respectively. Write  $T(X)$  for the term algebra of type  $T$  with variables in the set  $X$ . It is well known that any function  $v : X \rightarrow H$ , with  $H$  a  $S$ -algebra, may be extended to a unique homomorphism  $v : T(X) \rightarrow H$ .

Write  $\mathcal{SH}$  for the variety of  $S$ -algebras. Recall that  $\mathcal{SH}$  is said to have the *finite model property* (FMP) if for every  $\psi \in T(X)$  there is a  $S$ -algebra  $H$  and a homomorphism  $v : T(X) \rightarrow H$  such that if  $v(\psi) \neq 1$  then there is a  $S$ -finite algebra  $L$  and a homomorphism  $w : T(X) \rightarrow L$  such that  $w(\psi) \neq 1$ . Let us prove algebraically that  $\mathcal{SH}$  has the FMP.

If  $M$  is a bounded distributive lattice and  $N \subseteq M$ , we write  $\langle N \rangle$  to indicate the bounded sublattice generated by  $N$ . In particular the bottom and the top of  $\langle N \rangle$  and  $M$  are the same. Recall that if  $M$  is a finite distributive lattice then  $M$  is a Heyting algebra. Moreover,  $M$  is a  $S$ -algebra. If  $\{M_i\}_i$  is a family of  $S$ -algebras we write  $\rightarrow_i$  for the implication in  $M_i$  and  $S^i$  for the successor in  $M_i$ .

Note that for any sublattice  $L$  of a Heyting algebra  $H$ , if  $x, y$  and  $x \rightarrow y \in L$ , then  $x \rightarrow y$  is the relative pseudocomplement of  $x$  with respect to  $y$  in  $L$ . This holds because for every  $z \in L$ ,  $z \wedge x \leq y$  iff  $z \leq x \rightarrow y$ , and this property completely characterizes the relative pseudocomplement. The following lemma is a particular instance of the previous remark.

**Lemma 1.** *Let  $M_1$  be a finite distributive lattice and  $M_2$  a  $S$ -algebra such that  $M_1$  is a bounded sublattice of  $M_2$ . If  $x, y, x \rightarrow_2 y \in M_1$  then  $x \rightarrow_2 y = x \rightarrow_1 y$ .*

**Lemma 2.** *Let  $M_1$  be a finite bounded lattice and  $M_2$  a  $S$ -algebra such that  $M_1$  is a bounded sublattice of  $M_2$ . If  $x, S^2(x) \in M_1$  then  $S^1(x) \leq S^2(x)$ .*

*Proof.* Let  $x, S^2(x) \in M_1$ . For every  $y \in M_1$  we have that  $S^1(x) \leq y \vee (y \rightarrow_1 x)$ . In particular it holds for  $y = S^2(x)$ . Hence we have that

$$S^1(x) \leq S^2(x) \vee (S^2(x) \rightarrow_1 x). \tag{1}$$

As  $x, S^2(x), S^2(x) \rightarrow_2 x = x \in M_1$ , by Lemma 1 we have that  $S^2(x) \rightarrow_1 x = S^2(x) \rightarrow_2 x = x$ . Thus by equation (1) we conclude that  $S^1(x) \leq S^2(x) \vee x = S^2(x)$ .  $\square$

If  $H$  is a Heyting algebra and  $a, b \in H$  with  $a \leq b$ , we write  $[a, b]$  for the set  $\{x \in H : a \leq x \leq b\}$ . We say that  $[a, b]$  as sublattice of  $H$  is Boolean if for every  $x \in [a, b]$  there is a  $x^c \in [a, b]$  such that  $x \wedge x^c = a$  and  $x \vee x^c = b$ .

Next lemma is a particular case of the following observation. Since for any interval  $[a, b]$  in a Heyting algebra and for any  $x, y, z \in [a, b]$  we have  $z \wedge x \leq y$  iff  $z \leq x \rightarrow y$  iff  $z \leq b \wedge (x \rightarrow y)$  and  $b \wedge (x \rightarrow y) \in [a, b]$ , we have that the lattice  $[a, b]$  is a Heyting algebra in its own right, with residuum  $x \rightarrow_* y := b \wedge (x \rightarrow y)$ .

**Lemma 3.** *Let  $H$  be a Heyting algebra and  $a, b \in H$  with  $a \leq b$  such that  $[a, b]$  as sublattice of  $H$  is Boolean. If  $x \in [a, b]$  then  $x^c = b \wedge (x \rightarrow a)$ .*

**Lemma 4.** *If  $H$  is a  $S$ -algebra and  $a \in H$  then  $[a, S(a)]$  as sublattice of  $H$  is Boolean. In particular, for every  $x \in [a, S(a)]$  the complement of  $x$ , for which we write  $x^a$ , coincides with  $(x \rightarrow a) \wedge S(a)$ .*

*Proof.* Let  $x \in [a, S(a)]$ . A direct computation proves that  $x \wedge x^a = x \wedge a \wedge S(a) = a$  and  $x \vee x^a = x \vee ((x \rightarrow a) \wedge S(a)) = (x \vee (x \rightarrow a)) \wedge (x \vee S(a)) = S(a)$ .  $\square$

**Definition 1.** *Let  $\psi \in T(X)$ ,  $H$  a  $S$ -algebra and  $v : T(X) \rightarrow H$  a homomorphism. Let  $\rightarrow$  and  $S$  be the implication and the successor of  $H$  respectively. If  $\psi_1, \dots, \psi_n$  are the subformulas of  $\psi$ , for  $i = 1, \dots, n$  we define  $\hat{a}_i$  as  $v(\psi_i)$  and then we consider the sets  $A = \{\hat{a}_1, \dots, \hat{a}_n\} \subseteq H$ ,  $L_0 = \langle A \rangle$  and  $B = \{a \in A : S(a) \in A\}$ . Considering a list  $a_1, \dots, a_k$  for the elements of  $B$  (in case that  $B \neq \emptyset$ ), we define recursively the sets*

$$K_i = \{(x \rightarrow a_i) \wedge S(a_i) : x \in L_{i-1} \cap [a_i, S(a_i)]\},$$

$$L_i = \langle L_{i-1} \cup K_i \rangle,$$

for  $i = 1, \dots, k$ .

Note that every  $a_i, S(a_i) \in L_0$  and that every  $L_i$  is a finite distributive lattice,  $K_i \subseteq L_i$  and  $L_{i-1} \subseteq L_i$ .

**Lemma 5.** *Let  $H$ ,  $A$ ,  $B$  and  $L_i$ , for  $i = 0, \dots, k$  be as in Definition 1, and assume that  $B \neq \emptyset$ . Then, for every  $i = 1, \dots, k$ ,  $L_i \cap [a_i, S(a_i)]$  as a sublattice of  $L_i$  is Boolean. In particular, for every  $x \in [a_i, S(a_i)] \cap L_i$  we have that the complement of  $x$  in  $[a_i, S(a_i)] \cap L_i$  is  $x^{a_i}$ . Moreover,  $x^{a_i} = (x \rightarrow_i a_i) \wedge S(a_i)$ .*

*Proof.* For  $i = 1, \dots, k$  define  $B_i = L_i \cap [a_i, S(a_i)]$ , and let  $z \in B_i$ . Then  $z$  can be written as  $\bigvee_l \bigwedge_m x_{lm}$ , for finitely many  $x_{lm} \in L_{i-1} \cup K_i$ . Note that  $z = \bigvee_l \bigwedge_m z_{lm}$ , with  $z_{lm} = (x_{lm} \vee a_i) \wedge S(a_i)$ , so  $z_{lm} \in B_i$ . Using that  $z_{lm} \in [a_i, S(a_i)]$ , by the Lemma 4 we have that  $(z_{lm})^{a_i}$  is the complement of  $z_{lm}$  in the Boolean algebra  $[a_i, S(a_i)]$ . In the following we will prove that every  $(z_{lm})^{a_i} \in B_i$ .

If  $x_{lm} \in L_{i-1}$  then  $z_{lm} \in L_{i-1}$ . Hence  $z_{lm} \in L_{i-1} \cap [a_i, S(a_i)]$ , so  $(z_{lm})^{a_i} = (z_{lm} \rightarrow a_i) \wedge S(a_i) \in K_i \subseteq L_i$  and in consequence it belongs to  $B_i$ .

If  $x_{lm} \in K_i$  then  $x_{lm} = (x \rightarrow a_i) \wedge S(a_i)$ , for some  $x \in L_{i-1} \cap [a_i, S(a_i)]$ . Thus  $z_{lm} = (x \rightarrow a_i) \wedge S(a_i) = x^{a_i}$ , so  $(z_{lm})^{a_i} = (x^{a_i})^{a_i} = x \in L_{i-1} \cap [a_i, S(a_i)] \subseteq B_i$ .

We have proved that  $(z_{lm})^{a_i}$  is the complement of  $z_{lm}$  in  $B_i$ . An easy computation proves that  $\bigwedge_l \bigvee_m (z_{lm})^{a_i}$  is the complement of  $z$  in  $B_i$ , and hence  $B_i$  is a Boolean algebra. Besides as  $B_i$  is a Boolean sublattice of  $L_i$ , we conclude that  $z^{a_i} = (z \rightarrow_i a_i) \wedge S(a_i)$  (by Lemma 3).  $\square$

**Proposition 1.** *With the notation and hypothesis of Lemma 5, it holds that, for every  $i, j = 1, \dots, k$  such that  $i \leq j$ , we have that  $L_j \cap [a_i, S(a_i)]$  as sublattice of  $L_j$  is Boolean. In particular, for every  $x \in L_j \cap [a_i, S(a_i)]$  we have that the complement of  $x$  in  $L_j \cap [a_i, S(a_i)]$  is equal to  $x^{a_i}$ . Moreover,  $x^{a_i} = (x \rightarrow_i a_i) \wedge S(a_i)$ .*

*Proof.* Fix a natural number  $i$ ,  $i \leq k$ . We will prove by induction that the property holds for every  $j$  such that  $i \leq j \leq k$ . The case  $j = i$  follows from Lemma 5. Suppose that  $L_h \cap [a_i, S(a_i)]$  is a Boolean algebra for some  $h$  such that  $i \leq h < k$ . We will show that  $L_{h+1} \cap [a_i, S(a_i)]$  is a Boolean algebra.

A direct computation proves that the function  $f_h : L_{h+1} \cap [a_{h+1}, S(a_{h+1})] \rightarrow L_{h+1} \cap [a_i, S(a_i)]$ , given by  $f_h(x) = (x \vee a_i) \wedge S(a_i)$ , is a homomorphism of lattices. Let  $z \in L_{h+1} \cap [a_i, S(a_i)]$ , so  $z$  can be written as  $\bigvee_l \bigwedge_m x_{lm}$ , for finitely many  $x_{lm} \in L_h \cup K_{h+1}$ . In particular  $z = \bigvee_l \bigwedge_m z_{lm}$ , with  $z_{lm} = (x_{lm} \vee a_i) \wedge S(a_i)$ . To prove that  $L_{h+1} \cap [a_i, S(a_i)]$  is a Boolean algebra it is enough to prove that  $z_{lm}$  has complement in  $L_{h+1} \cap [a_i, S(a_i)]$ .

If  $x_{lm} \in L_h$  then  $z_{lm} \in L_h \cap [a_i, S(a_i)]$ . By inductive hypothesis we have that  $L_h \cap [a_i, S(a_i)]$  is a Boolean algebra, so  $z_{lm}^{a_i} \in L_h \cap [a_i, S(a_i)] \subseteq L_{h+1} \cap [a_i, S(a_i)]$ .

We consider the case  $x_{lm} \in K_{h+1}$ . In particular,  $x_{lm} \in L_{h+1} \cap [a_{h+1}, S(a_{h+1})]$ . Hence  $z_{lm} = f_h(x_{lm}) \in L_{h+1} \cap [a_i, S(a_i)]$ . We define the elements

$$\alpha = f_h(a_{h+1}), \omega = f_h(S(a_{h+1})), u = z_{lm} = f_h(x_{lm}), \bar{u} = f_h(x_{lm}^{a_{h+1}}),$$

$$v = (\omega^{a_i} \vee \bar{u}) \wedge \alpha^{a_i}.$$

The element  $v$  belongs to  $L_{h+1} \cap [a_i, S(a_i)]$ . It is clear that  $v \in [a_i, S(a_i)]$ . Besides as  $a_{h+1}, a_i, S(a_{h+1}), S(a_i) \in L_0$  we have that  $\alpha, \omega \in L_i$ , so  $\alpha, \omega \in L_i \cap [a_i, S(a_i)]$ . Using Lemma 5 we have that  $\alpha^{a_i}, \omega^{a_i} \in L_i \cap [a_i, S(a_i)] \subseteq L_{h+1} \cap [a_i, S(a_i)]$ . As  $\bar{u} \in L_{h+1} \cap [a_i, S(a_i)]$  we have that  $v \in L_{h+1} \cap [a_i, S(a_i)]$ . In the following we will prove that  $u \vee v = S(a_i)$  and  $u \wedge v = a_i$ .

Using that  $a_{h+1} \leq x_{lm} \leq S(a_{h+1})$  we have that

$$\alpha \leq u \leq \omega.$$

Then using that  $f_h$  is a homomorphism of lattices we have that

$$v \vee u = ((\omega^{a_i} \vee \bar{u}) \wedge \alpha^{a_i}) \vee u = (\omega^{a_i} \vee \bar{u} \vee u) \wedge (\alpha^{a_i} \vee u) = (\omega^{a_i} \vee \omega) \wedge (\alpha^{a_i} \vee u)$$

$$= S(a_i) \wedge (\alpha^{a_i} \vee u) \geq S(a_i) \wedge (\alpha^{a_i} \vee \alpha) = S(a_i) \wedge S(a_i) = S(a_i).$$

Thus  $u \vee v = S(a_i)$ . On the other hand,

$$v \wedge u = ((\omega^{a_i} \vee \bar{u}) \wedge \alpha^{a_i}) \wedge u = \alpha^{a_i} \wedge ((\omega^{a_i} \wedge u) \vee (\bar{u} \wedge u)) = \alpha^{a_i} \wedge ((\omega^{a_i} \wedge u) \vee \alpha)$$

$$\leq \alpha^{a_i} \wedge ((u^{a_i} \wedge u) \vee \alpha) = \alpha^{a_i} \wedge (a_i \vee \alpha) = \alpha^{a_i} \wedge \alpha = a_i.$$

Thus  $u \wedge v = a_i$ . Therefore  $L_{h+1} \cap [a_i, S(a_i)]$  is a Boolean algebra. □

**Theorem 6.**  $\mathcal{SH}$  has the FMP.

*Proof.* Let  $\psi \in T(X)$ ,  $H$  a  $S$ -algebra and  $v : T(X) \rightarrow H$  a homomorphism such that  $v(\psi) \neq 1$ . Let  $\rightarrow$  and  $S$  be the implication and the successor of  $H$  respectively. We will prove that there is a finite  $S$ -algebra  $L$  and  $w : T(X) \rightarrow L$  a homomorphism such that  $w(\psi) \neq 1$ .

Let  $\psi_1, \dots, \psi_n$  be all the subformulas of  $\psi$ . For  $i = 1, \dots, n$  we define  $\hat{a}_i = v(\psi_i)$ . In the following we will use the notation given in Definition 1.

If  $B = \emptyset$  then we can take  $L = L_0$ ; so let us assume in what follows that  $B$  is non-void.

Every  $L_i$  is a finite  $S$ -algebra. We will prove that  $S^1(a_1) = S(a_1)$ . As  $S(a_1) \in L_0$  we have that  $S(a_1) \in L_1$ . Thus by Lemma 2 it holds that  $S^1(a_1) \leq S(a_1)$ , so  $S^1(a_1) \in L_1 \cap [a_1, S(a_1)]$ . By Proposition 1 we have that

$$(S^1(a_1))^{a_1} = (S^1(a_1) \rightarrow_1 a_1) \wedge S(a_1) = a_1 \wedge S(a_1) = a_1. \tag{2}$$

Hence  $S^1(a_1) = S(a_1)$ .

In a similar way we can prove that  $S^2(a_2) = S(a_2)$ . Note that by Lemma 2 and Proposition 1 we have that  $S(a_1) = S^2(a_1)$ . Iterating this argument we obtain that  $L = L_k$  is a finite bounded sublattice of  $H$  that satisfies the following two conditions:

- (1) If  $a, b, a \rightarrow b \in L$  then  $a \rightarrow b = a \rightarrow_k b$  (by Lemma 1).
- (2) For every  $i = 1, \dots, k$ ,  $S(a_i) = S^k(a_i)$ .

Let  $V$  the set of propositional variables that appear in  $\psi$ . We define a function  $w : X \rightarrow L$  in the following way:

$$w(x) = \begin{cases} v(x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

This function may be extended to a unique homomorphism  $w : T(X) \rightarrow L$ . By an easy induction on formulas one can prove that  $w(\psi_i) = v(\psi_i)$ , for  $i = 1, \dots, n$ . Therefore  $w(\psi) = v(\psi) \neq 1$ .  $\square$

Take  $\alpha$  and  $\beta$  in  $T(X)$ . Note that an equation  $\alpha \approx \beta$  holds in a  $S$ -algebra  $H$  if and only if  $\alpha \rightarrow \beta \approx 1$  holds in  $H$ ; and the latter is equivalent to requiring that for any homomorphism  $v : T(X) \rightarrow H$ ,  $v(\alpha \rightarrow \beta) = 1$ .

**Corollary 7.** *The variety  $\mathcal{SH}$  is generated by its finite members.*

*Proof.* Let  $H$  be an  $S$ -algebra and let us assume that the equation  $\alpha \approx \beta$  does not hold in  $H$ . By the previous remark, this implies the existence of a homomorphism  $v : T(X) \rightarrow H$ , such that  $v(\alpha \rightarrow \beta) \neq 1$ . By Theorem 6, there are a finite  $S$ -algebra  $L$  and a homomorphism  $w : T(X) \rightarrow L$ , such that  $w(\alpha \rightarrow \beta) \neq 1$ .

Using the previous remark again, this implies that  $\alpha \approx \beta$  does not hold in the finite algebra  $L$ .  $\square$

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