Generalized variational procedure: An application to non-perturbative QCD

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We present a generalized variational procedure oriented to the algebraic solution of many body Hamiltonians expressed in bosonic and fermionic variables. The method specializes in the non-perturbative regime of the solutions. As an example, we focus on the application of the method to non-perturbative QCD.

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1. Introduction

Variational procedures have proved to be useful tools in the treatment of various quantum many-body problems, from molecular to hadron physics.[1,2] As an example, we shall mention the case of the low-energy hadronic spectrum, as described by QCD[3] in its non-perturbative regime, where one has to deal, simultaneously, with confined fermions (quarks and anti-quarks) and bosons (gluons). To find there an effective method to diagonalize the QCD Hamiltonian is not at all trivial. One often recurs to trial states, as the coherent state defined in,[4–7] which give some insight on the vacuum structure of QCD. In[8] a first step in this direction was reported. The Coulomb interaction was approximated by a contact interaction and a semi-analytic solution of the QCD Hamiltonian was found. Based on these results, we aim at developing a variational method to diagonalize the complete QCD Hamiltonian[9] within the basis constructed in.[8]

The paper is organized as follows: In section II a boson system is considered. The transformation to new quasi-particles is presented and the nature of the new vacuum state is discussed. In section III the same is done for a fermionic many-body system. In section IV we show the relation between the present method and the coherent state used in a variational treatment of QCD.[6,7] Finally, in section V the conclusions are drawn.

2. A system of many bosons

Let us define the creation and annihilation operators, $\boldsymbol{b}^{\dagger}_{\alpha a}$ and $\boldsymbol{b}^{\alpha a}$, where the two indexes are to be associated with different degrees of freedom. In the case of QCD, the index a may refer to color and α to flavor, spin and orbital degrees of freedom. Lower and upper indices denote the difference under transformations of the creation and annihilation operators. These operators obey the commutation relation

$$\left[\boldsymbol{b}^{\alpha a}, \boldsymbol{b}^{\dagger}_{\beta b}\right] = \delta_{\alpha\beta}\delta_{ab} \quad . \tag{1}$$

We introduce the following general ansatz for the transformation to new creation and annihilation operators

$$\begin{aligned} \boldsymbol{P}^{\dagger}_{\alpha,a} &= \frac{1}{\sqrt{2}}\sum_{\beta}\left(M_{\alpha}{}^{\beta}\boldsymbol{b}^{\dagger}_{\beta a} + N_{\alpha}{}^{\beta}\boldsymbol{b}_{\beta a}\right) \\ \boldsymbol{P}^{\alpha,a} &= \frac{1}{\sqrt{2}}\sum_{\beta}\left(M_{\beta}{}^{\alpha}\boldsymbol{b}^{\beta a} - N_{\beta}{}^{\alpha}\boldsymbol{b}^{\dagger\beta a}\right) \quad . \end{aligned} \tag{2}$$

The indices α and β refer to the spatial quantum numbers, *except the magnetic projection of total spin*, and a is a short hand notation for the sum of the color index and the magnetic projection of the spin. By construction the matrices M and N transform under raising and lowering of the indices in the same way as the creation and annihilation operators, to preserve the transformation properties of the $\boldsymbol{P}$ operators. In QCD the indices α and β run over the values which are allowed by the Coulomb condition (i.e: restricted to transversal space-components) and they can be divided into several other indexes, as done in.[8,10] For example,

$$\alpha \rightarrow \Xi(N, L, 1)J \quad , \tag{3}$$

where Ξ denotes either the $\mathcal{E}$ (electric) or $\mathcal{M}$ (magnetic) modes, N is the principal quantum number (when the harmonic oscillator is used as a basis), L the orbital angular momentum and J the total spin. For the boson operators we use the phase convention

$$\boldsymbol{b}^{\alpha a} = (-1)^{\varphi(\alpha)+\varphi(a)} \boldsymbol{b}_{\bar{\alpha}\bar{a}} \tag{4}$$

under raising and lowering indices, where the bar over an index denotes its conjugate component [a].[11–13] With the phase transformation (4), we have

$$\begin{aligned} M_\beta{}^\alpha &= (-1)^{\varphi(\alpha)+\varphi(\beta)} M^{\bar{\beta}}{}_{\bar{\alpha}} \\ N_\beta{}^\alpha &= (-1)^{\varphi(\alpha)+\varphi(\beta)} N^{\bar{\beta}}{}_{\bar{\alpha}} \quad . \end{aligned} \tag{5}$$

Next, we investigate the properties of the $\boldsymbol{P}$ operators under commutation. They are boson operators, which gives conditions to the M and N matrices. Then,

$$\begin{aligned} \delta_{\alpha\beta}\delta_{ab} \ &= \ \left[\boldsymbol{P}^{\alpha a}, \boldsymbol{P}^\dagger_{\beta b}\right] \\ &= \\ &\tfrac{1}{2}\textstyle\sum_{\gamma_1\gamma_2} M_{\gamma_1}{}^\alpha M_\beta{}^{\gamma_2} \left[\boldsymbol{b}^{\gamma_1 a}, \boldsymbol{b}^\dagger_{\gamma_2 b}\right] \\ &+\tfrac{1}{2}\textstyle\sum_{\gamma_1\gamma_2} M_{\gamma_1}{}^\alpha N_\beta{}^{\gamma_2} \left[\boldsymbol{b}^{\gamma_1 a}, \boldsymbol{b}_{\gamma_2 b}\right] \\ &-\tfrac{1}{2}\textstyle\sum_{\gamma_1\gamma_2} N_{\gamma_1}{}^\alpha M_\beta{}^{\gamma_2} \left[\boldsymbol{b}^{\dagger\gamma_1 a}, \boldsymbol{b}^\dagger_{\gamma_2 b}\right] \\ &-\tfrac{1}{2}\textstyle\sum_{\gamma_1\gamma_2} N_{\gamma_1}{}^\alpha N_\beta{}^{\gamma_2} \left[\boldsymbol{b}^{\dagger\gamma_1 a}, \boldsymbol{b}_{\gamma_2 b}\right] \\ &= \\ &\delta_{ab}\textstyle\sum_\gamma \tfrac{1}{2}\left(M_\gamma{}^\alpha M_\beta{}^\gamma + N_\gamma{}^\alpha N_\beta{}^\gamma\right) \quad . \end{aligned} \tag{6}$$

which implies

$$\begin{aligned} \sum_\gamma M_\beta{}^\gamma M_\gamma{}^\alpha &= \delta_{\alpha\beta} \\ \sum_\gamma N_\beta{}^\gamma N_\gamma{}^\alpha &= \delta_{\alpha\beta} \quad . \end{aligned} \tag{7}$$

One is free to choose $N_\alpha{}^\beta = M_\alpha{}^\beta$, which we adopt from here on.

2.1. *The trial state for the bosonic many-body system*

For the trial state we define a new vacuum $\mid \tilde{0}\rangle$, which satisfies

$$\boldsymbol{P}^{\alpha a} \mid \tilde{0}\rangle = 0 \quad . \tag{8}$$

In order to calculate matrix elements of the boson creation and annihilation operators, Eq. (2) has to be inverted, giving

$$\begin{aligned} \left(\boldsymbol{b}^\dagger_{\beta a} + \boldsymbol{b}_{\beta a}\right) &= \sqrt{2}\sum_\alpha M_\beta{}^\alpha \boldsymbol{P}^\dagger_{\alpha b} \\ \left(\boldsymbol{b}_{\beta a} - \boldsymbol{b}^\dagger_{\beta a}\right) &= \sqrt{2}\sum_\alpha M^\alpha_{\ \beta} \boldsymbol{P}_{\alpha a} \quad . \end{aligned} \tag{9}$$

[a] In this notation, the angular momentum projection M goes into $-M$ and in $SU(3)$ the color-hypercharge and color-isospin (Y, T, T_z) goes into $\rightarrow (-Y, T, -T_z)$.

and, from these equations one gets

$$\begin{aligned} \boldsymbol{b}^{\dagger}_{\beta b} &= \frac{1}{\sqrt{2}} \sum_{\alpha} \left(M_{\beta}{}^{\alpha} \boldsymbol{P}^{\dagger}_{\alpha b} - M^{\alpha}_{\ \beta} \boldsymbol{P}_{\alpha b} \right) \\ \boldsymbol{b}_{\beta b} &= \frac{1}{\sqrt{2}} \sum_{\alpha} \left(M_{\beta}{}^{\alpha} \boldsymbol{P}^{\dagger}_{\alpha b} + M^{\alpha}_{\ \beta} \boldsymbol{P}_{\alpha b} \right) \end{aligned} \tag{10}$$

The above transformation is more general. Under scaling, and preserving the commutation relations, we write

$$\begin{aligned} \boldsymbol{b}^{\dagger}_{\alpha a} &\rightarrow \sqrt{\lambda_{\alpha}} \boldsymbol{b}^{\dagger}_{\alpha a} \\ \boldsymbol{b}^{\alpha a} &\rightarrow \frac{1}{\sqrt{\lambda_{\alpha}}} \boldsymbol{b}^{\alpha a} \quad , \end{aligned} \tag{11}$$

The scaling λ_{α} may depend on the index a, a choice leading to a trial state which does not have definite color or spin. A further generalization of the transformation is possible, to include a displacement of the operators

$$\begin{aligned} \boldsymbol{b}^{\dagger}_{\alpha a} &\rightarrow \sqrt{\lambda_{\alpha}} \boldsymbol{b}^{\dagger}_{\alpha a} - \eta_{\alpha a} \\ \boldsymbol{b}^{\alpha a} &\rightarrow \frac{1}{\sqrt{\lambda_{\alpha}}} \boldsymbol{b}^{\alpha a} - \eta^{\alpha a} \quad . \end{aligned} \tag{12}$$

The $\eta_{\alpha a}$ are tensors and transform in the same way as the $\boldsymbol{b}$-operators.

2.2. *Nature of the boson vacuum state*

With the redefinition (11), Eq. (9) changes to

$$\left(\sqrt{\lambda_{\alpha}} \boldsymbol{b}^{\dagger}_{\alpha a} - \frac{1}{\sqrt{\lambda_{\alpha}}} \boldsymbol{b}_{\alpha a} \right) = -\sqrt{2} \sum_{\beta} M^{\beta}_{\ \alpha} \boldsymbol{P}_{\beta a} \quad . \tag{13}$$

Applying it to the new vacuum state $\mid \tilde{0} \rangle$, gives

$$\left(\sqrt{\lambda_{\alpha}} \boldsymbol{b}^{\dagger}_{\alpha a} - \frac{1}{\sqrt{\lambda_{\alpha}}} \boldsymbol{b}_{\alpha a} \right) \mid \tilde{0} \rangle = 0 \quad . \tag{14}$$

or

$$\boldsymbol{b}_{\alpha a} \mid \tilde{0} \rangle = \lambda_{\alpha} \boldsymbol{b}^{\dagger}_{\alpha a} \mid \tilde{0} \rangle \quad . \tag{15}$$

The annihilation operators act as derivatives to the creation operators, i.e.,

$$\boldsymbol{b}_{\alpha a} \rightarrow \frac{\partial}{\partial \boldsymbol{b}^{\dagger \alpha a}} \quad . \tag{16}$$

This provides, for the component $(\alpha \ a)$, the expression

$$\mid \tilde{0} \rangle \sim e^{\frac{1}{2} \lambda_{\alpha} \boldsymbol{b}^{\dagger}_{\alpha a} \boldsymbol{b}^{\dagger \alpha a}} \mid 0 \rangle \quad . \tag{17}$$

and it leads to the solution

$$\mid \tilde{0} \rangle \sim e^{\frac{1}{2} \sum_{\alpha} \lambda_{\alpha} \left(\boldsymbol{b}^{\dagger}_{\alpha} \cdot \boldsymbol{b}^{\dagger \alpha} \right)} \mid 0 \rangle \quad , \tag{18}$$

where $\left(\boldsymbol{b}^\dagger_\alpha \cdot \boldsymbol{b}^{\dagger\alpha}\right) = \sum_a \boldsymbol{b}^\dagger_{\alpha a}\boldsymbol{b}^{\alpha a}$. The introduction of the parameters λ_α gives us a further freedom in the variational procedure. The $\boldsymbol{P}$-operators can be cast into the form

$$
\begin{aligned}
\boldsymbol{P}^\dagger_{\alpha a} &= \frac{1}{\sqrt{2}}\sum_\beta M_\alpha{}^\beta \left(\sqrt{\lambda_\beta}\boldsymbol{b}^\dagger_{\beta a} + \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}_{\beta a}\right) \\
\boldsymbol{P}_{\alpha a} &= \frac{1}{\sqrt{2}}\sum_\beta M^\beta{}_\alpha \left(\sqrt{\lambda_\beta}\boldsymbol{b}_{\beta a} - \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}^\dagger_{\beta a}\right) \quad .
\end{aligned} \tag{19}
$$

This allows to vary $M_\alpha{}^\beta$ (and it complex conjugate $M^\alpha_{\ \beta}$), and λ_α. Using the more general definition of (12) does not change equations (13) and (14). Thus, the structure of the trial state will stay the same, because the differential equation in terms of the $\boldsymbol{b}$-operators will be the same. Eq. (10) will change to

$$
\begin{aligned}
\boldsymbol{b}^\dagger_{\beta b} &= \frac{1}{\sqrt{2\lambda_\beta}}\sum_\alpha \left(M_\beta{}^\alpha \boldsymbol{P}^\dagger_{\alpha b} - M^\alpha{}_\beta \boldsymbol{P}_{\alpha b}\right) + \frac{\eta_{\beta b}}{\sqrt{\lambda_\beta}} \\
\boldsymbol{b}_{\beta b} &= \sqrt{\frac{\lambda_\beta}{2}}\sum_\alpha \left(M_\beta{}^\alpha \boldsymbol{P}^\dagger_{\alpha b} + M^\alpha{}_\beta \boldsymbol{P}_{\alpha b}\right) + \sqrt{\lambda_\beta}\eta_{\beta b}
\end{aligned} \tag{20}
$$

and the equations for the $\boldsymbol{P}$-operators change to

$$
\begin{aligned}
\boldsymbol{P}^\dagger_{\alpha a} &= \frac{1}{\sqrt{2}}\sum_\beta M_\alpha{}^\beta \left(\sqrt{\lambda_\beta}\boldsymbol{b}^\dagger_{\beta a} + \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}_{\beta a}\right) \\
&\quad + \sqrt{2}\sum_\beta M_\alpha{}^\beta \eta_{\beta a} \\
\boldsymbol{P}_{\alpha a} &= \frac{1}{\sqrt{2}}\sum_\beta M^\beta{}_\alpha \left(\sqrt{\lambda_\beta}\boldsymbol{b}_{\beta a} - \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}^\dagger_{\beta a}\right) \quad .
\end{aligned} \tag{21}
$$

For the annihilation operator no contribution of $\eta_{\alpha a}$ enters its definition. The parameters to vary are:

$$
M_\alpha{}^\beta \ , \ \lambda_\alpha \text{ and } \eta_{\alpha a} \quad . \tag{22}
$$

and the use of $\eta_{\alpha a}$ allows us to shift the operators by their vacuum expectation values, as done in presence of spontaneously broken symmetries.

3. A system of many fermions

In this section, we extend the above consideration to fermions. We use the notation

$$
\boldsymbol{b}^\dagger_{\alpha(1,0)a,jm} \ , \ \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},jm} \quad . \tag{23}
$$

The $\boldsymbol{b}$-operators refer to quarks and the $\boldsymbol{d}$-operators to anti-quarks. They transform differently under $SU(3)$. While the quarks transform with respect to color as an (1,0) irreducible representation (irrep), and the anti-quarks as the irrep-(0,1). The same transformation properties apply for the annihilation operators

$$\boldsymbol{b}^{\alpha(1,0)a,jm} \ , \ \boldsymbol{d}^{\alpha(0,1)\bar{a},jm} \quad . \tag{24}$$

The indices α, β refer to numbers like the principal quantum number and other orbital indices, except color and spin. They can be treated as cartesian. Thus, the phase factors $\varphi(\alpha)$ of the previous sections are set to zero, i.e. $(-1)^{\varphi(\alpha)} = +1$. In the Appendix we derive the properties of the creation and annihilation operators under raising and lowering indices, using the phase convention of.[11,12] Here, we resume the results:

$$\begin{aligned}
\boldsymbol{b}^{\dagger\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j+m}\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,j-m} \\
\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j-m}\boldsymbol{b}^{\dagger\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{b}^{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j-m}\boldsymbol{b}_{\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{b}_{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j+m}\boldsymbol{b}^{\alpha(1,0)a,j-m} \quad ,
\end{aligned} \tag{25}$$

with $\chi_a = \frac{\lambda-\mu}{3} + \frac{Y}{2} + T_z$, Y is the hypercharge and T_z the third component of the isospin. A similar property holds for the anti-particle operators, i.e.,

$$\begin{aligned}
\boldsymbol{d}^{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j-m}\boldsymbol{d}_{\alpha(1,0)a,j-m} \\
\boldsymbol{d}_{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j+m}\boldsymbol{d}^{\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{d}^{\dagger\alpha(1,0)a,jm} &= (-1)^{\chi_a+j+m}\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j-m}\boldsymbol{d}^{\dagger\alpha(1,0)a,j-m} \quad .
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\left\{\boldsymbol{b}^{\alpha(1,0)a,j_1m_1}, \boldsymbol{b}^{\dagger}_{\beta(1,0)b,j_2m_2}\right\} &= \delta_{ab}\delta_{\alpha\beta}\delta_{j_1j_2}\delta_{m_1m_2} \\
\left\{\boldsymbol{d}^{\alpha(0,1)\bar{a},j_1m_1}, \boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{b},j_2m_2}\right\} &= \delta_{ab}\delta_{\alpha\beta}\delta_{j_1j_2}\delta_{m_1m_2} \quad .
\end{aligned} \tag{27}$$

(all other anti-commutators vanish). To perform the mapping onto new fermion operators, we follow the same procedure as for (2). These new fermion operators

$\boldsymbol{P}^\dagger$ and $\boldsymbol{P}$ are given by [b]

$$\begin{aligned}
\boldsymbol{P}^\dagger_{\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_\beta M_\alpha{}^\beta \left(\boldsymbol{b}^\dagger_{\beta(1,0)a,jm} + \boldsymbol{d}_{\beta(1,0)a,jm}\right) \\
\boldsymbol{P}^{\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_\beta M_\beta{}^\alpha \left(\boldsymbol{b}^{\beta(1,0)a,jm} + \boldsymbol{d}^{\dagger\beta(1,0)a,jm}\right) \\
\boldsymbol{D}^\dagger_{\alpha(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_\beta M^\beta_{\ \alpha} \left(\boldsymbol{d}^\dagger_{\beta(0,1)\bar{a},jm} + \boldsymbol{b}_{\beta(0,1)\bar{a},jm}\right) \\
\boldsymbol{D}^{\alpha(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_\beta M^\alpha_{\ \beta} \left(\boldsymbol{d}^{\beta(0,1)\bar{a},jm} + \boldsymbol{b}^{\dagger\beta(0,1)\bar{a},jm}\right) \quad ,
\end{aligned} \tag{28}$$

Lowering the indices of the annihilation operators gives (see the Appendix and (25)).

$$\begin{aligned}
\boldsymbol{P}_{\alpha(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_\beta M^\beta_{\ \alpha} \left(\boldsymbol{b}_{\beta(0,1)\bar{a},jm} - \boldsymbol{d}^\dagger_{\beta(0,1)\bar{a},jm}\right) \\
\boldsymbol{D}_{\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_\beta M_\alpha{}^\beta \left(\boldsymbol{d}_{\beta(1,0)a,jm} - \boldsymbol{b}^\dagger_{\beta(1,0)a,jm}\right) \quad .
\end{aligned} \tag{29}$$

One reason for this ansatz is that there is no mixing of color (both the quark creation-operator and the anti-quark annihilation operator with lower indexes belong to the same irrep). Note that these definitions preserve anti-commutation relations.

3.1. *The trial state for the fermionic many-body system*

With respect to the new vacuum, we require that

$$\begin{aligned}
\boldsymbol{P}_{\alpha(0,1)\bar{a},jm}|\tilde{0}\rangle &= 0 \\
\boldsymbol{D}_{\alpha(1,0)a,jm}|\tilde{0}\rangle &= 0 \quad .
\end{aligned} \tag{30}$$

[b]For simplicity, we keep the same notation of the previous section, and the fermionic or bosonic character of the operators will be specified when needed.

By inversion we get

$$\begin{aligned}
\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm} + \boldsymbol{d}_{\beta(1,0)a,jm} &= \sqrt{2}\sum_{\alpha} M_{\beta}{}^{\alpha} \boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm} \\
\boldsymbol{b}_{\beta(0,1)\bar{a},jm} - \boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm} &= \sqrt{2}\sum_{\alpha} M^{\alpha}{}_{\beta} \boldsymbol{P}_{\alpha(0,1)\bar{a},jm} \\
\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm} + \boldsymbol{b}_{\beta(0,1)\bar{a},jm} &= \sqrt{2}\sum_{\alpha} M^{\alpha}{}_{\beta} \boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm} \\
\boldsymbol{d}_{\beta(1,0)a,jm} - \boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm} &= \sqrt{2}\sum_{\alpha} M_{\beta}{}^{\alpha} \boldsymbol{D}_{\alpha(1,0)a,jm} \quad ,
\end{aligned} \tag{31}$$

leading to the expressions

$$\begin{aligned}
\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_{\alpha} M_{\beta}{}^{\alpha} \left[\boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm} - \boldsymbol{D}_{\alpha(1,0)a,jm}\right] \\
\boldsymbol{b}_{\beta(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_{\alpha} M^{\alpha}{}_{\beta} \left[\boldsymbol{P}_{\alpha(0,1)\bar{a},jm} + \boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm}\right] \\
\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_{\alpha} M^{\alpha}{}_{\beta} \left[\boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm} - \boldsymbol{P}_{\alpha(0,1)\bar{a},jm}\right] \\
\boldsymbol{d}_{\beta(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_{\alpha} M_{\beta}{}^{\alpha} \left[\boldsymbol{D}_{\alpha(1,0)a,jm} + \boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm}\right] \quad .
\end{aligned} \tag{32}$$

A more general transformation can be obtained introducing a displacement, like in the boson case,

$$\begin{aligned}
\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} &\longrightarrow \sqrt{\lambda_{\alpha}}\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} \\
\boldsymbol{b}^{\alpha(1,0)a,jm} &\longrightarrow \frac{1}{\sqrt{\lambda_{\alpha}}}\boldsymbol{b}^{\alpha(1,0)a,jm}
\end{aligned} \tag{33}$$

for the quarks and similarly for the anti-quarks:

$$\begin{aligned}
\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} &\longrightarrow \sqrt{\lambda_{\alpha}}\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} \\
\boldsymbol{d}^{\alpha(0,1)\bar{a},jm} &\longrightarrow \frac{1}{\sqrt{\lambda_{\alpha}}}\boldsymbol{d}^{\alpha(0,1)\bar{a},jm} \quad .
\end{aligned} \tag{34}$$

A further generalization may be defined by introducing Grassman numbers $\eta_{\alpha(1,0)a,jm}$ and $\eta_{\alpha(0,1)\bar{a},jm}$ such

$$\begin{aligned}
\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} &\rightarrow \sqrt{\lambda_{\alpha}}\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} - \eta_{\alpha(1,0)a,jm} \\
\boldsymbol{b}_{\alpha(0,1)\bar{a},jm} &\rightarrow \frac{1}{\sqrt{\lambda_{\alpha}}}\boldsymbol{b}_{\alpha(0,1)\bar{a},jm} - \eta_{\alpha(0,1)\bar{a},jm}
\end{aligned} \tag{35}$$

for the quark part and similarly for the anti-quark part:

$$
\begin{aligned}
\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} &\rightarrow \sqrt{\lambda_\alpha}\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} - \eta_{\alpha(0,1)\bar{a},jm} \\
\boldsymbol{d}_{\alpha(1,0)a,jm} &\rightarrow \frac{1}{\sqrt{\lambda_\alpha}}\boldsymbol{d}_{\alpha(1,0)a,jm} - \eta_{\alpha(1,0)a,jm}
\end{aligned}
\tag{36}
$$

The $\eta_{\alpha(1,0)a,jm}$ ($\eta_{\alpha(0,1)\bar{a},jm}$) are tensors and transform in the same way as the $\boldsymbol{d}$- and $\boldsymbol{b}$-operators with the lower index.

3.2. *Nature of the new fermion vacuum state*

With the redefinitions (33) and (34), we obtain

$$
\begin{array}{c}
\sqrt{\lambda_\alpha}\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} - \frac{1}{\sqrt{\lambda_\alpha}}\boldsymbol{b}_{\alpha(0,1)\bar{a},jm} \\
= \\
-\sqrt{2}\sum_\beta M^{\beta}_{\ \alpha}\boldsymbol{P}_{\beta(0,1)\bar{a},jm} \quad ; \\
\sqrt{\lambda_\alpha}\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} - \frac{1}{\sqrt{\lambda_\alpha}}\boldsymbol{d}_{\alpha(1,0)a,jm} \\
= \\
-\sqrt{2}\sum_\beta M_{\alpha}^{\ \beta}\boldsymbol{D}_{\beta(1,0)a,jm} \quad .
\end{array}
\tag{37}
$$

leading to the equations

$$
\begin{aligned}
\left(\sqrt{\lambda_\alpha}\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} - \frac{1}{\sqrt{\lambda_\alpha}}\boldsymbol{b}_{\alpha(0,1)\bar{a},jm}\right)|\tilde{0}\rangle = 0 \\
\left(\sqrt{\lambda_\alpha}\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} - \frac{1}{\sqrt{\lambda_\alpha}}\boldsymbol{d}_{\alpha(1,0)a,jm}\right)|\ \tilde{0}\rangle = 0 \quad ,
\end{aligned}
\tag{38}
$$

which imply

$$
\begin{aligned}
\boldsymbol{b}_{\alpha(0,1)\bar{a},jm}|\tilde{0}\rangle &= +\lambda_\alpha\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a}}|\tilde{0}\rangle \\
\boldsymbol{d}_{\alpha(1,0)a,jm}|\tilde{0}\rangle &= +\lambda_\alpha\boldsymbol{b}^{\dagger}_{\alpha(1,0)a}|\tilde{0}\rangle \quad .
\end{aligned}
\tag{39}
$$

The structure of the fermion vacuum state can be determined in the same way we use for bosons, except for the restrictions imposed by the Pauli Principle. Denoting by $|\ 0\rangle$ the vacuum state of the $\boldsymbol{b}$- and $\boldsymbol{d}$-operators and by $\mathcal{N}$ the normalization of the new vacuum state $|\ \tilde{0}\rangle$, whose expression is $\Pi_{\alpha am}\left(1/\sqrt{1+\lambda_\alpha^2}\right)$, but not relevant for the following discussion, we have

$$
\begin{aligned}
\mid \tilde{0}\rangle &= \mathcal{N} \exp\left[\sum_{\alpha} \lambda_\alpha (\boldsymbol{d}^{\dagger\alpha} \cdot \boldsymbol{b}^\dagger_\alpha)\right]|0\rangle \\
&= \mathcal{N} \exp\left[\sum_{\alpha a m} \lambda_\alpha \boldsymbol{d}^{\dagger\alpha(1,0)a,jm} \boldsymbol{b}^\dagger_{\alpha(1,0)a,jm}\right]|0\rangle \\
&= \mathcal{N} \Pi_{\alpha a m} \exp\left[\lambda_\alpha \boldsymbol{d}^{\dagger\alpha(1,0)a,jm} \boldsymbol{b}^\dagger_{\alpha(1,0)a,jm}\right]|0\rangle \\
&= \mathcal{N} \prod_{\alpha a m}\left[1 + (-1)^{\chi_a + j - m} \lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},jm} \boldsymbol{b}^\dagger_{\alpha(1,0)a,j-m}\right]|0\rangle \quad .
\end{aligned}
\tag{40}
$$

(In the last step we changed m to $-m$.) In the last line we expanded each exponential up to products of two terms, taking into account that $\left(\boldsymbol{b}^\dagger_{\alpha(1,0)a,j-m}\right)^2 = \left(\boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},jm}\right)^2 = 0$. This ansatz corresponds to a condensate of quark-antiquark-pairs, as expected. These pairs are coupled to color-spin zero, thus, the trial state has definite color-spin zero. Now we apply the annihilation operator $\boldsymbol{b}_{\alpha(0,1)\bar{a}}$ to this ansatz obtaining

$$
\begin{aligned}
\boldsymbol{b}_{\alpha(0,1)\bar{a},jm} \mid \tilde{0}\rangle &= \boldsymbol{b}_{\alpha(0,1)\bar{a},jm} \mathcal{N} \prod_{\alpha' a' m'}\left[1 + (-1)^{\chi_a + j - m'} \lambda_{\alpha'} \boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}',jm'} \boldsymbol{b}^\dagger_{\alpha'(1,0)a',j-m'}\right] \mid 0\rangle \\
&= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1 + (-1)^{\chi_a + j - m')} \lambda_{\alpha'} \boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'jm'} \boldsymbol{b}^\dagger_{\alpha'(1,0)a'j-m'}\right]\right\} \\
&\times \boldsymbol{b}_{\alpha(0,1)\bar{a},jm}\left[1 + (-1)^{\chi_a + j - m} \lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},jm} \boldsymbol{b}^\dagger_{\alpha(1,0)a,j-m}\right] \mid 0\rangle \\
&= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1 + (-1)^{\chi_a + j - m'} \lambda_{\alpha'} \boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'jm'} \boldsymbol{b}^\dagger_{\alpha'(1,0)a'j-m'}\right]\right\} \\
&\times (-1)^{(\chi_a + j - m) + (\chi_a + j + m)} \lambda_\alpha \boldsymbol{b}^{\alpha(1,0)a,j-m} \boldsymbol{b}^\dagger_{\alpha(1,0)a,j-m} (-1) \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a}jm} \mid 0\rangle \\
&= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1 + (-1)^{\chi_a + j - m'} \lambda_{\alpha'} \boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'jm'} \boldsymbol{b}^\dagger_{\alpha'(1,0)a'j-m'}\right]\right\} \\
&\times \lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a}jm} \mid 0\rangle \quad .
\end{aligned}
\tag{41}
$$

Since a fermion creation-operator commutes with the product of two fermion creation operators, we have

$$\begin{aligned}
\boldsymbol{b}_{\alpha(0,1)\bar{a}jm} \mid \tilde{0}\rangle &= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1+(-1)^{\chi_a+j-m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'m'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'j-m'}\right]\right\} \\
&\times \lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a}jm}\left[1+(-1)^{\chi_a+j-m}\lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},jm}\boldsymbol{b}^\dagger_{\alpha(1,0)a,j-m}\right] \mid 0\rangle \\
&= \lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a}jm}\left\{\mathcal{N}\prod_{\alpha'a'm'}\left[1+(-1)^{\chi_a+j-m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'jm'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'j-m'}\right]\right\} \mid 0\rangle \\
&= \lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a}jm} \mid \tilde{0}\rangle \quad .
\end{aligned} \tag{42}$$

We repeat the above steps with the application of the annihilation operator $\boldsymbol{d}$

$$\begin{aligned}
\boldsymbol{d}_{\alpha(1,0)a,jm} \mid \tilde{0}\rangle &= \boldsymbol{d}_{\alpha(0,1)a,jm}\mathcal{N}\prod_{\alpha'a'm'}\left[1+(-1)^{\chi_a+j+m')}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}',j-m'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a',jm'}\right] \mid 0\rangle \\
&= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1+(-1)^{\chi_a+j+m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'j-m'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'jm'}\right]\right\} \\
&\times \boldsymbol{d}_{\alpha(1,0)a,jm}\left[1+(-1)^{\chi_a+j+m}\lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},j-m}\boldsymbol{b}^\dagger_{\alpha(1,0)a,jm}\right] \mid 0\rangle \\
&= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1+(-1)^{\chi_a+j+m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'j-m'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'jm'}\right]\right\} \\
&\times (-1)^{2(\chi_a+j+m)}\lambda_\alpha \boldsymbol{d}^{\alpha(0,1)\bar{a},j-m}\boldsymbol{d}^\dagger_{\alpha(0,1)a,j-m}\boldsymbol{b}^\dagger_{\alpha(1,0)ajm} \mid 0\rangle \\
&= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1+(-1)^{\chi_a+j+m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'j-m'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'jm'}\right]\right\} \\
&\times (+1)\lambda_\alpha \boldsymbol{b}^\dagger_{\alpha(1,0)ajm} \mid 0\rangle \quad .
\end{aligned} \tag{43}$$

Using again that a fermion creation operator commutes with the product of two fermion creation operators, we have

$$\begin{aligned}
\boldsymbol{d}_{\alpha(1,0)a,jm} \mid \tilde{0}\rangle &= \mathcal{N}\left\{\prod_{(\alpha',a'm')\neq(\alpha,am)}\left[1+(-1)^{\chi_a+j+m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'-m'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'jm'}\right]\right\} \\
&\times \lambda_\alpha \boldsymbol{b}^\dagger_{\alpha(1,0)a,jm}\left[1+(-1)^{\chi_a+j+m}\lambda_\alpha \boldsymbol{d}^\dagger_{\alpha(0,1)\bar{a},j-m}\boldsymbol{b}^\dagger_{\alpha(1,0)a,jm}\right] \mid 0\rangle \\
&= +\lambda_\alpha \boldsymbol{b}^\dagger_{\alpha(1,0)a,jm}\left\{\mathcal{N}\prod_{\alpha'a'm'}\left[1+(-1)^{\chi_a+j+m'}\lambda_{\alpha'}\boldsymbol{d}^\dagger_{\alpha'(0,1)\bar{a}'jm'}\boldsymbol{b}^\dagger_{\alpha'(1,0)a'j-m'}\right]\right\} \mid 0\rangle \\
&= +\lambda_\alpha \boldsymbol{b}^\dagger_{\alpha(1,0)ajm} \mid \tilde{0}\rangle \quad .
\end{aligned} \tag{44}$$

This proves that our ansatz satisfies the operator equation for the new fermionic vacuum. The vacuum has definite color and spin, when we assume that the indices of the transformation matrix do not depend on color nor on the spin quantum numbers. With this, the $\boldsymbol{P}$ and $\boldsymbol{D}$-operators are given by

$$
\begin{aligned}
\boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M_{\alpha}{}^{\beta}\left(\sqrt{\lambda_\beta}\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm} + \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{d}_{\beta(1,0)a,jm}\right) \\
\boldsymbol{P}_{\alpha(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M^{\beta}{}_{\alpha}\left(\frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}_{\beta(0,1)\bar{a},jm} - \sqrt{\lambda_\beta}\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm}\right) \\
\boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M^{\beta}{}_{\alpha}\left(\sqrt{\lambda_\beta}\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm} + \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}_{\beta(0,1)\bar{a},jm}\right) \\
\boldsymbol{D}_{\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M_{\alpha}{}^{\beta}\left(\frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{d}_{\beta(1,0)a,jm} - \sqrt{\lambda_\beta}\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm}\right) \quad . \qquad (45)
\end{aligned}
$$

Adding a displacement (by adding a Grassman variable), the equations for the $\boldsymbol{P}$-operators change to

$$
\begin{aligned}
\boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M_{\alpha}{}^{\beta}\left(\sqrt{\lambda_\beta}\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm} + \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{d}_{\beta(1,0)a,jm}\right) \\
&\quad -\sqrt{2}\sum_{\beta} M_{\alpha}{}^{\beta}\eta_{\beta(1,0)a,jm} \\
\boldsymbol{P}_{\alpha(0,1)\bar{a}jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M^{\beta}{}_{\alpha}\left(\frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}_{\beta(0,1)\bar{a},jm} - \sqrt{\lambda_\beta}\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm}\right) \\
\boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M^{\beta}{}_{\alpha}\left(\sqrt{\lambda_\beta}\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm} + \frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{b}_{\beta(0,1)\bar{a},jm}\right) \\
&\quad -\sqrt{2}\sum_{\beta} M^{\beta}{}_{\alpha}\eta_{\beta(0,1)\bar{a},jm} \\
\boldsymbol{D}_{\alpha(1,0)ajm} &= \frac{1}{\sqrt{2}}\sum_{\beta} M_{\alpha}{}^{\beta}\left(\frac{1}{\sqrt{\lambda_\beta}}\boldsymbol{d}_{\beta(1,0)a,jm} - \sqrt{\lambda_\beta}\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm}\right) \quad . \qquad (46)
\end{aligned}
$$

In total, we have now the following parameters to vary:

$$
M_{\alpha}{}^{\beta} \ , \ \lambda_\alpha \text{ and } \eta_{\alpha(1,0)a,jm} \ , \ \eta_{\alpha(0,1)\bar{a},jm} \quad . \qquad (47)
$$

Finally, we give the expressions of the former creation and annihilation operators in terms of the new ones, including the information on λ_α, $\eta_{\alpha(1,0)a,jm}$ and $\eta_{\alpha(0,1)\bar{a},jm}$:

$$
\begin{aligned}
\boldsymbol{b}^{\dagger}_{\beta(1,0)a,jm} &= \frac{1}{\sqrt{2\lambda_\beta}} \sum_\alpha {M_\beta}^\alpha \left[\boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm} - \boldsymbol{D}_{\alpha(1,0)a,jm}\right] \\
&\quad + \frac{\eta_{\beta(1,0)a,jm}}{\sqrt{\lambda_\beta}} \\
\boldsymbol{b}_{\beta(0,1)\bar{a},jm} &= \sqrt{\frac{\lambda_\beta}{2}} \sum_\alpha {M^\alpha}_\beta \left[\boldsymbol{P}_{\alpha(0,1)\bar{a},jm} + \boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm}\right] \\
&\quad + \sqrt{\lambda_\beta}\eta_{\beta(0,1)\bar{a},jm} \\
\boldsymbol{d}^{\dagger}_{\beta(0,1)\bar{a},jm} &= \frac{1}{\sqrt{2\lambda_\beta}} \sum_\alpha {M^\alpha}_\beta \left[\boldsymbol{D}^{\dagger}_{\alpha(0,1)\bar{a},jm} - \boldsymbol{P}_{\alpha(0,1)\bar{a},jm}\right] \\
&\quad + \frac{\eta_{\beta(0,1)\bar{a},jm}}{\sqrt{\lambda_\beta}} \\
\boldsymbol{d}_{\beta(1,0)a,jm} &= \sqrt{\frac{\lambda_\beta}{2}} \sum_\alpha {M_\beta}^\alpha \left[\boldsymbol{D}_{\alpha(1,0)a,jm} + \boldsymbol{P}^{\dagger}_{\alpha(1,0)a,jm}\right] \\
&\quad + \sqrt{\lambda_\beta}\eta_{\beta(1,0)a,jm} \quad . \qquad (48)
\end{aligned}
$$

4. Relation to the coherent state as used in[6,7]

In[6,7] a monopole gas for the description of the QCD ground state was proposed and thermodynamical properties extracted. The trial state of this monopole gas, depending on several parameters, has the form of a coherent state. In this section we show that this trial state can be recast in the many-body language of the previous sections. The trial state introduced in[6,7] has the form

$$
\langle A \mid \tilde{0} \rangle = \mathcal{N} e^{-\frac{1}{2}\int d^3x \int d^3y A(x)\omega(x-y)A(y)} \quad , \qquad (49)
$$

where $\mathcal{N}$ is a normalization. We start with the expression of the field components $A(x)_a$ in the basis of the operators $\boldsymbol{b}^\dagger$ and $\boldsymbol{b}$

$$
\begin{aligned}
A(x)_a &= \sum_\alpha \phi^\alpha(x) \frac{1}{\sqrt{2}} \left(\boldsymbol{b}^{\dagger}_{\alpha a} + \boldsymbol{b}_{\alpha a}\right) \\
A^\dagger(x)_a \;=\; A^a(x) &= \sum_\alpha \phi_\alpha(x) \frac{1}{\sqrt{2}} \left(\boldsymbol{b}^{\dagger\alpha a} + \boldsymbol{b}^{\alpha a}\right) \quad , \qquad (50)
\end{aligned}
$$

The amplitude $\phi^\alpha(x)$ is to be associated with the factor $\xi_{\bar{\alpha}}$ of,[6] and

$$
\phi^*_\alpha(x) = \phi^\alpha(x) \;=\; (-1)^{\varphi(\alpha)} \phi_{\bar{\alpha}} \quad . \qquad (51)
$$

In the quantization of the field $A_a(x)$ there appears an additional factor $1/\sqrt{\Omega_\alpha}$, with Ω_α being the frequency of the solution $\phi^\alpha(x)$. We adopt the notation that this factor is included in the function. In order to preserve gauge invariance,[5] $\phi^\alpha(x)$ has to contain a combination of perturbative A-fields with a monopole solution, i.e., in general, different boson operators $\boldsymbol{b}^{\dagger}_{\alpha a}$ have to be defined. For simplicity, we use one generic expression.

With this, we have

$$
\begin{gathered}
A^{\dagger}_a(x)\omega(x-y)A_a(y) = \\
\frac{1}{\sqrt{2}}\sum_a\sum_\alpha \phi_\alpha(x)\left(\boldsymbol{b}^{\dagger\alpha a}+\boldsymbol{b}^{\alpha a}\right)\omega(x-y) \\
\sum_\beta \phi^\beta(y)\left(\boldsymbol{b}^{\dagger}_{\beta a}+\boldsymbol{b}_{\beta a}\right)\omega(x-y) \\
= \\
\frac{1}{\sqrt{2}}\sum_a\sum_{\alpha\beta}\phi_\alpha(x)\omega(x-y)\phi^\beta(y) \\
\left(\boldsymbol{b}^{\alpha a}\boldsymbol{b}^{\dagger}_{\beta a}+\boldsymbol{b}^{\dagger\alpha a}\boldsymbol{b}_{\beta a}+\boldsymbol{b}^{\dagger\alpha a}\boldsymbol{b}^{\dagger}_{\beta a}+\boldsymbol{b}^{\alpha a}\boldsymbol{b}_{\beta a}\right) \quad .
\end{gathered}
\tag{52}
$$

where, in lowering indexes and contracting them we have used the relation

$$
\begin{aligned}
\phi_\alpha(x)\boldsymbol{b}^{\alpha a} &= \phi_\alpha(x)(-1)^{\varphi(\alpha)}\boldsymbol{b}^{a}_{\bar{\alpha}} \\
&= \phi^{\bar{\alpha}}(x)\boldsymbol{b}^{a}_{\bar{\alpha}} \quad ,
\end{aligned}
\tag{53}
$$

With this, (52) can be rewritten as

$$
\begin{gathered}
\sum_a A^{\dagger}_a(x)\omega(x-y)A_a(y) = \\
\frac{1}{\sqrt{2}}\sum_a\sum_{\alpha\beta}\phi^{\bar{\alpha}}(x)\omega(x-y)\phi^\beta(y) \\
\left(\boldsymbol{b}^{a}_{\bar{\alpha}}\boldsymbol{b}^{\dagger}_{\beta a}+\boldsymbol{b}^{\dagger a}_{\bar{\alpha}}\boldsymbol{b}_{\beta a}+\boldsymbol{b}^{\dagger a}_{\bar{\alpha}}\boldsymbol{b}^{\dagger}_{\beta a}+\boldsymbol{b}^{a}_{\bar{\alpha}}\boldsymbol{b}_{\beta a}\right) \quad .
\end{gathered}
\tag{54}
$$

In terms of the scalar products,

$$
\sum_a \boldsymbol{b}^{a}_{\bar{\alpha}}\boldsymbol{b}^{\dagger}_{\beta a} = \left(\boldsymbol{b}_{\bar{\alpha}}\cdot\boldsymbol{b}^{\dagger}_{\beta}\right) \quad .
\tag{55}
$$

it reads

$$
\begin{gathered}
\sum_a A^{\dagger}_a(x)\omega(x-y)A_a(y) = \\
\frac{1}{\sqrt{2}}\sum_{\alpha\beta}\phi^{\bar{\alpha}}(x)\omega(x-y)\phi^\beta(y) \\
\left\{\left(\boldsymbol{b}_{\bar{\alpha}}\cdot\boldsymbol{b}^{\dagger}_{\beta}\right)+\left(\boldsymbol{b}^{\dagger}_{\bar{\alpha}}\cdot\boldsymbol{b}_{\beta}\right)+\left(\boldsymbol{b}^{\dagger}_{\bar{\alpha}}\cdot\boldsymbol{b}^{\dagger}_{\beta}\right)+\left(\boldsymbol{b}_{\bar{\alpha}}\cdot\boldsymbol{b}_{\beta}\right)\right\} \quad .
\end{gathered}
\tag{56}
$$

Because α is a dummy index, we can skip the bar over α in the sum.

By integrating Eq. (52) over x and y we obtain the same structure as in (56), by writing

$$
A^{\alpha\beta} = \int dxdy\phi^\alpha(x)\omega(x-y)\phi^\beta(y) \quad .
\tag{57}
$$

for the factors in front of the scalar products. Using the ansatz for monopoles given in,[6,7] the factors $A^{\alpha\beta}$ can be determined. What we have shown is that the coherent trial state, as used in,[6,7] can be cast into a standard language, like in conventional many body theories.

4.1. *Taking the product of the operators $\boldsymbol{P}^\dagger$ and $\boldsymbol{P}$*

The expression (54) for the exponent can finally be rewritten as a product of creation and annihilation operators by taking the product of two $\boldsymbol{P}$-operators as

$$\begin{aligned} &\textstyle\sum_a \sum_\alpha \lambda_\alpha \boldsymbol{P}^\dagger_{\alpha a} \boldsymbol{P}^{\alpha a} = \\ &\tfrac{1}{2} \textstyle\sum_a \sum_\alpha \lambda_\alpha \sum_{\beta_1\beta_2} \left(M_\alpha{}^{\beta_1} \boldsymbol{b}^\dagger_{\beta_1 a} + N_\alpha{}^{\beta_1} \boldsymbol{b}_{\beta_1 a} \right) \\ &\left(M^\alpha{}_{\beta_2} \boldsymbol{b}^{\beta_2 a} - N^\alpha{}_{\beta_2} \boldsymbol{b}^{\dagger \beta_2 a} \right) \quad . \end{aligned} \tag{58}$$

Performing the explicit multiplication gives

$$\begin{aligned} &\textstyle\sum_{\beta_1\beta_2} \Big\{ A^{\beta_1}_{\beta_2} \left(\boldsymbol{b}^\dagger_{\beta_1} \cdot \boldsymbol{b}^{\beta_2} \right) - B^{\beta_1}_{\beta_2} \left(\boldsymbol{b}^\dagger_{\beta_1} \cdot \boldsymbol{b}^{\dagger\beta_2} \right) \\ &+ B'^{\beta_1}_{\beta_2} \left(\boldsymbol{b}_{\beta_1} \cdot \boldsymbol{b}^{\beta_2} \right) - A'^{\beta_1}_{\beta_2} \left(\boldsymbol{b}_{\beta_1} \cdot \boldsymbol{b}^{\dagger\beta_2} \right) \Big\} \qquad , \end{aligned} \tag{59}$$

with

$$\begin{aligned} A^{\beta_1}_{\beta_2} &= \frac{1}{2} \sum_\alpha \lambda_\alpha M_\alpha{}^{\beta_1} M^\alpha{}_{\beta_2} \\ B^{\beta_1}_{\beta_2} &= \frac{1}{2} \sum_\alpha \lambda_\alpha M_\alpha{}^{\beta_1} N^\alpha{}_{\beta_2} \\ B'^{\beta_1}_{\beta_2} &= \frac{1}{2} \sum_\alpha \lambda_\alpha N_\alpha{}^{\beta_1} M^\alpha{}_{\beta_2} \\ A'^{\beta_1}_{\beta_2} &= \frac{1}{2} \sum_\alpha \lambda_\alpha N_\alpha{}^{\beta_1} N^\alpha{}_{\beta_2} \quad . \end{aligned} \tag{60}$$

In order to get the same structure as in (56) - (57), the upper indices in the $\boldsymbol{b}$ operators in (60) are lowered and the corresponding lower indices in the coefficient matrices are raised. In addition, the canonical transformation $\boldsymbol{b}^\dagger_{\alpha a} \to i\boldsymbol{b}^\dagger_{\alpha a}$, $\boldsymbol{b}^{\alpha a} \to \frac{1}{i}\boldsymbol{b}^{\alpha a}$ is performed, which maintains the commutation relations. This transformation changes the sign in front of the products of two creation or two annihilation operators, but not in the product of a creation with an annihilation operator. With this, we have

$$\begin{aligned} &\textstyle\sum_{\beta_1\beta_2} \Big\{ A^{\beta_1\beta_2} \left(\boldsymbol{b}^\dagger_{\beta_1} \cdot \boldsymbol{b}_{\beta_2} \right) + B^{\beta_1\beta_2} \left(\boldsymbol{b}^\dagger_{\beta_1} \cdot \boldsymbol{b}^\dagger_{\beta_2} \right) \\ &+ B'^{\beta_1\beta_2} \left(\boldsymbol{b}_{\beta_1} \cdot \boldsymbol{b}_{\beta_2} \right) + A'^{\beta_1\beta_2} \left(\boldsymbol{b}_{\beta_1} \cdot \boldsymbol{b}^\dagger_{\beta_2} \right) \Big\} \qquad , \end{aligned} \tag{61}$$

We choose the particular relations

$$A^{\beta_1\beta_2} = B^{\beta_1\beta_2} \ = \ B'^{\beta_1\beta_2} \ = \ A'^{\beta_1\beta_2} \quad , \tag{62}$$

thus, leading to (56) with the integration (57) performed. (62) demonstrates that the ansatz of a monopole gas is contained in the many-body trial state, but eliminating the conditions (62) permits a more general structure, which is of great advantage.

The QCD trial state can now be formally written as

$$e^{\sum_\alpha \lambda_\alpha \left(\boldsymbol{P}^\dagger_\alpha \cdot \boldsymbol{P}^\alpha \right)} \mid \tilde{0}\rangle \quad . \tag{63}$$

Because $\boldsymbol{P}^{\alpha a} \mid \tilde{0}\rangle = 0$, only the first term in the exponential contributes. Thus, the trial state *has been reduced to the vacuum state* $\mid \tilde{0}\rangle$,

The matrix elements $M_\alpha^{\ \beta}$ $(= N_\alpha^{\ \beta})$ can be dealt with in two ways:
i) Simply as parameters which are determined by minimizing the expectation value of the Hamiltonian with respect to a trial state, or
ii) determine $M_\alpha^{\ \beta}$ via the $A^{\alpha\beta}$ of Eq. (57) using also the relations (60)-(62), i.e., the matrix elements depend then on the parameters of the monopole gas. The above relation may be affected by spin mixing, because when the indices α and β contain the total spin j, the matrices M_α^β will mix the spin and the trial state does not have definite spin, though the new vacuum may have definite spin. If we want to avoid this, α and β should not depend on the total spin.

5. Conclusions

In this contribution we have developed a variational procedure which includes simultaneously fermions and bosons. We use explicitly the case of non-perturbative QCD,[6,7] as a working example, but it can be generalized to any system of fermions and bosons. A series of trial states have been proposed, with increasing complexity. They include a simple unitary transformation plus a possible re-scaling of the boson and fermion operators and a shift of the boson operators.

We have shown that the trial state presented in,[6,7] where a coherent state written in terms of a QCD functional was used in order to describe the ground state of QCD as a monopole gas, can be recast into a standard many-body framework. We think that this connection may facilitate the use of such techniques in dealing with the low energy domain of QCD. Further work is in progress concerning the use of effective QCD-inspired Hamiltonians.

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Appendix

In this appendix we derive properties of the fermion creation operators under lowering and raising their indices. We start from the convention we have introduced in,[8] which is

$$\boldsymbol{b}^{\dagger -\xi\alpha(0,1)\bar{a},jm} = (-1)^{\frac{1}{2}+\xi}(-1)^{\chi_a+j-m}\boldsymbol{b}^{\dagger}_{+\xi\alpha(1,0)a,j-m} \quad . \tag{64}$$

(α, β correspond to Cartesian indices). In the above equation (64), a is the color index, j is the spin, m its projection and ξ is the pseudo-spin component. The pseudo-spin quantum number refers to the upper and lower level and for the upper level $\xi = \frac{1}{2}$ and for the lower level it is $\xi = -\frac{1}{2}$. The phase factor χ_a is always integer, while j and m are half integer.

The property of the annihilation operator, under raising and lowering indices, is obtained by the hermitian conjugation of (64)

$$\begin{aligned}
\left(\boldsymbol{b}^{\dagger -\xi\alpha(0,1)\bar{a},jm}\right)^{\dagger} &= \boldsymbol{b}_{-\xi\alpha(0,1)\bar{a},jm} \\
& \left[(-1)^{\frac{1}{2}+\xi+\chi_a+j-m}\boldsymbol{b}^{\dagger}_{\xi\alpha(1,0)a,j-m}\right]^{\dagger} \\
&= (-1)^{\frac{1}{2}+\xi+\chi_a+j-m}\boldsymbol{b}^{\xi\alpha(1,0)a,j-m} \quad .
\end{aligned} \tag{65}$$

Identifying the fermion creation operator with component $\pm\frac{1}{2}$ with the creation of a particle ($\boldsymbol{b}^{\dagger}$) and the annihilation of an anti-particle ($\boldsymbol{d}$), respectively, and similarly for the fermion annihilation operator , we arrive at the following expressions

$$\begin{aligned}
\boldsymbol{b}^{\dagger\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j+m}\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,j-m} \\
\boldsymbol{b}^{\dagger}_{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j-m}\boldsymbol{b}^{\dagger\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{b}^{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j-m}\boldsymbol{b}_{\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{b}_{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j+m}\boldsymbol{b}^{\alpha(1,0)a,j-m} \quad .
\end{aligned} \tag{66}$$

A similar property holds for the anti-particle operators, i.e.,

$$\begin{aligned}
\boldsymbol{d}^{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j-m}\boldsymbol{d}_{\alpha(1,0)a,j-m} \\
\boldsymbol{d}_{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j+m}\boldsymbol{d}^{\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{d}^{\dagger\alpha(1,0)a,jm} &= (-1)^{\chi_a+j+m}\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},j-m} \\
\boldsymbol{d}^{\dagger}_{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j-m}\boldsymbol{d}^{\dagger\alpha(1,0)a,j-m} \\
&.
\end{aligned} \tag{67}$$

Here, one has to include an additional change in sign due to the phase $(-1)^{\frac{1}{2}-\xi}$.

Note, that the resulting phase property is *the same* for the particle creation as for the anti-particle creation operator. The same holds for the annihilation operators.

With the help of the just obtained results, we can study the structure of the $\boldsymbol{D}$- and $\boldsymbol{D}^\dagger$-operators (see Section III). The creation operator $\boldsymbol{D}^\dagger$ corresponds to a particle-annihilation operator at negative energy. Thus we make a similar ansatz for it, as for the annihilation operator $\boldsymbol{P}$, but with a different sign, i.e.,

$$\boldsymbol{D}^{\dagger\alpha(1,0)a,jm} = \frac{1}{\sqrt{2}}\sum_\beta M_\beta{}^\alpha \left(\boldsymbol{d}^{\dagger\beta(1,0)a,jm} - \boldsymbol{b}^{\beta(1,0)a,jm}\right) \quad . \tag{68}$$

It can be easily shown that this anti-commutes with $\boldsymbol{P}^\dagger$. Now, we lower the index in (68), leading to

$$\begin{aligned}\boldsymbol{D}^{\dagger\alpha(1,0)a,jm} &= \frac{1}{\sqrt{2}}\sum_\beta M_\beta{}^\alpha \left[(-1)^{\chi_a+j+m}\boldsymbol{d}^\dagger_{\beta(0,1)\bar{a},j-m} - (-1)^{\chi_a+j-m}\boldsymbol{b}_{\beta(0,1)\bar{a},j-m}\right] \\ &= (-1)^{\chi_a+j+m}\frac{1}{\sqrt{2}}\sum_\beta M_\beta{}^\alpha \left(\boldsymbol{d}^\dagger_{\beta(0,1)\bar{a},j-m} + \boldsymbol{b}_{\beta(0,1)\bar{a},j-m}\right) \\ &= (-1)^{\chi_a+j+m}\boldsymbol{D}^\dagger_{\alpha(0,1)a,j-m} \quad .\end{aligned} \tag{69}$$

The matrices M satisfy $M_\alpha{}^\beta = M^\alpha{}_\beta$. From this we conclude that the correct ansatz for the new anti-particle operator is

$$\boldsymbol{D}^\dagger_{\alpha(0,1)a,jm} = \frac{1}{\sqrt{2}}\sum_\beta M^\beta{}_\alpha \left(\boldsymbol{d}^\dagger_{\beta(0,1)\bar{a},jm} + \boldsymbol{b}_{\beta(0,1)\bar{a},jm}\right) \quad . \tag{70}$$

A similar manipulation for the anti-particle annihilation operator

$$\boldsymbol{D}_{\alpha(1,0)a,jm} = \frac{1}{\sqrt{2}}\sum_\beta M_\alpha{}^\beta \left(\boldsymbol{d}_{\beta(1,0)a,jm} - \boldsymbol{b}^\dagger_{\beta(1,0)a,jm}\right) \tag{71}$$

yields

$$\boldsymbol{D}^{\alpha(0,1)\bar{a},jm} = \frac{1}{\sqrt{2}}\sum_\beta M^\alpha{}_\beta \left(\boldsymbol{d}^{\beta(0,1)\bar{a},jm} + \boldsymbol{b}^{\dagger\beta(0,1)\bar{a},jm}\right) \tag{72}$$

and

$$\boldsymbol{D}_{\alpha(1,0)a,jm} = (-1)^{\chi_a+j+m}\boldsymbol{D}^{\alpha(0,1)\bar{a},j-m} \quad . \tag{73}$$

Equivalent considerations are applied for the raising and lowering of the indices of the $\boldsymbol{P}^\dagger$ and $\boldsymbol{P}$ operators, with the result:

$$\begin{aligned}\boldsymbol{P}^\dagger_{\alpha(1,0)a,jm} &= (-1)^{\chi_a+j-m}\boldsymbol{P}^{\dagger\alpha(0,1)\bar{a},j-m}\\ \boldsymbol{P}_{\alpha(0,1)\bar{a},jm} &= (-1)^{\chi_a+j+m}\boldsymbol{P}^{\alpha(1,0)a,j-m}\end{aligned} \tag{74}$$

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