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ON ROOTED DIRECTED PATH GRAPHS

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Abstract. An asteroidal triple is a stable set of three vertices such that each pair is connected by a path avoiding the neighborhood of the third vertex. An asteroidal quadruple is a stable set of four vertices such that any three of them is an asteroidal triple.

Two non adjacent vertices are linked by a special connection if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain technical conditions. Cameron, Hoàng, and Lévêque [DIMAP Workshop on Algorithmic Graph Theory, 67–74, Electron. Notes Discrete Math., 32, Elsevier, 2009] proved that if a pair of non adjacent vertices are linked by a special connection then in any directed path model T the subpaths of T corresponding to the vertices forming the special connection have to overlap and they force T to be completely directed in one direction between these vertices. Special connections along with the concept of asteroidal quadruple play an important role to study rooted directed path graphs, which are the intersection graphs of directed paths in a rooted directed tree.

In this work we define other special connections; these special connections along with the ones defined by Cameron, Hoàng, and Lévêque are nine in total, and we prove that every one forces T to be completely directed in one direction between these vertices. Also, we give a characterization of rooted directed path graphs whose rooted models cannot be rooted on a bold maximal clique. As a by-product of our result, we build new forbidden induced subgraphs for rooted directed path graphs.

1. Introduction

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [4] states that a graph is chordal if and only if it is the (vertex) intersection graph of a family of subtrees of a tree.

Natural subclasses of chordal graphs are path graphs, directed path graphs, rooted directed path graphs and interval graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. A graph is a *rooted path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. A graph is an *interval graph* if it is the intersection graph of a family of subpaths of a path.

By definition we have the following inclusions between the different classes considered (and these inclusions are strict): interval $\subset$ rooted directed path $\subset$ directed path $\subset$ path $\subset$ chordal.

Lekkerkerler and Boland [5] proved that a chordal graph is an interval graph if and only if it contains no asteroidal triple. As a by-product, they found a characterization of interval graphs by forbidden induced subgraphs.

Panda [8] found a characterization of directed path graphs by forbidden induced subgraphs, and then Cameron, Hoáng and Lévêque [2] gave a characterization of directed paths graph in terms of forbidden asteroids. For this purpose, they introduce the concept of a special connection. Two non adjacent vertices are linked by a special connection if they have a common neighbor or they are the endpoints of two vertex-disjoint paths of length three satisfying certain technical conditions. Special connections are interesting when considering directed path graphs because if a and b, two non adjacent vertices of a directed path graph, are linked by a special connection, then in every directed path model, the subpaths of T corresponding to the vertices forming the special connection have to overlap and they force T to be completely directed in one direction between a and b.

Clearly, rooted directed path graphs contain no asteroidal quadruples linked by special connections. The converse was conjectured by Cameron, Hoáng and Lévêque, but in this original form the conjecture is incomplete, since they could not describe all the connections between two non adjacent vertices that force any tree to be completely directed in one direction between these vertices.

In this article we define some special connections, which along with the ones defined in [1] are nine in total, and we prove that each one forces T to be completely directed in one direction between these vertices. Therefore, if a_1, a_2, a_3 is a strong asteroidal and there is a special connection between a_1 and a_2 then no directed path model can be rooted on a maximal clique that contains a_3. Furthermore, we prove that the converse is true in case of leafage three, i.e., if the model cannot be rooted on a maximal clique that contains a_3 then one of the nine special connection links a_1 and a_2. As a by-product of our result, we build new forbidden induced subgraphs for rooted directed path graphs.

The paper is organized as follows: in Section 2 we give some definitions and background. In Section 3 we define special connections and prove that if a pair of non adjacent vertices are linked by a special connection, then in any directed path model T the subpaths of T corresponding to the vertices forming the special connection have to overlap and they force T to be completely directed in one direction between these vertices. In section 4 we give some properties of models that cannot be rooted on a bold maximal clique. Finally, in Section 5 we prove that G is a directed path graph with leafage three, and it has a strong asteroidal triple a_1, a_2, a_3 such that there is a special connection between a_1 and a_2 if and only if no directed path model can be rooted on the maximal clique that contains a_3.

2. Definitions and background

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G + G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and edges are $E(G) \cup E(G')$. Note that if T, T' are two trees such that $|V(T) \cap V(T')| = 0$, then $T + T'$ is a forest.

A clique in a graph G is a set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all maximal cliques* of G.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$. A vertex is *simplicial* if its closed neighborhood is a maximal clique. Two adjacent vertices x and y are *twins* if $N[x] = N[y]$.

A *strong asteroidal* of a graph G is a stable set $\{a_1, \ldots, a_n\}$ $(n \geq 2)$ of vertices of G such that $G \setminus N[a_i]$ is a connected graph for $i = 1, \ldots, n$.

A *clique tree* T of a graph G is a tree whose vertices are the elements of $\boldsymbol{C}(G)$ and such that for each vertex x of G, those elements of $\boldsymbol{C}(G)$ that contain x induce a subtree of T, which we will denote by T_x. Note that G is the intersection graph of the subtrees $(T_x)_{x \in V(G)}$. In this paper, whenever we talk about the intersection of subgraphs of a graph we mean that the vertex sets of the subgraphs intersect.

Given two non adjacent vertices a, b of G, and a clique tree T of G, $T(a, b)$ is defined as the subtree of T of minimum size that contains at least a vertex of T_a and T_b.

Gavril [4] proved that a graph is chordal if and only if it has a clique tree. Clique trees are called *models* of the graph.

Observe that if G has a strong asteroidal $a_1, \ldots, a_n$ then every clique tree has $N[a_i]$ as a leaf for $i = 1, \ldots, n$. So a_i is a simplicial vertex of G for $i = 1, \ldots, n$.

In [7] Monma and Wei introduced the notation UV, DV and RDV to refer to the classes of path graphs, directed path graphs and rooted directed path graphs, respectively. They also proved the following clique tree characterizations for these classes. A graph is a *path graph* or a *UV graph* if it admits a *UV-model*, i.e., a clique tree T such that T_x is a subpath of T for every $x \in V(G)$. A graph is a *directed path graph* or a *DV graph* if it admits a *DV-model*, i.e., a clique tree T whose edges can be directed such that T_x is a directed subpath of T for every $x \in V(G)$. A graph is a *rooted path graph* or an *RDV graph*, if it admits an *RDV-model*, i.e., a clique tree T that can be rooted and whose edges are directed from the root toward the leaves such that T_x is a directed subpath of T for every $x \in V(G)$.

It has been proved in [3] that if G is a DV-graph, then any UV-model of G can be directed to obtain a DV-model of G. We say that a DV-model T of a DV graph G can be *rooted* if T can be rooted on a vertex such that it becomes an RDV-model of G.

Let T be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to maximal cliques of G. In order to simplify the notation, we often write $X \in T$ instead of $X \in V(T)$, and $e \in T$ instead of $e \in E(T)$. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\cup_{X \in V(T')} X$.

Let T be a tree. For $V' \subseteq V(T)$, let $T[V']$ be the minimal subtree of T containing V'. Then for $X, Y \in V(T)$, $T[X, Y]$ is the subpath of T between X and Y. Let $T[X, Y) = T[X, Y] \setminus Y$, $T(X, Y] = T[X, Y] \setminus X$, and $T(X, Y) = T[X, Y] \setminus \{X, Y\}$.

Note that some of these paths may be empty or reduced to a single vertex when X and Y are equal or adjacent. If $X \in V(T)$ and $e \in E(T)$ with $e = AB$ and $A \in T[X, B]$, then let $T[X, e] = T[X, B]$, $T[X, e) = T[X, A]$, $T(X, e] = T(X, B]$ and $T(X, e) = T(X, A]$. Given a vertex $X \in V(T(Y, Z))$, we say that there is a *vertex crossing X in $T[Y, Z]$* if $X' \cap X'' \neq \emptyset$, where X' and X'' are the two neighbors of X in $T[Y, Z]$.

Let T be a tree, we denote by $\ln(T)$ the number of leaves of T. The *leafage* of a chordal graph G, denoted by $\ell(G)$, is the minimum integer k such that G admits a model T with $\ln(T) = k$. Note that if G has a strong asteroidal $a_1, \ldots, a_n$ then $\ell(G) \geqq n$.

In a clique tree T, the *label* of an edge AB of T is defined as $lab(AB) = A \cap B$. We will say that e, e' in the same clique tree T are *twin edges* if $lab(e) = lab(e')$.

Let T be a DV-model of G, let Q be a vertex of T, and let e be an edge of T. Let T_1 and T_2 be the two connected components of $T - e$ where Q is in T_1. We say that vertices in $lab(e)$ have *the same end* with respect to Q if there exists a vertex Q' in T_1, possibly $Q' = Q$, such that for each $x \in lab(e)$, one endpoint of T_x is Q'.

We say that $X \in V(T)$ *dominates* $e \in E(T)$ if $lab(e) \subseteq X$. On the other hand, an edge e satisfying a given property P is *maximally farthest from a vertex C* if there is no edge e', different from e, satisfying this property and such that e is between C and e'.

3. Special connections

Let a and b be two non adjacent vertices of a graph G. We will define nine *types of connection* between these vertices. Observe that Types 1, 2 and 3 were already defined in [2].

- **Type 1:** there exists a path $P = a, x, b$ in G.
- **Type 2:** there exist two paths $P = a, y_1, y_2, b$ and $Q = a, x_1, x_2, b$ in G such that $\{x_1, y_1, y_2\}$ and $\{x_1, x_2, y_2\}$ are cliques of G.
- **Type 3:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and two vertices s_1, s_2 in G such that $\{x_1, x_2, y_1, y_2\}$, $\{x_1, y_1, y_2, s_1\}$ and $\{x_1, x_2, y_2, s_2\}$ are cliques of G. In this case it is said that $\{x_1, x_2, y_1, y_2, s_1, s_2\}$ induces an antenna.

  Next we define new special connections.
- **Type 4:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and vertices in G: t, u, z_i for $i \in \{1, \ldots, o\}$ such that $\{x_1, x_2, y_1, y_2, z_o\}$, $\{x_1, y_1, z_i, z_{i+1}\}_{i=1,\ldots,o-1}$, $\{y_1, z_1, u\}$ and $\{x_1, x_2, y_1, t\}$ are cliques of G (Figure 1a).
- **Type 5:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and vertices in G: s_1, s, t, t_i for $i \in \{1, \ldots, p\}$, u, z_i for $i \in \{1, \ldots, o\}$ such that $\{x_1, x_2, y_1, y_2, z_o, s_1\}$, $\{x_1, x_2, y_1, y_2, z_o, t_p\}$, $\{x_1, x_2, y_1, z_o, t_p, s\}$, $\{x_1, y_1, z_i, z_{i+1}\}_{i=1,\ldots,o-1}$, $\{y_1, z_1, u\}$, $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1,\ldots,p-1}$ and $\{x_1, y_1, t_1, t\}$ are cliques of G (Figure 1b).
- **Type 6:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and vertices in G: t, t_i for $i \in \{1, \ldots, p\}$, u, z_i for $i \in \{1, \ldots, o\}$ such that $\{x_1, x_2, y_1, y_2, z_o\}$, $\{x_1, x_2, y_1, z_o, t_p\}$, $\{x_1, y_1, z_i, z_{i+1}\}_{i=1,\ldots,o-1}$, $\{x_1, y_1,$

$z_o, t_i,\ t_{i+1}\}_{i=1,\ldots,p-1}$, $\{y_1, z_1, u\}$ and $\{x_1, y_1,\ t_1, t\}$ are cliques of G (Figure 1c).

- **Type 7:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and vertices in G: s, t, t_i for $i \in \{1, \ldots, p\}$, u, u', z_i for $i \in \{1, \ldots, o\}$, z'_i for $i \in \{1, \ldots, q\}$ such that $\{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$, $\{x_1, x_2, y_1, z_o, t_p, s\}$, $\{y_1, z_1, u\}$, $\{x_1, y_1, z_o, z'_i, z'_{i+1}, t_p\}_{i=1,\ldots,q-1}$, $\{x_1, y_1, z_o, z'_1, u'\}$, $\{x_1, y_1, z_i, z_{i+1}\}_{i=1,\ldots,o-1}$, $\{x_1, y_1, t_1, t\}$ and $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1,\ldots,p-1}$ are cliques of G (Figure 1d).
- **Type 8:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and vertices in G: t, t', t_i for $i \in \{1, \ldots, p\}$, t'_i for $i \in \{1, \ldots, r\}$, u, u', z_i for $i \in \{1, \ldots, o\}$, z'_i for $i \in \{1, \ldots, q\}$ such that $\{x_1, x_2,\ y_1, y_2,\ z_o, t_p,\ z'_q\}$, $\{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r\}$, $\{x_1, y_1, z_o, t_p, t'_1, t'\}$, $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1,\ldots,p-1}$, $\{y_1, z_1, u\}$, $\{x_1, y_1,\ z_o, z'_q,\ t_p, t'_i,\ t'_{i+1}\}_{i=1,\ldots,r-1}$, $\{x_1, y_1,\ z_o, z'_1,\ u'\}$, $\{x_1, y_1, z_i, z_{i+1}\}_{i=1,\ldots,o-1}$, $\{x_1, y_1,\ t_1, t\}$, and $\{x_1, y_1,\ z_o, t_p,\ z'_i,\ z'_{i+1}\}_{i=1,\ldots,q-1}$ are cliques of G (Figure 1e).
- **Type 9:** there exist two paths $P = a, y_1, y_2, b$, $Q = a, x_1, x_2, b$, and vertices in G: $s, s_1,\ t, t'$, t_i for $i \in \{1, \ldots, p\}$, t'_i for $i \in \{1, \ldots, r\}$, u, u', z_i for $i \in \{1, \ldots, o\}$, z'_i for $i \in \{1, \ldots, q\}$ such that $\{x_1, x_2, y_1, y_2, z_o, t_p, z'_q, t'_r, s_1\}$, $\{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r, s\}$, $\{x_1, y_1, z_o, t_p, t'_1, t'\}$, $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1,\ldots,p-1}$, $\{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}_{i=1,\ldots,r-1}$, $\{x_1, y_1, t_1, t\}$, $\{x_1, y_1, z_i, z_{i+1}\}_{i=1,\ldots,o-1}$, $\{y_1, z_1, u\}$, $\{x_1, y_1, z_o, z'_1, u'\}$ and $\{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}_{i=1,\ldots,q-1}$ are cliques of G (Figure 1f).

Theorem 1. *Let G be a DV graph, and let a, b be two non adjacent vertices of G that are linked by Type i with $1 \leq i \leq 9$. Then, for every T, DV-model of G, the subpath $T(a, b)$ is a directed path.*

Proof. Let Q_a be a maximal clique that contains a, and Q_b be a maximal clique that contains b.

(1) Types $1, 2, 3$. In [2] it was proved that if a and b are linked by Type i, with $i \in \{1, 2, 3\}$, then $T[Q_a, Q_b]$ is a directed path of T.

(2) We can assume that there is a special connection of Type 4, 5, 6, 7, 8 or 9.

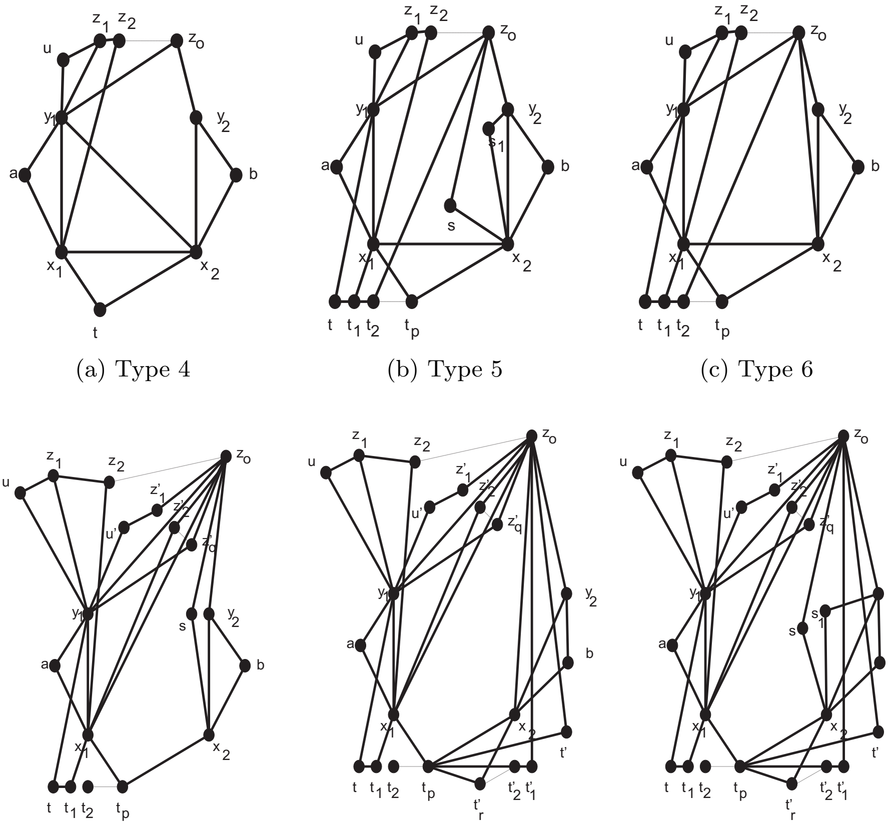
The edge of $T[Q_a, Q_b]$ incident to Q_a must have in its label S at least one vertex of $\{x_1, x_2\}$ ($\{y_1, y_2\}$), otherwise a and b are in two different components of $G \setminus S$ contradicting that a, x_1, x_2, b (a, y_1, y_2, b) is a path. Analogously, the edge incident to Q_b must have in its label at least one vertex of $\{y_1, y_2\}$ ($\{x_1, x_2\}$). Vertex a is not adjacent to x_2, and b is not adjacent to y_1 then $Q_a \cap \{x_1, x_2, y_1, y_2\} = \{x_1, y_1\}$ and $Q_b \cap \{x_1, x_2, y_1, y_2\} = \{x_2, y_2\}$.

(a) Suppose that the connection is of Type 4.

Let Q', Q_o, Q and Q_i be maximal cliques of G such that $Q' \supset \{x_1, x_2, y_1, t\}$, $Q_o \supset \{x_1, x_2, y_1, y_2, z_o\}$, $Q \supset \{y_1, z_1, u\}$, and $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$ for $i = 1, \ldots, o-1$.

We will prove that Q_a, Q', Q_o, Q_b appear in this order in T.

As a is not adjacent to x_2, b is not adjacent to x_1, and x_1, x_2 are vertices in $Q_o \cap Q'$, then we have $Q_a \notin T[Q_o, Q']$ and $Q_b \notin T[Q_o, Q']$. Observe that x_1 and y_1 are vertices in $(Q' \cap Q_a) - (Q_o \cap Q_b)$, x_2 and



(a) Type 4 (b) Type 5 (c) Type 6

(d) Type 7 (e) Type 8 (f) Type 9

Figure 1. Types 4 through 9. We leave out edges of cliques of size greater than or equal to four.

y_2 are in $(Q_o \cap Q_b) - (Q' \cap Q_a)$, $x_2 \in Q' \cap Q_b$, but neither x_1 or y_1 or y_2 are in $Q' \cap Q_b$. Thus Q_a, Q', Q_o, Q_b appear in this order in T. Since $x_1 \in Q_a \cap Q_o$ and $x_2 \in Q' \cap Q_b$ it follows that $T[Q_a, Q_b]$ is a directed path of T.

On the other hand $Q_i \notin T[Q', Q_o]$ for $i \neq o$, since x_2 is not adjacent to z_i for $i \neq o$. As x_1, y_1 and z_i are vertices in $Q_i \cap Q_{i+1}$ for $i = 1, \ldots, o-1$, x_1 and y_1 are in $Q_i \cap Q_a$, and z_1, y_1 are vertices in $Q_1 \cap Q$, then Q_a, Q', Q_o, Q_{o-1}, $\ldots$, Q_1, Q appear in this order in T. Vertex b is not adjacent to y_1, so $Q_b \notin T[Q_a, Q]$.

(b) Suppose that the connection is of Type 5.

Let $Q_o, Q'_o, Q'_p, Q_i, Q, Q'_i$ and Q' be maximal cliques of G such that $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, s_1\}$, $Q'_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p\}$, $Q'_p \supset \{x_1, x_2, y_1, z_o, t_p, s\}$, $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$ for $i = 1, \ldots, o-1$, $Q \supset$

$\{y_1, z_1, u\}$, $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$ for $i = 1, \ldots, p-1$, and $Q' \supset \{x_1, y_1, t_1, t\}$. We will prove that Q_a, Q', $Q'_1, \ldots,$ Q'_p, Q'_o, Q_o, Q_b appear in this order in T.

Vertex s_1 is not adjacent to t_p, and s is not adjacent to y_2, then $Q_o \notin T[Q'_o, Q'_p]$ and $Q'_p \notin T[Q'_o, Q_o]$ respectively. Observe that x_1, x_2, y_1, z_o and t_p are vertices in $Q'_o \cap Q'_p$ but $y_2 \notin Q'_o \cap Q'_p$. Also x_1, x_2, y_1, y_2 and z_o are in $Q'_o \cap Q_o$ but $t_p \notin Q'_o \cap Q_o$. Thus Q'_p, Q'_o, Q_o appear in this order in T. On the other hand, $Q'_i \notin T[Q'_p, Q_o]$ for $i \neq p$ since t_i is not adjacent to x_2. As $\{x_1, y_1, z_o, t_i\} \subset Q'_i \cap Q'_{i+1}$ and $\{x_1, y_1, z_o\} \subset Q'_i \cap Q_o$, but $t_i \notin Q_o$ for $i \neq p$, it follows that $Q'_1, \ldots,$ Q'_p, Q'_o, Q_o appear in this order in T. Note that $Q' \notin T[Q'_1, Q_o]$ since z_o is not adjacent to t; $\{x_1, y_1, t_1\} \subset Q' \cap Q'_1$, x_1 and y_1 are in $Q' \cap Q_o$ but $t_1 \notin Q_o$, so Q', $Q'_1, \ldots,$ Q'_p, Q'_o, Q_o appear in this order in T. Since a is not adjacent to t_i for $i = 1, \ldots, p$, and is also not adjacent to x_2 then $Q_a \notin T[Q', Q_o]$. Vertex b is not adjacent to t_i for $i = 1, \ldots, p$ and is also not adjacent to y_1, so $Q_b \notin T[Q', Q_o]$. As $x_1, y_1 \in (Q_a \cap Q') - (Q_o \cap Q_b)$ and $x_2, y_2 \in (Q_o \cap Q_b) - (Q_a \cap Q')$, it follows that Q_a, Q', $Q'_1, \ldots,$ Q'_p, Q'_o, Q_o, Q_b appear in this order in T. Since $x_1 \in Q_a \cap Q_o$ and $x_2 \in Q' \cap Q_b$ then $T[Q_a, Q_b]$ is a directed path of T.

Using the same argument of Case 2a, Q_a, Q', $Q'_1, \ldots,$ Q'_o, Q_o, $Q_{o-1}, \ldots,$ Q_1, Q appear in this order in T and also $Q_b \notin T[Q_a, Q]$.

(c) Suppose that the connection is of Type 6 or 7.

In case that the connection is of Type 6, let Q_o, Q'_p, Q_i, Q, Q'_i and Q' be maximal cliques of G such that $Q_o \supset \{x_1, x_2, y_1, y_2, z_o\}$, $Q'_p \supset \{x_1, x_2, y_1, z_o, t_p\}$, $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$ for $i = 1, \ldots, o-1$, $Q \supset \{y_1, z_1, u\}$, $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$ for $i = 1, \ldots, p-1$, and $Q' \supset \{x_1, y_1, t_1, t\}$.

In case that the connection is of Type 7, let $Q_o, Q'_p, Q''_i, Q'', Q_i, Q'_i$ and Q' be maximal cliques of G such that $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$, $Q'_p \supset \{x_1, x_2, y_1, z_o, t_p, s\}$, $Q''_i \supset \{x_1, y_1, z_o, z'_i, z'_{i+1}, t_p\}$ for $i = 1, \ldots,$ $q-1$, $Q'' \supset \{x_1, y_1, z_o, z'_1, u'\}$, $Q' \supset \{x_1, y_1, t_1, t\}$, $Q \supset \{y_1, z_1, u\}$, $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$ for $i = 1, \ldots, o-1$, and $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$ for $i = 1, \ldots, p-1$.

In both cases, we will prove that Q_a, Q', $Q'_1, \ldots,$ Q'_p, Q_o, Q_b appear in this order in T.

We know that $\{x_1, x_2, y_1, z_o, t_p\} \subset Q_o \cap Q'_p$ if the connection is of Type 7, and $\{x_1, x_2, y_1, z_o\} \subset Q_o \cap Q'_p$ if the connection is of Type 6. But in both cases we have $y_2 \notin Q_o \cap Q'_p$. Vertex t_i is not adjacent to x_2 for $i \neq p$, then $Q'_i \notin T[Q_o, Q'_p]$. Observe that $\{x_1, y_1, z_o, t_i\} \subset Q'_i \cap Q'_{i+1}$, $\{x_1, y_1, z_o\} \subset Q'_i \cap Q_o$ but $t_i \notin Q_o$ for $i \neq p$. Thus $Q'_1, \ldots,$ Q'_p, Q_o appear in this order in T. Note that $Q' \notin T[Q'_1, Q_o]$ since z_o is not adjacent to t. Vertices x_1, y_1 and t_1 are in $Q' \cap Q'_1$, x_1 and y_1 are $Q' \cap Q_o$ but $t_1 \notin Q_o$, so Q', $Q'_1, \ldots,$ Q'_p, Q_o appear in this order in T. Vertex a is not adjacent to t_i for $i = 1, \ldots, p$, and is also not adjacent to x_2, then $Q_a \notin T[Q', Q_o]$. Since b is not adjacent to t_i for $i = 1, \ldots, p$ and b is not adjacent to y_1 so $Q_b \notin T[Q', Q_o]$. As $x_1, y_1 \in (Q_a \cap Q') - (Q_o \cap Q_b)$ and $x_2, y_2 \in (Q_o \cap Q_b) - (Q_a \cap Q')$, then Q_a, Q', $Q'_1, \ldots$, Q'_p, Q_o, Q_b appear in this order in T. Since $x_1 \in Q_a \cap Q_o$ and $x_2 \in Q' \cap Q_b$ it follows that $T[Q_a, Q_b]$ is a directed path of T.

On the other hand, in case that the connection is of Type 6, using the same argument of Case 2a, we have that Q_a, Q', $Q'_1, \ldots$, Q_o, $Q_{o-1}, \ldots$, Q_1, Q appear in this order in T. And as b is not adjacent to y_1 then $Q_b \notin T[Q_a, Q]$.

In case that the connection is of Type 7, as z'_i is not adjacent to z_j for $j \in \{1, \ldots, o-1\}$ and $i \in \{1, \ldots, q\}$ it follows that $Q'''_1, Q''_i \notin T[Q_j, Q_{j+1}]$. Also $Q''_i \notin T[Q'_1, Q_o]$ since z'_i is not adjacent to t_j for $i \in \{1, \ldots, q-1\}$ and $j \in \{1, \ldots, p-1\}$. Observe that $\{x_1, y_1, z_o, t_p, z'_i\} \subset Q''_i \cap Q''_{i+1}$ for $i \neq q$, $\{x_1, y_1, z_o, t_p\} \subset Q''_i \cap Q_o$, $\{x_1, y_1\} \subset Q_a \cap Q''_i$, and $\{x_1, y_1, z_o\} \subset Q''_i \cap Q_{o-1}$ then Q_a, $Q'_1, \ldots$, Q_o, $Q''_q, \ldots$, Q''_1, $Q_{o-1}, \ldots$, Q appear in this order in T, and also $Q_b \notin T[Q_a, Q]$.

(d) Suppose that the connection is of Type 8 or 9.

In case that the connection is of Type 8, let Q_o, Q'_p, Q^{iv}, Q'_i, Q'''_i, Q', Q'', Q, Q_i, and Q''_i be maximal cliques of G such that $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$, $Q'_p \supset \{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r\}$, $Q^{iv}\{x_1, y_1, z_o, t_p, t'_1, t'\}$, $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$ for $i = 1, \ldots, p-1$, $Q'''_i \supset \{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}$ for $i = 1, \ldots, r-1$, $Q' \supset \{x_1, y_1, t_1, t\}$, $Q'' \supset \{x_1, y_1, z_o, z'_1, u'\}$, $Q \supset \{y_1, z_1, u\}$, $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$ for $i = 1, \ldots, o-1$, and $Q''_i \supset \{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}$ for $i = 1, \ldots, q-1$.

In case that the connection is of Type 9, let Q_o, Q'_p, Q^{iv}, Q'_i, Q'', Q'''_i, Q', Q_i, Q and Q''_i be maximal cliques of G such that $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p, z'_q, t'_r, s_1\}$, $Q'_p \supset \{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r, s\}$, $Q^{iv} \supset \{x_1, y_1, z_o, t_p, t'_1, t'\}$, $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$ for $i = 1, \ldots, p-1$, $Q'' \supset \{x_1, y_1, z_o, z'_1, u'\}$, $Q'''_i \supset \{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}$ for $i = 1, \ldots, r-1$, $Q' \supset \{x_1, y_1, t_1, t\}$, $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$ for $i = 1, \ldots, o-1$, $Q \supset \{y_1, z_1, u\}$, and $Q''_i \supset \{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}$ for $i = 1, \ldots, q-1$.

In both cases, we will prove that Q_a, Q', $Q'_1, \ldots$, Q'_{p-1}, Q^{iv}, $Q'''_1, \ldots$, Q'''_{r-1}, Q'_p, Q_o, Q_b appear in this order in T.

We know that $\{x_1, x_2, y_1, z_o, t_p, z'_q\} \subset Q_o \cap Q'_p$ but $y_2 \notin Q_o \cap Q'_p$. As t'_i is not adjacent to x_2 for $i \neq r$, so $Q'''_i \notin T[Q_o, Q'_p]$.

Observe that $\{x_1, y_1, z_o, z'_q, t'_i\} \subset Q'''_i \cap Q'''_{i+1}$, $\{x_1, y_1, z_o, z'_q, t_p, t'_r\} \subset Q'''_{r-1} \cap Q'_p$ but $t'_r \notin Q_o$, $\{x_1, y_1, z_o, z'_q, t_p\} \subset Q'''_i \cap Q'_p$ but $t'_i \notin Q'_p$ for $i \neq r$. Then $Q'''_1, \ldots$, Q'''_{r-1}, Q'_p, Q_o appear in this order in T.

On the other hand, $Q^{iv} \notin T[Q'_1, Q_o]$ since z'_q is not adjacent to t'. Vertices x_1, y_1, z_o, t_p and t'_1 are in $Q^{iv} \cap Q'''_1$, x_1, y_1, z_o and t_p are in $Q^{iv} \cap Q_o$ but $t'_1 \notin Q_o$ it follows that Q^{iv}, $Q'''_1, \ldots$, Q'''_{r-1}, Q'_p, Q_o appear in this order in T.

As t_i is not adjacent to t_p for $i \neq p-1$ then $Q'_i \notin T[Q^{iv}, Q_o]$. Observe that $\{x_1, y_1, z_o, t_i\} \subset Q'_i \cap Q'_{i+1}$, $\{x_1, y_1, z_o\} \subset Q'_i \cap Q_o$ but $t_i \notin Q_o$ for $i \neq p$ and $\{x_1, y_1, z_o, t_p, t'_1\} \subset Q'_{p-1} \cap Q^{iv}$, $\{x_1, y_1, z_o, t_p\} \subset Q'_{p-1} \cap Q_o$ and $t_1 \notin Q_o$. Thus $Q'_1, \dots, Q'_{p-1} Q^{iv}$, $Q'''_1, \dots, Q'''_{r-1}$, Q'_p, Q_o appear in this order in T.

On the other hand, $Q' \notin T[Q'_1, Q_o]$ since z_o is not adjacent to t; $\{x_1, y_1, t_1\} \subset Q' \cap Q'_1$, $\{x_1, y_1\} \subset Q' \cap Q_o$ but $t_1 \notin Q_o$, so Q', $Q'_1, \dots, Q'_{p-1}$, Q^{iv}, $Q'''_1, \dots, Q'''_{r-1}$, Q'_p, Q_o appear in this order in T. Since a is not adjacent to t_i for $i = 1, \dots, p$ and a is not adjacent to x_2, then $Q_a \notin T[Q', Q_o]$. Vertex b is not adjacent to t_i for $i = 1, \dots, p$ and is also not adjacent to y_1, so $Q_b \notin T[Q', Q_o]$. As $x_1, y_1 \in (Q_a \cap Q') - (Q_o \cap Q_b)$ and $x_2, y_2 \in (Q_o \cap Q_b) - (Q_a \cap Q')$, then Q_a, Q', $Q'_1, \dots, Q'_{p-1}$, Q^{iv}, $Q'''_1, \dots, Q'''_{r-1}$, Q'_p, Q_o, Q_b appear in this order in T. Since $x_1 \in Q_a \cap Q_o$ and $x_2 \in Q' \cap Q_b$ it follows that $T[Q_a, Q_b]$ is a directed path of T.

Using the same argument of Case 2c for Type 7, we have that Q_a, Q', $\dots$, Q_o, $Q''_q, \dots$, Q''_1, Q'', $Q_{o-1}, \dots$, Q appear in this order in T, and also $Q_b \notin T[Q_a, Q]$. $\square$

We will say that there is a *special connection* between two non adjacent vertices a and b if

(1) there exists a connection of Type 1 between a and b; or
(2) there exist two induced paths in G, $P = a, y_1, \dots, y_n, b$ and $Q = a, x_1, \dots, x_m, b$, such that
  (a) $P \cap Q = \{a, b\}$
  (b) if $\{x_i, x_{i+1}, y_j, y_{j+1}\}$ is a clique for $i \in \{1, \dots, m-1\}$ and $j \in \{1, \dots, n-1\}$ then there is a connection of Type $k \in \{3, 4, 5, 6, 7, 8, 9\}$ between x_{i-1} and y_{j+2} for $i \neq 1, j \neq n-1$, or between a and y_{j+2} for $j \neq n-1$, or between x_{i-1} and b for $i \neq 1$, or between a and b, or between y_{j-1} and x_{i+2} for $j \neq 1, i \neq m-1$, or between a and x_{i+2} for $i \neq m-1$ or between y_{j-1} and b for $j \neq 1$.
  (c) if $\{x_i, x_{i+1}, y_j, y_{j+1}\}$ is not a clique then there is a special connection of Type 2 between x_{i-1} and y_{j+2} for $i \neq 1, j \neq n-1$, or between a and y_{j+2} for $j \neq n-1$, or between x_{i-1} and b for $i \neq 1$, or between a and b, or between y_{j-1} and x_{i+2} for $j \neq 1, i \neq m-1$, or between a and x_{i+2} for $i \neq m-1$, or between y_{j-1} and b for $j \neq 1$.

4. Properties of models that cannot be rooted on a bold maximal clique

If G is a DV graph that has a strong asteroidal triple a_1, a_2, a_3, then $G \setminus N[a_i]$ is a connected graph for $i = 1, 2, 3$. Hence for every T, a DV-model of G, $N[a_i]$ for $i = 1, 2, 3$ must be a leaf of T. Let C_i be the closest vertex to $N[a_i]$ such that it has degree at least three, for $i = 1, 2, 3$. If $|V(T[N[a_i], C_i])| > 2$, we will denote by $e_i = A_i B_i$ the edge in $T[N[a_i], C_i]$, with A_i the neighbor of $N[a_i]$ and $B_i \neq N[a_i]$.

If there exists an edge dominated by e_i then we choose e'_i to be maximally farthest from e_i. We denote this edge by $e'_i = A'_iB'_i$ with $B'_i \in T[A'_i, C_i]$.

We will say that a DV graph G is *minimally non rooted on a maximal clique* H if no DV-model T of G can be rooted on H but for every $x \in V(G) \setminus H$, $G \setminus x$ has a DV-model that can be rooted on H.

In what follows, if T has three leaves we will denote by C the vertex of degree exactly three in T.

Lemma 1. *Let G be a DV graph that has a strong asteroidal triple a_1, a_2, a_3 and is minimal non rooted on $N[a_3]$.*

(1) Let T be a DV-model of G. Then:
  - *(a) For all e edge in $T[N[a_i], C_i]$ for $i = 1, 2$, there are at least two vertices $x, y \in lab(e)$ such that T_x and T_y have different end towards C_i.*
  - *(b) There are not twin edges in $T[N[a_i], C_i]$ for $i = 1, 2, 3$.*
  - *(c) If $|V(T[N[a_i], C_i])| > 2$ for $i = 1, 2$ then there is a dominated edge by e_i that is not in $T[N[a_i], C_i]$.*

(2) If T is a DV-model of G that has three leaves, then T does not have twin edges, one in $T[N[a_i], C]$ for $i = 1, 2$ and the other in $T[N[a_3], C]$.

Proof. (1) (a) Suppose by contradiction that every vertex x in $lab(e)$ has the same end to C_i. Let $e = AB \in T[N[a_i], C_i]$ for $i = 1, 2$ with $B \in T[A, C_i]$ and $T' = T - E(T[N[a_i], B])$. All vertices of $lab(e)$ are twins in $G_{T'}$. Let T'' be a DV-model of $G_{T'}$. Since a_1, a_2, a_3 is a strong asteroidal triple, $N[a_3]$ and $N[a_j]$ for $j \neq i, 3$ are leaves of T'', and by minimality we have that T'' can be rooted on $N[a_3]$. For $x \in lab(e)$, $T''_x = T''[Z, W]$ and $W \in T''[Z, N[a_3]]$. Let $\overline{T} = T'' + ZA + T[A, N[a_i]]$. It is easy to check that $\overline{T}$ is a DV-model of G that can be rooted on $N[a_3]$, a contradiction.

(b) Suppose by contradiction that there are two twin edges in $T[N[a_i], C_i]$ for $i \in \{1, 2, 3\}$. Let $e = AB$ and $e' = A'B'$ be twin edges with A, B, A', B' appearing in this order in $T[N[a_i], C_i]$, and $T' = T - E(T[A, B']) + AB'$. By minimality, there is T'' a DV-model of $G_{T'}$ that can be rooted on $N[a_3]$. Let $\widetilde{e} = \widetilde{A}\ \widetilde{B'}$ be an equivalent edge of AB' in T''. Thus, it is possible to build a DV-model of G from T'' by adding $T(A, B')$ as follows: $T'' - \widetilde{A}\widetilde{B'} + \widetilde{A}T(A', B)\widetilde{B'}$. Clearly, this DV-model can be rooted on $N[a_3]$, a contradiction.

(c) Suppose by contradiction that e_i cannot dominate an edge outside of $T[N[a_i], C_i]$, i.e., $e'_i \in T[N[a_i], C_i]$. Let $T' = T - T[N[a_i], A'_i)$. By the choice of e'_i, it is clear that A'_i is always a leaf in every DV-model of $G_{T'}$. By minimality, there is T'' a DV-model of $G_{T'}$ that can be rooted on $N[a_3]$. It is easy to see that $T'' + T[A'_i, N[a_i]]$ is a DV-model of G that can be rooted on $N[a_3]$, a contradiction.

(2) Suppose by contradiction that e and e' are twin edges, one in $T[N[a_i], C]$ and the other in $T[C, N[a_3]]$ for $i = 1, 2$. Let $e = AB \in T[N[a_i], C]$ and $e' = A'B' \in T[C, N[a_3]]$ with $B \in T[A, C]$ and $B' \in T[C, A']$. Let

$T' = T - \{e, e'\} + AB' + BA'$. It is a DV-model of G. Since T cannot be rooted on $N[a_3]$, there is a vertex crossing by C in $T[N[a_1], N[a_2]]$, and then $B \neq C$. As there is a vertex crossing by C in $T[N[a_i], N[a_3]]$ then there is no vertex crossing by C in $T[N[a_j], C]$ for $j \neq i, 3$ because G is a DV graph. Hence T' can be rooted on $N[a_3]$, a contradiction. □

Theorem 2. *Let G be a DV graph that has a strong asteroidal triple a_1, a_2, a_3 and is minimal non rooted on $N[a_3]$. If T is a DV-model of G that has three leaves, two twin edges one in $T[N[a_1], C]$ and the other in $T[N[a_2], C]$, then there is a special connection of Type 1 or Type 2 between a_1 and a_2.*

Proof. Since T has twin edges it follows that $|V(T[N[a_i], C])| \geq 3$ for some $i \in \{1, 2\}$. If there exists a vertex $x \in N[a_1] \cap N[a_2]$ then there is a special connection of Type 1 between a_1 and a_2.

Suppose that there is not a vertex in this condition. By Lemma 1a, and by the position of twin edges in T it follows that $N[a_1]A_1$ cannot be a dominated edge of e_2 if it exists, and $N[a_2]A_2$ cannot be a dominated edge of e_1 if it exists.

Let $e \in T[N[a_1], C]$ and $e' \in T[C, N[a_2]]$ be twin edges such that their distance is maximum in T. As $N[a_i]A_i$ is not dominated by e_j with $\{i, j\} = \{1, 2\}$, and by the choice of e'_i, it is clear that $e'_1 \in T[e', A_2]$ and $e'_2 \in T[e, A_1]$. But by the election of e and e' to maximum distance in T, e'_1 must be e' and e'_2 must be e. Then $lab(e) = lab(e') \subset A_1 \cap A_2$.

On the other hand, by Lemma 1a there are two vertices $x, y \in lab(e)$ such that $T_x = T[Ix, Dx]$, $T_y = T[Iy, Dy]$, $Ix \neq Iy$, $Dx \neq Dy$ with $Dx = N[a_2]$ and $Iy = N[a_1]$. Cleary, $x \notin N[a_1]$ and $y \notin N[a_2]$. Also by Lemma 1a, there are vertices $y_1 \in lab(N[a_1]A_1)$ and $x_1 \in lab(N[a_2]A_2)$ such that $Dy_1 \neq Dy = A_2$ and $Ix_1 \neq Ix = A_1$. Clearly, $x_1 \notin N[a_1]$ and $y_1 \notin N[a_2]$. Observe that $y_1 \notin lab(e'_2)$ and $x_1 \notin lab(e'_1)$. Hence, there is a special connection of Type 2 between a_1 and a_2. More clearly, $P = a_1, y, x_1, a_2$ and $Q = a_1, y_1, x, a_2$ are paths in G, and $\{x, y, y_1\}$, $\{y, x, x_1\}$ are cliques of G. □

Corollary 1. *Let G be a DV graph that has a strong asteroidal triple a_1, a_2, a_3 and is minimal non rooted on $N[a_3]$. If there exists T a DV-model of G with three leaves such that e'_i is in $T[N[a_j], C]\{i, j\} = \{1, 2\}$, then there is a special connection of Type 1 or Type 2 between a_1 and a_2.*

Proof. If $e'_1 = N[a_2]A_2$ or $e'_2 = N[a_1]A_1$, by Lemma 1a there is $x \in N[a_1] \cap N[a_2]$. Hence, there is a special connection of Type 1 between a_1 and a_2. Otherwise, as $e'_1 \in T[A_2, C]$ and $e'_2 \in T[A_1, C]$ then $lab(e'_1) = lab(e'_2)$. So by Theorem 2 there is a special connection of Type 1 or Type 2 between a_1 and a_2. □

Claim 1. *Let G be a DV graph with a strong asteroidal triple a_1, a_2, a_3 and such that no DV-model of G can be rooted on $N[a_3]$. If there exists T a DV-model of G with three leaves, and $e = AB \in T[N[a_3], C]$ is dominated by $e' \in T(N[a_1], C]$ with $B \in T[C, A]$, then no edge of $T(e', C]$ can have in its label vertices with the same end towards $N[a_1]$.*

By way of contradiction, suppose that there is $e'' \in T(e', C]$ such that all vertices in $lab(e'')$ have the same end towards $N[a_1]$. Let A_1 be the end of these vertices. Let $e'' = A''B''$ with $B'' \in T[A'', C]$. As $lab(e) \subset lab(e')$ and $e'' \in T(e', C]$ then $lab(e) \subset lab(e'') \subset A_1$. Also, the vertices in $lab(e'')$ have A_1 as a leaf. Let $T' = T - \{e'', e\} + B''A_1 + AA''$. It is clear that T' is a DV-model of G. Observe that there is no vertex crossing by A_1 in $T'[N[a_1], C]$ because A_1 is a leaf of each vertex in $lab(e'')$. Also, there is no vertex crossing by C in $T'[N[a_2], B]$ since there is no vertex crossing by C in $T[N[a_2], N[a_3]]$. Clearly, T' is a DV-model of G that can be rooted on $N[a_3]$, a contradiction. This proves Claim 1.

Claim 2. *Let G be a DV graph with a strong asteroidal triple a_1, a_2, a_3 and such that no DV-model of G can be rooted on $N[a_3]$. Let T be a DV-model of G with three leaves, and let $X, Y \in V(T)$ be such that one and only one of them is in $T[N[a_i], C]$ for $i = 1, 2$ and the other is in $T[N[a_3], C]$. If $e = AB$ is an edge in $T[X, C]$ with $B \in T[A, C]$, which is dominated by $D \in T[Y, C]$, then $\forall e' \in T[C, D']$, $lab(e') \not\subseteq B$ whenever $DD' \in E(T[Y, C])$.*

By way of contradiction, suppose that there is $e' \in T[C, D']$ such that $lab(e') \subset B$. Let $e' = A''B''$ be such that $B'' \in T[C, A'']$. Clearly $T' = T - \{e, e'\} + A''B + AB''$ is a model of G.

Suppose that X is in $T[N[a_1], C]$. Observe that $e' \in T[C, D']$, and A'' may be D'. Since e is dominated by D then $lab(e) \subset B''$. Thus T' is a DV-model of G. Clearly, T' does not have a vertex crossing by C in $T'[N[a_1], N[a_2]]$ since there is no vertex crossing by C in $T[N[a_3], N[a_2]]$. Hence, T' can be rooted on $N[a_3]$, a contradiction.

The proof is the same if Y is in $T[N[a_1], C]$. This proves Claim 2.

Election 1 of vertices in label of edges:

Let T be a DV-model of G and A, B be vertices that appear in this order in T. Let $e(1)$ be the edge in $T[A, B]$ incident to A.

- Take a vertex $w_1 \in lab(e(1))$ such that T_{w_1} is the shortest towards B. Let $T_{w_1} = T[Iw_1, Dw_1]$ with $A \in T[Iw_1, Dw_1]$. If $B \notin T[A, Dw_1]$ then we repeat the following process, $i > 0$:
- Let $e(i+1)$ be the edge in $T[Dw_i, B]$ incident to Dw_i and $w_{i+1} \in lab(e(i+1))$ such that $T_{w_{i+1}}$ is the shortest towards B, if $w_{i+1} \in Iw_i$ (for $i = 1$, take A instead of Iw_i) then $w_i = w_{i+1}$, we continue until cover all $T[A, B]$.

Observe that $T_{w_i} \not\subseteq T_{w_{i+1}}$.

The preceding election of vertices is a technical tool in order to define special connections of Type 4, 5, ..., or 9.

5. Proof of the main theorem

Finally, in this section we give the result that is the goal of this article, a characterization of rooted directed path graphs whose rooted models cannot be rooted on a bold maximal clique.

Theorem 3. *Let G be a DV graph with a strong asteroidal triple a_1, a_2, a_3 and leafage three. There is a special connection between a_1 and a_2 if and only if no DV-model of G can be rooted on $N[a_3]$.*

Proof. $\Rightarrow$ By Theorem 1.

$\Leftarrow$ Suppose that G is the smallest graph such that no DV-model of G can be rooted on $N[a_3]$. Since $l(G) = 3$ and $G \setminus N[a_i]$ is a connected graph for $i = 1, 2, 3$ then $N[a_i]$ is a leaf in every model of G. Let T be a DV-model of G that reaches the leafage, and C be the vertex of degree three in T. Since T cannot be rooted on $N[a_3]$ then there is a vertex crossing by C in $T[N[a_1], N[a_2]]$. By Lemma 1b, we can assume that there are not two edges with the same label in $T[N[a_i], C]$ for all $i \in \{1, 2, 3\}$. Clearly if $T[N[a_i], C]$ has exactly two vertices for $i = 1, 2$ then there exists a vertex $x \in N[a_1] \cap N[a_2]$, so there is a special connection of Type 1 between a_1 and a_2.

Now, we can assume that $T[N[a_i], C]$, for some $i \in \{1, 2\}$, has at least three vertices.

Suppose that $T[N[a_1], C]$ has at least three vertices. Thus there exists $e_1 \in T[N[a_1], C]$, and by Lemma 1c there exists $e'_1 \notin T[N[a_1], C]$. If e'_1 is in $T[N[a_2], C]$ then by Corollary 1 there is a special connection of Type 1 or 2 between a_1 and a_2.

Suppose that it is in $T[N[a_3], C]$. As T is a DV-model of G, and there is a vertex crossing by C in $T[N[a_1], N[a_2]]$ if $CN[a_2] \notin E(T)$ then $e'_2 = N[a_1]A_1$. Therefore, by Corollary 1, there is a special connection of Type 1 or Type 2 between a_1 and a_2.

Consider $CN[a_2] \in E(T)$.

By Lemma 1a, $|lab(N[a_1]A_1)|$ and $|lab(N[a_2]C)|$ are greater than one. Let $x_1, y_1 \in lab(N[a_1]A_1)$ and $x_2, y_2 \in lab(N[a_2]C)$ be such that $|\{Q \in C(G) : x_i \in Q\}| > |\{Q \in C(G) : y_i \in Q\}| > 1$, $|\{Q \in C(G) : x_i \in Q\}|$ is maximum, and $|\{Q \in C(G) : y_i \in Q\}|$ is minimum for $i = 1, 2$. Observe that if $x_1 \in N[a_2]$ or $x_2 \in N[a_1]$ then there is a special connection of Type 1 between a_1 and a_2.

In what follows, we suppose that $x_1 \notin N[a_2]$ and $x_2 \notin N[a_1]$. Let X_i be the leaf of T_{x_i} and Y_i be the leaf of T_{y_i} different from $N[a_i]$ for $i = 1, 2$ respectively. Observe that $X_2, Y_2 \in T[N[a_1], N[a_2]]$ but X_1, Y_1 may be in $T[C, N[a_3]]$.

First of all, we know that $lab(e'_1) \subset A_1$. As $lab(e'_1) \nsubseteq N[a_1]$, since $G \setminus N[a_1]$ is a connected graph, there is a vertex $v \in lab(e'_1) \cap A_1 - N[a_1]$.

In what follows, we will analyze several cases taking into account if $\{x_1, x_2, y_1, y_2\}$ is a clique of G or not. We will study two situations depending on whether there is an edge $e \in T[N[a_1], X_2]$ such that $lab(e) \subset C$.

Case 0: $T_{x_1} \cap T_{x_2} \neq \emptyset$ but $T_{y_2} \cap T_{x_1} = \emptyset$ and $T_{y_1} \cap T_{x_2} = \emptyset$. Clearly $x_1 \notin C$. Let $P = a_1, y_1, v, y_2, a_2$ and $Q = a_1, x_1, x_2, a_2$ be paths in G. Then there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{y_1, v, x_1\}$, $\{v, x_1, x_2\}$, $\{v, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, y_2 and between y_1, a_2; see Figure 2.

- There is not an edge $e \in T[N[a_1], X_2]$ such that $lab(e) \subseteq C$.

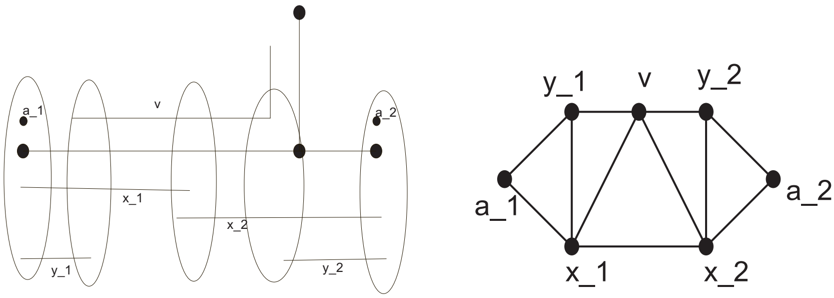


FIGURE 2. Case 0: Type 2 between a_1, y_2 and y_1, a_2.

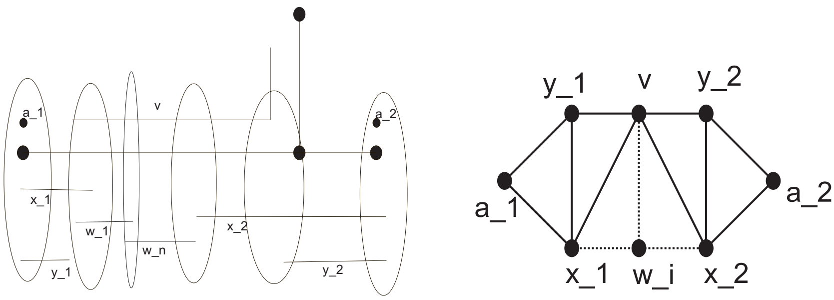


FIGURE 3. Case 1.1: Type 2 between a_1, w_2; w_{n-1}, a_2 and Type 1 between w_i, w_{i+2}.

Case 1: $T_{x_1} \cap T_{x_2} = \emptyset$. By the choice of x_1, each vertex in $lab(e_1')$ must have A_1 as a leaf. Clearly, there is a path $P = a_1, y_1, v, y_2, a_2$ in G between a_1 and a_2. We will need another path Q in G between a_1 and a_2. Observe that: a) by the election of x_1, for all $\overline{e} \in T[X_1, X_2] lab(\overline{e}) \cap N[a_1] = \emptyset$; b) by Claim 1, as $e_1' \in T[C, N[a_3]]$ is a dominated edge by $e_1 \in T(N[a_1], C]$, $\forall \overline{e} \in T[X_1, X_2]$, the vertices in its label cannot have the same end to $N[a_1]$. Since $v \in lab(\overline{e})$ and its end is A_1, it follows that there is $w \in lab(\overline{e}) - A_1$.

As was mentioned above, we need to search another path and Claim 1 will provide its vertices. We choose vertices through Election 1, we take $A = X_1$, $B = X_2$ and $w_i \notin A_1$ for $i = 1, \ldots, n$. Clearly, $Q = a_1, x_1, w_1, \ldots, w_n$, x_2, a_2 is a path in G different from P between a_1 and a_2. Observe that w_n may be in C. In this last case, as $lab(e) \nsubseteq C$ for all $e \in T[C_{a_1}, X_2]$ it follows that there exists a vertex $w' \in lab(e(n)) - C$ that has A_1 as one of its leaves. Recall that w_n was chosen in the label of $e(n)$.

Next, we will study if there is a clique in G of size four with two vertices of P and two vertices of Q.

Case 1.1: There is not a clique in G of size four with two vertices of P and two vertices of Q. Then y_1 and w_1 are not adjacent vertices; also

y_2 and w_n are not adjacent vertices. Therefore, there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{y_1, v, x_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n-1}$, $\{w_n, v, x_2\}$, $\{v, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2 and w_{n-1}, a_2, and of Type 1 between w_i, w_{i+2} with $i \in \{1, \dots, n-2\}$; see Figure 3.

Case 1.2: There is only one clique in G of size four with two vertices of P and two vertices of Q. First, suppose that $\{x_1, y_1, w_1, v\}$ is the clique. Since there is one and only one clique of size four, y_2 and w_n are not adjacent vertices. On the other hand, y_1 and w_1 are adjacent vertices and $w_1 \notin A_1$, then A_1 must be separated by a vertex s_1 in direction to $N[a_1]$. By the choice of y_1, $s_1 \notin N[a_1]$ so it is a simplicial vertex of G. Let s_2 be a separator vertex of X_1 to C such that $|\{Q \in C(G) : s_2 \in Q\}|$ is minimum. By the election of w_1, it is clear that $s_2 \notin Dw_1$, then it is not adjacent vertex to w_2. Therefore there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n-1}$, $\{w_n, v, x_2\}$, $\{v, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G, and $\{x_1, y_1, w_1, v, s_1, s_2\}$ induces an antenna.

Observe that there is a special connection of Type 3 between a_1 and w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \dots, n-2\}$, and of Type 2 between w_{n-1}, a_2; see Figure 4.

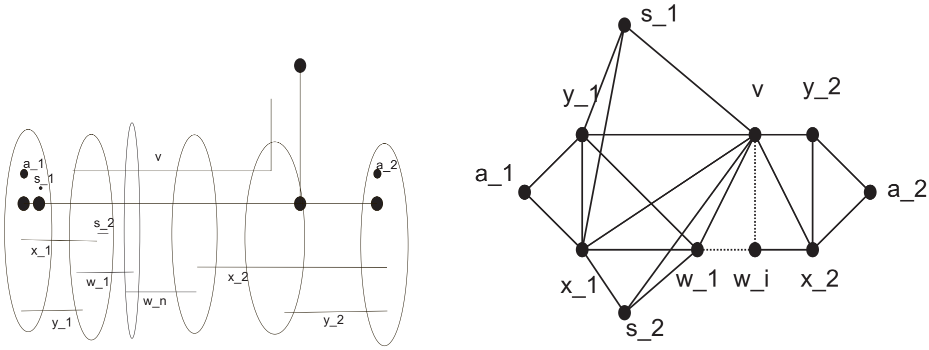


FIGURE 4. Case 1.2: Type 3 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 2 between w_{n-1}, a_2.

Case 1.3: There is one only one clique in G of size four with two vertices of P and two vertices of Q. Now, suppose that $\{x_2, y_2, w_n, v\}$ is the clique. As there is only one clique of size four, y_1 and w_1 are not adjacent vertices. We will analyze two situations depending on whether w_n is in C.

(1) $w_n \notin C$. Let s_3 be a separator vertex of C to $N[a_3]$. Clearly s_3 is not adjacent to w_n. Let s_4 be a separator vertex of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : s_4 \in Q\}|$ is minimum. By the election of w_n, s_4 is not

adjacent vertex to w_{n-1}. Then there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{y_1, v, x_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\ldots,n-1}$ and $\{x_2, y_2, a_2\}$ are cliques of G, and $\{x_2, y_2, w_n, v, s_3, s_4\}$ induces an antenna.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 3 between w_{n-1}, a_2; see Figure 5.

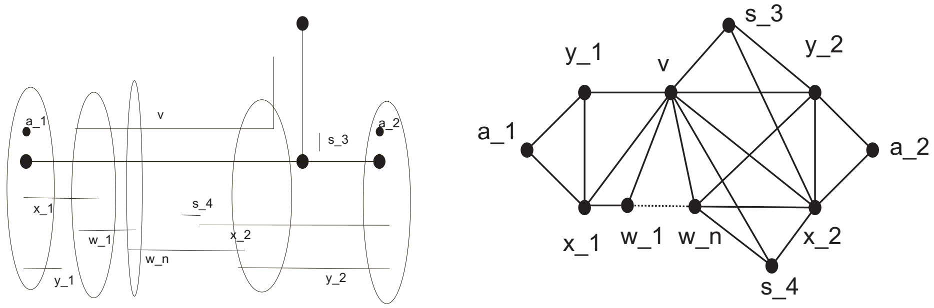


FIGURE 5. Case 1.3.1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 3 between w_{n-1}, a_2.

(2) $w_n \in C$. Hence there is a vertex in $lab(e(n)) - C$ that has A_1 as one of its leaves. Let w' be the vertex such that $|\{Q \in C(G) : w' \in Q\}|$ is maximum. Let W' be the other leaf of w'.

If $w' \in X_2$ then we take $P = a_1, y_1, w', x_2, a_2$ and $Q = a_1, x_1, w_1, \ldots, w_n, y_2, a_2$ paths in G.

In case that $y_2 \notin W'$ then there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 2 between w_{n-1}, a_2; see Figure 6.

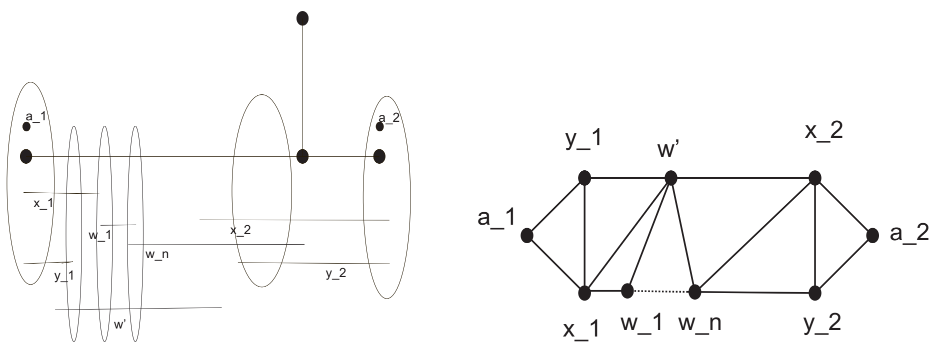


FIGURE 6. Case 1.3.2: Type 2 between a_1, w_2 and w_{n-1}, a_2; Type 1 between w_i, w_{i+2}.

In case that $y_2 \in W'$, by the same argument used in 1 taking w_n, w' instead of w_n, v, there is a special connection of Type 3 between w_{n-1}, a_2; see Figure 7.

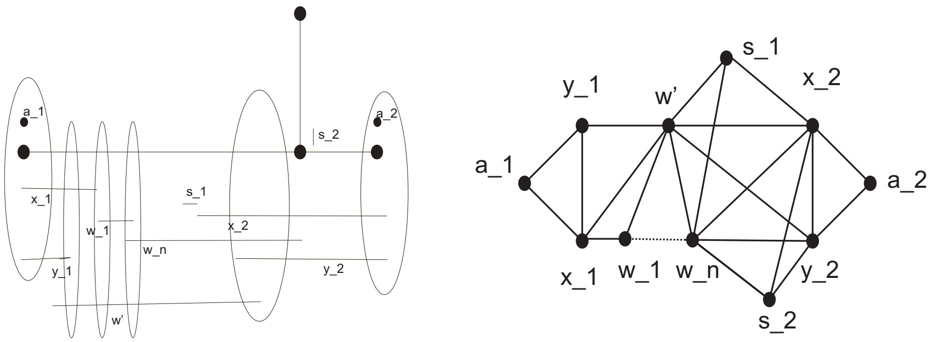


FIGURE 7. Case 1.3.2: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2} and Type 3 between w_{n-1}, a_2.

Now, suppose that $w' \notin X_2$. Then we choose vertices in label of edges of $T[W', X_2]$ that are not in C through Election 1 with $A = W'$ and $B = X_2$. Let t_i be these vertices for $i = 1, \ldots, m$ such that t_1 is the first vertex chosen. Observe that by the choice of w', $t_1 \notin A_1$ and by the election of w_n, $t_1 \notin lab(e(n))$, then t_1 is not adjacent to w_{n-1}. Let $P = a_1, y_1, w', t_1, \ldots, t_m, x_2, a_2$ and $Q = a_1, x_1, w_1, \ldots, w_n, y_2, a_2$ be paths in G. Note that $\{w_n, t_m, y_2, x_2\}$ and $\{y_1, x_1, w', w_1\}$ may be cliques. Clearly, $\{y_1, x_1, w', w_1\}$ is not a clique because $\{y_1, x_1, v, w_1\}$ is not a clique. In case that $\{w_n, t_m, x_2, y_2\}$ is not a clique then there is a special connection between a_1 and a_2. More clearly $\{a_1, y_1, x_1\}$, $\{w', x_1, w_1\}$, $\{w', w_i, w_{i+1}\}_{i=1,\ldots,n-1}$, $\{w', t_1, w_n\}$, $\{t_i, t_{i+1}, w_n\}_{i=1,\ldots,m}$, $\{t_m, x_2, w_n\}$, $\{x_2, w_n, y_2\}$, $\{x_2, y_2, a_2\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w', t_2 and between t_i, t_{i+2} with $i \in \{1, \ldots, m-2\}$, and of Type 2 between t_{m-1}, a_2; see Figure 8.

In case that $\{w_n, t_m, y_2, x_2\}$ is a clique, let s_3 be a separator vertex of C to $N[a_3]$. Clearly, s_3 is not adjacent to t_m since $t_m \notin C$. Let s_4 be a separator vertex of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : s_4 \in Q\}|$ is minimum. By the election of t_m, s_4 is not adjacent vertex to t_{m-1}. Hence, there is a special connection between a_1 and a_2. More clearly, $\{a_1, y_1, x_1\}$, $\{w', x_1, w_1\}$, $\{w', w_i, w_{i+1}\}_{i=1,\ldots,n-1}$, $\{w', t_1, w_n\}$, $\{t_i, t_{i+1}, w_n\}_{i=1,\ldots,m}$, $\{x_2, y_2, a_2\}$ are cliques of G, and $\{t_m, x_2, y_2, w_n, s_3, s_4\}$ induces an antenna.

Observe that there is a special connection of Type 3 between t_{m-1} and a_2; see Figure 9.

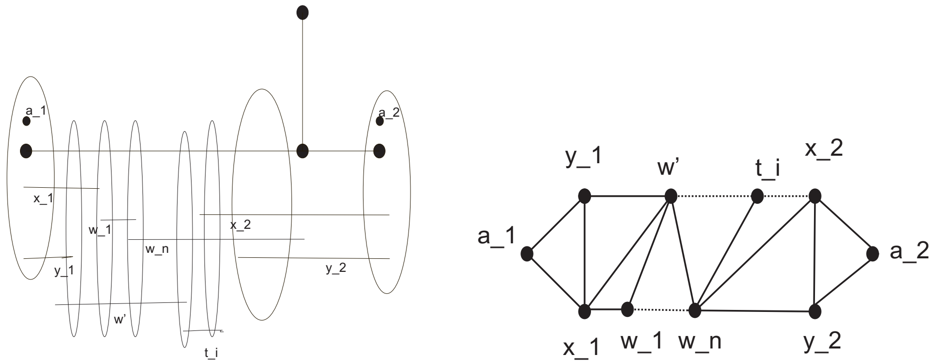


FIGURE 8. Case 1.3.2: Type 2 between a_1, w_2 and t_{m-1}, a_2; Type 1 between w', t_2 and t_i, t_{i+2}.

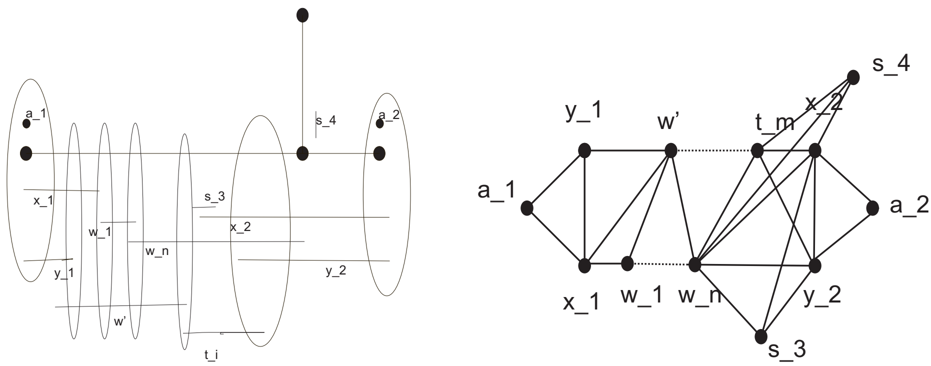


FIGURE 9. Case 1.3.2: Type 2 between a_1, w_2; Type 1 between w', t_2 and t_i, t_{i+2}; Type 3 between t_{m-1}, a_2.

Case 1.4: There are two cliques in G of size four with two vertices of P and two vertices of Q, and they are $\{x_1, y_1, w_1, v\}$ and $\{x_2, y_2, w_n, v\}$. In this situation, we obtain a combination of the previous cases.

Case 2: $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$ but $T_{y_2} \cap T_{x_1} = \emptyset$, or $T_{x_1} \cap T_{x_2} \cap T_{y_2} \neq \emptyset$ but $T_{y_1} \cap T_{x_2} = \emptyset$. In both situations it is clear that there are two paths $P = a_1, y_1, v, y_2, a_2$ and $Q = a_1, x_1, x_2, a_2$ in G between a_1 and a_2. Next, we will study whether there is a clique in G of size four with two vertices of P and two vertices of Q.

Case 2.1: $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$ but $T_{y_2} \cap T_{x_1} = \emptyset$. Clearly $\{x_1, y_1, v, x_2\}$ is a clique of G. By the election of x_1, every vertex of $lab(e'_1)$ has A_1 as a leaf. It is necessary to study two situations depending on whether T_{x_2} and T_v have the same end to $N[a_1]$—or, more clearly, on whether A_1 is a leaf of both of them.

First, we suppose that T_{x_2} and T_v have the same leaf in direction to $N[a_1]$, i.e., both of them have A_1 as a leaf. By Claim 1, for each $\overline{e} \in T[X_1, Y_2]$ the vertices in its label cannot have the same end to $N[a_1]$. As T_v and T_{x_2} have the same leaf A_1 to $N[a_1]$, we choose vertices $w_i \in lab(\overline{e}) - A_1$ through Election 1 with $i = 1, \ldots, n$. Let $P' = a_1, y_1, x_2, a_2$ and $Q' = a_1, x_1, w_1, \ldots, w_n, y_2, a_2$ be paths in G between a_1 and a_2. Observe that $\{x_1, x_2, y_1, w_1\}$ may be a clique. In case that $w_1 \notin Y_1$, there is not a clique of size four with two vertices of each path. Hence, there is a special connection between a_1, a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{x_1, x_2, y_1\}$, $\{x_1, x_2, w_1\}$, $\{w_i, w_{i+1}, x_2\}_{i=1,\ldots,n-1}$, $\{w_n, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G.

Note that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and between w_n, a_2; see Figure 10.

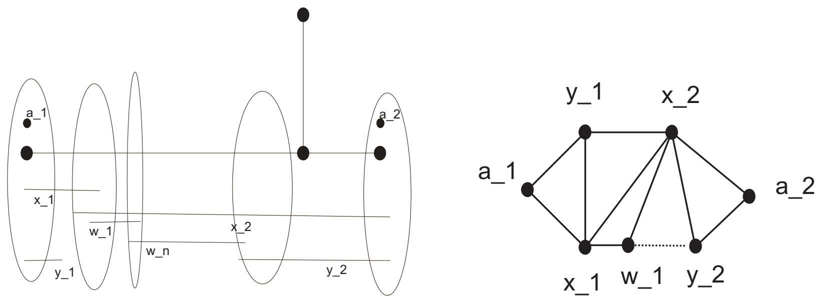


FIGURE 10. Case 2.1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2} and w_n, a_2.

In case that $w_1 \in Y_1$, as $w_1 \notin A_1$ then $Y_1 \neq A_1$. Let s_1 be a separator vertex of $X_2 = A_1$ to $N[a_1]$ such that $|\{Q \in C(G) : s_1 \in Q\}|$ is minimum. By the choice of y_1, s_1 is a simplicial vertex of G. As $w_1 \in Y_1$ then $X_1 \neq Dw_1$. Let s_2 be a separator vertex of X_1 to C_2 such that $|\{Q \in C(G) : s_2 \in Q\}|$ is minimum. By the election of w_1, s_2 is not adjacent to w_2. Hence there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{w_i, w_{i+1}, x_2\}_{i=1,\ldots,n}$, $\{w_n, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G, and $\{x_1, y_1, w_1, x_2, s_1, s_2\}$ induces an antenna.

Note that there is a special connection of Type 3 between a_1, w_2; of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and between w_n, a_2; see Figure 11.

Finally, we can assume that T_{x_2} and T_v do not have the same leaf in direction to $N[a_1]$. By the election of x_1, it is clear that $v \notin N[a_1]$. As $x_2 \notin N[a_1]$ and $X_2 \neq A_1$, we have that $x_2 \notin A_1$. Observe that $Y_1 \neq A_1$ since $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$. Let s_1 and s_2 be vertices such that s_1 is a separator vertex of A_1 to $N[a_1]$, s_2 is a separator vertex of X_1 to C and $|\{Q \in C(G) : s_2 \in Q\}|$ is minimum. By the choice of y_1, $s_1 \notin N[a_1]$ so it is a simplicial vertex of G. If s_2 is adjacent to y_2, let $P' = a_1, y_1, x_2, a_2$

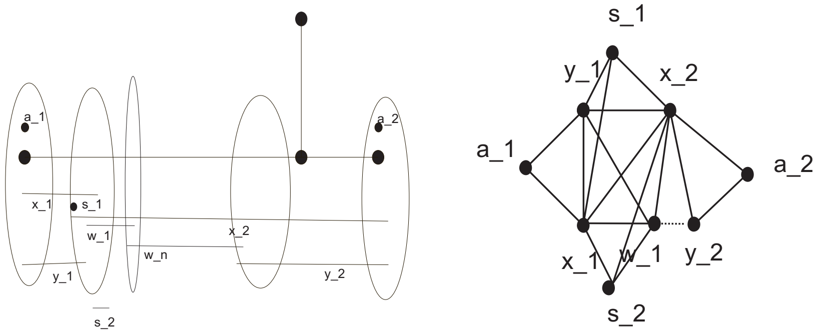


FIGURE 11. Case 2.1: Type 3 between a_1, w_2; Type 1 between w_i, w_{i+2} and w_n, a_2.

and $Q' = a_1, x_1, s_2, y_2, a_2$ be paths in G between a_1 and a_2. Clearly, there is not a clique of size four with two vertices of each path. Then there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{x_1, x_2, s_2\}$, $\{x_1, x_2, y_1\}$, $\{s_2, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G.

Note that there is a special connection of Type 2 between a_1, y_2, and of Type 1 between s_2, a_2; see Figure 12.

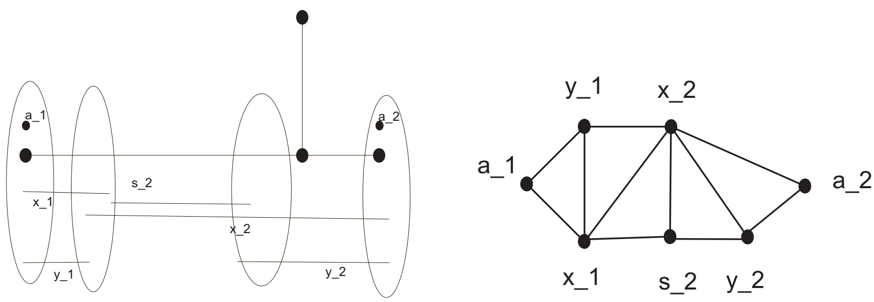


FIGURE 12. Case 2.1: Type 3 between a_1, y_2; Type 1 between v, a_2.

If s_2 is not adjacent to y_2, let $P = a_1, y_1, v, y_2, a_2$ and $Q = a_1, x_1, x_2, a_2$ be paths in G between a_1 and a_2. Clearly there is a special connection between a_1 and a_2. More clearly, $\{y_1, x_1, v, x_2, s_1, s_2\}$ induces an antenna and $\{a_1, x_1, y_1\}$, $\{v, x_2, y_2\}$, $\{x_2, y_2, a_2\}$ are cliques of G.

Note that there is a special connection of Type 3 between a_1, y_2, and of Type 1 between v, a_2; see Figure 13.

Case 2.2: $T_{x_1} \cap T_{x_2} \cap T_{y_2} \neq \emptyset$ but $T_{y_1} \cap T_{x_2} = \emptyset$. Clearly, there are two paths in G between a_1 and a_2: $P = a_1, x_1, x_2, a_2$ and $Q = a_1, y_1, v, y_2, a_2$. On the other hand, $\{x_1, x_2, y_2, v\}$ is a clique of G. We will analyze if x_1 is or is not in C.

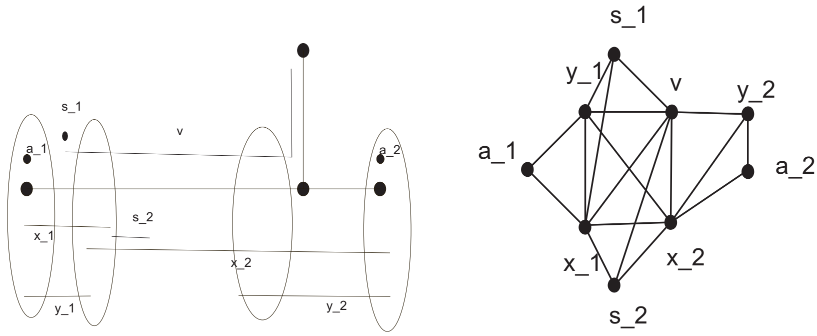


FIGURE 13. Case 2.1: Type 2 between a_1, y_2; Type 1 between s_2, a_2.

First, $x_1 \notin C$. Clearly $Y_2 \neq C$. By the election of x_1, every vertex in $lab(e_1')$ has A_1 as a leaf. Let s_2 be a separator vertex of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : s_2 \in Q\}|$ is minimum.

If $s_2 \in Y_1$ then we can change the paths in order to make Type 2 appear. Let $P' = a_1, y_1, s_2, x_2, a_2$ and $Q' = a_1, x_1, y_2, a_2$ be paths in G. More clearly, $\{a_1, x_1, y_1\}$, $\{s_2, x_1, y_1\}$, $\{s_2, x_1, x_2\}$, $\{x_1, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G.

Note that there is a special connection of Type 1 between a_1, s_2, and of Type 2 between y_1, a_2; see Figure 14.

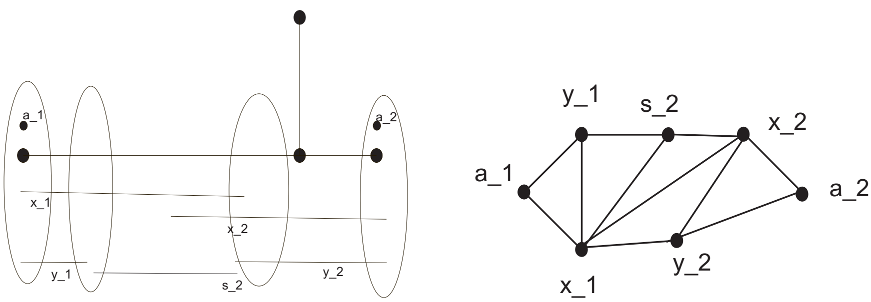


FIGURE 14. Case 2.2: Type 1 between a_1, s_2; Type 2 between y_1, a_2.

If $s_2 \notin Y_1$, let s_1 be a separator vertex of C to $N[a_3]$; then there is a special connection between a_1, a_2. More clearly, $\{y_2, x_2, v, x_1, s_1, s_2\}$ induces an antenna and $\{a_1, x_1, y_1\}$, $\{v, x_1, y_1\}$, $\{x_2, y_2, a_2\}$ are cliques of G.

Observe that there is a special connection of Type 1 between a_1, v, and of Type 3 between y_1, a_2; see Figure 15.

Finally, $x_1 \in C$. We know that $\forall e \in T[Y_1, X_2]$, $lab(e) \nsubseteq C$. We choose $w_i \in lab(e) - C$ through Election 1 with $A = Y_1$ and $B = X_2$. Let $P' = a_1, x_1, y_2, a_2$ and $Q' = y_1, w_1, \ldots, w_n, x_2, a_2$ be paths in G between a_1 and a_2.

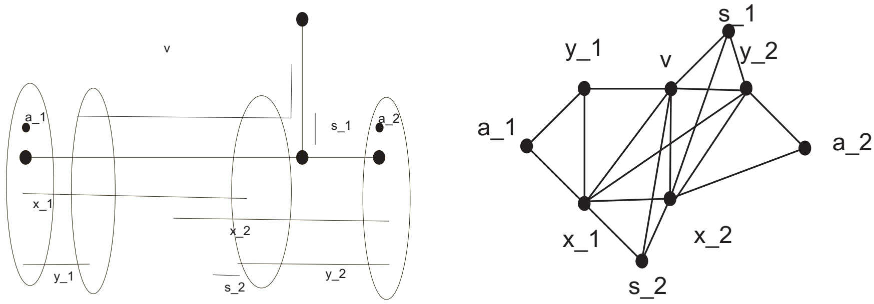


FIGURE 15. Case 2.2: Type 1 between a_1, v; Type 3 between y_1, a_2.

If w_n is not adjacent to y_2 then there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{x_1, y_1, w_1\}$, $\{w_i, w_{i+1}, x_1\}_{i=1,\ldots,n-1}$, $\{w_n, x_1, x_2\}$, $\{x_1, x_2, y_2\}$ and $\{x_2, y_2, a_2\}$ are cliques of G.

Observe that there is a special connection of Type 1 between a_1, w_1, between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 2 between w_{n-1}, a_2; see Figure 16.

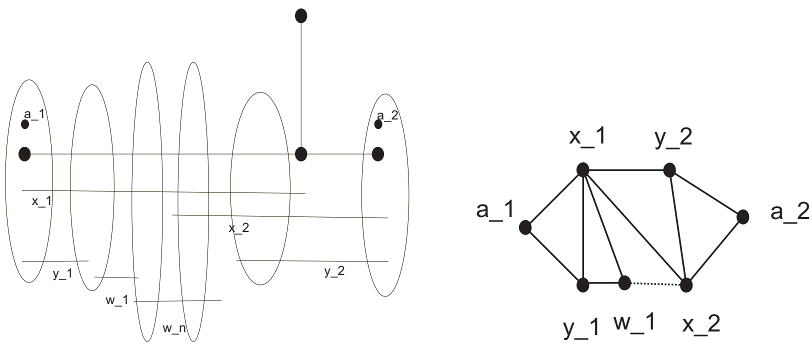


FIGURE 16. Case 2.2: Type 1 between a_1, w_1; between w_i, w_{i+2}; Type 2 between w_{n-1}, a_2.

If w_n is adjacent to y_2 then $Y_2 \neq C$ by the election of $w_n \notin C$. Clearly, there is a clique of size four with two of each path, it is $\{x_1, x_2, y_2, w_n\}$. Let s_1 and s_2 be vertices such that s_1 is a separator of C to $N[a_3]$, s_2 is a separator of X_2 to $N[a_1]$ and $|\{Q \in C(G) : s_i \in Q\}|$ is minimum for $i = 1, 2$. By the election of w_i, s_2 is not adjacent to w_{n-1}. Hence there is a special connection between a_1 and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{x_1, y_1, w_1\}$, $\{w_i, w_{i+1}, x_1\}_{i=1,\ldots,n-1}$, $\{x_2, y_2, a_2\}$ are cliques of G, and $\{w_n, x_1, x_2, y_2, s_1, s_2\}$ induces an antenna.

Observe that there is a special connection of Type 1 between a_1, w_1, between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 3 between w_{n-1}, a_2; see Figure 17.

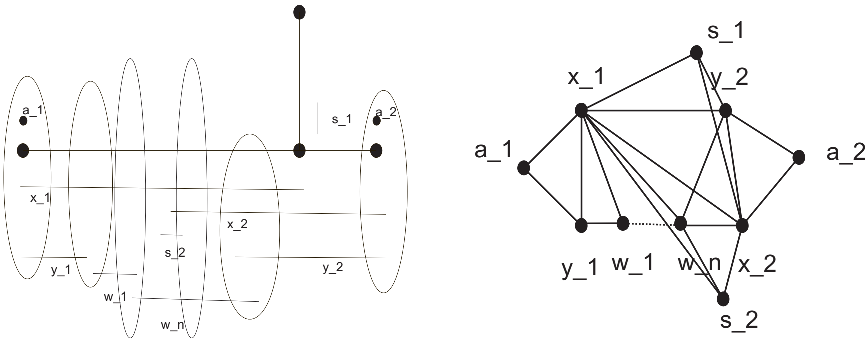


FIGURE 17. Case 2.2: Type 1 between a_1, w_1 and w_i, w_{i+2}; Type 3 between w_{n-1}, a_2.

Case 3: $T_{x_1} \cap T_{x_2} \cap T_{y_1} \cap T_{y_2} \neq \emptyset$. Clearly, there are two paths in G between a_1 and a_2. Let $P = a_1, y_1, y_2, a_2$ and $Q = a_1, x_1, x_2, a_2$ be these paths. Also $\{x_1, y_1, x_2, y_2\}$ is a clique of G which has two vertices of P and two vertices of Q. We will study two situations depending on whether x_1 is in C.

First, $x_1 \notin C$. Since $X_2 \neq N[a_1]$ then $Y_1 \neq A_1$. Let s_1 be a separator vertex of X_2 to $N[a_1]$, s_2 be a separator vertex of X_1 to $N[a_2]$ such that $|\{Q \in C(G) : s_i \in Q\}|$ is minimum for $i = 1, 2$. By the election of y_i for $i = 1, 2$, $s_2 \notin N[a_2]$ and $s_1 \notin N[a_1]$. Hence there is a special connection of Type 3 between a_1 and a_2; see Figure 18.

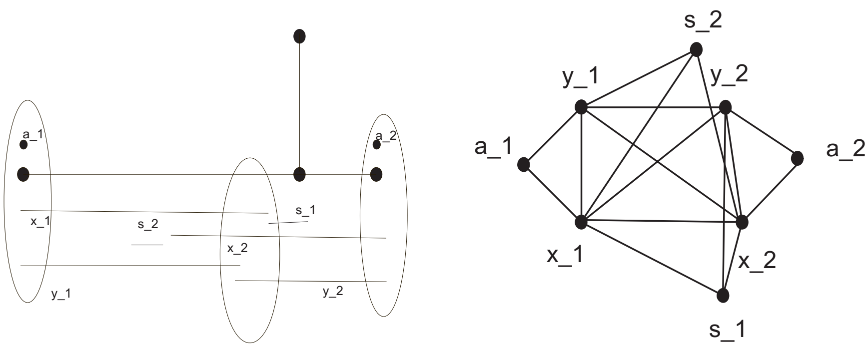
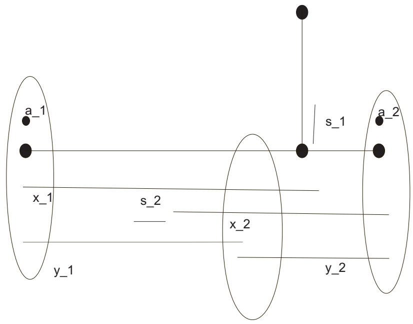


FIGURE 18. Case 3: Type 3 between a_1, a_2.

We now suppose that $x_1 \in C$. Since there is not an edge whose label is contained in C, then $y_1 \notin C$. Hence $Y_2 \neq C$. Observe that $X_2 \neq Y_1$ but Y_1 may be Y_2. Let s_2 be a separator vertex of X_2 to $N[a_1]$, s_1 be a separator vertex of C to $N[a_2]$ such that $|\{Q \in C(G) : s_i \in Q\}|$ is minimum for $i = 1, 2$. As seen on Case 1.2, $s_2 \notin N[a_1]$ and $s_1 \notin N[a_2]$. Hence there is a special connection of Type 3 between a_1 and a_2; see Figure 19.



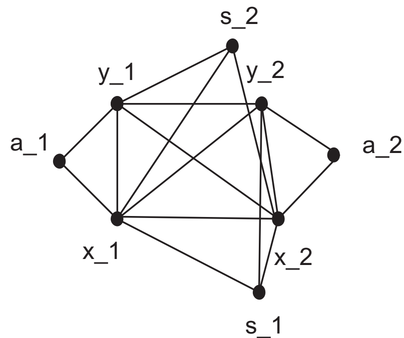


FIGURE 19. Case 3: Type 3 between a_1, a_2.

In both cases, $\{a_1, x_1, y_1\}$, $\{a_2, x_2, y_2\}$ are cliques of G, and $\{s_1, s_2, x_1, x_2, y_1, y_2\}$ induces an antenna.

- There is an edge $e \in T[N[a_1], X_2]$ such that $lab(e) \subseteq C$.

Case 1: $T_{x_1} \cap T_{x_2} = \emptyset$. By the election of x_1, each vertex in $lab(e_1')$ must have A_1 as a leaf. Clearly, there is a path $P = a_1, y_1, v, y_2, a_2$ in G between a_1 and a_2. We will need another path Q in G between a_1 and a_2. Observe that: a) by the election of x_1, for all $\overline{e} \in T[X_1, X_2]$, $lab(\overline{e}) \cap N[a_1] = \emptyset$; b) by Claim 1 as $e_1' \in T[C, N[a_3]]$ is a dominated edge by $e_1 \in T(N[a_1], C]$, $\forall \overline{e} \in T[X_1, X_2]$, the vertices in its label cannot have the same ends to $N[a_1]$. Hence as $v \in lab(\overline{e})$ for all $\overline{e} \in T[X_1, X_2]$, there is $w \in lab(\overline{e}) - A_1$.

As was mentioned above, we need to search vertices for another path. We choose vertices through Election 1 taking $A = X_1$, $B = X_2$, and $w_i \notin A_1$ for $i = 1, \ldots, n$. Clearly, $Q = a_1, x_1, w_1, \ldots, w_n, x_2, a_2$ is a path in G different from P between a_1 and a_2. As there is an edge in $T[X_1, X_2]$ whose label is contained in C then w_n is in C. Clearly $\{x_2, y_2, w_n, v\}$ is a clique of G, and $\{x_1, y_1, v, w_1\}$ may be a clique. Let $W'Dw_n \in E(T)$ be such that $Dw_n \in T[C, W']$. Let u be a separator vertex of W' to $N[a_3]$, i.e., $u \in W' - Dw_n$. Let $e(n) = XY$ be the edge which was chosen w_n with $Y \in T[X, C]$. Observe that Y may be X_2. On the other hand, $e(n)$ is dominated by Dw_n because of the choice of w_n, which is the shortest vertex to C, and $lab(e(n)) \subset C$. We will analyze if there is another edge $\widetilde{e} \in T[Y, X_2]$ such that $lab(\widetilde{e}) \subset C$.

$\square$ Suppose that another edge does not exist. By Claim 2, since $e(n)$ is an edge dominated by $Dw_n \in T[C, N[a_3]]$, then for every edge in $T[W', C]$ its label is not contained in Y. Hence we choose vertices in label of edges $\overline{e} \in T[W', C]$ that are not in Y, through Election 1 taking $A = W'$ and $B = C$. Let $z_i \notin Y$ be such that $T_{z_i} = T[Iz_i, Dz_i]$ with $Iz_i \in T(Y, Dz_i]$ for $i = 1, \ldots, o$, and z_o is the last vertex chosen. By the choice of $z_i \notin Y$ for $i = 1, \ldots, o$, it follows that $e_1' \notin T[W', C]$.

Now, we will analyze two situations depending on the position of Iz_o in $T(Y, C]$.

First, we consider $Iz_o \in T(X_2, C]$. Let t be a separator vertex of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : t \in Q\}|$ is minimum. Observe that $t \notin X$. Also t is not adjacent to y_2 or z_o. Then, there is a special connection between a_1 and a_2. More clearly, in case that $\{x_1, y_1, v, w_1\}$ is not a clique it follows that $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\ldots,n-1}$, $\{x_2, y_2, a_2\}$, $\{v, w_n, x_2, y_2, z_o\}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\ldots,o-1}$, $\{u, v, z_1\}$ and $\{v, w_n, x_2, t\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 4 between w_{n-1} and a_2; see Figure 20.

In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, by the same argument used in Case 1.2, there are two vertices s_1, s_2 in G such that $\{x_1, y_1, v, w_1, s_1, s_2\}$ induces an antenna.

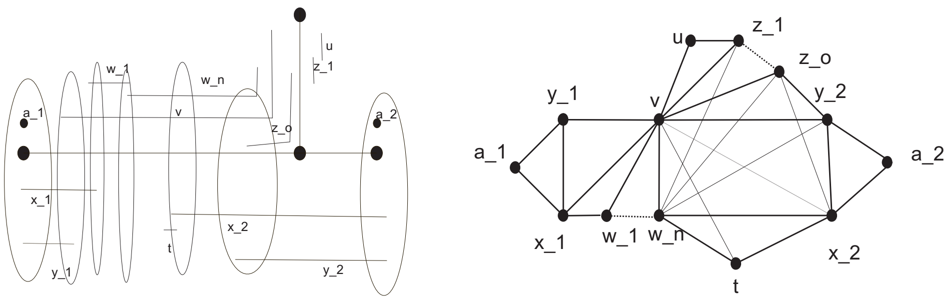


FIGURE 20. Case 1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 4 between w_{n-1} and a_2.

Finally, we consider $Iz_o \in T(Y, X_2]$. Let $e(o) = A_oB_o$ be the edge which z_o was chosen, with $B_o \in T[C, A_o]$. Let Z' be the vertex of T such that $Iz_oZ' \in E(T)$ and $Iz_o \in T[Z', X_2]$. Observe that $Z' \neq X$ since $Iz_o \neq Y$. Also, by the election of z_o, $e(o)$ is a dominated edge by I_{z_o}, then by Claim 2 for every e' edge in $T[Z', X_2]$ its label is not contained in B_o. Hence we choose vertices through Election 1 in label of edges of $T[Z', X_2]$ such that they are not in B_o. Let $t_i \notin B_o$ be the vertices chosen with $i \in \{1, \ldots, p\}$, and t_p be the last vertex chosen. It is clear that t_p may be in C. But there is not an edge different from $e(n)$ such that it is contained in C, then $t_p \notin C$. Clearly, t_p may or may not be adjacent to y_2.

$\Diamond$ t_p is adjacent to y_2. As $t_p \notin C$ there is a separator vertex of C to $N[a_3]$. Let s_1 be the separator of C to $N[a_3]$ minimizing $|\{Q \in C(G) : s_1 \in Q\}|$. As z_o was chosen instead of s_1 then z_{o-1} is not adjacent to s_1. Let s be a separator of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : s \in Q\}|$ is minimum. By the election of t_p, it is clear that s is not adjacent to t_{p-1}. Let t be a separator of It_1 to $N[a_1]$ such that $|\{Q \in C(G) : t \in Q\}|$ is minimum. Observe that $t \notin X$. There is a special connection between a_1 and a_2. More clearly, in case that $\{x_1, y_1, w_1, v\}$ is not a clique then $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\ldots,n-1}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\ldots,o-1}$, $\{x_2, y_2, a_2\}$, $\{u, v, z_1\}$, $\{v, w_n, z_o, x_2, t_p, s\}$, $\{t_p, x_2, y_2, z_o, w_n, v\}$, $\{t, w_n, v, t_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\ldots,p-1}$ and $\{v, w_n, x_2, y_2, z_o, s_1\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 5 between w_{n-1} and a_2; see Figure 21.

In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, as seen on Case 1.2 there are two vertices s_1', s_2 in G such that $\{x_1, y_1, v, w_1, s_1', s_2\}$ induces an antenna.

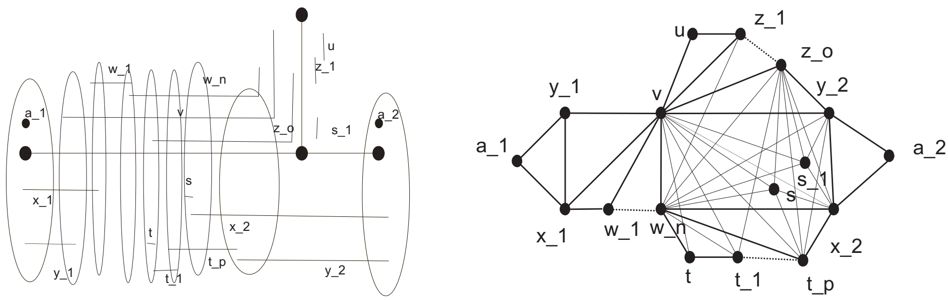


FIGURE 21. Case 1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; of Type 5 between w_{n-1}, a_2.

$\Diamond$ t_p is not adjacent to y_2. As before, consider only the separator of It_1 to $N[a_1]$. Then there is a special connection between a_1 and a_2. More clearly, in case that $\{x_1, y_1, w_1, v\}$ is not a clique then $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\ldots,n}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\ldots,o-1}$, $\{x_2, y_2, a_2\}$, $\{u, v, z_1\}$, $\{t, v, w_n, t_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\ldots,p-1}$, $\{t_p, x_2, v, w_n, z_o\}$ and $\{v, w_n, x_2, y_2, z_o\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 6 between w_{n-1} and a_2; see Figure 22.

In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, as seen on Case 1.2 there are two vertices s_1, s_2 in G such that $\{x_1, y_1, v, w_1, s_1, s_2\}$ induces an antenna.

$\Box$ Suppose that there is an edge in $T[B, X_2]$ which is contained in C. Let $\widetilde{e}$ be the nearest C. Observe that $lab(e(n)) \subset lab(\widetilde{e})$, but by our assumption $lab(\widetilde{e}) \nsubseteq lab(e(n))$; let $m \in lab(\widetilde{e}) - lab(e(n))$ be such that T_m is the shortest to $N[a_3]$ with Dm its leaf in $T[C, N[a_3]]$. Clearly, $m \neq w_n, v$. Observe that for all edges $\widetilde{\widetilde{e}}$ in $T(\widetilde{e}, X_2]$, $lab(\widetilde{\widetilde{e}}) \nsubseteq C$. Therefore there are vertices in the label of edges of $T(\widetilde{e}, X_2]$ that are not in C.

In case that $Dm \in T[Dw_n, N[a_3]]$, $\widetilde{e}$ is dominated by Dw_n; then by Claim 2 there are vertices in the label of edges of $T[W', C]$ that are not in $\widetilde{B}$ with $\widetilde{e} = \widetilde{A}\widetilde{B}$ and $\widetilde{B} \in T[\widetilde{A}, C]$. Let z_i be these vertices for $i = 1, \ldots, o$

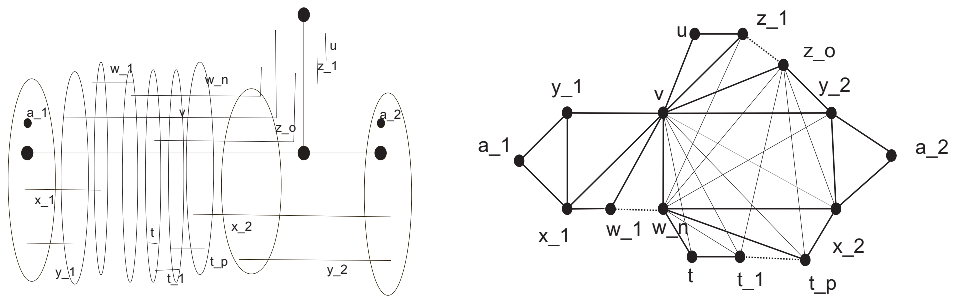


FIGURE 22. Case 1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 6 between w_{n-1}, a_2.

chosen through Election 1. Therefore $I_{z_o} \notin \widetilde{B}$. Also, by the election of $\widetilde{e}$, the vertices t_i chosen as before are not in C, in particular the vertex $t_p \notin C$. Hence we get situations described previously, i.e., Type 4 or Type 5 or Type 6. More clearly, in case that $\{x_1, y_1, w_1, v\}$ is not a clique then if $Iz_o \in T(X_2, C]$, $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n-1}$, $\{x_2, y_2, a_2\}$, $\{v, w_n, x_2, y_2, z_o\}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\dots,o-1}$, $\{u, v, z_1\}$, and $\{v, w_n, x_2, t\}$ are cliques of G. If $Iz_o \in T(\widetilde{B}, X_2]$ then in case that t_p is adjacent to y_2, $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{x_2, y_2, a_2\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n-1}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\dots,o-1}$, $\{u, v, z_1\}$, $\{v, w_n, z_o, x_2, t_p, s\}$, $\{t_p, x_2, y_2, z_o, w_n, v\}$, $\{t, w_n, v, t_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\dots,p-1}$ and $\{v, w_n, x_2, y_2, z_o, s_1\}$ are cliques of G. In case that t_p is not adjacent to y_2 then $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{x_2, y_2, a_2\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\dots,o-1}$, $\{u, v, z_1\}$, $\{t, v, w_n, t_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\dots,p-1}$, $\{t_p, x_2, v, w_n, z_o\}$ and $\{v, w_n, x_2, y_2, z_o\}$ are cliques of G. In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, as seen on Case 1.2 there are two vertices s'_1, s_2 in G such that $\{x_1, y_1, v, w_1, s'_1, s_2\}$ induces an antenna.

In case that $Dm \in T[C, Dw_n)$, $e(n)$ is dominated by Dw_n; then by Claim 2 there are vertices chosen through Election 1 that are not in Y. As before, let z_i be these vertices for $i = 1, \dots, o$. If $z_o \notin \widetilde{B}$ then we get situations described previously. If $z_o \in \widetilde{B}$, as Dm dominates $\widetilde{e}$, no edge of $T[M', C]$ ($DmM' \in E(T)$ with $M' \in T[M, Dw_n]$) is dominated by $\widetilde{B}$, then $e(o) \notin T[C, M']$. It is clear that $Iz_o \notin T[X_2, C]$. Also $t_p = m$. In this case $t_p \in C$. By Claim 2 as $\widetilde{e}$ is a dominated edge by $Dt_p = Dm$, in the label of edges of $T[C, M']$ there are vertices that are not in $\widetilde{B}$. Let z'_i be vertices chosen in the label of edges in $T[C, M']$ that are not in $\widetilde{B}$ through Election 1 taking $A = M'$, $B = C$ for $i = 1, \dots, q$, and with z'_q the last vertex chosen. Let u' be adjacent to z'_1 but not adjacent to z'_2. If $z'_q \in T(X_2, C]$ then let s be a separator of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : s \in Q\}|$ is minimum. We obtain a special connection between a_1 and a_2. More clearly, in case that $\{x_1, y_1, v, w_1\}$ is not a clique then $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$,

$\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n-1}$, $\{x_2, y_2, a_2\}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\dots,o-1}$, $\{u, v, z_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\dots,p-1}$, $\{t, v, w_n, t_1\}$, $\{u', z'_1, z_o, w_n, v\}$, $\{s, t_p, w_n, v, z_o, x_2\}$, $\{z'_q, x_2, y_2, v, w_n, t_p, z_o\}$ and $\{z'_i, z'_{i+1}, v, w_n, t_p, z_o\}_{i=1,\dots,q-1}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \dots, n-2\}$, and of Type 7 between w_{n-1} and a_2; see Figure 23.

In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, as seen on Case 1.2 there are two vertices s'_1, s_2 in G such that $\{x_1, y_1, v, w_1, s'_1, s_2\}$ induces an antenna.

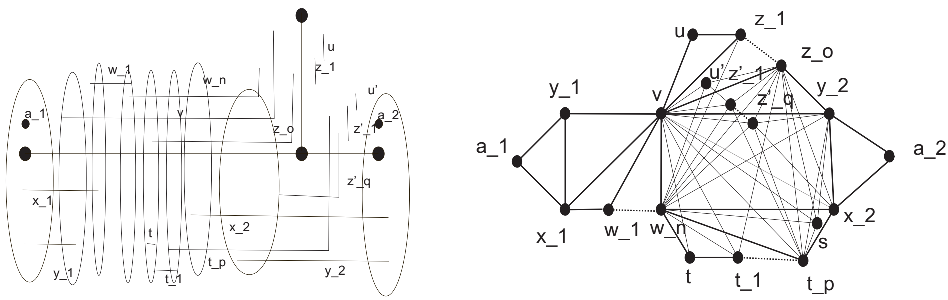


FIGURE 23. Case 1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 7 between w_{n-1}, a_2.

If $z'_q \in T(\widetilde{B}, X_2]$, as the edge $e(q)$ where z'_q was chosen is dominated by Iz'_q, it follows by Claim 2 that there are vertices t'_i that are not in B_q, with $e(q) = A_qB_q$ and $B_q \in T[C, A_q]$, for $i = 1, \dots, r$. Also by the election of $\widetilde{e}$, they are not in C. Let t'_r be the last vertex chosen. Observe that t'_r may be adjacent to y_2.

If t'_r is not adjacent to y_2 then there is a special connection between a_1 and a_2. More clearly, in case that $\{x_1, y_1, w_1, v\}$ is not a clique then $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\dots,n-1}$, $\{x_2, y_2, a_2\}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\dots,o-1}$, $\{u, v, z_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\dots,p-1}$, $\{t, v, w_n, t_1\}$, $\{t', t'_1, t_p, v, w_n, z_o\}$, $\{t'_i, t'_{i+1}, z_o, z'_q, t_p, v, w_n\}_{i=1,\dots,r-1}$, $\{t'_r, t_p, v, w_n, z_o, z'_q, x_2\}$, $\{t_p, v, w_n, z_o, z'_q, x_2, y_2\}$, $\{t_p, v, w_n, z_o, z'_i, z'_{i+1}\}_{i=1,\dots,q-1}$ and $\{u', z'_1, z_o, w_n, v\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \dots, n-2\}$, and of Type 8 between w_{n-1} and a_2; see Figure 24.

In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, as seen on Case 1.2 there are two vertices s_1, s_2 in G such that $\{x_1, y_1, v, w_1, s_1, s_2\}$ induces an antenna.

If t'_r is adjacent to y_2, let s be a separator of X_2 to $N[a_1]$ such that $|\{Q \in C(G) : s \in Q\}|$ is minimum. By the election of t'_r, it is clear that s is not adjacent to t'_{r-1}; recall that $t'_r \notin C$. Then, let s_1 be a separator vertex of C to $N[a_3]$ such that $|\{Q \in C(G) : s_1 \in Q\}|$ is minimum.

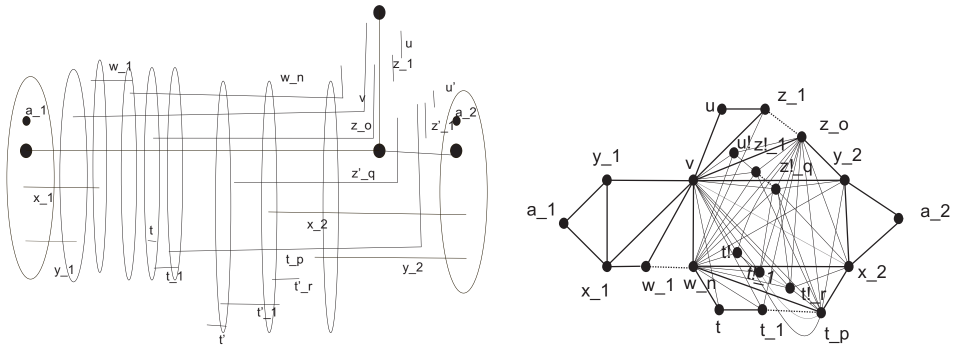


FIGURE 24. Case 1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 8 between w_{n-1}, a_2.

Observe that $s_1 \neq z'_q$ since $z'_q \in T(\widetilde{B}, X_2]$. Hence there is a special connection between a_1 and a_2. More clearly, in case that $\{x_1, y_1, w_1, v\}$ is not a clique then $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_1, v, w_1\}$, $\{w_i, v, w_{i+1}\}_{i=1,\ldots,n-1}$, $\{x_2, y_2, a_2\}$, $\{z_i, z_{i+1}, v, w_n\}_{i=1,\ldots,o-1}$, $\{u, v, z_1\}$, $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1,\ldots,p-1}$, $\{t, v, w_n, t_1\}$, $\{t', t'_1, t_p, v, w_n, z_o\}$, $\{t'_i, t'_{i+1}, z_o, z'_q, t_p, v, w_n\}_{i=1,\ldots,r-1}$, $\{t'_r, t_p, v, w_n, z_o, z'_q, x_2, y_2, s_1\}$, $\{t_p, v, w_n, z_o, z'_i, z'_{i+1}\}_{i=1,\ldots,q-1}$, $\{t_p, v, w_n, z_o, z'_1, u'\}$ and $\{s, t'_r, z'_q, t_p, z_o, v, w_n, x_2\}$ are cliques of G.

Observe that there is a special connection of Type 2 between a_1, w_2, of Type 1 between w_i, w_{i+2} with $i \in \{1, \ldots, n-2\}$, and of Type 9 between w_{n-1} and a_2; see Figure 25.

In case that $\{x_1, y_1, v, w_1\}$ is a clique of G, as seen on Case 1.2 there are two vertices s'_1, s_2 in G such that $\{x_1, y_1, v, w_1, s'_1, s_2\}$ induces an antenna.

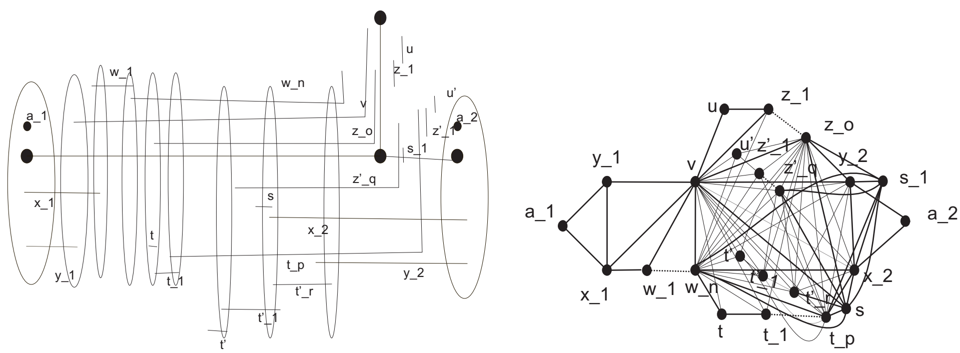


FIGURE 25. Case 1: Type 2 between a_1, w_2; Type 1 between w_i, w_{i+2}; Type 9 between w_{n-1}, a_2.

Case 2: $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$ but $T_{y_2} \cap T_{x_1} = \emptyset$, or $T_{x_1} \cap T_{x_2} \cap T_{y_2} \neq \emptyset$ but $T_{y_1} \cap T_{x_2} = \emptyset$. By our assumption, there is an edge in $T[N[a_1], X_2]$ whose label is contained in C. Hence $T_{x_1} \cap T_{x_2} \cap T_{y_1} = \emptyset$.

Clearly, there are two paths in G between a_1 and a_2; $P = a_1, x_1, x_2, a_2$ and $Q = a_1, y_1, v, y_2, a_2$. On the other hand, $\{x_1, x_2, y_2, v\}$ is a clique of G. In this situation, x_1 may be in $lab(e'_1)$. We know that there is an edge in $T[Y_1, X_2]$ whose label is contained in C. Let $\widetilde{e} = \widetilde{A}\widetilde{B}$ be the nearest C with $\widetilde{B} \in T[\widetilde{A}, C]$, and $m \in lab(\widetilde{e})$ such that T_m is the shortest to $N[a_3]$ and Dm its leaf in $T[C, N[a_3]]$. Observe that m is not v. Moreover, $Dm \neq B'_1$; otherwise $T' = T - \{e'_1, \widetilde{e}\} + \widetilde{A}B'_1 + A'_1\widetilde{B}$ is a DV-model that can be rooted on $N[a_3]$, a contradiction.

On the other hand, we choose w_i in label of edges in $T[Y_1, X_2]$ with the Election 1 taking $A = Y_1$ and $B = X_2$. Let w_n be the last vertex chosen, and Dw_n be the leaf of w_n to $N[a_3]$. Observe that w_n may be x_1 or m.

If $m = x_1$, let X'_1X_1 be the edge of T with $X'_1 \in T[X_1, N[a_3]]$. As X_1 dominates $\widetilde{e}$ it follows by Claim 2 that for all edges $\overline{e} \in T[C, X'_1]$, $lab(\overline{e}) \nsubseteq \widetilde{B}$. Then we choose vertices in the label of edges in $T[C, X']$ through Election 1 with $A = X'_1$ and $B = C$ such that they are not in $\widetilde{B}$. Let z_i be these vertices chosen for $i = 1, \ldots, o$, and z_o be the last vertex chosen. As in Case 1, we will analyze whether Iz_o is in $T(\widetilde{B}, X_2]$, and we obtain a special connection of Type 4, 5, or 6, taking x_1 instead of w_i in Case 1. More clearly, $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_2, y_2, a_2\}$, $\{v, x_1, x_2, y_2, z_o\}$, $\{z_i, z_{i+1}, v, x_1\}_{i=1,\ldots,o-1}$, $\{u, v, z_1\}$ and $\{v, x_1, x_2, t\}$ are cliques of G; so there is a special connection of Type 4 between y_1, a_2, and of Type 1 between a_1, v; see Figure 26.

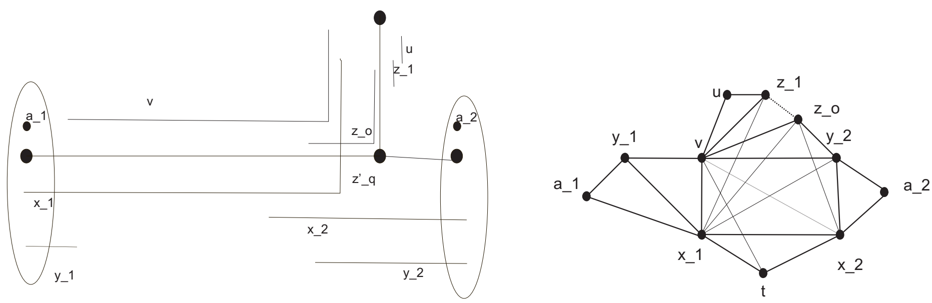


FIGURE 26. Case 2: Type 4 between y_1, a_2 and Type 1 between a_1, v.

Or $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_2, y_2, a_2\}$, $\{z_i, z_{i+1}, v, x_1\}_{i=1,\ldots,o-1}$, $\{u, v, z_1\}$, $\{v, x_1, z_o, x_2, t_p, s\}$, $\{t_p, x_2, y_2, z_o, x_1, v\}$, $\{t, x_1, v, t_1\}$, $\{t_i, t_{i+1}, v, x_1, z_o\}_{i=1,\ldots,p-1}$ and $\{v, x_1, x_2, y_2, z_o, s_1\}$ are cliques of G; so there is a special connection of Type 5 between y_1, a_2 and of Type 1 between a_1, v. Or $\{a_1, x_1, y_1\}$, $\{x_1, v, y_1\}$, $\{x_2, y_2, a_2\}$, $\{z_i, z_{i+1}, v, x_1\}_{i=1,\ldots,o-1}$, $\{u, v, z_1\}$, $\{t, v, x_1, t_1\}$, $\{t_i, t_{i+1}, v, x_1, z_o\}_{i=1,\ldots,p-1}$, $\{t_p, x_2, v, x_1, z_o\}$ and $\{v, x_1, x_2,$

$y_2, z_o\}$ are cliques of G; so there is a special connection of Type 6 between y_1, a_2, and of Type 1 between a_1, v; see Figure 27.

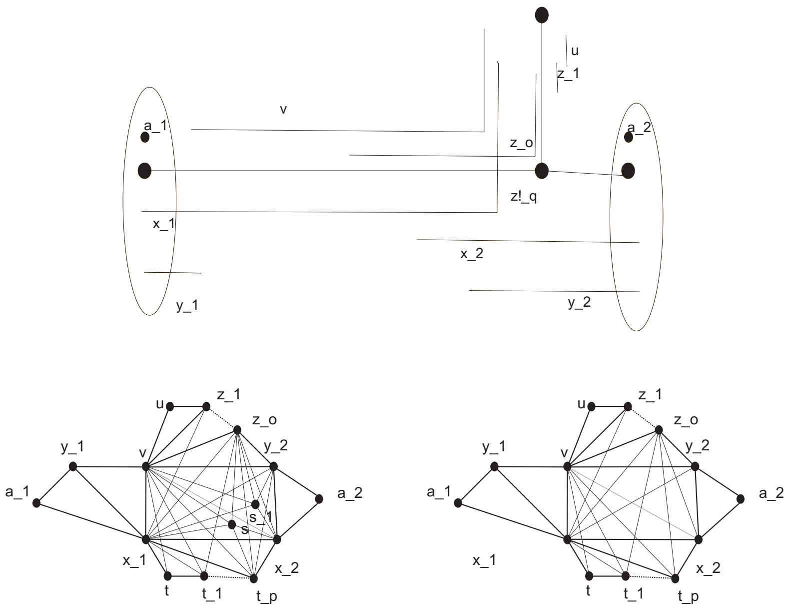


FIGURE 27. Case 2: Type 5 between y_1, a_2 and Type 1 between a_1, v or Type 6 between y_1, a_2 and Type 1 between a_1, v.

If $m = w_n$, let $P' = a_1, y_1, w_1, \ldots, w_n, x_2, a_2$ and $Q' = a_1, x_1, y_2, a_2$ be paths in G between a_1 and a_2. Hence there is a special connection of Type 4 or Type 5 or Type 6 between w_{n-1} and a_2. More clearly, $\{a_1, x_1, y_1\}$, $\{w_1, x_1, y_1\}$, $\{w_i, w_{i+1}, x_1\}_{i=1,\ldots,n-1}$, $\{x_2, y_2, a_2, \}$, $\{t, x_2, w_n, x_1\}$, $\{z_o, x_1, w_n, x_2, y_2\}$, $\{z_i, z_{i+1}, w_n, x_1\}_{i=1,\ldots,o-1}$ and $\{z_1, x_1, u\}$ are cliques of G. Or $\{a_1, x_1, y_1\}$, $\{w_1, x_1, y_1\}$, $\{w_i, w_{i+1}, x_1\}_{i=1,\ldots,n-1}$, $\{x_2, y_2, a_2, \}$, $\{t_p, y_2, x_2, z_o, w_n, x_1\}$, $\{x_2, t_p, z_o, s, w_n, x_1\}$, $\{z_o, t_i, t_{i+1}, x_1, w_n\}_{i=1,\ldots,p-1}$, $\{t, w_n, x_1, t_1\}$, $\{z_o, x_1, w_n, x_2, y_2\}$, $\{z_i, z_{i+1}, w_n, x_1\}_{i=1,\ldots,o-1}$ and $\{z_1, x_1, u\}$ are cliques of G. Or $\{a_1, x_1, y_1\}$, $\{w_1, x_1, y_1\}$, $\{w_i, w_{i+1}, x_1\}_{i=1,\ldots,n-1}$, $\{x_2, y_2, a_2, \}$, $\{y_2, x_2, z_o, w_n, x_1\}$, $\{x_2, t_p, z_o, w_n, x_1\}$, $\{z_o, t_i, t_{i+1}, x_1, w_n\}_{i=1,\ldots,p-1}$, $\{t, w_n, x_1, t_1\}$, $\{z_i, z_{i+1}, w_n, x_1\}_{i=1,\ldots,o-1}$ and $\{z_1, x_1, u\}$ are cliques of G; see Figure 28.

If $m \neq x_1, w_n$, let $P' = a_1, y_1, w_1, \ldots, w_n, x_2, a_2$ and $Q' = a_1, x_1, y_2, a_2$ be paths in G between a_1 and a_2. Hence there is a special connection of Type 4 or Type 5 or Type 6 or Type 7 or Type 8 or Type 9 between w_{n-1} and a_2 considering the analysis of Case 1 when there is another edge contained in C.

Case 3: $T_{x_1} \cap T_{x_2} \cap T_{y_1} \cap T_{y_2} \neq \emptyset$. Clearly, there are two paths in G between a_1 and a_2. Let $P = a_1, y_1, y_2, a_2$ and $Q = a_1, x_1, x_2, a_2$ be paths

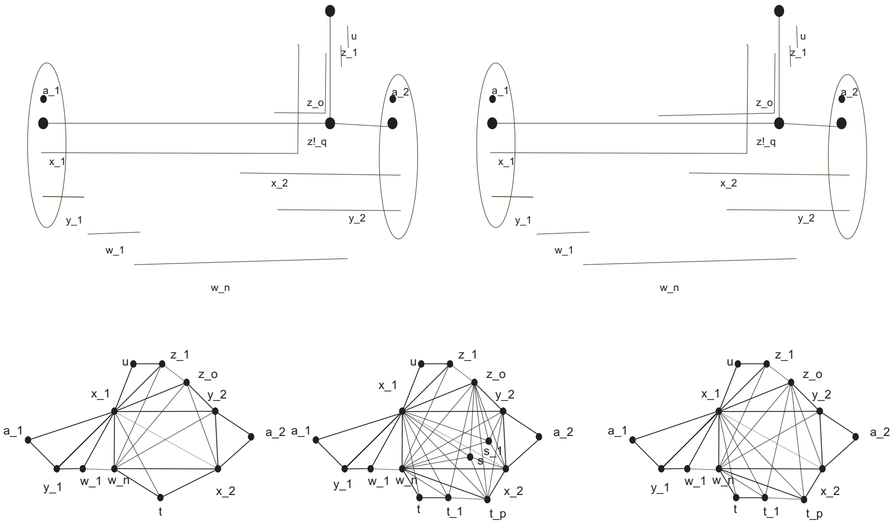


FIGURE 28. Case 2: Type 4 or Type 5 or Type 6 between w_{n-1}, a_2 and Type 1 between a_1, w_1.

in G. Also $\{x_1, y_1, x_2, y_2\}$ is a clique of G. By the existence of an edge $e \in T[N[a_1], X_2]$ such that $lab(e) \subset C$, we have $x_1 \in C$ and $y_1 \in C$. Observe that $N[a_1]A_1$ has a label contained in C, and may have another edge. On the other hand, x_1 and y_1 cannot be both vertices of $lab(e'_1)$, otherwise $T' = T - \{N[a_1]A_1, e'_1\} + N[a_1]B'_1 + A'_1A_1$ is a DV-model of G rooted on $N[a_3]$, a contradiction. Hence $y_1 \notin lab(e'_1)$; moreover, $y_1 \notin B'_1$. Let $Y'_1Y_1 \in E(T)$ be such that $Y'_1 \in T[Y_1, N[a_3]]$, and let $\widetilde{e} = \widetilde{A}\widetilde{B}$ be the closest edge to C dominated by Y_1. Clearly $lab(\widetilde{e}) \subset C$. By Claim 2, for all $e' \in T[C, Y'_1]$ we have $lab(e') \nsubseteq \widetilde{B}$. Cleary, there are vertices $z_i \in lab(e') - \widetilde{B}$ which were chosen through Election 1, and if z_o is the last vertex chosen then analyzing where Iz_o is we obtain the situations described in Case 1, i.e., there is a special connection of Type 4 or Type 5 or Type 6 taking $w_n = y_1$ and $v = x_1$. More clearly, $\{a_1, x_1, y_1\}$, $\{x_2, y_2, a_2,\}$, $\{t, x_2, y_1, x_1\}$, $\{z_o, x_1, y_1, x_2, y_2\}$, $\{z_i, z_{i+1}, y_1, x_1\}_{i=1,\dots,o-1}$ and $\{z_1, x_1, u\}$ are cliques of G. Or $\{a_1, x_1, y_1\}$, $\{x_2, y_2, a_2,\}$, $\{t_p, y_2, x_2, z_o, y_1, x_1\}$, $\{x_2, t_p, z_o, s, y_1, x_1\}$, $\{z_o, t_i, t_{i+1}, x_1, y_1\}_{i=1,\dots,p-1}$, $\{t, y_1, x_1, t_1\}$, $\{z_o, x_1, y_1, x_2, y_2\}$, $\{z_i, z_{i+1}, y_1, x_1\}_{i=1,\dots,o-1}$ and $\{z_1, x_1, u\}$ are cliques of G. Or $\{a_1, x_1, y_1\}$, $\{x_2, y_2, a_2,\}$, $\{y_2, x_2, z_o, y_1, x_1\}$, $\{x_2, t_p, z_o, y_1, x_1\}$, $\{z_o, t_i, t_{i+1}, x_1, y_1\}_{i=1,\dots,p-1}$, $\{t, y_1, x_1, t_1\}$, $\{z_i, z_{i+1}, y_1, x_1\}_{i=1,\dots,o-1}$ and $\{z_1, x_1, u\}$ are cliques of G; see Figure 29. □

The following corollary allows us to construct forbidden induced subgraphs for rooted directed path graphs different from those described in [1].

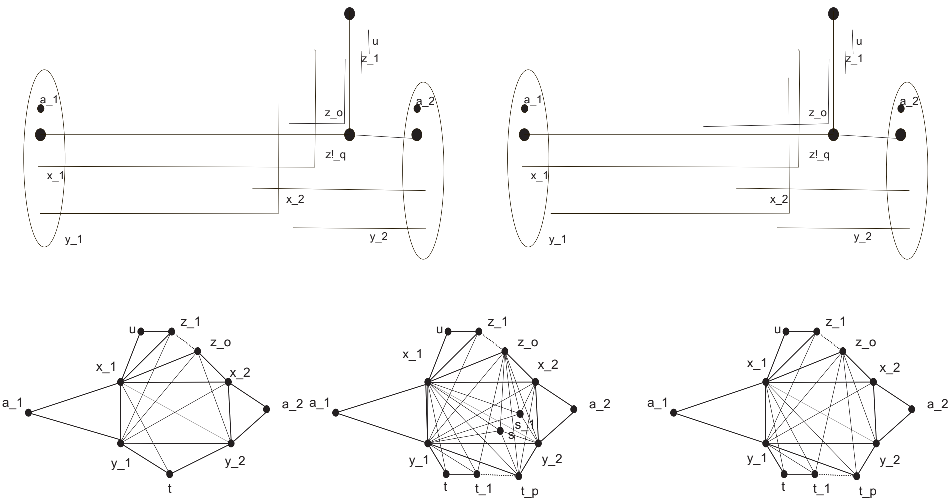


FIGURE 29. Case 3: Type 4 or Type 5 or Type 6 between a_1, a_2.

Corollary 2. *Let G be a DV graph with an asteroidal quadruple $\{a_1, a_2, a_3, a_4\}$. If a_1, a_2 and a_3, a_4 are linked by a special connection then G is not an RDV graph.*

Proof. Let Q_{a_i} be a clique that contains a_i for $i = 1, 2, 3, 4$ and T be a DV-model of G. Since a_1, a_2, a_3, a_4 is an asteroidal quadruple, $T[Q_{a_1}, Q_{a_2}, Q_{a_3}, Q_{a_4}]$ has four leaves. By Theorem 1, $T(a_1, a_2)$ and $T(a_3, a_4)$ are directed path, then T cannot be rooted. Therefore G is not an RDV graph. $\square$

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