

# Lie symmetries of the Shigesada–Kawasaki–Teramoto system

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## Abstract

The Shigesada–Kawasaki–Teramoto system, which consists of two reaction-diffusion equations with variable cross-diffusion and quadratic nonlinearities, is considered. The system is the most important case of the biologically motivated model proposed by Shigesada *et al.* (*J. Theor. Biol.* **79**(1979) 83–99). A complete description of Lie symmetries for this system is derived. It is proved that the Shigesada–Kawasaki–Teramoto system admits a wide range of different Lie symmetries depending on coefficient values. In particular, the Lie symmetry operators with highly unusual structure are unveiled and applied for finding exact solutions of the relevant nonlinear system with cross-diffusion.

**Keywords:** reaction-diffusion system; cross-diffusion; Lie symmetry; exact solution.

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# 1 Introduction

In 1979 Shigesada *et al.* [1] proposed a mathematical model to describe the densities of two biological species, which takes into account the heterogeneity of the environment and nonlinear dispersive movements of the individuals of these populations. The model was developed on the basis of Morisita's phenomelological theory of environmental density and has the form

$$\begin{aligned} u_t &= [(d_1 + d_{11}u + d_{12}v)u]_{xx} + (W_x u)_x + u(a_1 - b_1u - c_1v), \\ v_t &= [(d_2 + d_{21}u + d_{22}v)v]_{xx} + (W_x v)_x + v(a_2 - b_2u - c_2v), \end{aligned} \quad (1)$$

where the functions  $u$  and  $v$  arising in system (1) give the densities of two competing species in space and time,  $d_1$  and  $d_2$  denote the diffusion coefficients,  $d_{12}v$  and  $d_{21}u$  are so-called cross-diffusion pressures,  $d_{11}u$  and  $d_{22}v$  are intra-diffusion pressures,  $a_1$  and  $a_2$  are the intrinsic growth coefficients,  $b_1$  and  $c_2$  denote the coefficients of intra-specific competitions,  $b_2$  and  $c_1$  denote the coefficients of inter-specific competitions. The function  $W(x)$  is so-called environmental potential, which is assumed to be known. Nevertheless, the authors of [1] assumed that the environmental potential may be a non-constant function, system (1) with  $W(x) = const$  is usually referred as Shigesada–Kawasaki–Teramoto (SKT) system (model). It was shown by numerical simulations that system (1) possesses solutions describing coexistence of the species by the spatial segregation of habitat [1]. This kind of coexistence results from the mutual interferences of the species and the heterogeneity of the environment and means a steady-state segregation of densities of two competing species. The existence of the steady-state segregation clearly depends on the initial distributions of  $u$  and  $v$  and the parameter values of (1).

It is worth to note that the SKT system with  $d_{ij} = 0$  ( $i = 1, 2, j = 1, 2$ ) and  $W(x) = const$  reduces to the well-known diffusive Lotka–Volterra (DLV) system

$$\begin{aligned} u_t &= d_1 u_{xx} + u(a_1 - b_1u - c_1v), \\ v_t &= d_2 v_{xx} + v(a_2 - b_2u - c_2v). \end{aligned} \quad (2)$$

Starting from the pioneer works [2, 3], the conditions of existence, uniqueness and global stability/instability of solutions for the DLV system and the SKT system were investigated extensively by many scholars (see [4–10] and the papers cited therein). Although there are not many papers devoted to finding exact solutions (especially in explicit form) of these nonlinear systems. To the best of our knowledge, there are only a few papers, in which exact solutions of the SKT system were found [11–13]. The list of such references for the DLV systems is wider [14–19] and the results are summarized in the very recent book [20]. Note that a wide range of exact solutions of the Lotka–Volterra type systems with power diffusivities (without cross-diffusion) were constructed in in [21, 22] (see also [20] and references cited therein).

In this paper system (1) with  $W(x) = const$ , i.e., the SKT system

$$\begin{aligned} u_t &= [(d_1 + d_{11}u + d_{12}v)u]_{xx} + u(a_1 - b_1u - c_1v), \\ v_t &= [(d_2 + d_{21}u + d_{22}v)v]_{xx} + v(a_2 - b_2u - c_2v) \end{aligned} \quad (3)$$

is under study. Depending on signs of the parameters  $a_k$ ,  $b_k$  and  $c_k$  ( $k = 1, 2$ ) the SKT system (3) can describe different types of species interactions (competition, mutualism and prey–predator interaction). Hereafter  $d_{ij}$  ( $i = 1, 2, j = 1, 2$ ) are assumed to be real constants and  $d_{12}^2 + d_{21}^2 \neq 0$ , i.e., we consider only the systems with cross-diffusion. Moreover, we always assume that both equations involve diffusion, i.e., both equations are the second-order PDEs.

A complete description of Lie symmetries for this system with  $d_{ij} = 0$  ( $i = 1, 2, j = 1, 2$ ), i.e., DLV system, is derived in [15] and extended on the case of three-component DLV systems in [17].

It should be noted that the system (3) with  $d_{12} = d_{21} = 0$  is a particular case of the general reaction-diffusion (RD) system with variable diffusivities. Lie symmetry of such system was completely described in [21, 23].

The paper is organized as follows. In Section 2, a problem of the Lie symmetry classification of system (3) is completely solved. It is proved that the SKT system admits a wide range of different Lie symmetries depending on coefficient values. In particular, the Lie symmetry operators with highly unusual structure are unveiled. These operators are nonlinear with respect to the dependent variables  $u$  and  $v$ , hence they do not occur in RD systems without cross-diffusion. In Section 3, we present some examples of exact solutions and their possible interpretation. Finally, we summarize the results obtained and present some conclusions in the last section.

## 2 Lie symmetry classification of the SKT system (3)

It is easily checked that the system (3) with arbitrary coefficients is invariant under the operators

$$P_x = \frac{\partial}{\partial x} \equiv \partial_x, P_t = \frac{\partial}{\partial t} \equiv \partial_t. \quad (4)$$

Hereafter we call this algebra the trivial Lie algebra (other terminology used for this algebra is the ‘principal algebra’ and the ‘kernel of maximal invariance algebras’) of the SKT system (3). Thus, we aim to find all coefficients arising in the nonlinear system (3) that lead to extensions of its trivial Lie algebra (4). Because the SKT system (3) contains 12 parameters it is a non-trivial task and the result obtained is highly non-trivial.

Taking into account the known results for the DLV system [15], we consider only the systems with cross-diffusion ( $d_{12}^2 + d_{21}^2 \neq 0$ ), which cannot be reduced to the systems without cross diffusion ( $d_{12} = d_{21} = 0$ ) by any point (local) transformations.

It is convenient to separate the results obtained into three parts depending on the coefficients arising in the SKT system (3). Thus, three theorems will be presented for the following cases:

1. System (3) with both standard diffusion and cross-diffusion, and non-vanish reaction term(s).

2. System (3) with cross-diffusion diffusion only, i.e.,  $d_1 = d_2 = d_{11} = d_{22} = 0$ , and non-vanish reaction term(s).
3. System (3) with both standard diffusion and cross-diffusion, however reaction terms vanish, i.e.,  $a_k = b_k = c_k = 0$  ( $k = 1, 2$ ).

Notably that the nonlinear system (3) without reaction terms may describe the crystal growth processes (see [24, 25] for details).

Our main result can be formulated in form of Theorems 1–3 presenting the complete Lie symmetry classification of system (3) in each of the cases listed above.

**Theorem 1** *All possible maximal algebras of invariance (up to equivalent representations generated by transformations of the form (5)) of the SKT system (3) are presented in Table 1. Any other system of the form (3) which is invariant with respect to (w.r.t.) the three- and higher-dimensional maximal algebra of invariance (MAI) is reduced by a substitution of the form*

$$\begin{aligned}
t^* &= \alpha_{00}e^{\alpha_0 t}, & x^* &= \alpha_{01}x, \\
u^* &= \alpha_{10} + \alpha_{11}u + \alpha_{12}v + \alpha_{13}e^{\alpha_1 t}u, \\
v^* &= \alpha_{20} + \alpha_{21}u + \alpha_{22}v + \alpha_{23}e^{\alpha_2 t}v
\end{aligned} \tag{5}$$

to one of those given in Table 1 (constants  $\alpha_{ij}$  are determined by the form of the system in question).

**The sketch of the proof.** We can rewrite the system (3) in equivalent form:

$$\begin{aligned}
u_t &= d_1 u_{xx} + 2d_{11} u u_{xx} + d_{12} v u_{xx} + d_{12} u v_{xx} + \\
&\quad 2d_{11} u_x^2 + 2d_{12} u_x v_x + a_1 u - b_1 u^2 - c_1 u v, \\
v_t &= d_2 v_{xx} + 2d_{22} v v_{xx} + d_{21} u v_{xx} + d_{21} v u_{xx} + \\
&\quad 2d_{22} v_x^2 + 2d_{21} u_x v_x + a_2 v - c_2 v^2 - b_2 u v.
\end{aligned} \tag{6}$$

The proof of the theorem is based on the classical Lie scheme (see, e.g., [26–28]); however, it is highly non-trivial and cumbersome because the system (3) contains twelve arbitrary coefficients. Here we give an outline of how the proof proceeds. According to the Lie approach, the system (3) is considered as a manifold  $\mathcal{M} = \{S_1 = 0, S_2 = 0\}$ :

$$\begin{aligned}
S_1 &\equiv -u_t + d_1 u_{xx} + 2d_{11} u u_{xx} + d_{12} v u_{xx} + d_{12} u v_{xx} + \\
&\quad 2d_{11} u_x^2 + 2d_{12} u_x v_x + a_1 u - b_1 u^2 - c_1 u v = 0, \\
S_2 &\equiv -v_t + d_2 v_{xx} + 2d_{22} v v_{xx} + d_{21} u v_{xx} + d_{21} v u_{xx} + \\
&\quad 2d_{22} v_x^2 + 2d_{21} u_x v_x + a_2 v - c_2 v^2 - b_2 u v = 0
\end{aligned} \tag{7}$$

in the space of the following variables:

$$t, x, u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xx},$$

where subscripts  $t$  and  $x$  to the functions  $u$  and  $v$  denote differentiation with respect to these variables.

System (3) is invariant under the transformations generated by the infinitesimal operator

$$X = \xi^0(t, x, u, v)\partial_t + \xi^1(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v, \quad (8)$$

when the following invariance conditions are satisfied:

$$X_2 S_1|_{\mathcal{M}} = 0, X_2 S_2|_{\mathcal{M}} = 0. \quad (9)$$

The operator  $X_2$  is the second prolongation of the operator  $X$ , i.e.,

$$\begin{aligned} X_2 = X &+ \rho_t^1 \frac{\partial}{\partial u_t} + \rho_t^2 \frac{\partial}{\partial v_t} + \rho_x^1 \frac{\partial}{\partial u_x} + \rho_x^2 \frac{\partial}{\partial v_x} + \sigma_{tx}^1 \frac{\partial}{\partial u_{tx}} + \sigma_{tx}^2 \frac{\partial}{\partial v_{tx}} + \\ &\sigma_{tt}^1 \frac{\partial}{\partial u_{tt}} + \sigma_{tt}^2 \frac{\partial}{\partial v_{tt}} + \sigma_{xx}^1 \frac{\partial}{\partial u_{xx}} + \sigma_{xx}^2 \frac{\partial}{\partial v_{xx}}, \end{aligned} \quad (10)$$

where the coefficients  $\rho$  and  $\sigma$  with relevant subscripts are calculated by well-known formulae (see, e.g., [27]).

Substituting (10) into (9) and eliminating the derivatives  $u_t$  and  $v_t$  using (7), we can split this relation into separate parts for the derivatives  $u_x, v_x, u_{xx}, v_{xx}, u_x v_x$ . Finally, after the relevant calculations, we obtain the following system of determining equations (DEs):

$$\xi_x^0 = \xi_u^0 = \xi_v^0 = \xi_u^1 = \xi_v^1 = 0, \quad (11)$$

$$d_{21}v\eta_{uu}^1 + (d_2 + d_{21}u + 2d_{22}v)\eta_{uu}^2 + 2(d_{21} - d_{11})\eta_u^2 = 0, \quad (12)$$

$$(d_1 + 2d_{11}u + d_{12}v)\eta_{uu}^1 + d_{12}u\eta_{uu}^2 + 2d_{11}(\eta_u^1 + \xi_t^0 - 2\xi_x^1) + 2d_{12}\eta_u^2 = 0, \quad (13)$$

$$(d_1 + 2d_{11}u + d_{12}v)\eta_{vv}^1 + d_{12}v\eta_{vv}^2 + 2(d_{12} - d_{22})\eta_v^1 = 0, \quad (14)$$

$$d_{21}v\eta_{vv}^1 + (d_2 + d_{21}u + 2d_{22}v)\eta_{vv}^2 + 2d_{21}\eta_v^1 + 2d_{22}(\eta_v^2 + \xi_t^0 - 2\xi_x^1) = 0, \quad (15)$$

$$(d_1 + 2d_{11}u + d_{12}v)\eta_{uv}^1 + d_{12}u\eta_{uv}^2 + (2d_{11} - d_{21})\eta_v^1 + d_{12}(\eta_v^2 + \xi_t^0 - 2\xi_x^1) = 0, \quad (16)$$

$$d_{21}v\eta_{uv}^1 + (d_2 + d_{21}u + 2d_{22}v)\eta_{uv}^2 + d_{21}(\eta_u^1 + \xi_t^0 - 2\xi_x^1) + (2d_{22} - d_{12})\eta_u^2 = 0, \quad (17)$$

$$d_{12}u\eta_u^1 - (d_1 - d_2 + (2d_{11} - d_{21})u + (d_{12} - 2d_{22})v)\eta_v^1 - d_{12}u\eta_v^2 - d_{12}\eta^1 - d_{12}u(\xi_t^0 - 2\xi_x^1) = 0, \quad (18)$$

$$d_{21}v\eta_u^1 - (d_1 - d_2 + (2d_{11} - d_{21})u + (d_{12} - 2d_{22})v)\eta_u^2 - d_{21}v\eta_v^2 + d_{21}\eta^2 + d_{21}v(\xi_t^0 - 2\xi_x^1) = 0, \quad (19)$$

$$d_{21}v\eta_v^1 - d_{12}u\eta_u^2 - 2d_{11}\eta^1 - d_{12}\eta^2 - (d_1 + 2d_{11}u + d_{12}v)(\xi_t^0 - 2\xi_x^1) = 0, \quad (20)$$

$$d_{21}v\eta_v^1 - d_{12}u\eta_u^2 + d_{21}\eta^1 + 2d_{22}\eta^2 + (d_2 + d_{21}u + 2d_{22}v)(\xi_t^0 - 2\xi_x^1) = 0, \quad (21)$$

$$2d_{21}v\eta_{xu}^1 + 2(d_2 + d_{21}u + 2d_{22}v)\eta_{xu}^2 + 2d_{21}\eta_x^2 - d_{21}v\xi_{xx}^1 = 0, \quad (22)$$

$$2(d_1 + 2d_{11}u + d_{12}v)\eta_{xu}^1 + 2d_{12}u\eta_{xu}^2 + 4d_{11}\eta_x^1 + 2d_{12}\eta_x^2 + \xi_t^1 - (d_1 + 2d_{11}u + d_{12}v)\xi_{xx}^1 = 0, \quad (23)$$

$$2(d_1 + 2d_{11}u + d_{12}v)\eta_{xv}^1 + 2d_{12}u\eta_{xv}^2 + 2d_{12}\eta_x^1 - d_{12}u\xi_{xx}^1 = 0, \quad (24)$$

$$2d_{21}v\eta_{xv}^1 + 2(d_2 + d_{21}u + 2d_{22}v)\eta_{xv}^2 + 2d_{21}\eta_x^1 + 4d_{22}\eta_x^2 + \xi_t^1 - (d_2 + d_{21}u + 2d_{22}v)\xi_{xx}^1 = 0, \quad (25)$$

$$\eta_t^1 + (a_1u - b_1u^2 - c_1uv)\eta_u^1 + (a_2v - b_2uv - c_2v^2)\eta_v^1 - (a_1 - 2b_1u - c_1v)\eta^1 + c_1u\eta^2 - (d_1 + 2d_{11}u + d_{12}v)\eta_{xx}^1 - d_{12}u\eta_{xx}^2 - (a_1u - b_1u^2 - c_1uv)\xi_t^0 = 0, \quad (26)$$

$$\eta_t^2 + (a_2v - c_2v^2 - b_2uv)\eta_v^2 + (a_1u - b_1u^2 - c_1uv)\eta_u^2 + b_2v\eta^1 - (a_2 - b_2u - 2c_2v)\eta^2 - d_{21}v\eta_{xx}^1 - (d_2 + d_{21}u + 2d_{22}v)\eta_{xx}^2 - (a_2v - b_2uv - c_2v^2)\xi_t^0 = 0. \quad (27)$$

The system of DEs (11)—(27) is very cumbersome and one needs to establish how the functions  $\eta^1$  and  $\eta^2$  depend on the variables  $u$  and  $v$ . It is well-known that this dependence is linear in the case of RD equations [29] and systems [21, 30, 31]. It turns out that the cross-diffusion terms in RD systems may lead to a completely different result. In order to prove this, we need to examine differential consequences of equations (18)—(21) w.r.t.  $u$  and  $v$ :

$$d_{12}u\eta_{uu}^1 - (d_1 - d_2 + (2d_{11} - d_{21})u + (d_{12} - 2d_{22})v)\eta_{uv}^1 - d_{12}u\eta_{uv}^2 - (2d_{11} - d_{21})\eta_v^1 - d_{12}\eta_v^2 - d_{12}(\xi_t^0 - 2\xi_x^1) = 0, \quad (28)$$

$$d_{12}u\eta_{uv}^1 - (d_1 - d_2 + (2d_{11} - d_{21})u + (d_{12} - 2d_{22})v)\eta_{vv}^1 - d_{12}u\eta_{vv}^2 + 2(d_{22} - d_{12})\eta_v^1 = 0, \quad (29)$$

$$d_{21}v\eta_{uu}^1 - (d_1 - d_2 + (2d_{11} - d_{21})u + (d_{12} - 2d_{22})v)\eta_{uu}^2 - d_{21}v\eta_{uv}^2 + 2(d_{21} - d_{11})\eta_u^2 = 0, \quad (30)$$

$$d_{21}v\eta_{uv}^1 - (d_1 - d_2 + (2d_{11} - d_{21})u + (d_{12} - 2d_{22})v)\eta_{uv}^2 - d_{21}v\eta_{vv}^2 + d_{21}\eta_u^1 - (d_{12} - 2d_{22})\eta_u^2 + d_{21}(\xi_t^0 - 2\xi_x^1) = 0, \quad (31)$$

$$d_{21}v\eta_{uv}^1 - d_{12}u\eta_{uu}^2 - 2d_{11}\eta_u^1 - 2d_{12}\eta_u^2 - 2d_{11}(\xi_t^0 - 2\xi_x^1) = 0, \quad (32)$$

$$d_{21}v\eta_{vv}^1 - d_{12}u\eta_{uv}^2 - (2d_{11} - d_{21})\eta_v^1 - d_{12}\eta_v^2 - d_{12}(\xi_t^0 - 2\xi_x^1) = 0, \quad (33)$$

$$d_{21}v\eta_{uv}^1 - d_{12}u\eta_{uu}^2 + d_{21}\eta_u^1 - (d_{12} - 2d_{22})\eta_u^2 + d_{21}(\xi_t^0 - 2\xi_x^1) = 0, \quad (34)$$

$$d_{21}v\eta_{vv}^1 - d_{12}u\eta_{uv}^2 + 2d_{21}\eta_v^1 + 2d_{22}\eta_v^2 + 2d_{22}(\xi_t^0 - 2\xi_x^1) = 0. \quad (35)$$

The next crucial step is to remove all the first-order derivatives from (28)—(35) using the system of DEs (11)—(27). In fact, if one subtracts equation (30) from (12), equation (35) from (15), equation (31) from (17), equation (34) from (17) and adds (13) to (32), (14) to (29), (16) to (28), (16) to (33) then the linear algebraic system

$$\begin{aligned} (d_1 + 2d_{11}u + d_{12}v)\eta_{uu}^1 + d_{21}v\eta_{uv}^1 &= 0, \\ d_{12}u\eta_{uu}^1 + (d_2 + d_{21}u + 2d_{22}v)\eta_{uv}^1 &= 0, \\ d_{12}u\eta_{uv}^1 + (d_2 + d_{21}u + 2d_{22}v)\eta_{vv}^1 &= 0, \\ (d_1 + 2d_{11}u + d_{12}v)\eta_{uv}^1 + d_{21}v\eta_{vv}^1 &= 0, \\ (d_1 + 2d_{11}u + d_{12}v)\eta_{uu}^2 + d_{21}v\eta_{uv}^2 &= 0, \\ d_{12}u\eta_{uu}^2 + (d_2 + d_{21}u + 2d_{22}v)\eta_{uv}^2 &= 0, \\ d_{12}u\eta_{uv}^2 + (d_2 + d_{21}u + 2d_{22}v)\eta_{vv}^2 &= 0, \\ (d_1 + 2d_{11}u + d_{12}v)\eta_{uv}^2 + d_{21}v\eta_{vv}^2 &= 0 \end{aligned}$$

to find the function  $\eta_{uu}^1$ ,  $\eta_{uv}^1$ ,  $\eta_{vv}^1$ ,  $\eta_{uu}^2$ ,  $\eta_{uv}^2$  and  $\eta_{vv}^2$  is obtained. Because this system is overdetermined (8 equations for 6 functions), we have found that its solution only is

$$\eta_{uu}^1 = \eta_{uv}^1 = \eta_{vv}^1 = \eta_{uu}^2 = \eta_{uv}^2 = \eta_{vv}^2 = 0, \quad (36)$$

provided  $d_1^2 + d_2^2 + d_{11}^2 + d_{22}^2 \neq 0$ .

The case  $d_1 = d_2 = d_{11} = d_{22} = 0$  is special and will be examined separately.

Thus, solving equations (11) and (36), one immediately obtains

$$\begin{aligned} \xi^0 &= \xi^0(t), \quad \xi^1 = \xi^1(t, x), \\ \eta^1 &= r^1(t, x)u + q^1(t, x)v + p^1(t, x), \\ \eta^2 &= q^2(t, x)u + r^2(t, x)v + p^2(t, x), \end{aligned} \quad (37)$$

where  $\xi^0, \xi^1, r^1(t, x), q^1(t, x), p^1(t, x), q^2(t, x), r^2(t, x)$  and  $p^2(t, x)$  are arbitrary smooth functions at the moment. Taking into account (37), equations (12) and (14) from the system of DEs, are simplified:

$$[d_{21} - d_{11}]q^2(t, x) = 0, \quad (38)$$

$$[d_{12} - d_{22}]q^1(t, x) = 0. \quad (39)$$

Finally, two different cases should be examined

- 1.**  $d_{22} \neq d_{12}$ .      **2.**  $d_{22} = d_{12}, d_{21} = d_{11}$ .

The third possible case  $d_{21} \neq d_{11}$  is reduced to the first case by the substitution  $u \rightarrow v, v \rightarrow u$  and the relevant renaming.

We do not present further calculations because it is rather a standard routine to find the coefficients of the infinitesimal operator (8) provided they have the form (37). The detailed analysis of case **1** shows that the systems and MAIs listed in cases **3—7, 10—12** and **15** of Table 1 are obtained, while the same routine for case **2** produces the systems and MAIs listed in cases **1, 2, 8—9, 13—14** and **16**.

Finally, we note that each SKT systems listed in Table 1 is a representative of some other systems, which are reduced to one by the point (local) substitutions indicated in the last column of Table 1. The explicit forms of these substitutions are

1.  $u^* = v, \quad v^* = u.$
2.  $u^* = e_1 u, \quad v^* = e_2 v.$
3.  $u^* = u + \frac{d_1}{2d_{11}}, \quad v^* = v.$
4.  $u^* = u, \quad v^* = v + \frac{d_1}{d_{12}}.$
5.  $u^* = u + \frac{d_1 - d_2}{d_{11}}, \quad v^* = v.$
6.  $u^* = u, \quad v^* = v + \frac{d_2 - d_1}{d_{12}}.$
7.  $u^* = u, \quad v^* = bu + cv.$
8.  $u^* = u, \quad v^* = d_{11}u + d_{12}v.$
9.  $u^* = bu + cv, \quad v^* = d_{11}u + d_{12}v.$
10.  $t^* = \frac{1}{a}e^{at}, \quad u^* = e^{-at}u, \quad v^* = e^{-at}v.$
11.  $u^* = ue^{-a_1 t}, v^* = v.$
12.  $u^* = u, v^* = ve^{-a_2 t}.$

It can be easily seen that all the substitutions listed above can be united in the form (5) (of course, constants  $\alpha_{ij}$  must be correctly-specified by the system in question).

The sketch of the proof is now completed.  $\square$

**Remark 1** *In cases 5, 9, 10, 11, 13—16 of Table 1, the term  $a_1 u$  are removable by the substitution 11 from (40). However, we keep this term because one has a clear biological interpretation*



Table 1: MAIs of the SKT system (3)

	Systems	Restrictions	Basic operators of MAI	Substitution from (40)
1.	$u_t = [(u + \frac{c}{b}v)u]_{xx} + u(a - bu - cv)$ $v_t = [(u + \frac{c}{b}v)v]_{xx} + v(2a - bu - cv)$	$abc \neq 0$	$P_t, P_x, Q_1$	1, 2
2.	$u_t = [(1 + \frac{b}{a}u + \frac{c}{a}v)u]_{xx} + u(a - bu - cv)$ $v_t = [(2 + \frac{b}{a}u + \frac{c}{a}v)v]_{xx} - v(a + bu + cv)$	$abc \neq 0$	$P_t, P_x, Q_2$	1, 2
3.	$u_t = [(d_{11}u + d_{12}v)u]_{xx} - u(b_1u + c_1v)$ $v_t = [(d_{21}u + d_{22}v)v]_{xx} - v(b_2u + c_2v)$	$b_1^2 + c_1^2 + b_2^2 + c_2^2 \neq 0$	$P_t, P_x, D_1$	3, 4, 10
4.	$u_t = [(d_{11}u + d_{12}v)u]_{xx} + a_1u$ $v_t = [(d_{21}u + d_{22}v)v]_{xx} + a_2v$	$a_1 \neq a_2$	$P_t, P_x, D_2$	3, 4
5.	$u_t = [(d_1 + v)u]_{xx} + u(a_1 - c_1v)$ $v_t = [(d_2 + d_{22}v)v]_{xx} + v(a_2 - c_2v)$	$d_1^2 + d_2^2 \neq 0$	$P_t, P_x, u\partial_u$	1, 2
6.	$u_t = [(d_1 + v)u]_{xx} - b_1u^2$ $v_t = [(d_2 + d_{22}v)v]_{xx} - b_2uv$	$d_1^2 + d_2^2 \neq 0$	$P_t, P_x, D_3$	1, 2, 9
7.	$u_t = d_{12}[uv]_{xx} + u(a_1 - b_1u)$ $v_t = [v^2]_{xx} + v(a_2 - b_2u)$		$P_t, P_x, D_4$	1, 2, 3, 4
8.	$u_t = [(u + d_{12}v)u]_{xx} + au$ $v_t = [(u + d_{12}v)v]_{xx} + 2av$	$a \neq 0$	$P_t, P_x, D_2, Q_3$	1, 2
9.	$u_t = [(1 + v)u]_{xx} + u(a_1 - cv)$ $v_t = [(1 + v)v]_{xx} + v(a_2 - cv)$		$P_t, P_x, u\partial_u, Q_4$	1, 2, 5, 6, 8
10.	$u_t = d_{12}[uv]_{xx} + u(a_1 - c_1v)$ $v_t = [v^2]_{xx} + v(a_2 - c_2v)$	$a_2 \neq 0$ $c_1 \neq c_2$	$P_t, P_x, u\partial_u, Q_5$	1, 2, 3, 4
11.	$u_t = d_{12}[uv]_{xx} + u(a_1 - c_1v)$ $v_t = [v^2]_{xx} - c_2v^2$	$c_1 \neq c_2$	$P_t, P_x, u\partial_u, D_5$	1, 2, 3, 4
12.	$u_t = d_{12}[uv]_{xx} - b_1u^2$ $v_t = [v^2]_{xx} - b_2uv$		$P_t, P_x, D_1, D_4$	1, 2, 3, 4, 9, 10
13.	$u_t = [uv]_{xx} + u(a_1 - cv)$ $v_t = [v^2]_{xx} + v(a_2 - cv)$	$a_2c \neq 0$	$P_t, P_x, u\partial_u, Q_4, Q_5$	1, 2, 7
14.	$u_t = [uv]_{xx} + u(a_1 - cv)$ $v_t = [v^2]_{xx} - cv^2$	$c \neq 0$	$P_t, P_x, u\partial_u, D_5, Q_6$	1, 2, 7
15.	$u_t = d_{12}[uv]_{xx} + a_1u$ $v_t = [v^2]_{xx} + a_2v$	$a_2 \neq 0$ $d_{12} \neq 1$	$P_t, P_x, u\partial_u, D_4, Q_5$	1, 2
16.	$u_t = [uv]_{xx} + a_1u$ $v_t = [v^2]_{xx} + a_2v$	$a_2 \neq 0$	$P_t, P_x, u\partial_u, D_4, Q_4, Q_5$	1, 2, 8

(the rate of the birth (or mortality) for species/cells). We also note that some parameters in the systems listed in Table 1 can be reduced to  $\pm 1$  by scaling (see substitutions 2 and 10 in (40)).

**Remark 2** In Tables 1–3, the following designations for the Lie symmetry operators are introduced:

$$\begin{aligned}
D_0 &= 2t\partial_t + x\partial_x; \\
D_1 &= t\partial_t - (u\partial_u + v\partial_v); \\
D_2 &= x\partial_x + 2(u\partial_u + v\partial_v); \\
D_3 &= 2t\partial_t + x\partial_x - 2u\partial_u; \\
D_4 &= x\partial_x + 2v\partial_v; \\
D_5 &= t\partial_t + a_1tu\partial_u - v\partial_v; \\
D_6 &= t\partial_t - u\partial_u; \\
Q_1 &= e^{-at}(\partial_t + a(u - \frac{c}{b}v)\partial_u + 2av\partial_v); \\
Q_2 &= e^{-at}(\partial_t + 2au\partial_u + a(\frac{a}{c} - \frac{b}{c}u + v)\partial_v); \\
Q_3 &= e^{-at}(\partial_t + a(u - d_{12}v)\partial_u + 2av\partial_v); \\
Q_4 &= e^{(a_1-a_2)t}v\partial_u; \\
Q_5 &= e^{-a_2t}(\partial_t + a_1u\partial_u + a_2v\partial_v); \\
Q_6 &= e^{a_1t}v\partial_u; \\
R &= t\partial_t + \left(\frac{5d_2-4d_1}{3(d_1-d_2)}u + \frac{2d_1-d_2}{3(d_1-d_2)}v - \frac{2d_1-d_2}{3}\right)\partial_u + \left(\frac{4d_2-5d_1}{3(d_1-d_2)}v + \frac{d_1-2d_2}{3(d_1-d_2)}u + \frac{d_1-2d_2}{3}\right)\partial_v; \\
Z_1 &= \frac{e^x}{u-v}(\partial_u - \partial_v), \quad Z_2 = \frac{e^{-x}}{u-v}(\partial_u - \partial_v); \\
Z_3 &= \frac{\cos x}{u-v}(\partial_u - \partial_v), \quad Z_4 = \frac{\sin x}{u-v}(\partial_u - \partial_v); \\
Z_5 &= \frac{x}{u-v}(\partial_u - \partial_v), \quad Z_6 = \frac{1}{u-v}(\partial_u - \partial_v).
\end{aligned}$$

Now we examine system (3) under the condition  $d_1 = d_2 = d_{11} = d_{22} = 0$ , which reduces the system to the form

$$\begin{aligned}
u_t &= d_{12}[uv]_{xx} + u(a_1 - b_1u - c_1v), \\
v_t &= d_{21}[uv]_{xx} + v(a_2 - b_2u - c_2v).
\end{aligned} \tag{41}$$

Assuming  $d_{12}d_{21} \neq 0$ , we transform (41) to the form

$$\begin{aligned}
u_t &= [uv]_{xx} + u(a_1 - b_1u - c_1v), \\
v_t &= [uv]_{xx} + v(a_2 - b_2u - c_2v)
\end{aligned} \tag{42}$$

by the simple transformation

$$u^* = d_{21}u, \quad v^* = d_{12}v \tag{43}$$

(in system (42), stars next to  $u$  and  $v$  are skipped). In the case of the coefficients arising in system (42) the system of DEs (11)–(27) takes an essentially different form. As a result, equation (36) is not obtainable. An analysis of the system of DEs is omitted here. The final result can be formulated as follows.

Table 2: MAIs of system (42)

	Systems	Restrictions	Basic operators of MAI	Substitution from (44)
1.	$u_t = [uv]_{xx} - u(b_1u + c_1v)$ $v_t = [uv]_{xx} - v(b_2u + c_2v)$	$b_1^2 + c_2^2 \neq 0$	$P_t, P_x, D_1$	1
2.	$u_t = [uv]_{xx} + a_1u$ $v_t = [uv]_{xx} + a_2v$	$a_1 \neq a_2$	$P_t, P_x, D_2$	
3.	$u_t = [uv]_{xx} - uv$ $v_t = [uv]_{xx} - uv$		$P_t, P_x, D_1,$ $Z_1, Z_2$	1, 2
4.	$u_t = [uv]_{xx} + uv$ $v_t = [uv]_{xx} + uv$		$P_t, P_x, D_1,$ $Z_3, Z_4$	1, 2

**Theorem 2** *All possible MAIs (up to representations generated by transformations of the form (44)) of system (42) with non-vanish reaction term(s) are presented in Table 2. Any other system of the form (42), which is invariant w.r.t. the three- and higher-dimensional Lie algebra is reduced by a substitution of the form*

1.  $t^* = \frac{1}{a}e^{at}, \quad u^* = e^{-at}u, \quad v^* = e^{-at}v.$
2.  $t^* = bt, \quad x^* = \sqrt{b}x, \quad b > 0$

(44)

( $a$  and  $b$  are arbitrary non-zero constants) to one of those given in Table 2.

Finally, we examine system (3) without reaction terms, i.e., the cross-diffusion system

$$\begin{aligned} u_t &= [(d_1 + d_{11}u + d_{12}v)u]_{xx}, \\ v_t &= [(d_2 + d_{21}u + d_{22}v)v]_{xx}. \end{aligned}$$
(45)

It is worth to note that system (45) with arbitrary coefficients is invariant under the three-dimensional trivial algebra spanned by the basic operators  $P_t, P_x$  and  $D_0$ . All possible extensions of this trivial algebra admitting by (45) are presented in the following statement.

**Theorem 3** *All possible MAIs (up to equivalent representations generated by transformations of the form (46)) of system (45) are presented in Table 3. Any other system of the form (45) which is invariant w.r.t. the four- and higher-dimensional Lie algebra is reduced by a substitution of the form*

$$t^* = \alpha_1 t, \quad u^* = \alpha_2 + \alpha_3 u + \alpha_4 v, \quad v^* = \alpha_5 + \alpha_6 u + \alpha_7 v$$
(46)

to one of those given in Table 3 (constants  $\alpha_i$  are determined by the system in question).

Table 3: MAIs of system (45)

	Systems	Restrictions	Basic operators of MAI	Substitution from (47)
1.	$u_t = [(d_{11}u + d_{12}v)u]_{xx}$ $v_t = [(d_{21}u + d_{22}v)v]_{xx}$	$(d_{11} - d_{21})^2 + (d_{12} - d_{22})^2 \neq 0$ $d_{11}^2 + d_{21}^2 \neq 0$ $d_{12}^2 + d_{22}^2 \neq 0$	$P_t, P_x, D_0, D_1$	1, 2
2.	$u_t = [(d_1 + d_{11}u)u]_{xx}$ $v_t = [(d_2 + u)v]_{xx}$	$d_{11} \neq 1$ $d_1 \neq 2d_2d_{11}$	$P_t, P_x, D_0, v\partial_v$	1, 3
3.	$u_t = [(d_1 + u + v)u]_{xx}$ $v_t = [(d_2 + u + v)v]_{xx}$	$d_1 \neq d_2$	$P_t, P_x, D_0, R$	4
4.	$u_t = d_{11}[u^2]_{xx}$ $v_t = [uv]_{xx}$	$d_{11} \neq 1$	$P_t, P_x, D_0$ $v\partial_v, D_6$	1, 7
5.	$u_t = [(1 + u)u]_{xx}$ $v_t = [(1 + u)v]_{xx}$		$P_t, P_x, D_0,$ $v\partial_v, u\partial_v$	1, 5, 6
6.	$u_t = [u^2]_{xx}$ $v_t = [uv]_{xx}$		$P_t, P_x, D_0, D_6,$ $v\partial_v, u\partial_v$	1, 5, 7
7.	$u_t = [uv]_{xx}$ $v_t = [uv]_{xx}$		$P_t, P_x, D_0,$ $D_1, Z_5, Z_6$	4

In Table 3, the following substitutions are used:

1.  $u^* = v, \quad v^* = u;$
2.  $u^* = u, \quad v^* = v + \frac{d_1}{d_{12}};$
3.  $u^* = d_{21}u, \quad v^* = v;$
4.  $u^* = d_{11}u, \quad v^* = d_{12}v;$
5.  $t^* = d_1t, \quad d_1u^* = d_{11}u + d_{12}v, \quad v^* = v;$
6.  $t^* = (2d_2 - d_1)t, \quad (2d_2 - d_1)u^* = d_{11}u + d_1 - d_2, \quad v^* = v;$
7.  $u^* = d_1 + d_{21}u, \quad v^* = v.$

It should be stressed that MAIs of the nonlinear systems

$$\begin{aligned} u_t &= [uv]_{xx} + buv, \\ v_t &= [uv]_{xx} + buv \end{aligned} \tag{48}$$

( $b = \pm 1$  in the cases 3 and 4 of Table 2 and  $b = 0$  in the case 7 of Table 3) contain the Lie symmetry operators, which are **nonlinear** w.r.t. to the dependent variables  $u$  and  $v$ . It is new property of RD systems with cross-diffusion, which not occurs for the standard RD systems (see Lie symmetries in [21, 30, 31] and papers cited therein).

**Remark 3** *System (48) can be simplified as follows*

$$\begin{aligned} u_t &= [(u - w)u]_{xx} + bu(u - w), \\ w_t &= 0 \end{aligned} \tag{49}$$

by the transformation  $w = u - v$ . The symmetry operators mentioned above will be transformed to those, which are again nonlinear w.r.t. the depended variable(s).

### 3 Example of exact solutions

Here we consider the systems

$$\begin{aligned} u_t &= [uv]_{xx} - uv, \\ v_t &= [uv]_{xx} - uv \end{aligned} \tag{50}$$

and

$$\begin{aligned} u_t &= [uv]_{xx} + uv, \\ v_t &= [uv]_{xx} + uv \end{aligned} \tag{51}$$

from cases 3 and 4 of Table 2, which admit the most nontrivial algebras of invariance and are important subcases (up to local transformations) of the SKT system (3).

Let us construct exact solutions of the nonlinear systems (50) and (51) using the Lie symmetries  $X_1 = \lambda_1 Z_1 + \lambda_2 Z_2$  and  $X_2 = \lambda_1 Z_3 + \lambda_2 Z_4$ , respectively (hereafter  $\lambda_1$  and  $\lambda_2$  are arbitrary constants,  $\lambda_1^2 + \lambda_2^2 \neq 0$ ).

It is well-known that Lie symmetry operators with a complicated structure can be used for finding nontrivial solutions from the very simple ones (probably the first examples for RD systems were presented in [32], see also [30]). Here we show how it can be realized for the nonlinear system (50) (exact solutions of (51) can be obtained in a quite similar way).

Operator  $X_1$  generates the one-parameter Lie group:

$$\begin{aligned} u^* &= \frac{u+v}{2} + \frac{1}{2}\sqrt{(u-v)^2 + 4p(\lambda_1 e^x + \lambda_2 e^{-x})}, \\ v^* &= \frac{u+v}{2} - \frac{1}{2}\sqrt{(u-v)^2 + 4p(\lambda_1 e^x + \lambda_2 e^{-x})}, \end{aligned} \quad (52)$$

when  $u \geq v$ ;

$$\begin{aligned} u^* &= \frac{u+v}{2} - \frac{1}{2}\sqrt{(u-v)^2 + 4p(\lambda_1 e^x + \lambda_2 e^{-x})}, \\ v^* &= \frac{u+v}{2} + \frac{1}{2}\sqrt{(u-v)^2 + 4p(\lambda_1 e^x + \lambda_2 e^{-x})}, \end{aligned} \quad (53)$$

when  $u < v$  (here  $p$  is an arbitrary parameter).

In order to construct a nontrivial solution  $(u^*, v^*)$  of system (50), one needs to know a simple exact solution, which can be easily derived by setting  $u_x = v_x = 0$ . It means that the ODE system

$$\begin{aligned} u_t &= -uv, \\ v_t &= -uv \end{aligned}$$

is obtained instead of (50) and its general solution is

$$\begin{aligned} u &= \frac{\alpha_1 e^{\alpha_1 t}}{\alpha_2 + e^{\alpha_1 t}}, \\ v &= -\frac{\alpha_1 \alpha_2}{\alpha_2 + e^{\alpha_1 t}}, \end{aligned} \quad (54)$$

where  $\alpha_1$  and  $\alpha_2$  are arbitrary constants. Thus, solution (54) of system (50) can be generalized via transformations (52)–(53) to the five-parameter family of solutions

$$\begin{aligned} u(t, x) &= \frac{\alpha_1 e^{\alpha_1 t} - \alpha_1 \alpha_2}{2(\alpha_2 + e^{\alpha_1 t})} \pm \frac{1}{2}\sqrt{\alpha_1^2 + 4p(\lambda_1 e^x + \lambda_2 e^{-x})}, \\ v(t, x) &= \frac{\alpha_1 e^{\alpha_1 t} - \alpha_1 \alpha_2}{2(\alpha_2 + e^{\alpha_1 t})} \mp \frac{1}{2}\sqrt{\alpha_1^2 + 4p(\lambda_1 e^x + \lambda_2 e^{-x})}. \end{aligned} \quad (55)$$

Using the same algorithm, the following family of exact solutions of system (51) was derived:

$$\begin{aligned} u(t, x) &= \frac{\alpha_1 + \alpha_1 \alpha_2 e^{\alpha_1 t}}{2(1 - \alpha_2 e^{\alpha_1 t})} \pm \frac{1}{2}\sqrt{\alpha_1^2 + 4p(\lambda_1 \cos x + \lambda_2 \sin x)}, \\ v(t, x) &= \frac{\alpha_1 + \alpha_1 \alpha_2 e^{\alpha_1 t}}{2(1 - \alpha_2 e^{\alpha_1 t})} \mp \frac{1}{2}\sqrt{\alpha_1^2 + 4p(\lambda_1 \cos x + \lambda_2 \sin x)}. \end{aligned} \quad (56)$$

Notably, one may set  $\alpha_2 = \pm 1$  in the solutions (55)–(56) without losing a generality while the case  $\alpha_2 = 0$  leads to steady-state solutions.

**Remark 4** All possible steady-state solutions of the nonlinear systems (50) and (51) can be easily derived. As a result one obtains

$$\begin{aligned} u(x) &= \frac{f(x)}{g(x)}, & v(x) &= g(x) \neq 0, \\ u(x) &= h(x), & v(x) &= 0, \\ u(x) &= 0, & v(x) &= h(x), \end{aligned} \tag{57}$$

where  $g(x)$  and  $h(x)$  are arbitrary smooth functions, while the function  $f(x)$  is the general solution of the linear ODE  $f'' \mp f = 0$ . Obviously, transformations (52)–(53) do not generate new solutions from (57).

It should be emphasized that each solution of the form (56) satisfy the zero flux conditions on a correctly-specified space interval. For example, the exact solution with  $\lambda_2 = 0$  satisfy the zero Neumann conditions

$$u_x|_{x=0} = 0, \quad v_x|_{x=0} = 0, \quad u_x|_{x=\pi} = 0, \quad v_x|_{x=\pi} = 0 \tag{58}$$

at the interval  $(0, \pi)$ . This property is important for possible applications because zero flux at boundary is a typical requirement for biologically motivated models. It should be also noted that systems (50) and (51) are two canonical forms of a more general system. In fact, the substitution

$$t^* = \frac{\ln t}{ab}, \quad x^* = \frac{x}{\sqrt{b}} \quad (a \neq 0, b > 0), \quad u^* = \frac{atu}{d_2}, \quad v^* = \frac{atv}{d_1}$$

transforms systems (50) and (51) to the form

$$\begin{aligned} u_{t^*}^* &= d_1[u^*v^*]_{x^*x^*} + u^*(ab + b_1v^*), \\ v_{t^*}^* &= d_2[u^*v^*]_{x^*x^*} + v^*(ab + b_2u^*), \end{aligned} \tag{59}$$

where  $b_1 = \pm bd_1$ ,  $b_2 = \pm bd_2$ . Now one realizes that system (59) involves the logistic type terms, which are very common in the mathematical biology models.

Finally, we present an example for deriving exact solutions via the most common procedure, which is often called the Lie symmetry reduction. Let us take a linear combination of the Lie symmetries  $P_x$ ,  $Z_3$  and  $Z_4$ , i.e.,

$$X = \partial_x + \frac{\lambda_1 \cos x + \lambda_2 \sin x}{u - v} (\partial_u - \partial_v). \tag{60}$$

Obviously, (60) is again a Lie symmetry, which produces the ansatz

$$\begin{aligned} u &= \frac{\varphi_1(t) \pm \sqrt{\varphi_1^2(t) + 4\varphi_2(t) + 4(\lambda_1 \sin x - \lambda_2 \cos x)}}{2}, \\ v &= \frac{\varphi_1(t) \mp \sqrt{\varphi_1^2(t) + 4\varphi_2(t) + 4(\lambda_1 \sin x - \lambda_2 \cos x)}}{2}, \end{aligned} \tag{61}$$

where  $\varphi_1(t)$  and  $\varphi_2(t)$  are new unknown functions. Formally speaking, one should take the upper signs if  $u \geq v$ , otherwise the lower signs should be used, however it is not essential because system (51) is invariant under the discrete transformation  $u \rightarrow v$ ,  $v \rightarrow u$ .

Ansatz (61) reduces the nonlinear system (51) to the ODE system

$$\begin{aligned}\varphi_1' + 2\varphi_2 &= 0, \\ \varphi_1\varphi_1' + 2\varphi_2' &= 0.\end{aligned}\tag{62}$$

In contrast to (51), system (62) is integrable because is equivalent to the system

$$\begin{aligned}\varphi_2 &= -\frac{1}{2}\varphi_1', \\ \varphi_1' &= \frac{1}{2}\varphi_1^2 + \beta,\end{aligned}\tag{63}$$

in which the general solution of the second equation is well-known. Thus, having the general solution of the reduced system (63) and using ansatz (61), three different solutions (depending on the sign of the constant  $\beta$ ) of system (51) were found:

$$\begin{aligned}u &= -\frac{1}{t} \pm \sqrt{\lambda_1 \sin x - \lambda_2 \cos x}, \\ v &= -\frac{1}{t} \mp \sqrt{\lambda_1 \sin x - \lambda_2 \cos x}, \\ u &= \alpha_1 \tan(\alpha_1 t) \pm \sqrt{\lambda_1 \sin x - \lambda_2 \cos x - \alpha_1^2}, \\ v &= \alpha_1 \tan(\alpha_1 t) \mp \sqrt{\lambda_1 \sin x - \lambda_2 \cos x - \alpha_1^2}, \\ u &= \alpha_1 \frac{1+\alpha_2 e^{2\alpha_1 t}}{1-\alpha_2 e^{2\alpha_1 t}} \pm \sqrt{\lambda_1 \sin x - \lambda_2 \cos x + \alpha_1^2}, \\ v &= \alpha_1 \frac{1+\alpha_2 e^{2\alpha_1 t}}{1-\alpha_2 e^{2\alpha_1 t}} \mp \sqrt{\lambda_1 \sin x - \lambda_2 \cos x + \alpha_1^2},\end{aligned}\tag{64}$$

where  $\alpha_1 = \pm\sqrt{\frac{|\beta|}{2}}$  and  $\alpha_2$  is an arbitrary constant. Notably, the last solution from (64) is a particular case of (56).

## 4 Conclusions

In this paper, the Lie symmetry classification problem of the Shigesada–Kawasaki–Teramoto system is completely solved. Solution of this problem was initiated in [13], however, the result derived therein is not complete because essential restriction on coefficients were applied, as a result all the symmetries derived in [13] can be extracted from Table 1. Here it is proved that the SKT system (3) admits a wide range of Lie symmetries depending on 12 coefficients arising in the system (see Tables 1–3). From the applicability point of view, the most interesting systems occur in Tables 1 and 2. For example, the systems listed in cases 1–4 have a quite general structure and one may expect that some SKT models with the correctly-specified coefficients (which are chosen from experimental data) are equivalent to these systems. One may note that several systems in Table 1 (cases 5–7, 9–16) contain the cross-diffusion term in the first equation



only. Such systems occurs when one of the cross-diffusion coefficients is much larger than the other (see the pioneering paper [9] for detail). They are called triangular and are extensively studied during the last decade (see [10] and references therein).

From the Lie symmetry point of view the most interesting systems occur in Tables 2 and 3. In particular, the Lie symmetry operators with highly unusual structure are unveiled for the nonlinear systems listed in cases 3–4 of Table 2 and in case 7 of Table 3. In fact, operators  $Z_1, \dots, Z_6$  are **nonlinear w.r.t. the dependent variables  $u$  and  $v$**  because they involve coefficients of the form  $\frac{f(x)}{u-v}$  ( $f(x)$  is a correctly-specified function). To the best of our knowledge, this is the first time when nonlinear Lie symmetry operators are found for RD systems. In the case of RD systems without cross-diffusion, all possible Lie symmetry operators are known (see [21] and references therein) and they are always linear w.r.t. the dependent variables. In the case of RD systems involving cross-diffusion, there is no a complete description of all possible Lie symmetry operators at the present time. However, the results obtained in [33] (the case of constant cross-diffusion), [34] (power-law coefficients of cross-diffusion), [35] (diffusion and cross-diffusion in the first equation and no any diffusion in the second) and [36] (Galilei-invariant systems with cross-diffusion) show that all Lie symmetries of RD systems found therein are linear w.r.t. to unknown functions. We foresee that new nonlinear Lie symmetry operators will be found for suitable generalizations of the SKT system (3). Notably, nonlinear Lie symmetry operators do not occur in the case of any single RD equation [37], however it was recently established that the RD equation with a correctly-specified gradient-dependent diffusivity and an arbitrary reaction term admits such operators and is linearizable [38] (see Theorem 1 therein).

Finally, the Lie symmetry classification is applied for finding exact solutions of the nonlinear systems, which are invariant under the operators mentioned above. Our purpose was to show how highly non-trivial Lie symmetries generate exact solutions, which may be useful in applications. In particular, we have shown that some exact solutions satisfy zero flux boundary conditions, which are typical requirements for solutions of biologically motivated models.

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