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Por

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Abstract

In these notes we show how to solve a large family of Linear Rational-Expectations Models using the (Generalized) Schur Decomposition. The solution method closely follows the one described by Klein (2000). After developing the general method, we use it to solve a standard macroeconomic model. We include a set of appendices in order to offer a self-contained exposition.

Resumen

En estas notas mostramos cómo resolver una importante familia de Modelos Lineales con Expectativas Racionales utilizando la Descomposición (Generalizada) de Schur. El método de solución sigue de cerca el descripto por Klein (2000). Luego de desarrollar el método general, lo utilizamos para resolver un modelo macroeconómico estándar. Incluimos un conjunto de apéndices con el objeto de ofrecer una exposición autocontenida.

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Contents

1 A Basic Linear Rational-Expectations Model **1**

1.1 The Model 1
1.2 A Simple Example: The Cagan Model 2
1.3 Some Remarks 2

2 Finding the solution to the Basic Model **3**

2.1 Solving the Model using the Schur Decomposition 3
2.2 Important Results 7
2.3 A Simple Example: Solving the Cagan Model 8
2.4 Impulse-Response Functions 10
2.5 Second Moments 10

3 A More General Model **13**

3.1 The Model and its Solution 13
3.2 A Simple Example 16

4 A Rational-Expectations Macroeconomic Model **18**

4.1 The Model 18
4.2 Log-linearization 20
4.3 Parametrization and Solution 22

5 Appendices **26**

6 References **42**

1 A Basic Linear Rational-Expectations Model

1.1 The Model

Consider the following model:

$$x_{t+1} = A_{xx}x_t + A_{xy}y_t + \varepsilon_{t+1} \tag{1}$$
$$\mathbb{E}_t y_{t+1} = A_{yx}x_t + A_{yy}y_t \qquad x_0 \text{ given} \tag{2}$$

where $t \in \{0, 1, 2, ...\}$ is the time period, x_t is an $n_x \times 1$ vector of *predetermined* variables, y_t is an $n_y \times 1$ vector of *nonpredetermined* variables, ε_{t+1} is an $n_x \times 1$ vector of exogenous *i.i.d.* shocks with zero means and constant variance-covariance matrix Σ_ε, and A_{xx}, A_{xy}, A_{yx} and A_{yy} are known matrices of dimensions $n_x \times n_x$, $n_x \times n_y$, $n_y \times n_x$ and $n_y \times n_y$, respectively.[1] We use $\mathbb{E}_t$ to denote the conditional expectation $\mathbb{E}(\cdot \mid \Omega_t)$, where Ω_t is the information set at time t. The information sets satisfy $\Omega_{t-1} \subseteq \Omega_t$, and Ω_t includes at least current and past values of x_t and y_t. Notice that there is an initial condition for predetermined variables but not for nonpredetermined ones.

Following Klein (2000), we call a variable predetermined, or *backward-looking*, if: *i*) its one-period-ahead forecast error is exogenous; and *ii*) its initial value is exogenously given.[2] From (1) we see that the one-period-ahead forecast error of x_t is $x_{t+1} - \mathbb{E}_t x_{t+1} = \varepsilon_{t+1}$, which is exogenous by assumption, and x_0 is exogenously given. In other words, the value in $t+1$ of a variable that is predetermined at t is a function only of variables known at time t, plus the impact of an exogenous shock that becomes known in $t+1$. The vector x_t may contain exogenous variables, like a serially correlated productivity level. Nonpredetermined variables, also called *forward-looking* or *jump* variables, differ from predetermined variables in that their one-period-ahead forecast errors and their initial values are endogenous. Unlike predetermined variables, the value taken by a nonpredetermined variable in $t+1$ can be affected by the realization of other endogenous variables in $t+1$.

We can combine equations (1) and (2) as follows:

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix} \qquad x_0 \text{ given} \tag{3}$$

where A is the square matrix of dimension $n \equiv n_x + n_y$ defined by:

$$A \equiv \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix}. \tag{4}$$

[1] The law of motion for x_t is sometimes written as $x_{t+1} = A_{xx}x_t + A_{xy}y_t + \Gamma\eta_{t+1}$, where Γ is a known $n_x \times n_x$ matrix, and η_{t+1} is an $n_x \times 1$ *i.i.d.* random vector with mean zero and covariance matrix $\mathbb{E}\{\eta_{t+1}\eta'_{t+1}\} = I$ (the $n_x \times n_x$ identity matrix). This is equivalent to defining $\varepsilon_{t+1} \equiv \Gamma\eta_{t+1}$ in (1). Then: $\mathbb{E}\varepsilon_{t+1} \equiv \mathbb{E}\{\Gamma\eta_{t+1}\} = \Gamma\mathbb{E}\{\eta_{t+1}\} = 0$, and$\Sigma_\varepsilon \equiv \mathbb{E}\{[\varepsilon_{t+1} - \mathbb{E}\varepsilon_{t+1}][\varepsilon_{t+1} - \mathbb{E}\varepsilon_{t+1}]'\} = \mathbb{E}\{\varepsilon_{t+1}\varepsilon'_{t+1}\} = \mathbb{E}\{\Gamma\eta_{t+1}\eta'_{t+1}\Gamma'\} = \Gamma\mathbb{E}\{\eta_{t+1}\eta'_{t+1}\}\Gamma' = \Gamma I\Gamma' = \Gamma\Gamma'$.

[2] This is a generalization of the definition given by Blanchard and Kahn (1980), who define a predetermined variable as one that has a given initial condition and satisfies $x_{t+1} = \mathbb{E}_t x_{t+1}$ (*i.e.*, $\varepsilon_{t+1} = x_{t+1} - \mathbb{E}_t x_{t+1} = 0\ \forall t$).

1.2 A Simple Example: The Cagan Model

Suppose the real demand for money is given by:[3]

$$m_t^d - p_t = -a(\mathbb{E}_t p_{t+1} - p_t) \tag{5}$$

where $p_t \equiv \ln P_t$, $m_t^d \equiv \ln M_t^d$, P_t is the price level in period t, M_t^d is the nominal money demand at the end of period t, and $a > 0$. That is, real money demand depends negatively on expected inflation.[4]

Denote the money supply by M. From (5), and imposing the equilibrium condition $M_t = M_t^d$, we obtain:

$$p_t = \alpha \mathbb{E}_t p_{t+1} + (1 - \alpha) m_t \tag{6}$$

where $m_t \equiv \ln M_t$, and $\alpha \equiv \frac{a}{1+a} \in (0, 1)$. Equation (6) shows that the (natural log of the) price level in period t is a weighted average of the expected (natural log of the) price level in period $t + 1$ and the (natural log of the) money supply in period t.

Money supply is exogenously determined by the monetary authority. Suppose the natural log of money supply follows an AR(1) process:

$$m_{t+1} = \rho m_t + \varepsilon_{t+1} \tag{7}$$

where ε_{t+1} is white noise and $|\rho| < 1$.

Equations (6) and (7) form a rational-expectations version of the Cagan Model. These equations can be written in form (3) as follows:

$$\begin{bmatrix} m_{t+1} \\ \mathbb{E}_t p_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\frac{1-\alpha}{\alpha} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} m_t \\ p_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \qquad m_0 \text{ given} \tag{8}$$

where m_t is predetermined and p_t is nonpredetermined. In this model, the unique predetermined variable is exogenous. In more general models, there can also be endogenous predetermined variables.

1.3 Some Remarks

A couple of remarks are in order.

Remark 1. The first-order form (3) may seem more restrictive than it really is. Higher-order models with lagged variables or current expectations of variables more than one period ahead can be reduced to first-order form (see Appendix

[3] This example closely follows the one in Söderlind (2001).

[4] This specification is obtained as a simplification of a money demand function of the form $\frac{M_t^d}{P_t} = L(\overset{+}{Y_t}, \overset{-}{i_t})$, where Y is real output and i is the nominal interest rate. From Fisher's parity condition we have: $i_t \cong r_t + (E_t p_{t+1} - p_t)$, where r is the real interest rate. Substituting this expression into the money demand equation, and assuming that Y and r are constant, we can write real money demand as a (negative) function of expected inflation.

A for an example). Models with lagged expectations of present and future variables can also be put in form (3) (see Appendix A for an example).[5]

Remark 2. One may wonder why we did not include constants C_x and C_y in equations (1) and (2), respectively. Suppose we did. Then, as long as the matrix $I - A$ is invertible, we could eliminate the constants by rewriting all variables as deviations from their nonstochastic steady-state values (see Appendix B).

2 Finding the solution to the Basic Model

2.1 Solving the Model using the Schur Decomposition

A solution to our problem is a sequence $\{x_t, y_t\}_{t=0}^{\infty}$ of functions of variables in Ω_t that satisfies (3) for all possible realizations of these variables. We are interested in *nonexplosive solutions* to (3), so we impose the following boundary conditions:

$$\lim_{i\to\infty} |\mathbb{E}_t x_{t+i}| < \infty, \quad \lim_{i\to\infty} |\mathbb{E}_t y_{t+i}| < \infty \tag{9}$$

Actually, we look for a *recursive representation* of the solution to (3). That is, we want matrices M and C so that the solution satisfies:

$$\begin{aligned} x_{t+1} &= Mx_t + \varepsilon_{t+1} && (10)\\ y_t &= Cx_t && (11) \end{aligned}$$

The solution method presented below follows closely the one described in Klein (2000). The idea is to use the Schur Decomposition to reduce the original system to a block-triangular system (with two blocks). Then we solve the new system recursively, solving the second block first and then using this solution to solve the first block.

Define the $n \times 1$ vector:

$$w_t \equiv \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{12}$$

Taking the conditional expectation of (3), and using (12), we can write:

$$\mathbb{E}_t w_{t+1} = Aw_t \tag{13}$$

Now we triangularize A using the (Complex) Schur Decomposition (see Appendix C).[6] That is, we find a complex unitary $n \times n$ matrix Z and a complex upper triangular $n \times n$ matrix T such that $A = ZTZ^H$, where Z^H is the conjugate transpose of Z (and then $Z^H Z = ZZ^H = I$, where I is the identity

[5] Binder and Pesaran (1995) show how to reduce a linear system of expectational difference equations with arbitrary leads and lags, and expectations taken with respect to information available at different times, to a second-order canonical form. Klein (2000) shows how to convert this second-order form into first-order form.

[6] Blanchard and Kahn (1980) decouple the system by means of the Jordan canonical form of the matrix A. From a computational point of view the Schur Decomposition is better since the Jordan decomposition tends to be numerically unstable. See Klein (2000), page 1406.

matrix). This triangularization has the property that the diagonal elements of T correspond to the eigenvalues of A.[7] Moreover, it is possible to reorder both T and Z such that the n_θ eigenvalues with modulus smaller than one come first, and the n_δ eigenvalues with modulus higher than one come last (where $n_\theta + n_\delta = n$).[8,9,10] We can also partition T accordingly (recall that T is upper triangular):

$$T = \begin{bmatrix} T_{\theta\theta} & T_{\theta\delta} \\ \mathbf{0} & T_{\delta\delta} \end{bmatrix} \tag{14}$$

where $T_{\theta\theta}$ and $T_{\delta\delta}$ are upper triangular matrices of dimension $n_\theta \times n_\theta$ and $n_\delta \times n_\delta$, respectively.

Define the auxiliary variables:

$$\begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \equiv Z^H \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{15}$$

where θ_t and δ_t are vectors of dimension $n_\theta \times 1$ and $n_\delta \times 1$, respectively.

From (12) and (13) we have:

$$\mathbb{E}_t \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} \equiv A \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{16}$$

Premultiplying by Z^H and using $A = ZTZ^H$:

$$\mathbb{E}_t Z^H \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} \equiv Z^H ZTZ^H \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

Using $Z^H Z = I$ and (15):

$$\mathbb{E}_t \begin{bmatrix} \theta_{t+1} \\ \delta_{t+1} \end{bmatrix} \equiv T \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix}$$

Finally, using (14) we obtain the block-diagonal system we were looking for:[11]

$$\mathbb{E}_t \begin{bmatrix} \theta_{t+1} \\ \delta_{t+1} \end{bmatrix} \equiv \begin{bmatrix} T_{\theta\theta} & T_{\theta\delta} \\ \mathbf{0} & T_{\delta\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \tag{17}$$

[7] They are also the eigenvalues of T itself, since T is triangular. Therefore, A and T have the same eigenvalues.

[8] Recall that the modulus of a complex number $w = a + bi$ is given by $\sqrt{a^2 + b^2}$. For real numbers the imaginary part is absent (*i.e.*, $b = 0$) so we get $\sqrt{a^2} = |a|$.

[9] Eigenvalues with modulus smaller than one are called *stable*, and those with modulus higher than one are called *unstable*. To avoid additional complicatios, we assume there are no eigenvalues with modulus equal to one. Some authors refer to eigenvalues with modulus smaller (higher) than one as eigenvalues that lie inside (outside) the unit circle. See Chapter 1 in Enders (1995) for an explanation.

[10] If A is invertible, $\mathbb{E}_t w_{t+1} = A w_t$ can be rewritten as follows: $w_t = B \mathbb{E}_t w_{t+1}$, where $B = A^{-1}$. The eigenvalues of B are the reciprocal of the eigenvalues of A (if λ is an eigenvalue of A, $\frac{1}{\lambda}$ is an eigenvalue of B). Therefore, stable eigenvalues of A are unstable eigenvalues of B, and viceversa.

[11] As Z^H is invertible, knowledge of x_t and y_t is equivalent to knowledge of θ_t and δ_t; the transformation does not affect the information set Ω_t. Also, the existence and uniqueness of a solution to (16) is equivalent to the existence and uniqueness of a solution to (17).

Remark. As noted in Klein (2000), for the above representation we do not really need that T is upper triangular; it is enough to have upper block triangularity. This is why the *Real* Schur Decomposition can also be used to solve our problem [12]

From the second block of (17) we get:

$$\mathbb{E}_t \delta_{t+1} \equiv T_{\delta\delta} \delta_t \tag{18}$$

Since the diagonal elements of $T_{\delta\delta}$ are the unstable eigenvalues, (18) implies that the boundary condition (9) will be violated unless:[13]

$$\delta_t = 0 \quad \forall t \tag{19}$$

From the first block (17) we obtain $\mathbb{E}_t \theta_{t+1} \equiv T_{\theta\theta} \theta_t$. Combining this expression with (19) we get the following system:

$$\mathbb{E}_t \theta_{t+1} \equiv T_{\theta\theta} \theta_t \tag{20}$$

where an initial condition for θ_t is still to be found.

Premultiply (15) by Z, and use $ZZ^H = I$, to get:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} \equiv Z \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix}$$

Partitioning Z conformably:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} \equiv \begin{bmatrix} Z_{x\theta} & Z_{x\delta} \\ Z_{y\theta} & Z_{y\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \tag{21}$$

where $Z_{x\theta}$, $Z_{x\delta}$, $Z_{y\theta}$, and $Z_{y\delta}$ are matrices of dimension $n_x \times n_\theta$, $n_x \times n_\delta$, $n_y \times n_\theta$, and $n_y \times n_\delta$, respectively.

Substituting (19) into (21) we get:

$$x_t = Z_{x\theta} \theta_t \tag{22}$$
$$y_t = Z_{y\theta} \theta_t \tag{23}$$

If $Z_{x\theta}$ is invertible, we can use (22) to get:

$$\theta_t = Z_{x\theta}^{-1} x_t \tag{24}$$

Remark. A necessary condition for invertibility is that $Z_{x\theta}$ is a square matrix, that is, $n_\theta = n_x$. Therefore, invertibility of $Z_{x\theta}$ requires that the number of stable eigenvalues equals the number of predetermined variables (or, equivalently, that the number of unstable eigenvalues equals the number of nonpredetermined variables). When $Z_{x\theta}$ has full rank we have $rank(Z_{x\theta}) = \min\{n_x, n_\theta\}$; in this

[12] See Golub and van Loan (1996), Theorem 7.4.1 (page 341), for a presentation of the Real Schur Decomposition.

[13] See Appendix D for a proof of this result.

case, $n_x = n_\theta$ is sufficient for invertibility.[14] Although not very common, it is possible to have $n_\theta = n_x$ and $Z_{x\theta}$ singular (see Appendix E for an example).

Using (24) and the initial condition for x_t we obtain:

$$\theta_0 = Z_{x\theta}^{-1} x_0 \tag{25}$$

which provides an initial condition for system (20).

From (1) we have $\mathbb{E}_t x_{t+1} = A_{xx} x_t + A_{xy} y_t$, and then, $\varepsilon_{t+1} = x_{t+1} - \mathbb{E}_t x_{t+1}$. Substituting (22) into the latter we get: $\varepsilon_{t+1} = Z_{x\theta}\theta_{t+1} - \mathbb{E}_t Z_{x\theta}\theta_{t+1} = Z_{x\theta}\theta_{t+1} - Z_{x\theta}\mathbb{E}_t\theta_{t+1}$. Then:

$$\varepsilon_{t+1} = Z_{x\theta}(\theta_{t+1} - \mathbb{E}_t\theta_{t+1}) \tag{26}$$

From (26) we get $\theta_{t+1} - \mathbb{E}_t\theta_{t+1} = Z_{x\theta}^{-1}\varepsilon_{t+1}$, and then, $\theta_{t+1} = \mathbb{E}_t\theta_{t+1} + Z_{x\theta}^{-1}\varepsilon_{t+1}$. Substituting (20) into the last expression we obtain:

$$\theta_{t+1} = T_{\theta\theta}\theta_t + Z_{x\theta}^{-1}\varepsilon_{t+1} \tag{27}$$

Therefore, the nonexplosive solution to system (17) is given recursively by (19), (27) and (25).

Now we want to go back to our original system in terms of x_t and y_t. Shift (22) one period ahead and use (27) to get: $x_{t+1} = Z_{x\theta}\theta_{t+1} = Z_{x\theta}(T_{\theta\theta}\theta_t + Z_{x\theta}^{-1}\varepsilon_{t+1})$. Using (24) to eliminate θ_t, and rearranging: $x_{t+1} = Z_{x\theta}T_{\theta\theta}Z_{x\theta}^{-1}x_t + \varepsilon_{t+1}$. Then:

$$x_{t+1} = Mx_t + \varepsilon_{t+1} \tag{28}$$

where

$$M \equiv Z_{x\theta}T_{\theta\theta}Z_{x\theta}^{-1} \tag{29}$$

Analogously, substituting (24) into (23) we obtain: $y_t = Z_{y\theta}Z_{x\theta}^{-1}x_t$. Then:

$$y_t = Cx_t \tag{30}$$

where

$$C \equiv Z_{y\theta}Z_{x\theta}^{-1} \tag{31}$$

Equations (28) and (30), together with the initial condition x_0, give the recursive representation of the solution to (3). The vector of predetermined variables evolves according to a VAR(1), and the vector of nonpredetermined variables is a linear function of the one on predetermined variables.

Notice that M and C do not depend on Σ_ε, the variance-covariance matrix of ε_t. That is, *certainty equivalence* holds.

[14] Blanchard and Kahn make the full-rank assumption. See Blanchard & Kahn (1980), page 1307.

2.2 Important Results

Assume $Z_{x\theta}$ has full rank, and then $rank(Z_{x\theta}) = \min\{n_x, n_\theta\}$. From the previous analysis we conclude:

Result 1. (*Blanchard and Kahn* (1980)*, Proposition* 1).
If the number of stable eigenvalues is equal to the number of predetermined variables, *i.e.* $n_\theta = n_x$, (or, equivalently, if the number of unstable eigenvalues is equal to the number of nonpredetermined variables, *i.e.* $n_\delta = n_y$), system (3) has a unique nonexplosive solution.

Remark. The system may also have unstable solutions.

We also have the following results, that we present without a formal proof:

Result 2. (*Blanchard and Kahn* (1980)*, Proposition* 2)
If the number of stable eigenvalues is smaller than the number of predetermined variabes, *i.e.* $n_\theta < n_x$ (or, equivalently, if the number of unstable eigenvalues is bigger than the number of nonpredetermined variables, *i.e.*, $n_\delta > n_y$), there is no solution satisfying both system (3) and the no-explosion condition (9).

Remark. The system may also have unstable solutions.

Result 3. (*Blanchard and Kahn* (1980)*, Proposition* 3).
If the number of stable eigenvalues is bigger than the number of predetermined variabes, *i.e.* $n_\theta > n_x$ (or, equivalently, if the number of unstable eigenvalues is smaller than the number of nonpredetermined variables, *i.e.*, $n_\delta < n_y$), there is an infinity of solutions satisfying both system (3) and the no-explosion condition (9).

We can provide some intuition for the results above. For the first result, recall that the full-rank assumption and $n_\theta = n_x$ imply that $Z_{x\theta}$ is invertible. Then, the unique solution to $Z_{x\theta}\theta_0 = x_0$ is $\theta_0 = Z_{x\theta}^{-1}x_0$. Hence, any vector of initial conditions for the predetermined variables determines a unique vector of initial conditions for the auxiliary variables, θ_0. From (23) we know $y_0 = Z_{y\theta}\theta_0$. Hence, θ_0 induces a unique vector of initial values for the nonpredetermined variables, y_0. Another way to think about this result is as follows: our original system (see (3)) has n equations and n variables, with $n = n_x + n_y$; accordingly, we need n restrictions to pin down a unique solution; we have n_x restrictions coming from the initial condition for the predetermined variables, and n_δ restrictions coming from the stability conditions (recall (19)); therefore, the total number of restrictions is $n_x + n_\delta$; hence, $n_x + n_\delta = n$ requires $n_\delta = n - n_x = n_y$; therefore, the stability conditions exactly pin down the initial values of the nonpredetermined variables, y_0. Notice that $n_\delta = n_y$ is equivalent to $n_\theta = n_x$, since $n = n_x + n_y = n_\theta + n_\delta$.

When $n_\theta < n_x$, the full-rank assumption implies: $rank(Z_{x\theta}) = n_\theta < n_x$. Hence, $Z_{x\theta}$ has more rows than columns, and system $Z_{x\theta}\theta_0 = x_0$ has more equations than unknowns. In this case, the system cannot be solved for arbitrary initial vectors x_0. Since $y_0 = Z_{y\theta}\theta_0$, we cannot solve for y_0 either. In terms of our original system, we have $n_x + n_\delta$ restrictions; but $n_\theta < n_x$ is equivalent to $n_\delta > n_y$, and then $n_x + n_\delta > n_x + n_y = n$; therefore, we do not have enough restrictions to pin down the initial values of the nonpredetermined variables, y_0.

When $n_\theta > n_x$, the full-rank assumption implies: $rank(Z_{x\theta}) = n_x < n_\theta$. Hence, $Z_{x\theta}$ has fewer columns than rows, and system $Z_{x\theta}\theta_0 = x_0$ has fewer equations than unknowns. In this case, the system has infinitely many solutions for any initial vectors x_0. In particular, $n_\theta - n_x$ components of θ_0 can be chosen arbitrarily. Since $y_0 = Z_{y\theta}\theta_0$, this implies that $n_\theta - n_x$ components of y_0 can be chosen arbitrarily. But $n_\theta - n_x = n_y - n_\delta$ (since $n = n_x + n_y = n_\theta + n_\delta$). Therefore, $n_y - n_\delta$ components of y_0 can be chosen arbitrarily. This is consistent with the fact that, for our original system, we have fewer restrictions than unknowns: $n_x + n_\delta < n_x + n_y$ (recall that $n_\theta > n_x \Leftrightarrow n_\delta < n_y$).

For examples of Results 2 and 3 see Appendix F.

2.3 A Simple Example: Solving the Cagan Model

Let's go back now to Cagan's model, and suppose that $\rho = 0.9$ and $\alpha = 0.5$. Then, A in (8) becomes:

$$A = \begin{bmatrix} 0.9 & 0 \\ -1 & 2 \end{bmatrix}$$

Using a computer, we find that the Schur decomposition of A gives:[15]

$$T = \begin{bmatrix} 0.9 & 1 \\ 0 & 2 \end{bmatrix}, \quad Z = \begin{bmatrix} -0.7399 & 0.6727 \\ -0.6727 & -0.7399 \end{bmatrix}, \quad Z^H = \begin{bmatrix} -0.7399 & -0.6727 \\ 0.6727 & -0.7399 \end{bmatrix}$$

Notice that A is triangular, and then its eigenvalues coincide with its diagonal elements. As required, T is upper triangular with the eigenvalues of A along the diagonal, and the stable eigenvalue comes first. Also, Z is unitary, so $Z^H = Z^{-1}$. In this particular example, both T and Z are real (and then Z^H coincides with the transpose of Z).

Partitioning T as in (14), and Z as in (21) we obtain:

$$\begin{aligned} T_{\theta\theta} &= 0.9, \;\; T_{\theta\delta} = 1, \;\; T_{\delta\theta} = 0, \;\; T_{\delta\delta} = 2 \\ Z_{x\theta} &= -0.7399, \;\; Z_{x\delta} = 0.6727, \;\; Z_{y\theta} = -0.6727, \;\; Z_{x\delta} = -0.7399 \end{aligned}$$

Substituting into (29) and (31):

$$\begin{aligned} M &\equiv Z_{x\theta} T_{\theta\theta} Z_{x\theta}^{-1} = -0.7399 \times 0.9 \times \frac{-1}{0.7399} = 0.9 \\ C &\equiv Z_{y\theta} Z_{x\theta}^{-1} = -0.6727 \times \frac{-1}{0.7399} = 0.909 \end{aligned}$$

[15] As a companion to Klein (2000), Paul Klein developed a Matlab routine, solab.m, that solves the type of models discussed in these notes. The code is available on line at http://paulklein.se/newsite/codes/codes.php

Finally, (28) and (30) give:[16]

$$\begin{aligned} m_{t+1} &= 0.9m_t + \varepsilon_{t+1} \\ p_t &= 0.9091m_t \end{aligned}$$

The Cagan Model is simple enough so that we can check the solution given above by direct calculation.

From (6) we know: $p_t = (1-\alpha)m_t + \alpha E_t p_{t+1}$. Iterating forward and applying the Law of Iterated Expectations we get:[17] $p_t = (1-\alpha)\sum_{s=0}^{T}\alpha^s \mathbb{E}_t m_{t+s} + \alpha^{T+1}\mathbb{E}_t p_{t+T+1}$. Letting $T \to \infty$, and imposing the condition $\lim\limits_{T\to\infty} \alpha^{T+1}\mathbb{E}_t p_{t+T+1} = 0$, we get:

$$p_t = (1-\alpha)\sum_{s=0}^{\infty}\alpha^s \mathbb{E}_t m_{t+s}$$

From (7) we know: $m_{t+1} = \rho m_t + \varepsilon_{t+1}$. Iterating forward we get: $m_{t+s} = \rho^s m_t + \sum_{i=1}^{s}\rho^{s-i}\varepsilon_{t+i}$. Then:

$$\mathbb{E}_t m_{t+s} = \rho^s m_t$$

since $\mathbb{E}_t \rho^s m_t = \rho^s m_t$ and $\mathbb{E}_t \varepsilon_{t+s} = 0\ \forall s \geq 1$.

Substituting $\mathbb{E}_t m_{t+s} = \rho^s m_t$ into the expression for p_t we get: $p_t = (1-\alpha)m_t \sum\limits_{s=0}^{\infty}(\alpha\rho)^s$. Then:

$$p_t = \frac{1-\alpha}{1-\alpha\rho}m_t$$

since $\sum\limits_{s=0}^{\infty}(\alpha\rho)^s = \frac{1}{1-\alpha\rho}$, because $|\alpha\rho| < 1$.

When $\alpha = 0.5$ and $\rho = 0.9$, we get: $\frac{1-\alpha}{1-\alpha\rho} = \frac{0.5}{0.55} \cong 0.9091$, which coincides with the result we found using the Schur Decomposition.

Remark. This simple model helps understand the problems that can arise when there are eigenvalues with modulus equal to one. When $\rho = 1$, the solution for the price level is $p_t = m_t$. From this expresion we get $\mathbb{E}_t p_{t+s} = \mathbb{E}_t m_{t+s} = m_t$. Then, $\lim\limits_{s\to\infty} |\mathbb{E}_t p_{t+s}| = |m_t| < \infty$, which shows that the solution satisfies our definition of stability. The problem, however, is that the conditional variance of the price level diverges to infinity, unless $\varepsilon_t = 0\ \forall t$ (which would happen when $\sigma_\varepsilon^2 = 0$). We have: $\mathrm{Var}_t\{p_{t+s}\} = \mathrm{Var}_t\{m_{t+s}\} = \mathrm{Var}_t\{m_t + \varepsilon_{t+1} + \varepsilon_{t+2} + ... + \varepsilon_{t+s}\} = s\sigma_\varepsilon^2 \to \infty$ as $s \to \infty$, unless $\sigma_\varepsilon^2 = 0$.

[16] Money supply, the only predetermined variable of the model, is exogenous. Therefore, the solution recovers the AR(1) process given in (7).

[17] When $\Omega_t \subseteq \Omega_{t+1}\ \forall t$, the Law of Iterated Expectations establishes that $\mathbb{E}_t\{\mathbb{E}_{t+s}p_{t+s}\} = \mathbb{E}_t p_{t+s}\ \forall s \in \{1, 2, ...\}$.

2.4 Impulse-Response Functions

Using the solution (28) and (30) we can trace the effects on x and y of a shock at time t, *i.e.*, we can derive the impulse-response functions. We start by writing x_t and y_t as functions of current and past shocks.

From (28), and the initial condition, we get:
$x_1 = Mx_0 + \varepsilon_1$
$x_2 = Mx_1 + \varepsilon_2 = M(Mx_0 + \varepsilon_1) + \varepsilon_2 = M^2x_0 + M\varepsilon_1 + \varepsilon_2$
$x_3 = Mx_2 + \varepsilon_3 = M(M^2x_0 + M\varepsilon_1 + \varepsilon_2) + \varepsilon_3 = M^3x_0 + M^2\varepsilon_1 + M\varepsilon_2 + \varepsilon_3$
...
$x_t = M^tx_0 + M^{t-1}\varepsilon_1 + M^{t-2}\varepsilon_2 + ... + M\varepsilon_{t-1} + \varepsilon_t$
Then:

$$x_t = M^tx_0 + \sum_{i=0}^{t-1} M^i\varepsilon_{t-i} \tag{32}$$

where $M^0 = I$.

Substituting (32) into (30) we obtain:

$$y_t = CM^tx_0 + \sum_{i=0}^{t-1} CM^i\varepsilon_{t-i} \tag{33}$$

From (32) and (33) we conclude that the impact of a shock in t, ε_t, on the current and future values of x and y is:

$$\begin{array}{rccccc} \text{Variable} & x_t & x_{t+1} & x_{t+2} & \ldots & x_{t+j} \\ \text{Impact of a shock in period } t & \varepsilon_t & M\varepsilon_t & M^2\varepsilon_t & \ldots & M^j\varepsilon_t \\ \\ \text{Variable} & y_t & y_{t+1} & y_{t+2} & \ldots & y_{t+j} \\ \text{Impact of a shock in period } t & C\varepsilon_t & CM\varepsilon_t & CM^2\varepsilon_t & \ldots & CM^j\varepsilon_t \end{array} \tag{34}$$

2.5 Second Moments

From (28) we know: $x_t = Mx_{t-1} + \varepsilon_t$. Iterating infinitely far into the past, and imposing $\lim_{s\to\infty} M^sx_{t-s} = 0$, we get the MA($\infty$) representation for the time series of the state vector:[18]

$$x_t = \sum_{i=0}^{\infty} M^i\varepsilon_{t-i} \tag{35}$$

Remark. The condition $\lim_{s\to\infty} M^sx_{t-s} = 0$ follows from the fact that the eigenvalues of M are all smaller than one in modulus. Actually, $\lim_{s\to\infty} M^s = 0 \Leftrightarrow$ all the eigenvalues of M have modulus smaller than one (See Sydsæter et al. (2005),

[18] The same result can be obtained using the lag operator, L, defined by $Lx_t = x_{t-1}$: $x_t = Mx_{t-1} + \varepsilon_t \Rightarrow x_t = MLx_t + \varepsilon_t \Rightarrow (I - ML)x_t = \varepsilon_t \Rightarrow x_t = (I - ML)^{-1}\varepsilon_t$, where $(I - ML)^{-1}\varepsilon_t = \sum_{i=0}^{\infty} M^i\varepsilon_{t-i}$. Lag operators are discussed in Enders (1995), Chapter 1.

Chapter 21, Result 21.7). And we know this condition holds because the eigenvalues of M coincide with the eigenvalues of $T_{\theta\theta}$: $Mx = \lambda x \Rightarrow Z_{x\theta}T_{\theta\theta}Z_{x\theta}^{-1}x = \lambda x \Rightarrow Z_{x\theta}^{-1}Z_{x\theta}T_{\theta\theta}Z_{x\theta}^{-1}x = Z_{x\theta}^{-1}\lambda x \Rightarrow T_{\theta\theta}Z_{x\theta}^{-1}x = \lambda Z_{x\theta}^{-1}x \Rightarrow T_{\theta\theta}w = \lambda w$. We also know that $\sum\limits_{i=0}^{\infty} M^i \varepsilon_{t-i}$ is well defined. See Chapter 2 in Lütkepohl (2005) for a good reference on these questions.

From (35) we can get the *unconditional expectation* of x_t :

$$\mathbb{E}x_t = 0 \tag{36}$$

Therefore, the *unconditional variance-covariance matrix* of x_t satisfies:
$\Sigma_x = \mathbb{E}\{(x_t - \mathbb{E}x_t)(x_t - \mathbb{E}x_t)'\} = \mathbb{E}\{x_t x_t'\}$
$\Sigma_x = \mathbb{E}\{(\sum_{i=0}^{\infty} M^i \varepsilon_{t-i})(\sum_{i=0}^{\infty} M^i \varepsilon_{t-i})'\}$
$\Sigma_x = \mathbb{E}\{\sum_{i=0}^{\infty} M^i \varepsilon_{t-i} \sum_{i=0}^{\infty} \varepsilon_{t-i}' M'^i\}$ since $M^{i\prime} = M'^i$
$\Sigma_x = \mathbb{E}\{\sum_{i=0}^{\infty} M^i \varepsilon_{t-i}\varepsilon_{t-i}' M'^i + \sum\sum_{i\neq j}^{\infty} M^i \varepsilon_{t-i}\varepsilon_{t-j}' M'^j\}$
$\Sigma_x = \sum_{i=0}^{\infty} M^i \mathbb{E}\{\varepsilon_{t-i}\varepsilon_{t-i}'\} M'^i + \sum\sum_{i\neq j}^{\infty} M^i \mathbb{E}\{\varepsilon_{t-i}\varepsilon_{t-j}'\} M'^j$
Then, using (36):

$$\Sigma_x = \sum_{i=0}^{\infty} M^i \Sigma_\varepsilon M'^i \tag{37}$$

since $\mathbb{E}\{\varepsilon_{t-i}\varepsilon_{t-j}'\} = 0 \ \forall i \neq j$.

We can further write:
$\Sigma_x = \Sigma_\varepsilon + \sum_{i=1}^{\infty} M^i \Sigma_\varepsilon M'^i = \Sigma_\varepsilon + M(\sum_{i=1}^{\infty} M^{i-1}\Sigma_\varepsilon M'^{i-1})M'$
$\Sigma_x = \Sigma_\varepsilon + M(\sum_{i=0}^{\infty} M^i \Sigma_\varepsilon M'^i)M'$
Then:

$$\Sigma_x = \Sigma_\varepsilon + M\Sigma_x M' \tag{38}$$

From (30) we obtain: $\mathbb{E}y_t = \mathbb{E}\{Cx_t\} = C\mathbb{E}x_t \Rightarrow$

$$\mathbb{E}y_t = 0 \tag{39}$$

Therefore, the *unconditional variance-covariance matrix* of y_t satisfies:
$\Sigma_y = \mathbb{E}\{(y_t - \mathbb{E}y_t)(y_t - \mathbb{E}y_t)'\} = \mathbb{E}\{y_t y_t'\}$
$\Sigma_y = \mathbb{E}\{(Cx_t)(Cx_t)'\} = \mathbb{E}\{Cx_t x_t' C'\}$
$\Sigma_y = C\mathbb{E}\{x_t x_t'\}C'$
Then:

$$\Sigma_y = C\Sigma_x C' \tag{40}$$

There are different methods to obtain the solution to (38). One way is to iterate on the following expression until convergence:

$$\Sigma_{x,j} = \Sigma_\varepsilon + M\Sigma_{x,j-1}M' \tag{41}$$

where the iteration could start by setting $\Sigma_{x,0} = \mathbf{0}$ or $\Sigma_{x,0} = I$.[19]

Alternatively, we can use the *vec* operator - that stacks the column vectors of a matrix - to write:

[19] The speed of convergence can be increased using the so-called doubling algorithm. See Chapter 4 in Uribe (2011) for a description.

$vec(\Sigma_x) = vec(\Sigma_\varepsilon + M\Sigma_x M')$
$vec(\Sigma_x) = vec(\Sigma_\varepsilon) + vec(M\Sigma_x M')$
$vec(\Sigma_x) = vec(\Sigma_\varepsilon) + (M \otimes M)vec(\Sigma_x)$
$(I - M \otimes M)vec(\Sigma_x) = vec(\Sigma_\varepsilon)$
Then:

$$vec(\Sigma_x) = (I - M \otimes M)^{-1} vec(\Sigma_\varepsilon) \tag{42}$$

where we have used $vec(A + B) = vec(A) + vec(B)$, and $vec(ABC) = (C' \otimes A)vec(B)$, with $\otimes$ denoting the Kronecker product operator.[20]

Remark 1. To get $vec(\Sigma_x)$ from (42) we need to be sure that $I - M \otimes M$ is invertible, and this will be true if and only if the eigenvalues of $I - M \otimes M$ are all different from zero (Sydsæter et al. (2005), Chapter 21, Result 21.6). We know the eigenvalues of $M \otimes M$ are products of the eigenvalues of M (Sydsæter et al. (2005), Chapter 23, Result 23.10), and we have argued that all the eigenvalues of M have modulus smaller than one. It follows that all the eigenvalues of $M \otimes M$ have modulus smaller than one. Now, the eigenvalues of $I - M \otimes M$ are of the form $1 - \lambda$, where λ is an eigenvalue $M \otimes M$. Since $|\lambda| < 1$, we know that $1 - \lambda$ cannot be zero. Hence, $I - M \otimes M$ is invertible.

Remark 2. Even though (42) gives an exact formula for Σ_x, (41) may be preferred in applications, since computing (42) can be less accurate and slower, especially when n_x is large (since it is necessary to invert an $n_x^2 \times n_x^2$ matrix).

Consider now the following variance-covariance matrix: $\mathbb{E}\{x_t x'_{t-j}\}$, $j > 0$. We have:
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{(\sum_{i=0}^{\infty} M^i \varepsilon_{t-i}) x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{(\sum_{i=0}^{j-1} M^i \varepsilon_{t-i} + \sum_{i=j}^{\infty} M^i \varepsilon_{t-i}) x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{(\sum_{i=0}^{j-1} M^i \varepsilon_{t-i} + M^j \sum_{i=j}^{\infty} M^{i-j} \varepsilon_{t-i}) x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{(\sum_{i=0}^{j-1} M^i \varepsilon_{t-i} + M^j \sum_{k=0}^{\infty} M^k \varepsilon_{(t-j)-k}) x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{(\sum_{i=0}^{j-1} M^i \varepsilon_{t-i} + M^j x_{t-j}) x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{(\sum_{i=0}^{j-1} M^i \varepsilon_{t-i}) x'_{t-j}\} + \mathbb{E}\{M^j x_{t-j} x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = \mathbb{E}\{\sum_{i=0}^{j-1} M^i \varepsilon_{t-i} \sum_{i=0}^{\infty} M^i \varepsilon_{(t-j)-i}\} + \mathbb{E}\{M^j x_{t-j} x'_{t-j}\}$
$\mathbb{E}\{x_t x'_{t-j}\} = M^j \mathbb{E}\{x_{t-j} x'_{t-j}\}$ (since the first term of the previous expression contains only cross-products)
Then:

$$\mathbb{E}\{x_t x'_{t-j}\} = M^j \Sigma_x \tag{43}$$

Also:
$\mathbb{E}\{y_t y'_{t-j}\} = \mathbb{E}\{Cx_t (Cx_{t-j})'\} = \mathbb{E}\{Cx_t x'_{t-j} C'\}$
$\mathbb{E}\{y_t y'_{t-j}\} = CE\{x_t x'_{t-j}\}C'$
Then:

$$\mathbb{E}\{y_t y'_{t-j}\} = CM^j \Sigma_x C' \tag{44}$$

[20] See Chapter 23 in Sydsæter et al. (2005) for definitions of the Kronecker product and the *vec* operators, and a list their basic properties.

3 A More General Model

3.1 The Model and its Solution

We generalize (3) as follows:

$$G\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = A\begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y\times 1} \end{bmatrix} \qquad x_0 \text{ given} \tag{45}$$

where G is an $n \times n$, possibly singular, matrix. In terms of partitioned matrices we get:

$$\begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix}\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y\times 1} \end{bmatrix} \tag{46}$$

We assume that x_t is predetermined, and therefore it has an exogenous one-period-ahead forecast error $\xi_{t+1} \equiv x_{t+1} - \mathbb{E}_t x_{t+1}$.[21]

Based on information available at time t, we can take expectations of (45) in order to get:

$$G\mathbb{E}_t\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = A\begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{47}$$

or, more compactly,

$$G\mathbb{E}_t w_{t+1} = Aw_t \tag{48}$$

Remark. The system presented in Klein (2000) and Blanchard and Kahn (1980) is of the form $G\mathbb{E}_t w_{t+1} = Aw_t + Dz_t$, where $\{z_t\}$ is a stable sequence of exogenous random vectors of dimension $n_z \times 1$, and D is a known matrix of dimension $n \times n_z$. In Appendix G we show how to rewrite this system in form (48) when the exogenous random vectors follow a VAR(1).

Notice that if G were invertible, we could premultiply (47) by G^{-1} and obtain a system in form (16). In this case we could solve the problem using the Schur Decomposition, as explained earlier. If G is singular, this approach is not possible, but we can still solve the problem applying the Generalized (Complex) Schur Decomposition to the matrix pair (A, G) (see Appendix C). That is, we find complex unitary $n\times n$ matrices Q and Z, and complex upper triangular $n\times n$ matrices S and T, such that $A = QTZ^H$ and $G = QSZ^H$. The decomposition is such that the generalized eigenvalues of (A, G) are of the form $\frac{t_{ii}}{s_{ii}}$, where s_{ii} and t_{ii} are the diagonal elements of S and T, respectively. Moreover, we can reorder S, T , Q and Z such that the n_θ generalized eigenvalues with $\left|\frac{t_{ii}}{s_{ii}}\right| < 1$

[21] Notice that, in this more general system, ε_{t+1} is not the one-period-ahead forecast error of x_t. From the first block of (46) we get: $G_{xx}x_{t+1} + G_{xy}E_t y_{t+1} = A_{xx}x_t + A_{xy}y_t + \varepsilon_{t+1}$. Taking expectations conditional on period-t information: $G_{xx}E_t x_{t+1} + G_{xy}E_t y_{t+1} = A_{xx}x_t + A_{xy}y_t$. Subtracting the second expression from the first one: $G_{xx}(x_{t+1} - E_t x_{t+1}) = \varepsilon_{t+1}$. Then: $\varepsilon_{t+1} = G_{xx}\xi_{t+1}$. Since G_{xx} may not be invertible, it is not generally possible to write $\xi_{t+1} = G_{xx}^{-1}\varepsilon_{t+1}$. We can still write $\xi_{t+1} = G_{xx}^{*}\varepsilon_{t+1}$, where G_{xx}^{*} is a generalized inverse of G_{xx}.

come first, and the n_δ generalized eigenvalues with modulus higher than one come last (where $n_\theta + n_\delta = n$).

Define the auxiliary variables:

$$\begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \equiv Z^H \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{49}$$

where θ_t and δ_t are vectors of dimension $n_\theta \times 1$ and $n_\delta \times 1$, respectively.

Premultiplying (47) by Q^H, and using $A = QTZ^H$ and $G = QSZ^H$, we get:[22]

$$Q^H QS\mathbb{E}_t Z^H \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} \equiv Q^H QTZ^H \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

Using $Q^H Q = I$ and (49):

$$S\mathbb{E}_t \begin{bmatrix} \theta_{t+1} \\ \delta_{t+1} \end{bmatrix} \equiv T \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix}$$

Partitioning S and T conformably with θ_t and δ_t we can write:

$$\begin{bmatrix} S_{\theta\theta} & S_{\theta\delta} \\ 0 & S_{\delta\delta} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \theta_{t+1} \\ \delta_{t+1} \end{bmatrix} \equiv \begin{bmatrix} T_{\theta\theta} & T_{\theta\delta} \\ 0 & T_{\delta\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \tag{50}$$

Notice that $S_{\theta\theta}$ is invertible: since $S_{\theta\theta}$ is triangular, its determinant is the product of its diagonal elements; moreover, since all the generalized eigenvalues with $\left|\frac{t_{ii}}{s_{ii}}\right| < 1$ are ordered first, s_{ii} cannot be zero for any i in this block; therefore, $|S_{\theta\theta}| \neq 0$, and $S_{\theta\theta}$ is invertible. Analogous reasoning shows that $T_{\delta\delta}$ is also invertible. Matrix $S_{\delta\delta}$, however, may not be invertible, since some of its diagonal elements could be zero (corresponding to infinite generalized eigenvalues). This will happen whenever G is singular and the matrix pair (A, G) is regular (see Appendix C).

Remark. As noted in Klein (2000), for the above representation we do not really need that S and T are upper triangular; it is sufficient to have upper block triangularity. This is why the Generalized *Real* Schur Decomposition can also be used to solve the system.[23]

From (50) we get:

$$S_{\delta\delta}\mathbb{E}_t\delta_{t+1} \equiv T_{\delta\delta}\delta_t \tag{51}$$

Since the pair $(S_{\delta\delta}, T_{\delta\delta})$ contains the unstable eigenvalues, the system will violate the no-explosion condition unless:[24]

[22] As Q^H is invertible, knowledge of x_t and y_t is equivalent to knowledge of θ_t and δ_t (the transformation does not affect the information set Ω_t). Hence, one system is equivalent to the other.

[23] See Golub and van Loan (1996), Theorem 7.7.2 (page 377), for a presentation of the Generalized Real Schur Decomposition.

[24] See Appendix D for a proof of this result.

$$\delta_t = 0 \quad \forall t \tag{52}$$

Substituting (52) into (50) we obtain: $S_{\theta\theta}\mathbb{E}_t\theta_{t+1} \equiv T_{\theta\theta}\theta_t$. Since $S_{\theta\theta}$ is invertible, we get:

$$\mathbb{E}_t\theta_{t+1} \equiv S_{\theta\theta}^{-1}T_{\theta\theta}\theta_t \tag{53}$$

where an initial condition for θ_t is still to be found.

Premultiplying (49) by Z, and using $ZZ^H = I$, we get:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} \equiv Z \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix}$$

Partitioning Z conformably:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} \equiv \begin{bmatrix} Z_{x\theta} & Z_{x\delta} \\ Z_{y\theta} & Z_{y\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} \tag{54}$$

where $Z_{x\theta}$, $Z_{x\delta}$, $Z_{y\theta}$, and $Z_{y\delta}$ are matrices of dimension $n_x \times n_\theta$, $n_x \times n_\delta$, $n_y \times n_\theta$, and $n_y \times n_\delta$, respectively.

Substituting (52) into (54) we get:

$$x_t = Z_{x\theta}\theta_t \tag{55}$$

$$y_t = Z_{y\theta}\theta_t \tag{56}$$

If $Z_{x\theta}$ is invertible, we can use (55) to get:

$$\theta_t = Z_{x\theta}^{-1}x_t \tag{57}$$

Remark. As noted earlier, a necessary condition for invertibility is that $n_\theta = n_x$. Therefore, invertibility of $Z_{x\theta}$ requires that the number of stable eigenvalues equals the number of predetermined variables (or, equivalently, that the number of unstable eigenvalues equals the number of nonpredetermined variables). When $Z_{x\theta}$ has full rank, $n_x = n_\theta$ is sufficient for invertibility.

Using (57) and the initial condition for x_t we obtain:

$$\theta_0 = Z_{x\theta}^{-1}x_0 \tag{58}$$

which provides an initial condition for system (53).

The one-period-ahead forecast error of x_t is $\xi_{t+1} = x_{t+1} - \mathbb{E}_t x_{t+1}$, which is exogenous by assumption. Using (55) we get: $\xi_{t+1} = Z_{x\theta}\theta_{t+1} - \mathbb{E}_t Z_{x\theta}\theta_{t+1} = Z_{x\theta}\theta_{t+1} - Z_{x\theta}\mathbb{E}_t\theta_{t+1}$. Then:

$$\xi_{t+1} = Z_{x\theta}(\theta_{t+1} - \mathbb{E}_t\theta_{t+1}) \tag{59}$$

From (59) we get $\theta_{t+1} - \mathbb{E}_t\theta_{t+1} = Z_{x\theta}^{-1}\xi_{t+1}$, and then, $\theta_{t+1} = \mathbb{E}_t\theta_{t+1} + Z_{x\theta}^{-1}\xi_{t+1}$. Substituting (53) into the last expression we obtain:

$$\theta_{t+1} = S_{\theta\theta}^{-1}T_{\theta\theta}\theta_t + Z_{x\theta}^{-1}\xi_{t+1} \tag{60}$$

Therefore, the nonexplosive solution to system (50) is given recursively by (52), (60) and (58).

Now we want to go back to our original system in terms of x_t and y_t. Substituting (60) into (55) we obtain: $x_{t+1} = Z_{x\theta}\theta_{t+1} = Z_{x\theta}(S_{\theta\theta}^{-1}T_{\theta\theta}\theta_t + Z_{x\theta}^{-1}\xi_{t+1})$. Using (57) to eliminate θ_t, and rearranging: $x_{t+1} = Z_{x\theta}S_{\theta\theta}^{-1}T_{\theta\theta}Z_{x\theta}^{-1}x_t + \xi_{t+1}$. Then:

$$x_{t+1} = Mx_t + \xi_{t+1} \tag{61}$$

where

$$M \equiv Z_{x\theta}S_{\theta\theta}^{-1}T_{\theta\theta}Z_{x\theta}^{-1} \tag{62}$$

Analogously, substituting (57) into (56) we obtain: $y_t = Z_{y\theta}Z_{x\theta}^{-1}x_t$. Then:

$$y_t = Cx_t \tag{63}$$

where

$$C \equiv Z_{y\theta}Z_{x\theta}^{-1} \tag{64}$$

Equations (61) and (63), together with the initial condition x_0, give the recursive representation of the solution to (45).

The Blanchard-Kahn resutls apply, with generalized eigenvalues taking the place of (standard) eigenvalues. The calculation of impulse-response functions and second moments apply as well, with the new definitions for matrices M and C, and with Σ_ξ in place of Σ_ε.[25]

3.2 A Simple Example

Consider the following simple model:

$$\begin{aligned} x_{t+1} &= \frac{1}{4}x_t + y_t + \varepsilon_{t+1} \\ y_t &= \frac{1}{2}x_t \end{aligned}$$

where ε_{t+1} is white noise and x_0 is given.

It is really easy to solve this model. Substitution of the second equation into the first gives the recursive solution for the predetermined variable: $x_{t+1} = \frac{1}{4}x_t + \frac{1}{2}x_t + \varepsilon_{t+1} \Rightarrow$

$$x_{t+1} = \frac{3}{4}x_t + \varepsilon_{t+1}$$

Hence, $M = 3/4$. The second equation is already in the required form, with $C = 1/2$.

[25] When G_{xx} is invertible, we have $\xi_{t+1} = G_{xx}^{-1}\varepsilon_{t+1}$. Then, $\Sigma_\xi = G_{xx}^{-1}\Sigma_\varepsilon G_{xx}^{-1\prime}$.

Now we can apply our general solution method to the model given above and confirm that it provides the correct solution. We start by rewriting the model in form (46):

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} 1/4 & 1 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

where $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1/4 & 1 \\ 1/2 & -1 \end{bmatrix}$. Notice that G is singular.

Using a computer we find that the Generalized Schur decomposition of A gives:[26]

$$S = \begin{bmatrix} 0.8944 & -0.4472 \\ 0 & 0 \end{bmatrix}; \quad T = \begin{bmatrix} 0.6708 & 0.7826 \\ 0 & 1.1180 \end{bmatrix}$$

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad Z = \begin{bmatrix} -0.8944 & 0.4472 \\ -0.4472 & -0.8944 \end{bmatrix}; \quad Z^H = \begin{bmatrix} -0.8944 & -0.4472 \\ 0.4472 & -0.8944 \end{bmatrix}$$

Then:

$$\begin{aligned} S_{\theta\theta} &= 0.8944,\ S_{\theta\delta} = -0.4472,\ S_{\delta\delta} = 0 \\ T_{\theta\theta} &= 0.6708,\ T_{\theta\delta} = 0.7826,\ T_{\delta\delta} = 1.118 \\ Z_{x\theta} &= -0.8944,\ Z_{x\delta} = 0.4472,\ Z_{y\theta} = -0.4472,\ Z_{y\delta} = -0.8944 \end{aligned}$$

Then, $M \equiv Z_{x\theta} S_{\theta\theta}^{-1} T_{\theta\theta} Z_{x\theta}^{-1} \Rightarrow M = -0.8944 \times \frac{1}{0.8944} \times 0.6708 \times \frac{1}{-0.8944} \Rightarrow M = \frac{0.6708}{0.8944} \Rightarrow$

$$M = \frac{3}{4}$$

Also, $C = Z_{y\theta} Z_{x\theta}^{-1} \Rightarrow C = -0.4472 \times \frac{1}{-0.8944} \Rightarrow$

$$C = \frac{1}{2}$$

Moreover, from $G_{xx} = 1$ and $\varepsilon_{t+1} = G_{xx}\xi_{t+1}$ we get:

$$\xi_{t+1} = \varepsilon_{t+1}$$

Then, $x_{t+1} = Mx_t + \xi_{t+1}$ and $y_t = Cx_t$ become:

$$\begin{aligned} x_{t+1} &= \frac{3}{4}x_t + \varepsilon_{t+1} \\ y_t &= \frac{1}{2}x_t \end{aligned}$$

which coincides with the solution we found earlier.

[26] We use the Matlab code developed by Paul Klein, solab.m, available at http://paulklein.se/newsite/codes/codes.php

4 A Rational-Expectations Macroeconomic Model

In this section we use the methods discussed above to solve a standard, infinite-horizon, stochastic, Real Business Cycle (RBC) model.[27]

4.1 The Model

There is a representative household with preferences over sequences of consumption and leisure. In any period $t \in \{0, 1, 2, ...\}$, the household chooses consumption (C_t), the supply of labor time (H_t), and investment ($I_t = K_{t+1} - (1-\delta)K_t$). Total time available is normalized to one, so leisure time is $1 - H_t$. The household owns the capital stock (K_t), which is rented to the representative firm. The budget constraint of the household establishes that total expenditure in consumption and investment must be financed with wages ($w_t H_t$) and the rents from capital ($r_t K_t$).[28] Therefore, household's decisions can be represented by the solution of the following utility-maximization problem:

$$
\begin{aligned}
\max_{\{C_t, H_t, K_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t + a \ln(1 - H_t)\right] \\
\text{s.t.} \quad & C_t + K_{t+1} - (1-\delta)K_t = w_t H_t + r_t K_t \\
& K_0 > 0 \ \textit{given}
\end{aligned}
$$

where we have assumed that preferences are time separable, with period utility function $U(C_t, 1 - H_t) = \ln C_t + a \ln(1 - H_t)$, discount factor $\beta \in (0, 1)$, and $a > 0$.

The representative firm's production function is Cobb-Douglas:

$$
Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}, \quad \theta \in (0, 1)
$$

Total factor productivity is stochastic and evolves as follows:

$$
\ln \lambda_{t+1} = (1 - \gamma) \ln \lambda + \gamma \ln \lambda_t + \epsilon_{t+1}
$$

where $\gamma \in (0, 1)$, $\lambda > 0$, and $\{\epsilon_{t+1}\}_{t=0}^{\infty}$ is a sequence of *i.i.d.* random variables with mean zero and variance σ_{ϵ}^2.

The firm generates output by hiring inputs in competitive markets, with the aim of maximizing profits. Using the production function to eliminate output, we can write the profit-maximization problem as follows:

$$
\max_{K_t, H_t} \ \lambda_t K_t^{\theta} H_t^{1-\theta} - w_t H_t - r_t K_t
$$

[27] We solve the standard model with divisible labor, as presented in Hansen (1985). This paper also develops a model with indivisible labor that can be solved with the same methodology presented here.

[28] The representative household owns the representative firm. Since profits will be zero, we do not need to include them as part of the household's income.

The combination of constant returns to scale with competitive markets implies that any solution of the profit-maximization problem must yield zero profits.

Given the stochastic process for total factor productivity, a competitive equilibrium for this economy is a sequence of stochastic allocations $\{C_t,\ I_t,\ H_t,\ K_{t+1},\ Y_t\}_{t=0}^{\infty}$, and a sequence of stochastic prices $\{r_t,\ w_t\}_{t=0}^{\infty}$, such that: (i) the allocation of the household maximizes expected utility, given the sequence of equilibrium prices; (ii) the allocation of the firm maximizes profits, given the sequence of equilibrium prices; (iii) the sequence of prices clears all markets. Market clearing requires that, at the equilibrium prices, the labor supplied by the household coincides with the labor demanded by the firm, the capital supplied for rent by the household coincides with the capital demanded by the firm, and the output supplied by the firm coincides with the household's demand for consumption and investment.

After finding the first-order conditions for the utility and profit maximization problems we can summarize the equilibrium conditions with the following set of nonlinear stochastic dynamic equations:

$$\ln \lambda_{t+1} = (1-\gamma)\ln \lambda + \gamma \ln \lambda_t + \epsilon_{t+1}$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}$$

$$w_t = (1-\theta)\tfrac{Y_t}{H_t}$$

$$r_t = \theta \tfrac{Y_t}{K_t}$$

$$Y_t = C_t + I_t$$

$$\tfrac{aC_t}{1-H_t} = w_t$$

$$\tfrac{1}{C_t} = \mathbb{E}_t\left\{\tfrac{\beta}{C_{t+1}}\left(r_{t+1} + 1 - \delta\right)\right\}$$

The system above has 8 equations to solve for the evolution of 8 variables (λ_t, K_{t+1}, Y_t, C_t, I_t, H_t, r_t, w_t). The first and second equations are simply the laws of motion for total factor productivity and the capital stock, respectively. The third one is the production function. The fourth and fifth equations come from the profit-maximization problem, and simply say that factor prices (in terms of output) must equal their marginal products. The sixth equation imposes market clearing in the output market.[29] The seventh equation comes from the utility-maximization problem, and establishes an intratemporal relation between consumption and labor. It can be interpreted as a labor supply equation, where H_t is an increasing function of w_t, for given C_t. Finally, the eigth equation is

[29] We are also imposing equilibrium in factor markets, since we are using H_t (K_t) to denote both the demand and supply of labor (capital).

the standard Euler equation obtained from utility maximization. It establishes the connection between current and future consumption that must be satisfied by any optimal saving/consumption plan.

4.2 Log-linearization

Since the system displayed above is nonlinear, we cannot really use the solution method described earlier. What we will do is take a log-linear approximation of the model around its nonstochastic steady state. This will produce a linear system that we can solve. The solution of this system will be an accurate approximation of the solution to the original system as long as the equilibrium allocations are close enough to the nonstochastic steady state.

We start by calculating the unique nonstochastic steady state of the model. In this steady state there are no shocks ($\varepsilon_t = 0\ \forall t$) and all variables are constant ($X_{t+1} = X_t$, for any variable X). Using an upper bar to denote steady state quantities, we get:[30]

$$\begin{aligned}
\overline{\lambda} &= \lambda \\
\overline{r} &= \frac{1}{\beta} - 1 + \delta \\
\overline{w} &= (1-\theta)\overline{\lambda}\left(\frac{\theta\overline{\lambda}}{\overline{r}}\right)^{\frac{\theta}{1-\theta}} \\
\overline{K} &= \frac{\theta\overline{w}}{(a+1-\theta)\overline{r} - a\theta\delta} \\
\overline{I} &= \delta\overline{K} \\
\overline{H} &= \left(\frac{\overline{r}}{\theta\overline{\lambda}}\right)^{\frac{1}{1-\theta}}\overline{K} \\
\overline{Y} &= \frac{\overline{r}}{\theta}\overline{K} \\
\overline{C} &= (\frac{\overline{r}}{\theta} - \delta)\overline{K}
\end{aligned}$$

Now we proceed to log-linearize the model around its nonstochastic steady state (Appendix H describes the method of log-linearization). For any variable X we define $\widehat{X}_t \equiv \ln X_t - \ln \overline{X}$, which measures the relative deviation of X_t

[30] This model is simple enough so that we can solve for the nonstochastic steady state analytically. In more complicated models the steady state has to be found numerically.

from $\overline{X}$.[31] Log-linearizing the original system we get (See Appendix H):

$$\widehat{\lambda}_{t+1} = \gamma\widehat{\lambda}_t + \varepsilon_{t+1}$$

$$\widehat{K}_{t+1} = \delta\widehat{I}_t + (1-\delta)\widehat{K}_t$$

$$0 = \widehat{\lambda}_t + \theta\widehat{K}_t + (1-\theta)\widehat{H}_t - \widehat{Y}_t$$

$$0 = \widehat{Y}_t - \widehat{H}_t - \widehat{w}_t$$

$$0 = \widehat{Y}_t - \widehat{K}_t - \widehat{r}_t$$

$$0 = \overline{Y}\widehat{Y}_t - \overline{C}\widehat{C}_t - \overline{I}\widehat{I}_t$$

$$0 = \widehat{w}_t - \frac{\overline{H}}{1-\overline{H}}\widehat{H}_t - \widehat{C}_t$$

$$\mathbb{E}_t\widehat{C}_{t+1} - \beta\overline{r}\mathbb{E}_t\widehat{r}_{t+1} = \widehat{C}_t$$

The system above can be written in matrix form, as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\beta\overline{r} & 0 \end{bmatrix} \begin{bmatrix} \widehat{\lambda}_{t+1} \\ \widehat{K}_{t+1} \\ \mathbb{E}_t\widehat{Y}_{t+1} \\ \mathbb{E}_t\widehat{C}_{t+1} \\ \mathbb{E}_t\widehat{I}_{t+1} \\ \mathbb{E}_t\widehat{H}_{t+1} \\ \mathbb{E}_t\widehat{r}_{t+1} \\ \mathbb{E}_t\widehat{w}_{t+1} \end{bmatrix} =$$

$$\begin{bmatrix} \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 & \delta & 0 & 0 & 0 \\ 1 & \theta & -1 & 0 & 0 & 1-\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \overline{Y} & -\overline{C} & -\overline{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \frac{-\overline{H}}{1-\overline{H}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{\lambda}_t \\ \widehat{K}_t \\ \widehat{Y}_t \\ \widehat{C}_t \\ \widehat{I}_t \\ \widehat{H}_t \\ \widehat{r}_t \\ \widehat{w}_t \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[31] Recall that, when X_t is sufficiently close to $\overline{X}$, $\ln X_t - \ln\overline{X} \cong \frac{X_t - \overline{X}}{\overline{X}}$.

Let G denote the matrix on the left-hand side and A denote the matrix on the right-hand side. Define the following column vectors:

$$x_t \equiv \begin{bmatrix} \widehat{\lambda}_t \\ \widehat{K}_t \end{bmatrix}, \quad y_t \equiv \begin{bmatrix} \widehat{Y}_t \\ \widehat{C}_t \\ \widehat{I}_t \\ \widehat{H}_t \\ \widehat{r}_t \\ \widehat{w}_t \end{bmatrix}, \; \varepsilon_{t+1} \equiv \begin{bmatrix} \epsilon_{t+1} \\ 0 \end{bmatrix}$$

Then we can write:

$$G \begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0} \end{bmatrix}$$

which is a system in form (45).

4.3 Parametrization and Solution

To solve the model numerically we need to assign particular values to its parameters. Following Hansen (1985) we set: $\theta = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\gamma = 0.95$, $\lambda = 1$, $a = 2$, and $\sigma_\epsilon = 0.00712$.[32] These figures imply: $\overline{\lambda} = 1$, $\overline{r} = 0.035$, $\overline{w} = 2.37$, $\overline{K} = 11.43$, $\overline{H} = 0.301$, $\overline{Y} = 1.114$, $\overline{I} = 0.286$, and $\overline{C} = 0.829$.

From our previous analyisis we know the solution is of the form $x_{t+1} = Mx_t + \xi_{t+1}$ and $y_t = Cx_t$.[33] Using a computer we get:[34]

$$M = \begin{bmatrix} 0.95 & 0 \\ 0.1162 & 0.9528 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.4874 & 0.1932 \\ 0.3981 & 0.5660 \\ 4.6468 & -0.8879 \\ 0.7616 & -0.2606 \\ 1.4874 & -0.8068 \\ 0.7258 & 0.4538 \end{bmatrix}$$

[32] Hansen calibrates the model in order to match quarterly data for the US over the period 1955.Q3 - 1984.Q1.

[33] Notice that $G_{xx} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and then $G_{xx}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then: $\xi_{t+1} = G_{xx}^{-1}\varepsilon_{t+1}$ gives: $\xi_{t+1} = \varepsilon_{t+1}$.

[34] We use the Matlab code developed by Paul Klein, solab.m, available at http://paulklein.se/newsite/codes/codes.php

Then, $x_{t+1} = Mx_t + \xi_{t+1}$ becomes:

$$\widehat{\lambda}_{t+1} = 0.95\widehat{\lambda}_t + \varepsilon_{t+1}$$

$$\widehat{K}_{t+1} = 0.1162\widehat{\lambda}_t + 0.9528\widehat{K}_t$$

And $y_t = Cx_t$ gives:

$$\widehat{Y}_t = 1.4874\widehat{\lambda}_t + 0.1932\widehat{K}_t$$

$$\widehat{C}_t = 0.3981\widehat{\lambda}_t + 0.5660\widehat{K}_t$$

$$\widehat{I}_t = 4.6468\widehat{\lambda}_t - 0.8879\widehat{K}_t$$

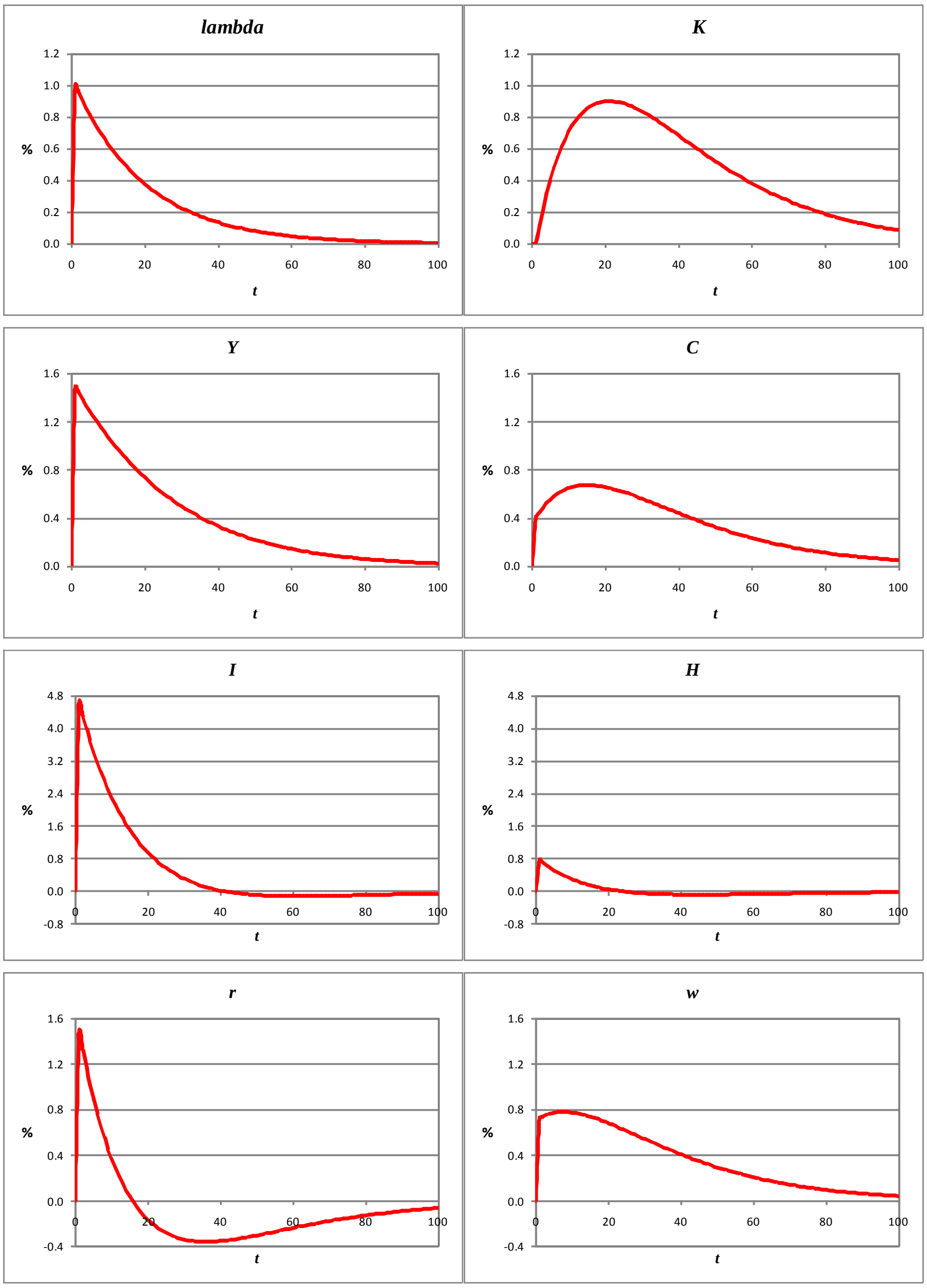
$$\widehat{H}_t = 0.7616\widehat{\lambda}_t - 0.2606\widehat{K}_t$$

$$\widehat{r}_t = 1.4874\widehat{\lambda}_t - 0.8068\widehat{K}_t$$

$$\widehat{w}_t = 0.7258\widehat{\lambda}_t + 0.4538\widehat{K}_t$$

The expressions above allow us to easily calculate the impulse response of each variable to a technology shock. Assuming that in period 0 the economy is in steady state and that a 1% technology shock occurs in period 1 (*i.e.*, $\epsilon_1 = 1$, $\epsilon_t = 0 \ \forall t \geq 2$) we get the figures displayed below (for each variable, time is in the horizontal axis and the %-deviation from the nonstochastic steady state is in the vertical axis).

Impulse Responses to a 1% Technology Shock



Finally, we can use the formulas described earlier to obtain second moments for all the variables. For example, using (42) and (40) we obtain the variance-covariance matrix for the nonpredetermined and predetermined variables:

$$\Sigma_x = \begin{bmatrix} \sigma_\lambda^2 & \sigma_{\lambda K} \\ \sigma_{K\lambda} & \sigma_K^2 \end{bmatrix} = \begin{bmatrix} 5.20 & 6.05 \\ 6.05 & 15.29 \end{bmatrix}$$

$$\Sigma_y = \begin{bmatrix} \sigma_Y^2 & \sigma_{YC} & \sigma_{YI} & \sigma_{YH} & \sigma_{Yr} & \sigma_{Yw} \\ \sigma_{CY} & \sigma_C^2 & \sigma_{CI} & \sigma_{CH} & \sigma_{Cr} & \sigma_{Cw} \\ \sigma_{IY} & \sigma_{IC} & \sigma_I^2 & \sigma_{IH} & \sigma_{Ir} & \sigma_{Iw} \\ \sigma_{HY} & \sigma_{HC} & \sigma_{HI} & \sigma_H^2 & \sigma_{Hr} & \sigma_{Hw} \\ \sigma_{rY} & \sigma_{rC} & \sigma_{rI} & \sigma_{rH} & \sigma_r^2 & \sigma_{rw} \\ \sigma_{wY} & \sigma_{wC} & \sigma_{wI} & \sigma_{wH} & \sigma_{wr} & \sigma_w^2 \end{bmatrix}$$

$$= \begin{bmatrix} 15.6 & 10.3 & 30.8 & 3.7 & 3.6 & 11.9 \\ 10.3 & 8.4 & 15.7 & 1.3 & -0.8 & 9.0 \\ 30.8 & 15.7 & 74.4 & 10.5 & 16.2 & 20.2 \\ 3.7 & 1.3 & 10.5 & 1.7 & 3.0 & 2.0 \\ 3.6 & -0.8 & 16.2 & 3.0 & 6.9 & 0.6 \\ 11.9 & 9.0 & 20.2 & 2.0 & 0.6 & 9.9 \end{bmatrix}$$

Remark. One way of evaluating the performance of a model is to compare the second moments generated by the model to empirical second moments obtained from time-series data. In our RBC model there is no long-run growth. It is not difficult to show, however, that our model is equivalent to one with constant long-run growth where all variables have been normalized by the growth component. Hence, if Z_t grows in the long-run at rate g, we can define a new variable, w_t, as follows: $z_t \equiv \frac{Z_t}{Z_0 e^{gt}}$. In a balanced-growth path, Z_t grows at rate g, and then z_t is constant at some value, $\overline{z}$. The relative deviation of z_t from $\overline{z}$ is then: $\widehat{z}_t \equiv \ln z_t - \ln \overline{z} = \ln(\frac{Z_t}{Z_0 e^{gt}}) - \ln \overline{z} = \ln Z_t - \ln Z_0 - gt - \ln \overline{z}$. This shows that the model-generated second moments obtained above are comparable to the second moments generated from data that has been linearly detrended (in logs). Linear detrending, however, is not the only possibility. In particular, it is standard among researchers to detrend macro variables using the Hodrick-Prescott (HP) filter. In this case, it would be inappropriate to compare the moments calculated above to their empirical counterparts. To do a meaningful comparison one would first need to determine the mapping from the moments of our "hatted" variables to the moments of HP-filtered data. For different ways of doing this analytically, see Burnside (1999) and Uhlig (1999). Another possibility is to generate simulated time series with the model, detrend the simulated data using the HP filter, and then calculate the second moments of the filtered simulated data. An advantage of the latter methodology is that, by simulating many time-paths for each variable, one can provide standard errors for the estimated moments.

5 Appendices

Appendix A

In this appendix we present an example to show how to reduce a higher-order model to the first-order form given in (3). We also provide an example of a model with lagged expectations that can be put in form (3).[35]

Example 1. A higher-order model.
Consider the following model:

$$Y_t + \alpha Y_{t-2} + \beta \mathbb{E}_t Y_{t+2} = \eta_t \tag{65}$$

where $\eta_t = \rho \eta_{t-1} + \nu_t$ and ν_t is an exogenous white-noise process. We have initial conditions for Y_{-2}, Y_{-1}, and η_0.

The model can be rewritten as follows:

$$\begin{bmatrix} \eta_{t+1} \\ Y_{t-1} \\ Y_t \\ \mathbb{E}_t Y_{t+1} \\ \mathbb{E}_t Y_{t+2} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{\beta} & -\frac{\alpha}{\beta} & 0 & -\frac{1}{\beta} & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ Y_{t-2} \\ Y_{t-1} \\ Y_t \\ \mathbb{E}_t Y_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{66}$$

Define:

$$x_t \equiv \begin{bmatrix} \eta_t \\ Y_{t-2} \\ Y_{t-1} \end{bmatrix}, \quad y_t \equiv \begin{bmatrix} Y_t \\ \mathbb{E}_t Y_{t+1} \end{bmatrix}, \quad \varepsilon_{t+1} \equiv \begin{bmatrix} \nu_{t+1} \\ 0 \\ 0 \end{bmatrix}$$

Then:

$$x_{t+1} = \begin{bmatrix} \eta_{t+1} \\ Y_{t-1} \\ Y_t \end{bmatrix}, \quad \mathbb{E}_t y_{t+1} = \mathbb{E}_t \begin{bmatrix} Y_{t+1} \\ \mathbb{E}_{t+1} Y_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_t Y_{t+1} \\ \mathbb{E}_t Y_{t+2} \end{bmatrix}$$

where we have used the law of iterated expectations to write $\mathbb{E}_t\{\mathbb{E}_{t+1} Y_{t+2}\} = \mathbb{E}_t Y_{t+2}$.

Notice that $x_0 = (\eta_0, Y_{-2}, Y_{-1})'$ is given, and $x_{t+1} - \mathbb{E}_t x_{t+1} = \varepsilon_{t+1}$ is exogenous. Therefore, x_t is a vector of predetermined variables.

Using the definitions given above, we can rewrite (66) in form (3) as follows:

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0_{2\times 1} \end{bmatrix}$$

where A is the 5×5 matrix given in (66), x_t is predetermined and y_t is nonpredetermined.

[35] We closely follow Examples B and D in Blanchard and Kahn (1980).

Example 2. A model with lagged expectations
Consider the following model:
(i) $Y_t = C_t + I_t + G_t$ with $G_{t+1} = \rho_g G_t + \varepsilon^g_{t+1}$.
(ii) $C_t = \alpha(Y_t + \mathbb{E}_t Y_{t+1}) + \vartheta_t$ with $\vartheta_{t+1} = \rho_\vartheta \vartheta_t + \varepsilon^\vartheta_{t+1}$.
(iii) $I_t = \beta(\mathbb{E}_t Y_{t+1} - \mathbb{E}_{t-1} Y_t) + \nu_t$ with $\nu_{t+1} = \rho_\nu \nu_t + \varepsilon^\nu_{t+1}$
where ε^g_{t+1}, $\varepsilon^\vartheta_{t+1}$, and ε^ν_{t+1} are *i.i.d.* zero-mean shocks, $\alpha, \beta > 0$, $\rho_g, \rho_\vartheta, \rho_\nu \in (0, 1)$, and G_0, ϑ_0, ν_0, and $\mathbb{E}_{-1} Y_0$ are given. Notice that the third equation includes the lagged expectation of a current variable.

Substituting (ii) and (iii) into (i), and rearranging, we get:

$$\mathbb{E}_t Y_{t+1} = \frac{1-\alpha}{\alpha+\beta} Y_t + \frac{\beta}{\alpha+\beta} \mathbb{E}_{t-1} Y_t - \frac{1}{\alpha+\beta}(G_t + \vartheta_t + \nu_t) \tag{67}$$

Define: $X_t \equiv \mathbb{E}_{t-1} Y_t \Rightarrow X_{t+1} = \mathbb{E}_t Y_{t+1}$. Then we can write:

$$\begin{bmatrix} G_{t+1} \\ \vartheta_{t+1} \\ \nu_{t+1} \\ X_{t+1} \\ \mathbb{E}_t Y_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_g & 0 & 0 & 0 & 0 \\ 0 & \rho_\vartheta & 0 & 0 & 0 \\ 0 & 0 & \rho_\nu & 0 & 0 \\ -\frac{1}{\alpha+\beta} & -\frac{1}{\alpha+\beta} & -\frac{1}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & \frac{1-\alpha}{\alpha+\beta} \\ -\frac{1}{\alpha+\beta} & -\frac{1}{\alpha+\beta} & -\frac{1}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & \frac{1-\alpha}{\alpha+\beta} \end{bmatrix} \begin{bmatrix} G_t \\ \vartheta_t \\ \nu_t \\ X_t \\ Y_t \end{bmatrix} + \begin{bmatrix} \varepsilon^g_{t+1} \\ \varepsilon^\vartheta_{t+1} \\ \varepsilon^\nu_{t+1} \\ 0 \\ 0 \end{bmatrix} \tag{68}$$

Define:

$$x_t \equiv \begin{bmatrix} G_t \\ \vartheta_t \\ \nu_t \\ X_t \end{bmatrix}, \quad y_t \equiv [Y_t], \quad \varepsilon_{t+1} \equiv \begin{bmatrix} \varepsilon^g_{t+1} \\ \varepsilon^\vartheta_{t+1} \\ \varepsilon^\nu_{t+1} \\ 0 \end{bmatrix}$$

Notice that $x_0 = (G_0, \vartheta_0, \nu_0, X_0)'$ is given, and $x_{t+1} - \mathbb{E}_t x_{t+1} = (\varepsilon^g_{t+1}, \varepsilon^\vartheta_{t+1}, \varepsilon^\nu_{t+1}, 0)'$ is exogenous. Therefore, x_t is a vector of predetermined variables.

We can rewrite (68) in form (3) as follows:

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

where A is the 5×5 matrix given in (68), x_t is predetermined, and y_t is non-predetermined. Notice that A has two rows that are equal, and therefore is singular.

Hence, we have managed to write a model with lagged expectations of a current variable in form (3).

Appendix B

In this appendix we show how to convert models of the form

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = C + A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix}$$

into form (3), by redefining variables as deviations from their nonstochastic steady-state levels.

In a nonstochastic steady state we have:

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = C + A \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$$

Assuming $I - A$ is invertible, the expression above gives:[36]

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = (I - A)^{-1} C$$

Define the deviations from the nonstochastic steady-state values as follows:

$$\begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix} \equiv \begin{bmatrix} x_t - \overline{x} \\ y_t - \overline{y} \end{bmatrix}$$

Then, subtracting $\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = C + A \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$ from the original system we get:

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} - \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = C + A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix} - C - A \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$$

$$\begin{bmatrix} x_{t+1} - \overline{x} \\ \mathbb{E}_t y_{t+1} - \overline{y} \end{bmatrix} = A \left(\begin{bmatrix} x_t \\ y_t \end{bmatrix} - \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right) + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix}$$

$$\begin{bmatrix} x_{t+1} - \overline{x} \\ \mathbb{E}_t \{ y_{t+1} - \overline{y} \} \end{bmatrix} = A \begin{bmatrix} x_t - \overline{x} \\ y_t - \overline{y} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix}$$

Then:

$$\begin{bmatrix} \widehat{x}_{t+1} \\ \mathbb{E}_t \widehat{y}_{t+1} \end{bmatrix} = A \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix}$$

which is the expression we were looking for.

[36] The eigenvalues of $I - A$ are of the form $1 - \lambda$, where λ is an eigenvalue of A. The invertibility of $I - A$ requires that the product of its eigenvalues be different from zero (since the product of the eigenvalues is equal to the determinant): $(1 - \lambda_1)(1 - \lambda_2)...(1 - \lambda_n) \neq 0$. Hence, if A has at least one eigenvalue equal to one, $I - A$ will be singular and the model will not have a well-defined nonstochastic steady state (unless $C = \mathbf{0}$, in which case $\overline{x} = \mathbf{0}$ and $\overline{y} = \mathbf{0}$).

Appendix C

In this appendix we present the (Complex) Schur Decomposition and the Generalized (Complex) Schur decomposition.

C.1 (Complex) Schur Decomposition

We start with some definitions.

Definitions: Conjugate Transpose, and Unitary Matrix.
Let A denote an $m \times n$ matrix of (possibly) complex numbers:

$$A = \begin{bmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i & ... & a_{1n} + b_{1n}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i & ... & a_{2n} + b_{2n}i \\ ... & ... & ... & ... \\ a_{m1} + b_{m1}i & a_{m2} + b_{m2}i & ... & a_{mn} + b_{mn}i \end{bmatrix}$$

The *conjugate transpose* of A, denoted A^H, is formed by transposing A and replacing each element with its complex conjugate:

$$A^H = \begin{bmatrix} a_{11} - b_{11}i & a_{21} - b_{21}i & ... & a_{m1} - b_{m1}i \\ a_{12} - b_{12}i & a_{22} - b_{22}i & ... & a_{m2} - b_{m2}i \\ ... & ... & ... & ... \\ a_{1n} - b_{1n}i & a_{2n} - b_{2n}i & ... & a_{mn} - b_{mn}i \end{bmatrix}$$

Notice that, if A is real, then A^H and A' denote the same matrix (where A' denotes the transpose of a real matrix).

A complex $n \times n$ matrix A is *unitary* if its conjugate transpose coincides with its inverse, *i.e.*, if $A^H = A^{-1}$. Therefore, when A is unitary, we have: $AA^H = A^H A = I$, where I is the $n \times n$ identity matrix.

As an example, consider the following 2×2 matrix:

$$A = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\ \\ \frac{1}{2} - \frac{1}{2}i & -\frac{1}{2} + \frac{1}{2}i \end{bmatrix}$$

Its conjugate transpose is:

$$A^H = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\ \\ \frac{1}{2} - \frac{1}{2}i & -\frac{1}{2} - \frac{1}{2}i \end{bmatrix}$$

It is easy to check that $AA^H = I$, where I is the 2×2 identity matrix. Therefore, $A^H = A^{-1}$, which shows that A is unitary.

Definitions: Eigenvalues and Eigenvectors.
Let A be a (possibly complex) $n \times n$ matrix. A (possibly complex) scalar λ is an *eigenvalue* of A if it satisfies $Ax = \lambda x$, for $x \neq \mathbf{0}$. The $n \times 1$ (possibly complex) vectors $x \neq \mathbf{0}$ that satisfy $Ax = \lambda x$ for a given λ are called the (right) eigenvectors of A (corresponding to λ).

Remarks. Notice that $Ax = \lambda x$ can be rewritten as follows: $(A - \lambda I)x = \mathbf{0}$. For a given λ, this homogeneous system will have a nontrivial solution ($x \neq \mathbf{0}$) if and only if $|A - \lambda I| = 0$. The determinant $|A - \lambda I|$ is an n-th order polynomial in λ, and is called the *characteristic polynomial for A*. Analogously, equation $|A - \lambda I| = \mathbf{0}$ is called the *characteristic equation for A*. From the fundamental theorem of algebra we know that the characteristic equation will have n roots. These roots may be either real or complex numbers, and need not be all different. For values of λ different from the roots of $|A - \lambda I| = \mathbf{0}$, the only solution to $Ax = \lambda x$ is $x = \mathbf{0}$. Therefore, there are n eigenvalues of A, which coincide with the n roots of the characteristic equation. We also have the result that, if A is a real matrix, then complex eigenvalues come in conjugate pairs. The eigenvalues can be computed using the so called QR method (see Golub and van Loan (1996)).

Theorem: (Complex) Schur Decomposition.
Let A be a complex $n \times n$ matrix. Then, there exists a complex, unitary $n \times n$ matrix Z such that

$$Z^H A Z = T \tag{69}$$

where T is a complex, upper triangular $n \times n$ matrix with the eigenvalues of A in its diagonal. Furthermore, Z can be chosen so that the eigenvalues of A appear in any order along the diagonal of T.

Proof. See Golub and van Loan (1996), Theorem 7.1.3, page 313.

Remark 1. Premultiplying (69) by Z and posmultiplying it by Z^H we obtain: $ZZ^H AZZ^H = ZTZ^H$. Then:

$$A = ZTZ^H \tag{70}$$

since $ZZ^H = I$, because Z is unitary.

Remark 2. Reordering the eigenvalues in the diagonal of T requires that Z be altered conformably so that (69) holds. This can be quite involved. Fortunately, there is software available to do it.

C.2 Generalized (Complex) Schur Decomposition

Definition: Generalized Eigenvalue Problem.
The generalized eigenvalue problem for a pair of (possibly complex) $n \times n$ matrices (A, B) is the problem of finding the (possibly complex) scalar λ and the (possibly complex) $n \times 1$ vectors $x \neq \mathbf{0}$ that satisfy $Ax = \lambda Bx$. The values of λ that satisfy the equation are the *generalized eigenvalues*, and the corresponding vectors $x \neq \mathbf{0}$ are the *generalized* (*right*) *eigenvectors*.

*Remarks***.** We can rewrite $Ax = \lambda Bx$ as follows: $(A - \lambda B)x = \mathbf{0}$ If there exists λ such that $|A - \lambda B| \neq \mathbf{0}$, the matrix pair (A, B) is said to be *regular*; otherwise it is called *singular*. If B is invertible, then the pair (A, B) is regular. In this case, we can reduce the generalized eigenvalue problem to a standard eigenvalue problem premultiplying $Ax = \lambda Bx$ by B^{-1} to get $B^{-1}Ax = \lambda x$, and the number of generalized eigenvalues is exactly equal to n. If B is noninvertible and (A, B) is regular, there are p finite generalized eigenvalues, and $n-p$ infinite generalized eigenvalues, where p is the degree of the polynomial $|A - \lambda B|$ (this includes the possibility that $p = 0$, *i.e.*, that there are no finite generalized eigenvalues).[37] Finally, if B is noninvertible and (A, B) is singular, then $|A - \lambda B| = \mathbf{0}$ for any λ, and there is an infinite number of generalized eigenvalues. When B is noninvertible, finding the generalized eigenvalues requires that the QZ method is used instead of the aforementioned QR method (see Golub and van Loan (1996)).

Theorem: Generalized (Complex) Schur Decomposition.
Let A and B be complex $n \times n$ matrices. Then, there exist complex, unitary $n \times n$ matrices Q and Z such that

$$Q^H AZ = T \tag{71}$$

$$Q^H BZ = S \tag{72}$$

where S and T are complex, upper triangular $n \times n$ matrices. The diagonal elements of T divided by the diagonal elements of S, $\frac{t_{ii}}{s_{ii}}$, are the generalized eigenvalues of the matrix pair (A, B). If $s_{ii} \neq 0$ and $t_{ii} \neq 0$, the generalized eigenvalue is finite; if $s_{ii} = 0$ and $t_{ii} \neq 0$, the generalized eigenvalue is infinite (by convention);[38] if $s_{ii} = 0$ and $t_{ii} = 0$, the matrix pair (A, B) is singular and there is an infinity of generalized eigenvalues. Furthermore, the pairs (s_{ii}, t_{ii}) can be arranged in any order.

Proof. See Golub and van Loan (1996), Theorem 7.7.1, page 377.

[37] There is some abuse of language in allowing generalized eigenvalues to become infinite. Strictly speaking, they have to belong to the complex field. Therefore, a situation with no finite eigenvalues would be described as one in which the set of generalized eigenvalues is empty.

[38] See the previous footnote.

Remark 1. Premultiplying (71) and (72) by Q, and posmultiplying them by Z^H, we get:

$$A = QTZ^H \quad (73)$$
$$B = QSZ^H \quad (74)$$

since Z and Q are unitary.

Remark 2. Reordering the pairs (s_{ii}, t_{ii}) requires to reorder Q and Z conformably. This can be complicated, but there is software available to do it.

Remark 3. To get some intuition why the Generalized Schur Decomopositon works, suppose S turns out to be invertible. From the generalized eigenvalue problem for the matrix pair (A, B) we have: $Ax = \lambda Bx$. Using (73) and (74) we obtain: $QTZ^H x = \lambda QSZ^H x$. Premultiplying both sides by Q^H we get: $TZ^H x = \lambda SZ^H x$. And then: $Tw = \lambda Sw$, where $w \equiv Z^H x$. Premultiplying by S^{-1} we get: $S^{-1}Tw = \lambda w$. This expression shows that λ is an eigenvalue of $S^{-1}T$. Since S is upper triangular, S^{-1} is also upper triangular, with diagonal elements $\frac{1}{s_{ii}}$. Since T is upper triangular, $S^{-1}T$ is upper triangular, with diagonal elements, $\frac{t_{ii}}{s_{ii}}$. Recall that the eigenvalues of an upper triangular matrix coincide with its diagonal elements. Therefore, $\frac{t_{ii}}{s_{ii}}$ are the eigenvalues of $S^{-1}T$. Hence, we have shown that the generalized eigenvalues of the matrix pair (A, B) are the diagonal elements of $S^{-1}T$, $\frac{t_{ii}}{s_{ii}}$.

Remark 4. From the theorem above we know that, if $s_{ii} = 0$ and $t_{ii} = 0$, the matrix pair (A, B) is singular. Therefore, if (A, B) is nonsingular (*i.e.*, regular), we cannot have $s_{ii} = 0$ and $t_{ii} = 0$.

Appendix D

In this appendix we show that (18) implies that the boundary condition (9) is violated unless $\delta_t = 0$ for each t.

From (18) we have:
(i) $\mathbb{E}_t \delta_{t+1} = T_{\delta\delta}\delta_t$
Then:
$\mathbb{E}_{t+1}\delta_{t+2} = T_{\delta\delta}\delta_{t+1}$
Then:
$\mathbb{E}_t\{\mathbb{E}_{t+1}\delta_{t+2}\} = \mathbb{E}_t T_{\delta\delta}\delta_{t+1}$
$\mathbb{E}_t\delta_{t+2} = T_{\delta\delta}\mathbb{E}_t\delta_{t+1}$, using the law of iterated expectations:
Then:
(ii) $\mathbb{E}_t\delta_{t+2} = (T_{\delta\delta})^2\delta_t$, using (i).
If we keep iterating forward we obtain:

$$\mathbb{E}_t\delta_{t+s} = (T_{\delta\delta})^s\delta_t \quad (75)$$

Since $T_{\delta\delta}$ is an upper-triangular matrix with diagonal elements t_{ii}, $(T_{\delta\delta})^s$ is an upper-triangular matrix with diagonal elements $\tau_{ii} = (t_{ii})^s$. Also, since $|t_{ii}| > 1$, we get: $|\tau_{ii}| = |t_{ii}|^s > 1$.

We can rewrite (75) as follows:

$$\begin{bmatrix} \mathbb{E}_t\delta_{1t+s} \\ \mathbb{E}_t\delta_{2t+s} \\ ..... \\ \mathbb{E}_t\delta_{n_\delta t+s} \end{bmatrix} = \begin{bmatrix} (t_{11})^s & \tau_{12} & ... & \tau_{1n_\delta} \\ 0 & (t_{22})^s & ... & \tau_{2n_\delta} \\ ... & ... & ... & ... \\ 0 & 0 & 0 & (t_{n_\delta n_\delta})^s \end{bmatrix} \begin{bmatrix} \delta_{1t} \\ \delta_{2t} \\ ..... \\ \delta_{n_\delta t} \end{bmatrix} \tag{76}$$

where τ_{ij}, the typical nondiagonal element of $(T_{\delta\delta})^s$, is a function of the elements of $T_{\delta\delta}$ (including powers of its diagonal elements).

From (76) we get:

$$\begin{aligned} &\mathbb{E}_t\delta_{1t+s} = (t_{11})^s\delta_{1t} + \tau_{12}\delta_{2t} + \tau_{13}\delta_{3t} + ... + \tau_{1n_\delta}\delta_{n_\delta t} \\ &\mathbb{E}_t\delta_{2t+s} = (t_{22})^s\delta_{2t} + \tau_{23}\delta_{3t} + ... + \tau_{2n_\delta}\delta_{n_\delta t} \\ &..... \\ &\mathbb{E}_t\delta_{n_\delta-1t+s} = (t_{n_\delta-1n_\delta-1})^s\delta_{n_\delta-1t} + \tau_{n_\delta-1n_\delta}\delta_{n_\delta t} \\ &\mathbb{E}_t\delta_{n_\delta t+s} = (t_{n_\delta n_\delta})^s\delta_{n_\delta t} \end{aligned} \tag{77}$$

Since $|t_{n_\delta n_\delta}|^s > 1$, the last line in (77) implies that $\mathbb{E}_t\delta_{n_\delta t+s}$ explodes when $s \to \infty$, unless $\delta_{n_\delta t} = 0$. Using this into the previous line we get: $\mathbb{E}_t\delta_{n_\delta-1t+s} = (t_{n_\delta-1n_\delta-1})^s\delta_{n_\delta-1t}$. Then $\mathbb{E}_t\delta_{n_\delta-1t+s}$ explodes when $s \to \infty$, unless $\delta_{n_\delta-1t} = 0$. If we keep doing this, we conclude that, for any $i \in \{1, 2, ..., n_\delta\}$, $\mathbb{E}_t\delta_{i\ t+s}$ explodes when $s \to \infty$, unless $\delta_{it} = 0$. Therefore, $\mathbb{E}_t\delta_{t+s}$ explodes when $s \to \infty$, unless $\delta_t = 0$.

We know from (20) that, when $\delta_t = 0$, $\mathbb{E}_t\theta_{t+1} = T_{\theta\theta}\theta_t$. Recall that the diagonal elements of $T_{\theta\theta}$ have modulus *smaller* than 1. Therefore, a procedure analogous to the one given above, allows us to conclude that $\mathbb{E}_t\theta_{t+s}$ *does not explode* as $s \to \infty$.

From (21) we know $x_t = Z_{x\theta}\theta_t + Z_{x\delta}\delta_t$ and $y_t = Z_{y\theta}\theta_t + Z_{y\delta}\delta_t$. Then: $\mathbb{E}_t x_{t+s} = Z_{x\theta}\mathbb{E}_t\theta_{t+s} + Z_{x\delta}\mathbb{E}_t\delta_{t+s}$ and $\mathbb{E}_t y_{t+s} = Z_{y\theta}\mathbb{E}_t\theta_{t+s} + Z_{y\delta}\mathbb{E}_t\delta_{t+s}$. Therefore, $\mathbb{E}_t x_{t+s}$ and $\mathbb{E}_t y_{t+s}$ will explode as $s \to \infty$ if $\mathbb{E}_t\delta_{t+s}$ does, violating the boundary condition (9).

For the more general model of Section 3, equation (51) gives: $S_{\delta\delta}\mathbb{E}_t\delta_{t+1} = T_{\delta\delta}\delta_t$. Since $S_{\delta\delta}$ may not be invertible, we cannot write $\mathbb{E}_t\delta_{t+1} = S_{\delta\delta}^{-1}T_{\delta\delta}\delta_t$ (in this case the analysis would be exactly as above). We know, however, that $T_{\delta\delta}$ *is* invertible. Therefore, we can premultiply (51) by $T_{\delta\delta}^{-1}$ to get: $T_{\delta\delta}^{-1}S_{\delta\delta}\mathbb{E}_t\delta_{t+1} = \delta_t$. Notice that $T_{\delta\delta}^{-1}S_{\delta\delta}$ is an upper-triangular matrix with diagonal elements that satisfy $\left|\frac{s_{ii}}{t_{ii}}\right| < 1$. Therefore, a reasoning similar to the one done before, shows that $\mathbb{E}_t\delta_{t+s}$ explodes as $s \to \infty$, unless $\delta_t = 0\ \forall t$.

Appendix E

In this appendix we give an example of a model in which the number of stable eigenvalues coincides with the number of predetermined variables but $Z_{x\theta}$ is not invertible.[39]

Consider the following model:

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \tag{78}$$

where x_t is predetermined and y_t is nonpredetermined.

Applying the Schur Decomposition we obtain:

$$A = ZTZ^H \Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Notice that there is one stable eigenvalue (0.5). Hence, the number of stable eigenvalues coincides with the number of predetermined variables. However, from

$$Z = \begin{bmatrix} Z_{x\theta} & Z_{x\delta} \\ Z_{y\theta} & Z_{y\delta} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

we obtain $Z_{x\theta} = [0]$, which is a singular 1×1 matrix. Therefore, even though the number of stable eigenvalues coincides with the number of predetermined variables, $Z_{x\theta}$ is not invertible. Hence, we cannot apply Result 1 in Section 2 to conclude that there is a unique stable solution. Actually, it is easy to see that a solution to (78) has to be unstable. From the first equation we obtain: $x_{t+1} = 2x_t + \varepsilon_{t+1}$. Iterating forward we get: $x_{t+j} = 2^j x_t + \sum_{i=0}^{j-1} 2^i \varepsilon_{t+j-i}$. Then: $\mathbb{E}_t x_{t+j} = 2^j x_t$. Then: $\lim_{j\to\infty} |\mathbb{E}_t x_{t+j}| = \infty$, unless $x_t = \mathbf{0}$. In particular, $\lim_{j\to\infty} |\mathbb{E}_0 x_j| = \infty$, unless $x_0 = \mathbf{0}$.

In terms of the transformed system (17), the problem is that the noninvertibility of $Z_{x\theta}$ does not allow us to invert $x_t = Z_{x\theta}\theta_t$ (see (22)) in order to get $\theta_t = Z_{x\theta}^{-1} x_t$. This, in turn, prevents us from using the initial condition x_0 to get a unique initial condition, θ_0, for the stable auxiliary variable. Without this initial condition, we cannot solve (17) uniquely.

[39] The example is taken from Klein (2000).

Appendix F

In this appendix we exemplify Results 2 and 3 in Section 2.

A model with no stable solutions.
Consider the Cagan Model again, but assume that $\rho > 1$. Then, we have one predetermined variable (m_t) and two unstable eigenvalues (ρ and $\frac{1}{\alpha}$)[40]. Money supply evolves according to $m_{t+1} = \rho m_t + \varepsilon_{t+1}$. Iterating forward and taking expectations we get: $\mathbb{E}_t m_{t+s} = \rho^s m_t$. Since $\rho > 1$, $\lim\limits_{s\to\infty} |\mathbb{E}_t m_{t+s}| = \infty$.

We can also show that the price sequence explodes. Keeping the assumption that $\rho > 1$ and $\alpha \in (0,1)$, asume $\alpha\rho \in (0,1)$ (*i.e.*, $\rho < \frac{1}{\alpha}$). In this case, the solution for p_t coincides with the one given in the text: $p_t = \frac{1-\alpha}{1-\alpha\rho} m_t$. Then: $p_{t+s} = \frac{1-\alpha}{1-\alpha\rho} m_{t+s} \Rightarrow \mathbb{E}_t p_{t+s} = \frac{1-\alpha}{1-\alpha\rho} \mathbb{E}_t m_{t+s} \Rightarrow \mathbb{E}_t p_{t+s} = \frac{1-\alpha}{1-\alpha\rho} \rho^s m_t$. Then, $\lim\limits_{s\to\infty} |\mathbb{E}_t p_{t+s}| = \infty$, unless $m_t = 0$.[41]

Hence, the system has no stable solution. This exemplifies Result 2 in Section 2.

A model with an infinite number of stable solutions.
Following Söderlind (2001), we modify equation (6) of the Cagan model as follows:

$$p_t = \alpha \mathbb{E}_t p_{t+1} + m_t, \qquad \alpha > 0. \tag{79}$$

The money supply equation (7) remains the same. We can write the system in matrix form as follows:

$$\begin{bmatrix} m_{t+1} \\ \mathbb{E}_t p_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} m_t \\ p_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \qquad m_0 \text{ given} \tag{80}$$

where m_t is predetermined, p_t is nonpredetermined, and $|\rho| < 1$.

Since A is triangular, its eigenvalues coincide with its diagonal elements. Suppose $\alpha > 1$. Then, both eigenvalues, ρ and $\frac{1}{\alpha}$, are stable. Therefore, we have a model with two stable eigenvalues and only one predetermined variable.

Iterating forward on $m_{t+1} = \rho m_t + \varepsilon_{t+1}$ we get: $\mathbb{E}_t m_{t+s} = \rho^s m_t$. Iterating forward on (79), using $\mathbb{E}_t m_{t+s} = \rho^s m_t$ and the law of iterated expectations, we get: $p_t = [1 + (\alpha\rho) + (\alpha\rho)^2 + ... + (\alpha\rho)^T] m_t + \alpha^{T+1} \mathbb{E}_t p_{t+T+1}$.

Suppose $|\alpha\rho| < 1$ and let $T \to \infty$. Imposing the condition $\lim\limits_{T\to\infty} \alpha^{T+1} \mathbb{E}_t p_{t+T+1} = 0$ we get the stable solution:

$$p_t^* = \frac{1}{1-\alpha\rho} m_t \tag{81}$$

[40] Recall that A is upper triangular and then its eigenvalues coincide with its diagonal elements, ρ and $\frac{1}{\alpha}$. Recall, also, that $\alpha \in (0,1)$.

[41] Recall that, to find the solution $p_t = \frac{1-\alpha}{1-\alpha\rho} m_t$, we imposed the condition $\lim\limits_{T\to\infty} \alpha^{T+1} \mathbb{E}_t p_{t+T+1} = 0$. This condition still holds, even though the price sequence explodes. We have: $\alpha^{T+1} \mathbb{E}_t p_{t+T+1} = \alpha^{T+1} \frac{1-\alpha}{1-\alpha\rho} \mathbb{E}_t m_{t+T+1} = \alpha^{T+1} \frac{1-\alpha}{1-\alpha\rho} \rho^{T+1} m_t = (\alpha\rho)^{T+1} \frac{1-\alpha}{1-\alpha\rho} m_t$. Then: $\lim\limits_{T\to\infty} \alpha^{T+1} \mathbb{E}_t p_{t+T+1} = \lim\limits_{T\to\infty} (\alpha\rho)^{T+1} \frac{1-\alpha}{1-\alpha\rho} m_t = 0$, since $\alpha\rho \in (0,1)$.

However, (81) is not the only stable solution. Consider an expression of the form:

$$p_t = p_t^* + b_t \tag{82}$$

where b_t is a "bubble". Substituting (82) in both sides of (79) we get:
$p_t^* + b_t = \alpha\mathbb{E}_t(p_{t+1}^* + b_{t+1}) + m_t$
$\frac{1}{1-\alpha\rho}m_t + b_t = \frac{\alpha}{1-\alpha\rho}\mathbb{E}_t m_{t+1} + \alpha\mathbb{E}_t b_{t+1} + m_t$
$\frac{1}{1-\alpha\rho}m_t + b_t = \frac{\alpha\rho}{1-\alpha\rho}m_t + \alpha\mathbb{E}_t b_{t+1} + m_t$
$\frac{1}{1-\alpha\rho}m_t + b_t = \frac{1}{1-\alpha\rho}m_t + \alpha\mathbb{E}_t b_{t+1}$
$b_t = \alpha\mathbb{E}_t b_{t+1}$
Then:

$$\mathbb{E}_t b_{t+1} = \frac{1}{\alpha} b_t \tag{83}$$

Hence, for any sequence of random variables $\{b_t\}$ that satisfies (83), we get a solution in form (82). Since $\alpha > 1$, (83) shows that the bubble is stable. Therefore, (82) gives an infinite number of stable solutions. This exemplifies Result 3 in Section 2.

To see what is going on in terms of the transformed system (17), recall from (22) that: $x_t = Z_{x\theta}\theta_t$. In our example, $x_t = m_t$, $Z_{x\theta}$ is 1×2, and θ_t is 2×1. Then we can write: $m_t = [z_1 \ z_2]\begin{bmatrix} \theta_{1t} \\ \theta_{2t} \end{bmatrix}$. For $t = 0$ we get: $m_0 = z_1\theta_{10} + z_2\theta_{20}$. Therefore, given the initial condition m_0, we can choose an infinite number of initial conditions θ_0 (*i.e.*, any pair of values for θ_{10} and θ_{20} that satisfies $m_0 = z_1\theta_{10} + z_2\theta_{20}$). Hence, system (17) has an infinite number of stable solutions.

Remark. When $\alpha < 1$, (83) shows that the bubble is unstable. In this case, choosing $b_t = 0 \ \forall t$ gives the unique stable solution. In other words, when $\alpha < 1$, the number of stable eigenvalues coincides with the number of predetermined variables, and Result 1 in Section 2 applies.

Appendix G

In this appendix we show how to reduce the system $G\mathbb{E}_t w_{t+1} = Aw_t + Dz_t$ to another system of the form $G\mathbb{E}_t w_{t+1} = Aw_t$, when the exogenous random vectors z_t follow a VAR(1): $z_{t+1} = Nz_t + u_t$ with $\mathbb{E}u_t = \mathbf{0}$ and $\mathbb{E}\{u_t u_t'\} = \Sigma_u$.

We start with the system:

$$\begin{aligned} G\mathbb{E}_t w_{t+1} &= Aw_t + Dz_t \\ \mathbb{E}_t z_{t+1} &= Nz_t \end{aligned}$$

which can be written in matrix form as follows:

$$\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_{t+1} \\ \mathbb{E}_t w_{t+1} \end{bmatrix} = \begin{bmatrix} N & \mathbf{0} \\ D & A \end{bmatrix} \begin{bmatrix} z_t \\ w_t \end{bmatrix}$$

The expression above can be written in form (48) as follows:

$$\widetilde{G}\mathbb{E}_t \widetilde{w}_{t+1} = \widetilde{A}\widetilde{w}_t$$

where

$$\widetilde{G} \equiv \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix}, \quad \widetilde{A} \equiv \begin{bmatrix} N & \mathbf{0} \\ D & A \end{bmatrix}, \quad \widetilde{w}_t \equiv \begin{bmatrix} z_t \\ w_t \end{bmatrix}$$

Partitioning G, A, D and w we can write:

$$\begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G_{xx} & G_{xy} \\ \mathbf{0} & G_{yx} & G_{yy} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_{t+1} \\ \mathbb{E}_t x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} N & \mathbf{0} & \mathbf{0} \\ D_x & A_{xx} & A_{xy} \\ D_y & A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \\ y_t \end{bmatrix}$$

The expression above can be written in form (47) as follows:

$$\begin{bmatrix} \widetilde{G}_{xx} & \widetilde{G}_{xy} \\ \widetilde{G}_{yx} & G_{yy} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \widetilde{x}_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{xx} & \widetilde{A}_{xy} \\ \widetilde{A}_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ y_t \end{bmatrix}$$

where

$$\begin{aligned} \widetilde{G}_{xx} &\equiv \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & G_{xx} \end{bmatrix}, \quad \widetilde{G}_{xy} \equiv \begin{bmatrix} \mathbf{0} \\ G_{xy} \end{bmatrix}, \quad \widetilde{G}_{yx} \equiv \begin{bmatrix} \mathbf{0} & G_{yx} \end{bmatrix} \\ \widetilde{A}_{xx} &\equiv \begin{bmatrix} N & \mathbf{0} \\ D_x & A_{xx} \end{bmatrix}, \quad \widetilde{A}_{xy} \equiv \begin{bmatrix} \mathbf{0} \\ A_{xy} \end{bmatrix}, \quad \widetilde{A}_{yx} \equiv \begin{bmatrix} D_y & G_{yx} \end{bmatrix} \end{aligned}$$

and the new vector of predetermined variables is

$$\widetilde{x}_t \equiv \begin{bmatrix} z_t \\ x_t \end{bmatrix}$$

Appendix H

In this appendix we describe the method of log-linearization and then we use it to log-linearize the RBC model presented in Section 4.

H.1 Log-linearization[42]

Consider the expression $y_t = f(x_t)$, where f is a differentiable function, and let $\overline{y} = f(\overline{x})$ be its steady-state value. We want to approximate the log-deviation $\ln y_t - \ln \overline{y}$ with a linear function of the log-deviation $\ln x_t - \ln \overline{x}$, for values of x_t close to $\overline{x}$.

From $y_t = f(x_t)$ we get: $\ln y_t = \ln f(x_t)$. We can rewrite this as follows: $\ln y_t = \ln f(e^{\ln x_t}) \equiv g(\ln x_t)$. Now we take a first-order Taylor approximation of $g(\ln x_t)$ around $\ln \overline{x}$:

$$g(\ln x_t) \cong g(\ln \overline{x}) + g'(\ln \overline{x})(\ln x_t - \ln \overline{x})$$

Using $g(\ln x_t) \equiv \ln f(e^{\ln x_t})$ we get:

$$\ln f(e^{\ln x_t}) \cong \ln f(e^{\ln \overline{x}}) + \frac{1}{f(e^{\ln \overline{x}})} f'(e^{\ln \overline{x}}) e^{\ln \overline{x}} (\ln x_t - \ln \overline{x})$$

We can rewrite the expression above as follows:

$$\begin{aligned} \ln f(x_t) &\cong \ln f(\overline{x}) + \frac{\overline{x}}{f(\overline{x})} f'(\overline{x})(\ln x_t - \ln \overline{x}) \\ &= \ln f(\overline{x}) + e_f(\overline{x})(\ln x_t - \ln \overline{x}) \end{aligned}$$

where $e_f(\overline{x}) \equiv \frac{\overline{x}}{f(\overline{x})} f'(\overline{x})$ is the elasticity of the function f evaluated at $\overline{x}$.

Recalling that $y_t = f(x_t)$ we get the expression we were looking for:

$$\ln y_t - \ln \overline{y} \cong e_f(\overline{x})(\ln x_t - \ln \overline{x})$$

Defining $\widehat{y}_t \equiv \ln y_t - \ln \overline{y}$ and $\widehat{x}_t \equiv \ln x_t - \ln \overline{x}$ we get:

$$\widehat{y}_t \cong e_f(\overline{x}) \widehat{x}_t$$

Notice that we can write: $\widehat{x}_t \equiv \ln x_t - \ln \overline{x} = \ln \left(\frac{x_t}{\overline{x}} \right) = \ln \left(1 + \frac{x_t - \overline{x}}{\overline{x}} \right)$. For values of x_t sufficiently close to $\overline{x}$ we know that $\ln \left(1 + \frac{x_t - \overline{x}}{\overline{x}} \right) \cong \frac{x_t - \overline{x}}{\overline{x}}$ (this follows from a first-order Taylor approximation of $m(x_t) \equiv \ln \left(1 + \frac{x_t - \overline{x}}{\overline{x}} \right)$ around $\overline{x}$). Then: $\widehat{x}_t \cong \frac{x_t - \overline{x}}{\overline{x}}$. Therefore, we can write:

$$\frac{y_t - \overline{y}}{\overline{y}} \cong e_f(\overline{x}) \frac{x_t - \overline{x}}{\overline{x}}$$

[42] An alternative approach to log-linearization is presented in Uhlig (1999).

In fact, we can derive the previous expression directly. First, write $y_t = f(x_t) = e^{\ln f(x_t)}$. Then: $\frac{y_t}{\overline{y}} = \frac{e^{\ln f(x_t)}}{e^{\ln f(\overline{x})}} = e^{\ln f(x_t) - \ln f(\overline{x})} \equiv h(x_t)$. A first-order Taylor approximation of $h(x_t)$ around $\overline{x}$ gives:

$$h(x_t) \cong h(\overline{x}) + h'(\overline{x})(x_t - \overline{x})$$

Using $h(x_t) \equiv e^{\ln f(x_t) - \ln f(\overline{x})}$ we get:

$$\begin{aligned} e^{\ln f(x_t) - \ln f(\overline{x})} &\cong e^0 + e^0 \frac{1}{f(\overline{x})} f'(\overline{x})(x_t - \overline{x}) \\ &= 1 + \frac{\overline{x}}{f(\overline{x})} f'(\overline{x}) \frac{x_t - \overline{x}}{\overline{x}} \\ &= 1 + e_f(\overline{x}) \frac{x_t - \overline{x}}{\overline{x}} \end{aligned}$$

Finally, using $e^{\ln f(x_t) - \ln f(\overline{x})} = \frac{y_t}{\overline{y}}$ we obtain: $\frac{y_t}{\overline{y}} \cong 1 + e_f(\overline{x}) \frac{x_t - \overline{x}}{\overline{x}} \Rightarrow \frac{y_t}{\overline{y}} - 1 \cong e_f(\overline{x}) \frac{x_t - \overline{x}}{\overline{x}} \Rightarrow$

$$\frac{y_t - \overline{y}}{\overline{y}} \cong e_f(\overline{x}) \frac{x_t - \overline{x}}{\overline{x}}$$

which is what we wanted to show.

The previous expression suggests a simple way of log-linearizing $y_t = f(x_t)$ through differentiation. Start by taking the natural logarithm: $\ln y_t = \ln f(x_t)$. Then totally differentiate, evaluating all derivatives at their steady-state levels: $\frac{dy_t}{\overline{y}} = \frac{f'(\overline{x})}{f(\overline{x})} dx_t \Rightarrow \frac{dy_t}{\overline{y}} = \frac{\overline{x}}{f(\overline{x})} f'(\overline{x}) \frac{dx_t}{\overline{x}} \Rightarrow \frac{dy_t}{\overline{y}} = e_f(\overline{x}) \frac{dx_t}{\overline{x}}$. Finally, interpret dx_t as $x_t - \overline{x}$ and dy_t as $y_t - \overline{y}$. Similarly, we can directly differentiate $y_t = f(x_t)$ to get $dy_t = f'(\overline{x}) dx_t$, and then divide by $\overline{y} = f(\overline{x})$ to obtain: $\frac{dy_t}{\overline{y}} = \frac{f'(\overline{x})}{f(\overline{x})} dx_t$. Multiplying and dividing the right-hand side by $\overline{x}$ we get: $\frac{dy_t}{\overline{y}} = \frac{\overline{x} f'(\overline{x})}{f(\overline{x})} \frac{dx_t}{\overline{x}} \Rightarrow \frac{dy_t}{\overline{y}} = e_f(\overline{x}) \frac{dx_t}{\overline{x}}$.

A similar procedure works for multivariate functions, $y_t = f(x_{1t}, x_{2t}, ..., x_{nt})$. In this case we get:

$$\widehat{y}_t \cong e_{f,1}(\overline{x}) \widehat{x}_{1t} + e_{f,2}(\overline{x}) \widehat{x}_{2t} + ... + e_{f,n}(\overline{x}) \widehat{x}_{nt}$$

where $e_{f,i}(\overline{x})$ is the partial elasticity of f with respect to its i-th argument, evaluated at $\overline{x}$.

In stochastic models, we usually have expressions of the form

$$y_t = \mathbb{E}_t\{f(x_{t+1})\}$$

Taking natural logarithms we get: $\ln y_t = \ln \mathbb{E}_t\{f(x_{t+1})\}$. Now we make the approximation $\ln \mathbb{E}_t\{f(x_{t+1})\} \cong \mathbb{E}_t\{\ln f(x_{t+1})\}$ to get:[43]

$$\ln y_t \cong \mathbb{E}_t\{\ln f(x_{t+1})\}$$

[43] Sin $\ln(\cdot)$ is a strictly concave function, we know from Jensen's Inequality that $\ln \mathbb{E}_t\{f(x_{t+1})\} > \mathbb{E}_t\{\ln f(x_{t+1})\}$.

Since $\ln \overline{y} = \ln f(\overline{x})$, the expression above gives: $\ln y_t - \ln \overline{y} \cong \mathbb{E}_t\{\ln f(x_{t+1})\} - \ln f(\overline{x}) \Rightarrow$

$$\ln y_t - \ln \overline{y} \cong \mathbb{E}_t\{\ln f(x_{t+1}) - \ln f(\overline{x})\}$$

Using $\ln f(x_{t+1}) \cong \ln f(\overline{x}) + e_f(\overline{x})(\ln x_{t+1} - \ln \overline{x})$ we can rewrite the expression above as follows: $\ln y_t - \ln \overline{y} \cong \mathbb{E}_t\{e_f(\overline{x})(\ln x_{t+1} - \ln \overline{x})\} \Rightarrow$

$$\widehat{y}_t \cong e_f(\overline{x})\mathbb{E}_t\{\widehat{x}_{t+1}\}$$

H.2 Log-linearizing the RBC model

The law of motion for total factor productivity is $\ln \lambda_{t+1} = (1-\gamma)\ln \lambda + \gamma \ln \lambda_t + \epsilon_{t+1}$, and $\overline{\lambda} = \lambda$. Then:

$$\begin{aligned} \widehat{\lambda}_{t+1} &\equiv \ln \lambda_{t+1} - \ln \overline{\lambda} \\ &= (1-\gamma)\ln \lambda + \gamma \ln \lambda_t + \epsilon_{t+1} - \ln \lambda \\ &= \gamma(\ln \lambda_t - \ln \lambda) + \epsilon_{t+1} \\ &= \gamma \widehat{\lambda}_t + \epsilon_{t+1} \end{aligned}$$

Notice that this case does not require any approximation. The same is true for the production function and the two first-order condtions for the firm: $Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}$, $w_t = (1-\theta)\frac{Y_t}{H_t}$, and $r_t = \theta \frac{Y_t}{K_t}$. Taking natural logs and subtracting the log of the steady-state values we get:

$$\begin{aligned} \widehat{Y}_t &= \widehat{\lambda}_t + \theta \widehat{K}_t + (1-\theta)\widehat{H}_t \\ \widehat{w}_t &= \widehat{Y}_t - \widehat{H}_t \\ \widehat{r}_t &= \widehat{Y}_t - \widehat{K}_t \end{aligned}$$

From $K_{t+1} = I_t + (1-\delta)K_t$ we get $dK_{t+1} = dI_t + (1-\delta)dK_t$. Dividing by $\overline{K} = \frac{\overline{I}}{\delta}$ we get: $\frac{dK_{t+1}}{\overline{K}} = \delta \frac{dI_t}{\overline{I}} + (1-\delta)\frac{dK_t}{\overline{K}} \Rightarrow$

$$\widehat{K}_{t+1} = \delta \widehat{I}_t + (1-\delta)\widehat{K}_t$$

From $Y_t = C_t + I_t$ we get $dY_t = dC_t + dI_t$. Dividing by $\overline{Y}$ we get: $\frac{dY_t}{\overline{Y}} = \frac{1}{\overline{Y}}dC_t + \frac{1}{\overline{Y}}dI_t$. Then: $\frac{dY_t}{\overline{Y}} = \frac{\overline{C}}{\overline{Y}}\frac{dC_t}{\overline{C}} + \frac{\overline{I}}{\overline{Y}}\frac{dI_t}{\overline{I}} \Rightarrow \widehat{Y}_t = \frac{\overline{C}}{\overline{Y}}\widehat{C}_t + \frac{\overline{I}}{\overline{Y}}\widehat{I}_t \Rightarrow$

$$\overline{Y}\widehat{Y}_t = \overline{C}\widehat{C}_t + \overline{I}\widehat{I}_t$$

From $\frac{aC_t}{1-H_t} = w_t$ we get: $\ln a + \ln C_t - \ln(1-H_t) = \ln w_t$. Differentiating we get: $\frac{dC_t}{\overline{C}} + \frac{1}{1-\overline{H}}dH_t = \frac{dw_t}{\overline{w}} \Rightarrow \frac{dC_t}{\overline{C}} + \frac{\overline{H}}{1-\overline{H}}\frac{dH_t}{\overline{H}} = \frac{dw_t}{\overline{w}} \Rightarrow$

$$\widehat{C}_t + \frac{\overline{H}}{1-\overline{H}}\widehat{H}_t = \widehat{w}_t$$

From $\frac{1}{C_t} = \mathbb{E}_t\{\frac{\beta}{C_{t+1}}(r_{t+1} + 1 - \delta)\}$ we get: $-\ln C_t = \ln \mathbb{E}_t\{\frac{\beta}{C_{t+1}}(r_{t+1} + 1 - \delta)\}$. Approximating the log of the expectation with the expectation fo the log we get: $-\ln C_t \cong \mathbb{E}_t\left\{\ln\beta - \ln C_{t+1} + \ln\left(r_{t+1} + 1 - \delta\right)\right\}$. Differentiating (and dropping the approximation sign): $-\frac{dC_t}{\overline{C}} = \mathbb{E}_t\{-\frac{dC_{t+1}}{\overline{C}} + \frac{1}{\overline{r}+1-\delta}dr_{t+1}\}$. Since $\frac{1}{\overline{r}+1-\delta} = \beta$ we get: $-\frac{dC_t}{\overline{C}} = \mathbb{E}_t\{-\frac{dC_{t+1}}{\overline{C}} + \beta dr_{t+1}\} \Rightarrow -\frac{dC_t}{\overline{C}} = \mathbb{E}_t\{-\frac{dC_{t+1}}{\overline{C}} + \beta\overline{r}\frac{dr_{t+1}}{\overline{r}}\} \Rightarrow$ $-\widehat{C}_t = \mathbb{E}_t\{-\widehat{C}_{t+1} + \beta\overline{r}\widehat{r}_{t+1}\} \Rightarrow$

$$\mathbb{E}_t\widehat{C}_{t+1} - \beta\overline{r}\mathbb{E}_t\widehat{r}_{t+1} = \widehat{C}_t$$

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