| | |
|---|---|
| Title | Equilibrium execution strategies with generalized price impacts (Financial Modeling and Analysis) |
| Author(s) | Ohnishi, Masamitsu; Shimoshimizu, Makoto |
| Citation | 数理解析研究所講究録 = RIMS Kokyuroku (2019), 2111: 84-106 |
| Issue Date | 2019-04 |
| URL | http://hdl.handle.net/2433/251978 |
| Right | |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

数理解析研究所講究録 第2111巻 2019年 84-106



# Equilibrium execution strategies with generalized price impacts*

Masamitsu OHNISHI†‡ and Makoto SHIMOSHIMIZU†

†Graduate School of Economics, Osaka University, Osaka, 560–0043, Japan

‡Center for Mathematical Modeling and Data Science, Osaka University, Osaka, 560–8531, Japan

January 25, 2019

**Abstract**

This paper examines the execution problem of large traders with generalized price impact model. Constructing a model in a discrete–time setting, we solve this problem by applying the backward induction method of the dynamic programming. In this model, we formulate the expected utility maximization problem of multiple large traders as a Markov game and derive an equilibrium execution strategy at a Markov perfect equilibrium. This model enables us to investigate how the execution strategies and trade performances of these large traders are affected by the existence of other traders. Moreover, we find that these equilibrium execution strategies become deterministic when the total execution volumes of non–large traders are deterministic. We also show, by some numerical examples, the comparative statics results with respect to several problem parameters.

## 1 Introduction

In the security market analysis, there is a growing awareness among academic researchers or practitioners that some kind of institutional traders called 'large trader' cause the 'price impact' through their own trades. A life insurance company, trust company, or a company who manages pension fund exhibit the typical examples of such traders of great importance. Large traders recognize these price impacts as 'liquidity risk.' They can reduce the liquidity risk by splitting their order into small size over the course of the trading epoch. Conversely, submitting the small pieces of order gradually may expose them to the price risk. Consequently, when large traders allocate large orders into (small) pieces, they have to pay attention to two distinct facets; the liquidity risk which arises owing to the large orders they submit and the price risk which corresponds to the price fluctuations in the future.

Various ways to trade are available to a preponderance of a trading market since the structure of trading systems diverges in different directions. As an example of a wide variety of electric

*This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

trading platforms, 'algorithmic trading' has emerged in recent years and the so–called high–frequency trading (HFT) with the computer system, which typifies the algorithmic trading, significantly influences the financial market. The development of the trading system facilitates an increasing number of studies encompassing a field such as market impact modeling, or optimal execution problem.

In this paper, we investigate execution problems pertaining to the interaction among large traders and non–large traders from a theoretical point of view. [2] propel to the forefront in investigations of this field, which address the optimization problem of minimizing the expected execution cost in a discrete–time framework via a dynamic programming approach. This analysis identifies the optimal execution volume as equally divided volume throughout the trading epochs. Notwithstanding a valuable insight into the execution problem, their model disregards any attitudes toward risk. Accordingly, [1] derives an optimal execution strategy by considering both the execution cost and the volatility risk, which entails the analysis with a mean–variance approach. As for [7] and [8], they construct models with the residual effect of the price impact, i.e. the transient price impact which dissipates over the trading time window. These papers solve an optimization problem of maximizing an expected utility payoff from the final wealth at the maturity, deriving an optimal execution strategy.

As a trend of the previous papers including those mentioned above, discussions regarding the behavior of an institutional trader have dominated research in recent years. These researches, however, do not incorporate into their model the existence of traders other than large traders, whom we call 'trading crowd' as [5]. Only a few existing researches concerned with execution problems have thoroughly investigated the price impact model with trading crowd. As [13] shows, small trades have statistically by far larger impacts on the price than that of large trades in a relative sense. These results infer that one should take into account a price impact caused by trading crowd when constructing a price impact model. The effect of the price impact caused by trading crowd on the execution price features the generalized price impact in our model.

Moreover, a multitude of large traders ordinarily exist in a real market. Nevertheless, most of the prior studies deal with the optimal execution problem of a single large trader model. The following example underlines the above fact; consider a security market where multiple institutional traders or brokers manage their trading execution ordered by their clients. The clients can split their whole order into some blocks to submit their orders on different institutional traders or brokers. Then, these institutional traders or brokers execute their orders in the same market to make a profit. Each of their orders is so large that the submissions of the other institutional traders or brokers can affect the execution price in comparison with the case that only a single large trader exists in the market. This situation substantiates the interaction of more than one large traders in the same security market.

Only a few papers delve into the interaction between more than one large traders; examples are [15], [14], [10], to mention only a few related papers. [15] analyze the interaction of two large traders on their execution strategies, which inspires the following two works. In [14], they formulate what they call a market impact game model. Their study unearths some features of a Nash equilibrium strategy, proving that a unique Nash equilibrium exists in a class of static and deterministic strategies in explicit form. They also prevail, via a rather direct method, that the equilibrium is also a Nash equilibrium in a broader class of dynamic strategies. Subsequently, [10] extends the above model to $n$–large trader model and constructs cost minimization problems in terms of a mean–variance and expected utility maximization problems of $n$–large traders. An important result of their analysis is that a Nash equilibrium exists in each problem, which is also in explicit form and is unique for the former problem. They also reveal that the Bachelier price model renders the Nash equilibrium obtained from each problem identical, where the price is composed of a Brownian motion as a term expressing the volatility of stock price. These studies are noteworthy since they theoretically highlight the interaction of execution strategies among multiple large traders. This kind of work is so novel in these research fields.

This paper explores an execution strategy in a multiple large trader Markov game model. These large traders maximize their expected utility payoff from his/her final wealth at the maturity. The methods by which we derive these strategies are the backward induction procedure of dynamic programming, which is equivalent to those introduced in [7] and [8]. Using a price impact caused by trading crowd is embedded in the construction of a price impact model, leading to a similar but different model of the previous research. Under appropriate model settings, our investigation shows that there exist an equilibrium execution strategy at a Markov perfect equilibrium in the second model. Our contribution to the field of the execution problem is that in general, the equilibrium execution strategies are not necessarily static nor deterministic. The execution strategies become deterministic when the execution volume of trading crowd are static and deterministic.

This paper proceeds as follows. In Section 2, we consider the maximization problem of the expected utility of multiple risk–averse large traders with Constant Absolute Risk Aversion (CARA) type utility (or negative exponential utility) from the wealth at the maturity. Then, we construct a Markov game model of multiple large traders, from which we derive an equilibrium execution strategy at a Markov perfect equilibrium. Section 3 displays the numerical examples of each model. Finally, Section 4 concludes.

## 2 $n$–Large Trader Stochastic Game Model

In the discrete time framework $t = 1, \ldots, T, T+1(, T \in \mathbb{Z}_+ := \{1, 2, \ldots\})$, we assume that there exists $n$–large traders in a trading market, denoted by $i = 1, \ldots, n(, n \in \mathbb{Z}_+ := \{1, 2, \ldots\})$. These large traders plan to purchase $Q^i (\in \mathbb{R})$ volume of one risky asset by the time $T + 1$. We also suppose that each large trader has the CARA utility function with the absolute risk aversion rate $R^i > 0$.

### 2.1 Market Model

We assume that $q^i_t (\in \mathbb{R})$ for $i \in \{1, \ldots, n\}$ represent the large amount of orders submitted by the large trader at time $t \in \{1, \ldots, T\}$. Then, $\overline{Q}_t$ denotes the number of shares remained to purchase by the large trader at time $t \in \{1, \ldots, T, T+1\}$. The positive and negative $Q_t$ stand for the acquisition and the liquidation of the risky asset, respectively. This leads to the similar setup for a selling problem. From this assumption, we have $\overline{Q}_1^i = Q$ and

$$\overline{Q}_{t+1}^i = \overline{Q}_t^i - q_t^i, \quad t = 1, \ldots, T, \quad i = 1, \ldots, n. \tag{2.1}$$

The market price (or quoted price) of the risky asset at time $t \in \{1, \ldots, T, T+1\}$ is represented by $p_t$. Since the large trader submit large amount of orders, the execution price becomes not $p_t$ but $\widehat{p}_t$ with the additive execution cost. Submitting one unit of (large) order at time $t \in \{1, \ldots, T\}$ causes the instantaneous price impact, which is denoted as $\lambda_t$. The execution of trading crowd also has an impact on the execution price. $\kappa_t$ represents the price impact per unit at time $t \in \{1, \ldots, T\}$ which stems from the submittion of trading crowd. We denote the total execution volume of trading crowd at time $t \in \{1, \ldots, T\}$ by a sequence of random variables $v_t$, which follows a normal distribution with mean $\mu_{v_t}$ and variance $\sigma_{v_t}^2$ for each time $t \in \{1, \ldots, T, T+1\}$, that is,

$$v_t \sim N(\mu_{v_t}, \sigma_{v_t}^2), \quad t = 1, \ldots, T. \tag{2.2}$$

Throughout this paper, we assume that the buy–trade and sell–trade of a large trader induce the same (instantaneous) price impact, although it would be different in the real market. This assumption is, however, justified from the statistical analysis of market data in [3]. In this work, they estimate the permanent and temporary price impact by conducting a robust linear regression of price changes on net order–flow. This estimation and the relevant statistics obtained by using several stock market data reveal that the linear assumption of the price impact is compatible with the real stock market and that the price impact caused by both buy and sell trades are same from the statistical point of view.

From these facts, we define the execution price in the form of the linear price impact model as follows:

$$\widehat{p}_t = p_t + \lambda_t\Big(\sum_{i=1}^{n} q_t^i\Big) + \kappa_t v_t, \quad t = 1, \ldots, T. \tag{2.3}$$

We assume that the two large traders cause same price impact per unit, $\lambda_t$ for simplicity. The generalization of the price impact caused by each large trader, that is, the dependency of $\lambda_t$ on $i \in \{1, \ldots, n\}$ is not what we want to explore and it leads to a substantially tedious extension. This dependence will not contribute to additional intriguing results or will not have significant influences on the execution strategies we obtain in the following. We henceforth conduct the subsequent formulation with $\lambda_t$ which is independent of each large trader $i$ from these reasons.

The residual price impact of past price $r_t$ at time $t \in \{1, \ldots, T, T+1\}$ is represented by means of a decay kernel function $G(t)$ of time $t \in \{1, \ldots, T, T+1\}$. We define this price impact with the exponantial decay kernel,

$$G(t) := e^{-\rho t}, \quad t = 1, \ldots, T, T+1. \tag{2.4}$$

With a deterministic price reversion rate $\alpha_t (\in [0, 1])$ and deterministic resilience speed $\rho(\in$

$[0, \infty)$), the dynamics of the residual effect of past price impact $r_t$ is given by

$$\begin{aligned}
r_{t+1} &:= \sum_{k=1}^{t}\{\lambda_k(\sum_{i=1}^{n} q_k^i) + \kappa_k v_k\}\alpha_k \mathrm{e}^{-\rho((t+1)-k)} \\
&= \mathrm{e}^{-\rho}\sum_{k=1}^{t-1}\{\lambda_k(\sum_{i=1}^{n} q_k^i) + \kappa_k v_k\}\alpha_k \mathrm{e}^{-\rho(t-k)} + \{\lambda_t(\sum_{i=1}^{n} q_t^i) + \kappa_t v_t\}\alpha_t \mathrm{e}^{-\rho} \\
&= [r_t + \{\lambda_t(\sum_{i=1}^{n} q_t^i) + \kappa_t v_t\}\alpha_t]\mathrm{e}^{-\rho}. \qquad (2.5)
\end{aligned}$$

Eq. (2.5) shows the recursiveness of the residual effect, i.e., $r_{t+1}$ depends on only $r_t$ and the transient price impact $\{(\lambda_t(\sum_{i=1}^{n} q_t^i)+\kappa_t v_t\}\alpha_t \mathrm{e}^{-\rho}$, which indicates that $r_t$ has a Markov property in this settings. The Markov property of this residual effect arises from the assumption of the exponantial decay kernel.

Some public news or information of the economic situation affect the price. Therefore, we define the independent random variable $\varepsilon_t$ at time $t \in \{1, \ldots, T\}$ as the effect of the public news/information about economic situation between $t$ and $t+1$, and assume that $\varepsilon_t$ follows the normal distribution with mean $\mu_{\varepsilon_t}$ and variance $\sigma^2_{\varepsilon_t}$, i.e.,

$$\varepsilon_t \sim N(\mu_{\varepsilon_t}, \sigma^2_{\varepsilon_t}), \quad t = 1, \ldots, T. \qquad (2.6)$$

We suppose in the following that the two stochastic process, $v_t$ and $\varepsilon_t$, $t \in \{1, \ldots, T\}$ are mutually independent. However, we can derive similar results without this assumption (that is, if they follow a bivariate normal distribution).

By the definition of $\varepsilon_t$, we can set the fundamental price $p_t^f := p_t - r_t$ as follows:

$$\begin{aligned}
&p_{t+1}^f = p_t^f + \{\lambda_t(\sum_{i=1}^{n} q_t^i) + \kappa_t v_t\}(1-\alpha_t) + \varepsilon_t \\
&(= p_{t+1} - r_{t+1} = p_t - r_t + \{\lambda_t(\sum_{i=1}^{n} q_t^i) + \kappa_t v_t\}(1-\alpha_t) + \varepsilon_t), \quad t = 1, \ldots, T. \qquad (2.7)
\end{aligned}$$

From (2.3), (2.5), and (2.7), the the dynamics of market price satisfies

$$p_{t+1} = p_t - (1-\mathrm{e}^{-\rho})r_t + \{\lambda_t(\sum_{i=1}^{n} q_t^i) + \kappa_t v_t\}\{\alpha_t \mathrm{e}^{-\rho} + (1-\mathrm{e}^{-\rho})\} + \varepsilon_t, \quad t = 1, \ldots, T. \qquad (2.8)$$

**Corollary 1** *In this context,* $\{\lambda_t(\sum_{i=1}^{n} q_t^i)+\kappa_t v_t\}(1-\alpha_t)$, $\{\lambda_t(\sum_{i=1}^{n} q_t^i)+\kappa_t v_t\}\alpha_t$, *and* $\{\lambda_t(\sum_{i=1}^{n} q_t^i)+\kappa_t v_t\}\alpha_t \mathrm{e}^{-\rho}$ *represent the parmanent impact, temporary impact, and transient impact, respectively.*

Let $w_t^i$, $i = 1, \ldots, n$ denote the wealth processes of each large trader. The dynamics of $w_t^i$ becomes

$$\begin{aligned}
w_{t+1}^i &= w_t^i - \widehat{p}_t q_t^i \\
&= w_t^i - \{p_t + \lambda_t(\sum_{i=1}^{n} q_t^i) + \kappa_t v_t\}q_t^i, \quad t = 1, \ldots, T, \quad i = 1, \ldots, n. \qquad (2.9)
\end{aligned}$$

## 2.2 Formulation as a Markov Game Model

In this subsection, we formulate the large trader's problem as a discrete–time Markov game model. Time elapses as $1, \ldots, T, T+1$. The state of the process at time $t \in \{1, \ldots, T, T+1\}$ is a $(2n+2)$–tuple, and is denoted as

$$s_t := ((w_t^1, \ldots, w_t^n), p_t, (\overline{Q}_t^1, \ldots, \overline{Q}_t^n), r_t) \in \mathbb{R}^{2n+2} =: S. \tag{2.10}$$

For $t \in \{1, \ldots, T\}$, an allowable action chosen at state $s_t$ is an execution volume $q_t^i \in \mathbb{R} =: A^i$, $i \in \{1, \ldots, n\}$ so that the set $A^1, \ldots, A^n$ of admissible actions is independent of the current state $s_t$.

When an action $q_t^i$ is chosen in a state $s_t$ at time $t \in \{1, \ldots, T\}$, a transition to a next state

$$s_{t+1} = h_t(s_t, (q_t^1, \ldots, q_t^n), (v_t, \varepsilon_t)). \tag{2.11}$$

occurs according to the law of motion precisely described in the previous subsection which is symbolically denoted by a (Borel measurable) system dynamics function $h_t$ $(: S \times (A^1 \times \cdots \times A^n) \times (\mathbb{R} \times \mathbb{R}) \longrightarrow S)$:

$$s_{t+1} = h_t(s_t, (q_t^1, \ldots, q_t^n), (v_t, \varepsilon_t)), \quad t = 1, \ldots, T. \tag{2.12}$$

A utility payoff (or reward) arises only in a terminal state $s_{T+1}$ at the end of horizon $T+1$ as

$$g_{T+1}^i(s_{T+1}) := \begin{cases} -\exp\{-Rw_{T+1}^i\} & \text{if } \overline{Q}_{T+1}^i = 0; \\ -\infty & \text{if } \overline{Q}_{T+1}^i \neq 0, \end{cases} \tag{2.13}$$

where $R^i$ for all $i \in \{1, \ldots, n\}$ represent the risk aversion rate and is larger than 0. The term $-\infty$ means a hard constraint enforcing the large trader to execute all of the remaining volume $\overline{Q}_T^i$ at the maturity $T$, that is, $q_T^i = \overline{Q}_T^i$ for all $i \in \{1, \ldots, n\}$.

The types of large traders could be defined by

$$(w_1^i, Q^i, R^i), \quad i = 1, \ldots, n, \tag{2.14}$$

and these are assumed to be their common knowledge. In the real market, large traders have little access to these information of the counterpart. We can, however, consider a plausible explanation for the assumption of Eq. (2.14) from the viewpoint of game theoretic analysis. In this model, our focus is placed on how the existence of the other large trader influences the execution strategy in comparison with a single large trader's (optimal) execution problem. We formulate this Markov game model as a dynamic game of complete information. Therefore, the above (hypothesized) definition and assumption associated with the notion of common knowledge are legitimate so that the solution concept of a Nash equilibrium in a non–cooperative game is (rationally or ideally) applicable in this model. The formulation of a generalized model as a dynamic game of incomplete information requires further intricate analysis, which is left for our future research.

If we define a (history–independent) one–stage decision rule $f_t$ at time $t \in \{1, \dots, T\}$ by a Borel measurable map from a state $s_t \in S = \mathbb{R}^{2n+2}$ to an action

$$q_t^i = f_t^i(s_t) \in A^i = \mathbb{R}, \quad i = 1, \dots, n, \tag{2.15}$$

then a Markov execution strategy $\pi$ is defined as a sequence of one–stage decision rules

$$\pi^i := (f_1^i, \dots, f_t^i, \dots, f_T^i), \quad i = 1, \dots, n. \tag{2.16}$$

We denote the set of all Markov execution strategies as $\Pi_{\mathrm{M}}^i$. Further, for $t \in \{1, \dots, T\}$, we define the sub execution strategy after time $t$ of a Markov execution strategy $\pi^i = (f_1^i, \dots, f_t^i, \dots, f_T^i) \in \Pi^i$ as

$$\pi_t^i := (f_t^i, \dots, f_T^i), \quad i = 1, \dots, n, \tag{2.17}$$

and the entire set of $\pi_t^i$ as $\Pi_{\mathrm{M},t}^i$. Hereafter, we consider only non–randomized Markov strategies in this paper.

By definition (2.13), the value function for $i \in \{1, \dots, n\}$ under an execution strategy $\pi^i$ becomes an expected utility payoff arising from the terminal wealth $w_{T+1}^i$ of the large trader with the absolute risk aversion $R^i$:

$$\begin{aligned} &V_1^i(\pi^1, \dots, \pi^n)[s_t] := \mathbb{E}_1^{\pi^1, \dots, \pi^n}[-\exp\{-R^i w_{T+1}^i\} \cdot 1_{\{\overline{Q}_{T+1}^i = 0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}^i \neq 0\}} \mid s_t], \\ &i = 1, \dots, n, \end{aligned} \tag{2.18}$$

where $1_A$ is the indicator function of the event $A$ and, for $t \in \{1, \dots, T\}$, $\mathbb{E}_1^{\pi^1, \dots, \pi^n}$ is a conditional expectation under substrategy profile $(\pi^1, \dots, \pi^n)$ from time $t \in \{1, \dots, T, T+1\}$.

Then, for $t \in \{1, \dots, T, T+1\}$ and $s_t \in S$, we further let

$$\begin{aligned} &V_t^i(\pi_t^1, \dots, \pi_t^n)[s_t] := \mathbb{E}_t^{\pi_t^1, \dots, \pi_t^n}[-\exp\{-R^i w_{T+1}^i\} \cdot 1_{\{\overline{Q}_{T+1}^i = 0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}^i \neq 0\}} \mid s_t], \\ &i = 1, \dots, n, \quad t = 1, \dots, T, T+1, \end{aligned} \tag{2.19}$$

be the expected utility payoff at time $t$ under substrategy profile $(\pi_t^1, \dots, \pi_t^n)$ from time $t \in \{1, \dots, T\}$.

What we seek here is an equilibrium execution strategy of these large traders. First, the definition of a Nash equilibrium in this model becomes as follows:

**Definition 1 (Nash Equilibrium)** $(\pi^{1*}, \dots, \pi^{n*}) \in \Pi_M^1 \times \cdots \times \Pi_M^n$ *is a Nash equilibrium starting from a fixed state* $s_1$ *if and only if*

$$\begin{aligned} &V_1^i(\pi^{1*}, \dots, \pi^{i*}, \dots, \pi^{n*})[s_1] \geq V_1^i(\pi^{1*}, \dots, \pi^i, \dots, \pi^{n*})[s_1], \\ &\forall \pi^i \in \Pi_M^i, \quad i = 1, \dots, n. \end{aligned} \tag{2.20}$$

We can define a refinement of a Nash equilibrium of this model as the notion of a Markov perfect equilibrium:

**Definition 2 (Markov Perfect Equilibrium)** $(\pi^{1*}, \ldots, \pi^{n*}) \in \Pi_M^1 \times \cdots \times \Pi_M^n$ *is a Markov perfect equilibrium if and only if*

$$\begin{aligned} &V_t^i(\pi_t^{1*}, \ldots, \pi_t^{i*}, \ldots, \pi_t^{n*})[s_t] \geq V_t^i(\pi_t^{1*}, \ldots, \pi_t^{i}, \ldots, \pi_t^{n*})[s_t], \\ &\forall \pi_t^i \in \Pi_{M,t}^i, \quad i = 1, \ldots, n, \quad \forall s_t \in S, \quad \forall t = 1, \ldots, T. \end{aligned} \tag{2.21}$$

Based on the following One Stage [Step, Shot] Deviation Principle, we obtain a Markov perfect equilibrium by backward induction procedure of dynamic programming from time $T$.

$$\begin{aligned} V_t^i(\boldsymbol{\pi}_t^*)[s_t] &= \sup_{q_t^i \in \mathrm{R}} \mathbb{E}\left[ V_{t+1}^i(\boldsymbol{\pi}_{t+1}^*)[h_t(s_t, (f_t^{1*}(s_t), \ldots, q_t^i, \ldots, f_t^{n*}(s_t)), (v_t, \varepsilon_t))] \,\middle|\, s_t \right] \\ &= \mathbb{E}\left[ V_{t+1}^i(\boldsymbol{\pi}_{t+1}^*)[h_t(s_t, (f_t^{1*}(s_t), \ldots, f_t^{i*}(s_t), \ldots, f_t^{n*}(s_t)), (v_t, \varepsilon_t))] \,\middle|\, s_t \right]; \\ &\quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \end{aligned} \tag{2.22}$$

where

$$\boldsymbol{\pi}_t^* := (\pi_t^{1*}, \ldots, \pi_t^{n*}) \in \Pi_{M,t}^1 \times \cdots \times \Pi_{M,t}^n. \tag{2.23}$$

## 2.3 Markov Perfect Equilibrium

**Theorem 1 (Markov Perfect Equilibrium)** *There exists a Markov perfect equilibrium* $(\pi^{1*}, \ldots, \pi^{n*}) \in \Pi_{\mathrm{M}}^1 \times \cdots \times \Pi_{\mathrm{M}}^n$ *at which for the large trader* $i \in \{1, \ldots, n\}$ *the following properties hold:*

1. *The execution volume* $q_t^{i*}, i = 1, \ldots, n$ *at the Markov perfect equilibrium are affine functions of each remaining execution volume,* $\overline{Q}_t^1, \ldots, \overline{Q}_t^n$*, and the cumulative residual effect* $r_t$ *at time* $t$*, i.e.,*

$$q_t^{i*} = f_t^i(s_t) = a_t^i + b_t^{i,1}\overline{Q}_t^1 + \cdots + b_t^{i,n}\overline{Q}_t^n + d_t^i r_t, \quad t = 1, \ldots, T, \quad i = 1, \ldots, n; \tag{2.24}$$

2. *The expected utility at the Markov perfect equilibrium* $(\pi^{1*}, \ldots, \pi^{n*})$ *for the large traders* $i = 1, \ldots, n$ *in the subgame starting from the state* $s_t (t = 1, \ldots, T)$ *have a functional form as follows :*

$$\begin{aligned} V_t^i(\pi_t^{1*}, \ldots, \pi_t^{n*})[s_t] = -\exp\{ &-R^i(w_t - p_t\overline{Q}_t^i - G_t^i(\overline{Q}_t^i)^2 - H_t^i\overline{Q}_t^i + I_t^i r_t \overline{Q}_t^i + J_t^i r_t^2 + L_t^i r_t \\ &+ \textstyle\sum_{j \neq i} M_t^{i,j}(\overline{Q}_t^j)^2 + \sum_{j \neq i} N_t^{i,j}\overline{Q}_t^j + \sum_{j \neq i} X_t^{i,j} r_t \overline{Q}_t^j + (\sum_{j \neq i} Y_t^i \overline{Q}_t^j)\overline{Q}_t^i \\ &+ \textstyle\sum_{k \neq i} \sum_{l \neq i,k} Y_t^{i,k,l}\overline{Q}_t^k\overline{Q}_t^l + Z_t^i)\}, \quad t = 1, \ldots, T, \quad i = 1, \ldots, n. \end{aligned} \tag{2.25}$$

*where*

$$a_t^i, b_t^{i,1}, \ldots, b_t^{i,n}, d_t^i, G_t^i, H_t^i, I_t^i, J_t^i, L_t^i, M_t^{i,j}, N_t^{i,j}, X_t^{i,j}, Y_t^{i,k,l}, Z_t^i, \quad t = 1, \ldots, T, \tag{2.26}$$

*are deterministic functions of time* $t$ *which are dependent on the problem parameters, and can be computed backwardly in time* $t$ *from maturity* $T$*.*

See the sketchy proof of this theorem in Appendix A.

From the theorem above, a Markov perfect equilibrium strategy $(\pi^{1*}, \ldots, \pi^{n*})$ depend on the state $((w_t^1, \ldots, w_t^n), p_t, (\overline{Q}_t^1, \ldots, \overline{Q}_t^n), r_t)$ of the game only through each remaining execution volume, $\overline{Q}_t^1, \ldots, \overline{Q}_t^n$, and the cumulative residual effect $r_t$, not through the wealth, $w_t^1, \ldots, w_t^n$, or market price $p_t$. In addition, from the difinition of residual effect $r_t$, the equilibrium executin volume at a Markov perfect equilibrium $q_t^{i*}$ for each $i \in \{1, \ldots, n\}$ include a nondeterministic term (random variable) only thorough $v_t$ in $r_t$, which indicates $v_t$ affects the equilibrium execution strategies. Therefore, we have the folllowing facts.

**Corollary 2** *If the orders of trading crowds $v_t$ for $t \in \{1, \ldots, T\}$, are deterministic, the optimal execution volumes $q_t^{i*}$ at time $t \in \{1, \ldots, T\}$ for $i \in \{1, \ldots, n\}$ also become deterministic functions of time in a class of the static (and non-randomized) execution strategy.*

These results are our contribution to the field of a market impact game and the different points compared with the existing research of equilibrium execution strategies such as [14] or [10]. Their works reveal that the equilibrium execution strategies are deterministic when minimizing the expected execution cost and considering a mean–variance optimization, that is, minimizing the mean–variance functional of trading costs. Moreover, they show that maximizing expected CARA utility of revenues defined by negative costs is consistent with mean–variance optimization over the class of deterministic strategies. As shown in our model, however, equilibrium execution strategies obtained by backward induction methods of dynamic programming are not always deterministic. It is mainly when the aggregate volumes of orders submitted by trading crowd are deterministic that the equilibrium execution strategies become also deterministic.

## 3 Comparative Statics

In this section, we illustrate some numerical examples and show some properties of the equilibrium execution strategies derived in Section 2 in the case of $n = 2$. The maturity is set as $T = 10$, and large trader $i = 1, 2$ plans to execute the volume $Q^i = 100,000$ within the time horizon $\{1, \ldots, T\}$ at the beginning. We conduct the following comparative statics assuming the time homogeneity of the time–dependent parameters $\mu_{v_t}, \sigma_{v_t}, \mu_{\varepsilon_t}, \sigma_{\varepsilon_t}, \alpha_t, \lambda_t, \kappa_t$. Here we further assume that there is no price impact caused by trading crowd, i.e., $\kappa_t = 0$, which is equivalent with setting $\mu_{v_t}$ and $\sigma_{v_t}$ as zero for all $t \in \{1, \ldots, T\}$. This assumption yields the explicit form of equilibrium execution volumes at a Markov perfect equilibrium (, which indicates the deterministic equilibrium execution volumes), since $\sigma_{v_t} = 0$ means that the submission of the trading crowd becomes deterministic. The benchmark values are set as follows:

$$\begin{aligned}
&\mu_{\varepsilon_t} \equiv 0; \quad \sigma_{\varepsilon_t} \equiv 0.02; \quad \alpha_t \equiv 0.5; \\
&\lambda_t \equiv 0.001; \quad \kappa_t \equiv 0; \quad \rho = 0.1; \quad R^i = 0.001, \quad \forall t = 1, \ldots, T, \quad i = 1, 2.
\end{aligned} \tag{3.1}$$

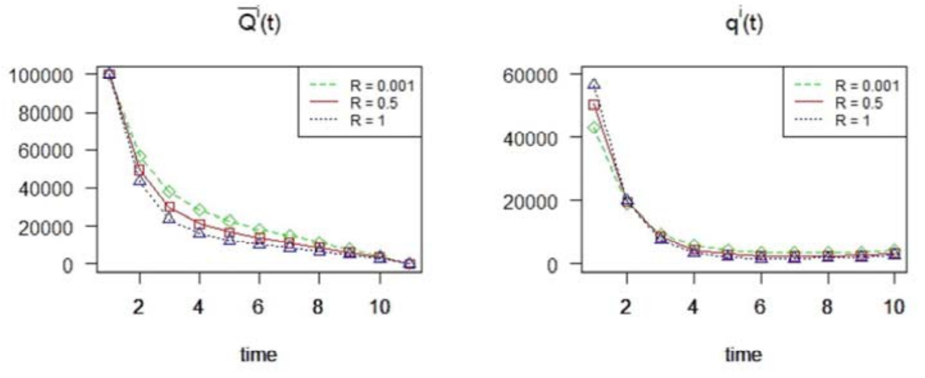
We also set $r_1 = 0$ since there exists no residual effects of past price at the beginning of trading, i.e., at time $t = 1$.

## 3.1 In the Case of Symmetric Large Trader

In this subsection, we deal with the case that the initial inventory and the risk aversion rate of each large trader are equal; $Q^1 = Q^2$, and $R^1 = R^2$. We set the same risk aversion rate as $R$ in this subsection.

### 3.1.1 The Effect of Risk Aversion

First, we demonstrate the difference of a large trader's execution volume with different risk aversion rates in the following three cases: $R = 0.001, R = 0.5$, and $R = 1$.




(a) Remaining Execution Volume $\overline{Q}_t^i(t = 1, \dots, T)$ (b) Execution Volume $q_t^i(t = 1, \dots, T)$

Figure 1: The Effect of Risk Aversion

Figure 6 shows that the more risk averse the large trader is, the faster he or she executes. That is because the more risk–averse trader tends to avoid the price risk of the fluctuation in the future as possible, which is consistent with intuitive understanding.

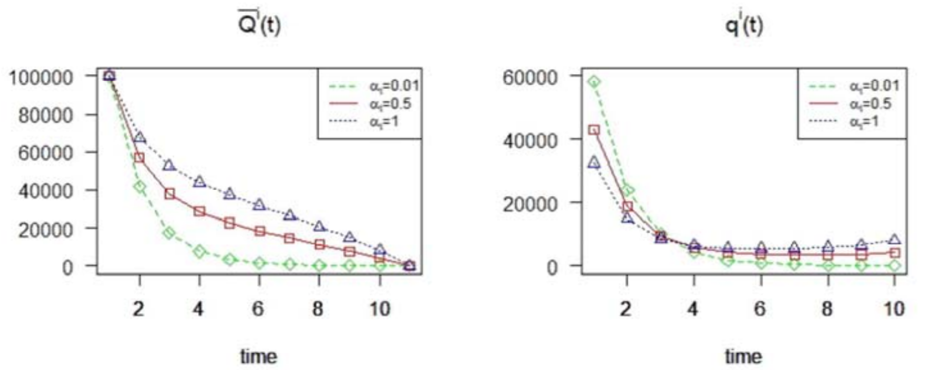
### 3.1.2 The Effect of $\alpha_t$

We examine the effect on the execution volume caused by the risk reversion rate $\alpha_t$ through the following three cases: $\alpha_t = 0.01, \alpha_t = 0.5$, and $\alpha_t = 1$.

As depicted in Figure 7, the large traders execute faster as $\alpha_t \to 0$. The reason for this is that the smaller $\alpha_t$ at time $t \in \{1, \dots, T\}$ coincides with the smaller price recovery at the next time $t + 1$, which infers that the large part of the price impact remains, leading to a higher execution price at the next trading.

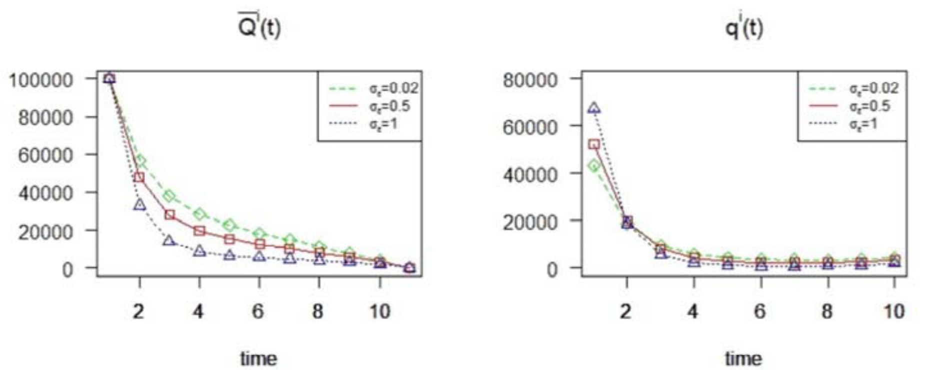
### 3.1.3 The Effect of $\sigma_{\varepsilon_t}$

The next example describes how $\sigma_{\varepsilon_t}$ affects the execution volume through the following three cases: $\sigma_{\varepsilon_t} = 0.02, \sigma_{\varepsilon_t} = 0.5$, and $\sigma_{\varepsilon_t} = 1$.




(a) Remaining Execution Volume $\overline{Q}_t^i(t = 1, \ldots, T)$ (b) Execution Volume $q_t^i(t = 1, \ldots, T)$

Figure 2: The Effect of $\alpha_t$



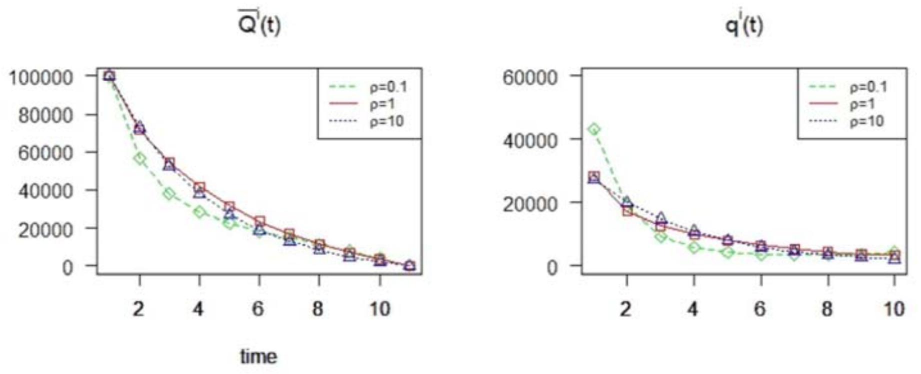

(a) Remaining Execution Volume $\overline{Q}_t^i(t = 1, \ldots, T)$ (b) Execution Volume $q_t^i(t = 1, \ldots, T)$

Figure 3: The Effect of $\sigma_{\varepsilon_t}$

Figure 8 illustrates that if $\sigma_{\varepsilon_t}$ is large, the large trader executes much faster, particular seen when $\sigma_{\varepsilon_t}$ is equal to 1. From these kinds of phenomena, we can intuitively interpret that the large trader takes into account the possibility that the market price of the asset might increase suddenly in the future since there exist much higher possibilities of fluctuations of the future price of a risky asset if the variance of the effect of the public news is large.

#### 3.1.4 The Effect of Resilience Speed

To examine the effect of the resilience speed on the execution volume, we draw, in the following, the figures of the three cases: $\rho = 0.1, \rho = 1$, and $\rho = 10$.




(a) Remaining Execution Volume $\overline{Q}_t^i(t = 1, \ldots, T)$ (b) Execution Volume $q_t^i(t = 1, \ldots, T)$

Figure 4: The Effect of the resilience speed

From Figure 9, we confirm that the large traders execute slower as $\rho$ increases. That is because the large $\rho$ reduces the residual effects, which makes the price at the next transaction lower. This postulates the following property of the equilibrium execution strategy at a Markov perfect equilibrium. Since $r_{t+1} \to 0$ as $\rho$ tends to $\infty$ from Eq. (2.5), the equilibrium execution volume at a Markov perfect equilibrium at time $t \in \{1, \ldots, T\}$ becomes an affine function of the remaining execution volumes of each large trader $\overline{Q}_t^1$ and $\overline{Q}_t^2$:

$$q_t^{i*} = f_t^i(s_t) = a_t^i + b_t^i \overline{Q}_t^i + c_t^i \overline{Q}_t^j, \quad t = 1, \ldots, T, \quad i, j = 1, 2, \quad i \neq j, \tag{3.2}$$

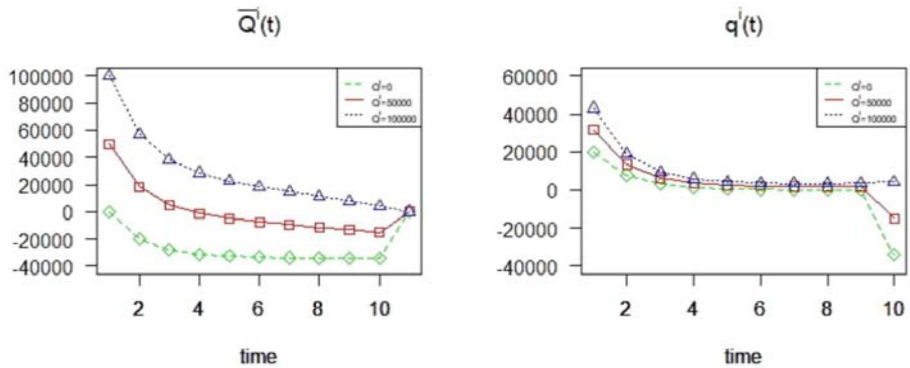
when $\rho$ goes to infinity. Hence, when the price impact is supposed to be permanent, $q_t^*$ is a deterministic function on time $t \in \{1, \ldots, T\}$ and depends on the state $s_t = ((w_t^1, w_t^2), p_t, (\overline{Q}_t^1, \overline{Q}_t^2), r_t)$ only through the remaining execution volumes $\overline{Q}_t^1$ and $\overline{Q}_t^2$. These facts resolve the deterministic equilibrium execution strategy into a one at a Markov perfect equilibrium in a more broader class of non–static execution strategy.

### 3.2 In the Case of Asymmetric Large Trader

We illustrate the situation where the position of each large trader at the beginning of the trading horizon or the risk aversion rate are different in this subsection; $Q^1 \neq Q^2$ or $R^1 \neq R^2$.

#### 3.2.1 The Effect of $Q^i$

We demonstrate the case that at the beginning one large trader $i$ plans to purchase $Q^i$ volumes of one risky asset which is not equal to (or less than) the quantity $Q^j$ to be bought by the large trader $j$. We present the three cases as follows: $Q^i = 100,000$, $Q^i = 50,000$, and $Q^i = 0$; whereas the remaining execution volume of the counterpart is fixed: $Q^j = 100,000$.




(a) Remaining Execution Volume $\overline{Q}_t^i(t = 1, \ldots, T)$ (b) Execution Volume $q_t^i(t = 1, \ldots, T)$

Figure 5: The Effect of $Q^i$

In [9], they consider about a possibility of a gain from a round trip trading in the case of a single large trader model. Referring to [5], an opportunity of an arbitrage in a weak sense is a round trip trading schedule ($\pi^i =$) $\boldsymbol{q}^i$ if

$$\mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}^i|s_1] - w_1^i = \mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}^i - w_1^i|s_1] = \mathbb{E}_1^{\boldsymbol{q}}\left[-\sum_{t=1}^{T}\widehat{p}_t q_t^i \middle| s_1\right] > 0 \tag{3.3}$$

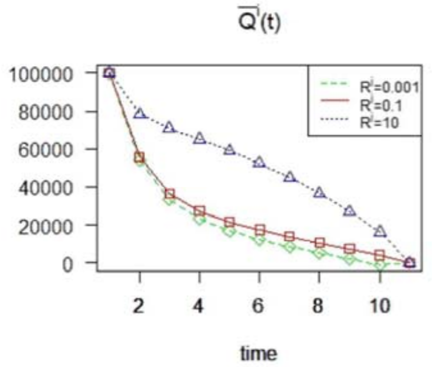
where for a sequence $\boldsymbol{q}^i := (q_1^i, \ldots, q_T^i) \in \mathbb{R}^T$, a static (and non–randomized) execution strategy $\pi^i = (f_1^i, \ldots, f_T^i) \in \Pi_M^i$ defined by $f_t^i(s_t) = q_t^i$ for any $s_t \in S = \mathbb{R}$ is a round trip trading schedule if $\sum_{t=1}^T q_t^i = 0$, and $\widehat{p}_t$ for $t \in \{1, \ldots, T\}$ is the execution price defined in Section 2. [9] shows that when trading crowd causes an price impact, either positive or negative one (which means $\mu_{v_t} \neq 0$), then there exist a round trip trade for a large trader which satisfies an arbitrage in a weak sense in the case of $n = 1$.

Figure 10 illustrates that if the initial volume $Q^i$ is equal to 0, there exists a round trip trading for trader $i$ which satisfy an arbitrage in a weak sense. When $Q^i$ is 50000, a round trip trading is included in the trajectory of the remaining execution volume. These results infer a new (essential) opportunity of a round trip trading. This insights also suggest a possibility that a volume $Q^{i*}$ exists for the large trader $i$ satisfying the following condition: if the initial volume $Q^i$ is smaller than $Q^{i*}$, the execution strategy of large trader $i$ becomes a round trip trading or partially includes a round trip trading which meets the requirement for a weak arbitrage.

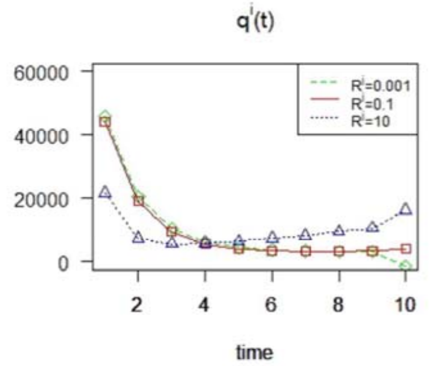
### 3.2.2 The Effect of $R^j$

We examine how the execution strategies are affected by the risk aversion of the opponent by illustrating the following three cases: $R^j = 0.001, R^j = 0.1$ and $R^j = 10$ while $R^i$ is always 0.1.

From Figure 11, we confirm that the large trader $i$ executes less volume at the beginning and liquidates his/her last position more at the terminal as the risk aversion rate of the counterpart




(a) Remaining Execution Volume $\overline{Q}_t^i(t = 1, \ldots, T)$




(b) Execution Volume $q_t^i(t = 1, \ldots, T)$

Figure 6: The Effect of $R^j$

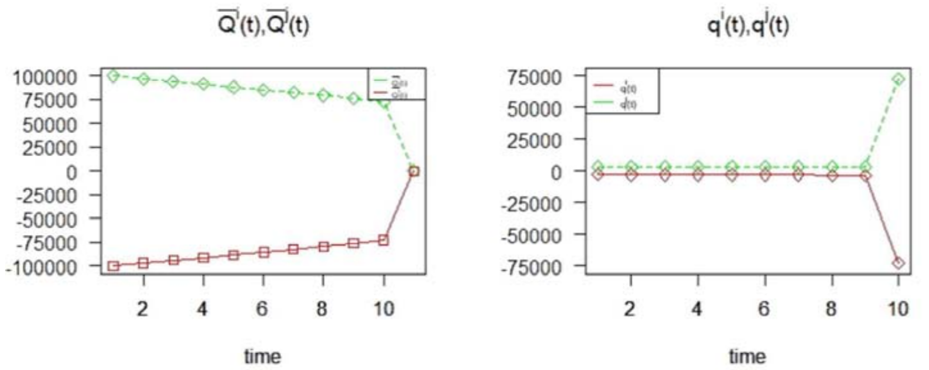
$j$ becomes larger. In the middle time of the trading epoch, the trajectory of the executed volume tends to increase, particularly seen when $R^j = 10$. These facts infer the following statements: as the counterpart is more risk–averse, the speed of the execution becomes slower at the beginning and the path of the execution volume of the large trader is gradually growing over the course of the execution process. Then, the large trader executes more volume at the end of the trading.

**Proposition 2** *If the price impact* (*or quoted price*) *is not affected by the public news effect for all the time, that is,* $\mu_{\varepsilon_t} = 0$ *and* $\sigma_{\varepsilon_t} = 0, (t \in \{1, \ldots, T\})$*, the execution strategies of the large trader* $i$ *do not depend on the risk aversion rate of the counterpart* $R^j$*. Then, a unique strategy is determined by other determining factors and parameters.*

#### 3.2.3 The Existence of Sell Trader

In the following, we illustrate the case that one large trader $j$ sells the quantity $Q^j$. The quantity $Q^j$ to be sold is the same as the volume $Q^i$ to be bought by the large trader $i$. In mathematical expression, $Q^j = -Q^i = -100000$.

Figure 12 indicates both the buy– and sell–traders remain patient with the other side's execution with each other before the maturity $T$, liquidating all their remaining positions at the terminal. A close consideration reveals that they don't execute their whole order simultaneously at the beginning of the trade, although it leads to canceling out the impact of their execution on market price. The following reasoning illustrates the above insights. The sell–trader can earn if he/she executes the sell–transaction after the execution price goes up because of a buy–side execution than the simultaneous execution at the beginning. In contrast, the buy–side trader can purchase the risky asset at a lower price after the sell–trader unwinds his/her position, which saves execution costs. These examinations imply that both sides of the large traders can produce a profit by executing slowly, which causes them to liquidate their positions much measuredly.

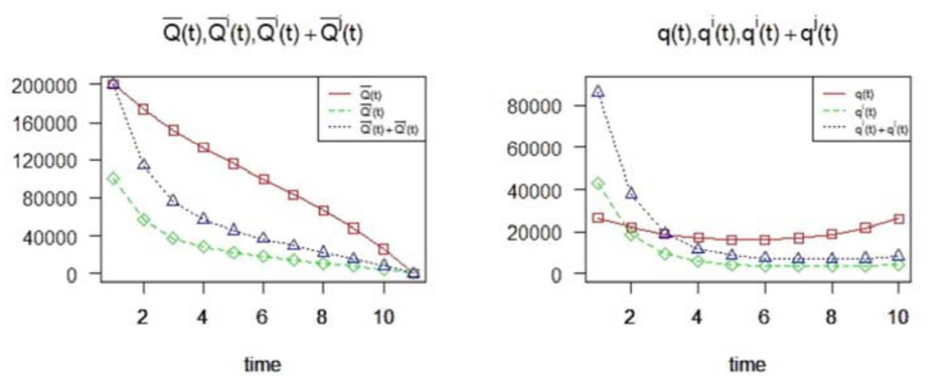



(a) Remaining Execution Volume $\overline{Q}_t^i(t=1,\ldots,T)$ (b) Execution Volume $q_t^i(t=1,\ldots,T)$

Figure 7: The Existence of Sell Trader

### 3.3 Comparison of the Case with Single Large Trader Case

The following figures show how the executions differ between in the single–large trader model and two–large trader model in our model. In order to compare the differences, the price impact coefficient of the trading crowd $v_t$ is set as 0, which is the same meaning as setting $\mu_{v_t}=0$ and $\sigma_{v_t}=0,(t=1,\ldots,T)$, in both models. Then, the volume of which a single large trader unwinds the position and the total volume of two large traders are deemed as same, i.e., $Q/2=Q^i=Q^j=100,000$ (that is, $Q=Q^i+Q^j=200,000$ and $Q^i=Q^j$).




(a) Remaining Execution Volumes $\overline{Q}_t,\overline{Q}_t^i$, and $\overline{Q}_t^1+\overline{Q}_t^2(t=1,\ldots,T)$ (b) Execution Volumes $q_t$, $q_t^i$, and $q_t^1+q_t^2(t=1,\ldots,T)$

Figure 8: Comparison of the Two Models

From Figure 13, the single large trader of the two–large trader model executes each order not too fast over the trading time horizon at first glance. The initial position is, however, quite different, the second model of which is half the volume of the first model ($Q = 200,000$ and $Q^i = 100,000$). It is not overstated that the large trader of the two large trader model unwinds his/her position twice as fast as that of the single large trader model. The interaction of the multiple large traders becomes a vital factor for the reason. The traders in the second model unwind their positions faster for fear that the price will be pushed up by the execution of the counterpart. This logic contrasts with the explanation in Subsection 3.2.2., where each large trader intends to acquire/liquidate their positions after the counterpart executes his/her order; if two large traders with equivalent initial inventories, either buying or selling, exist in a market, each large trader takes advantage of acquiring the risky asset at a lower price and liquidating at a higher price by executing faster than the counterpart.

## 4 Conclusion and Future Research

We constructed, in a (finite) discrete time framework, a model focusing on multiple large traders. They maximize the expected CARA utility arising from each large trader's wealth at the end of the trading epoch in a market with trading crowd. By constructing the generalized price impact model, the backward induction method of dynamic programming permitted us to derive a Markov perfect equilibrium strategy in this model. The most important results which emerged from this research is as follows: the aggregate execution volume of trading crowd has an impact on the execution of each large trader. All the intriguing results we have obtained from the numerical examples accounted for how various kinds of situations can influence the execution strategy of a large trader in our models. This kind of work concerned with an execution problem through the backward induction procedure of dynamic programming will be explored from a more in–depth and extensive perspective, which we can expect will also give us a more illuminating insight into all the other problems left in this field of research as follows.

In the above model, we have assumed that the price reversion rate and the resilience speed are deterministic and, due to the assumption, that the residual effect becomes deterministic if the total execution volumes of trading crowd are also deterministic, which gives rise to the deterministic optimal and equilibrium execution strategies in our model. This assumption makes the fundamental price of the risky asset observable for large traders before the trading time. The fundamental value of a risky asset is, however, unobservable and uncertain in a real market. Therefore, we can evolve the model built in this paper as an incomplete state information model, which leads to an analysis in a more realistic situation of the marketplace. Developing an incomplete state information model of either single– or multiple–large traders will (surely) contribute to some developments of a study involved in a trading market.

## Acknowledgements

The authors would like to thank Dr. Sato for closely examining our preliminary draft and the comments at the Japan Finance Conference, and Dr. Jaimungal and Dr. Al-Aradi for their helpful comments and suggestions at the Quantitative Methods in Finance 2018 Conference.


## References


[1] Almgren, R. and Chriss, N., Optimal execution of portfolio transactions. *Journal of Risk*, 2000, **3**, 5–39.

[2] Bertsimas, D. and Lo, A. W., Optimal control of execution costs. *Journal of Financ. Mark.*, 1998, **1**, 1–50.

[3] Cartea, Á. and Jaimungal, S., Incorporating order-flow into optimal execution. *Math. and Financ. Econ.*, 2016, **10**, 339–364.

[4] Guéant, O., *The Financial Mathematics of Market Liquidity: From optimal execution to market making*. 2016 (CRC Press: Boca Raton, Florida).

[5] Huberman, G. and Stanzl, W., Price manipulation and quasi-arbitrage. *Econometrica*, 2004, **72**, 1247–1275.

[6] KKunou, S. and Ohnishi, M., Optimal execution strategies with price impact. *RIMS Kokyuroku*, 2010, **1645**, 234–247.

[7] Kuno, S. and Ohnishi, M., Optimal execution in illiquid market with the absence of price manipulation. *J. of Math. Fin.*, 2015, **5**, 1–14.

[8] Kuno, S., Ohnishi, M., and Shimizu, P., Optimal off-exchange execution with closing price. *J. of Math. Fin.*, 2017, **7**, 54–64.

[9] Kuno, S., Ohnishi, M., and Shimoshimizu, M., Optimal execution strategies with generalized price impact models. *RIMS Kokyuroku*, 2018, **2078**, 77-83.

[10] Luo, X. and Schied, A., Nash equilibrium for risk-averse investors in a market impact game with transient price impact. arXiv preprint arXiv:1807.03813, 2018.

[11] Ohnishi, M., and Shimoshimizu, M., Optimal and equilibrium execution strategies with generalized price impact. Paper presented at the Nippon Finance Association 26th Annual Conference, Hitotsubashi University, 24 June, 2018.

[12] Ohnishi, M., and Shimoshimizu, M., Optimal and equilibrium execution strategies with generalized price impact. Paper presented at JAFEE International Conference on Financial Engineering on Collaboration with NUS-UTokyo, The University of Tokyo, 24-25 August 2018.

[13] Potters, M. and Bouchaud, J. P., More statistical properties of order books and price impact. *Physica A*, 2003, **324**, 133–140.

[14] Schied, A. and Zhang, T., A market impact game under transient price impact. *Math. Oper. Res.*, 2018, Published online in Articles in Advance 06 Sep 2018, 1–20.

[15] Schöneborn, T., Trade execution in illiquid markets: Optimal stochastic control and multi-agent equilibria. Doctoral dissertation, Technische Universität Berlin, 2008.

# Appendix

## A Proof of Theorem 2.1

In the following, we derive an equilibrium execution strategy for each large trader at a Markov perfect equilibrium by backward induction method of dynamic programming from maturity $T$ in the case of $n = 2$. The following proof describes the sketchy one of Theorem 4.1 for the case $\kappa_t \equiv 0$ or $v_t \equiv 0$, from the results of which we conduct the comparative statics shown in Section 5. We can derive the similar results if $\kappa_t \neq 0$ and $v_t$ satisfies $v_t \neq 0$ for all $t \in \{1, \dots, T\}$. First, at the time $t = T$, due to the (hard) constraint for each large trader to unwind all remainder of his/her position at time $t = T$,

$$\overline{Q}^i_{T+1} = \overline{Q}^i_T - q^i_T = 0, \quad i = 1, 2, \tag{A.1}$$

must be satisfied, which yields $\overline{Q}^i_T = q^i_T$ for $i = 1, 2$. Therefore, the expected utility payoff of the large trader $i = 1, 2$ (or the value function of large trader $i$) at the maturity becomes

$$\begin{aligned}
V^i_T(\pi^{1*}_T, \pi^{2*}_T)[s_T] &= \sup_{q^i_T \in \mathbb{R}} \mathbb{E}[V^i_{T+1}(\pi^{1*}_{T+1}, \pi^{2*}_{T+1})[s_{T+1}] | s_T] \\
&= \sup_{q^i_T \in \mathbb{R}} \mathbb{E}[-\exp\{R^i w^i_{T+1}\} | s_T] \\
&= \sup_{q^i_T \in \mathbb{R}} \mathbb{E}[-\exp\{R^i [w^i_T - \{p_T + \lambda_T (q^i_T + q^j_T)\} q^i_T]\} | s_T] \\
&= -\exp\{R^i [w^i_T - \{p_T + \lambda_T (\overline{Q}^i_T + \overline{Q}^j_T)\} \overline{Q}^i_T]\} \\
&= -\exp\{R^i [w^i_T - p_T \overline{Q}^i_T - \lambda_T (\overline{Q}^i_T)^2 - \lambda_t \overline{Q}^i_T \overline{Q}^j_T], \quad i, j = 1, 2, \quad i \neq j.
\end{aligned} \tag{A.2}$$

For $t = T - 1$, the value functions $V^i_{T-1}(\pi^{1*}_{T-1}, \pi^{2*}_{T-1})[s_{T-1}], i = 1, 2$ become the following functional form:

$$
\begin{aligned}
& V^i_{T-1}(\pi^{1*}_{T-1}, \pi^{2*}_{T-1})[s_{T-1}] \\
= & \sup_{q^i_{T-1}\in\mathbb{R}} \mathbb{E}[V^i_T(\pi^{1*}_T, \pi^{2*}_T)[s_T]|s_{T-1}] \\
= & \sup_{q^i_{T-1}\in\mathbb{R}} \mathbb{E}[-\exp\{-R^i[w^i_T - p_T\overline{Q}^i_T - \lambda_T(\overline{Q}^i_T)^2 - \lambda_T\overline{Q}^i_T\overline{Q}^j_T]\}|s_{T-1}] \\
= & \sup_{q^i_{T-1}\in\mathbb{R}} -\exp\{-R^i\{(1-\alpha^{T-1})\lambda_{T-1} + \lambda_T + \frac{1}{2}R^i\sigma^2_{\epsilon_{T-1}}\}(q^i_{T-1})^2 \\
& \quad + \{(-\lambda_{T-1}\alpha^{T-1} + 2\lambda_T + R^i\sigma^2_{\epsilon_{T-1}})\overline{Q}^i_{T-1} + \lambda_T\overline{Q}^j_{T-1} - (1-\mathrm{e}^{-\rho})r_{T-1} \\
& \quad - \{(1-\alpha^{T-1})\lambda_{T-1} + \lambda_T\}q^j_{T-1} + \mu_{\epsilon_{T-1}}\}q^i_{T-1} \\
& \quad + w_{T-1} - p_{T-1}\overline{Q}^i_{T-1} - (\lambda_T + \frac{1}{2}R^i\sigma^2_{\epsilon_{T-1}})(\overline{Q}^i_{T-1})^2 - \mu_{\epsilon_{T-1}}\overline{Q}^i_{T-1} + (1-\mathrm{e}^{-\rho})r_{T-1}\overline{Q}^i_{T-1} \\
& \quad - \lambda_T\overline{Q}^i_{T-1}\overline{Q}^j_{T-1} + (\lambda_T - \lambda_{T-1}\alpha^{T-1})\overline{Q}^i_{T-1}q^j_{T-1}\},
\end{aligned}
\tag{A.3}
$$

where

$$
\alpha^{T-1} := \alpha_{T-1}\mathrm{e}^{-\rho} + (1-\alpha_{T-1}). \tag{A.4}
$$

Thus, we can obtain the execution volume of the large trader $i = 1, 2$ for $t = T-1$ at the supremum of Eq. (A.3) by completing the square of the quadratic equation in the negative exponential function above as follows:

$$
q^{i*}_{T-1} = \frac{1}{2A^i_{T-1}}(B^i_{T-1}\overline{Q}^i_{T-1} + C^i_{T-1}\overline{Q}^j_{T-1} + D^i_{T-1}r_{T-1} + F^i_{T-1}q^j_{T-1} + S^i_{T-1}), \quad i, j = 1, 2, \quad i \neq j, \tag{A.5}
$$

where

$$
\begin{aligned}
A^i_{T-1} &:= (1-\alpha^{T-1})\lambda_{T-1} + \lambda_T + \frac{1}{2}R^i\sigma^2_{\varepsilon_{T-1}}, \\
B^i_{T-1} &:= -\lambda_{T-1}\alpha^{T-1} + 2\lambda_T + R^i\sigma^2_{\varepsilon_{T-1}}, \\
C^i_{T-1} &:= \lambda_T, \\
D^i_{T-1} &:= -(1-\mathrm{e}^{-\rho}), \\
F^i_{T-1} &:= -\{(1-\mathrm{e}^{-\rho})\lambda_{T-1} + \lambda_T\}, \\
S^i_{T-1} &:= \mu_{\varepsilon_{T-1}}.
\end{aligned}
\tag{A.6}
$$

Solving the simultaneous equation of Eq. (A.5) with respect to $q^1_{T-1}$ and $q^2_{T-1}$ yields the execution volumes at a Markov perfect equilibrium:

$$
\begin{aligned}
f^{i*}_{T-1}(s_{T-1}) &=: B^{i*}_{T-1}\overline{Q}^i_{T-1} + C^{i*}_{T-1}\overline{Q}^j_{T-1} + D^{i*}_{T-1}r_{T-1} + S^{i*}_{T-1} \\
&= a^i_{T-1} + b^i_{T-1}\overline{Q}^i_{T-1} + c^i_{T-1}\overline{Q}^j_{T-1} + d^i_{T-1}r_{T-1}, \quad i, j = 1, 2, \quad i \neq j,
\end{aligned}
\tag{A.7}
$$

where for $i, j = 1, 2, i \neq j$,,

$$
\begin{aligned}
&\delta^i_{T-1} := 2A^i_{T-1} - \frac{F^i_{T-1}F^j_{T-1}}{2A^j_{T-1}},\\
&B^{i*}_{T-1} := b^i_{T-1} = \frac{1}{\delta^i_{T-1}}(B^i_{T-1} + \frac{F^j_{T-1}C^i_{T-1}}{2A^i_{T-1}}), \quad C^{i*}_{T-1} := c^i_{T-1} = \frac{1}{\delta^i_{T-1}}(C^i_{T-1} + \frac{F^j_{T-1}B^i_{T-1}}{2A^i_{T-1}}),\\
&D^{i*}_{T-1} := d^i_{T-1} = \frac{1}{\delta^i_{T-1}}(D^i_{T-1} + \frac{F^j_{T-1}D^i_{T-1}}{2A^i_{T-1}}), \quad S^{i*}_{T-1} := a^i_{T-1} = \frac{1}{\delta^i_{T-1}}(S^i_{T-1} + \frac{F^j_{T-1}S^i_{T-1}}{2A^i_{T-1}}).
\end{aligned}
\tag{A.8}
$$

Then, we obtain the following expected utility payoff for large trader $i = 1, 2, i \neq j$ at the Markov perfect equilibrium:

$$
\begin{aligned}
V^i_{T-1}(\pi^{1*}_{T-1}, \pi^{2*}_{T-1})[s_{T-1}] = -\exp\{&-R^i(w^i_{T-1} - p_{T-1}\overline{Q}^i_{T-1} - G^i_{T-1}(\overline{Q}^i_{T-1})^2 - H^i_{T-1}\overline{Q}^i_{T-1}\\
&+ I^i_{T-1}r_{T-1}\overline{Q}^i_{T-1} + J^i_{T-1}r^2_{T-1} + L^i_{T-1}r_{T-1} + M^i_{T-1}(\overline{Q}^j_{T-1})^2\\
&+ N^i_{T-1}\overline{Q}^j_{T-1} + X^i_{T-1}r_{T-1}\overline{Q}^j_{T-1} + Y^i_{T-1}\overline{Q}^i_{T-1}\overline{Q}^j_{T-1} + Z^i_{T-1})\},
\end{aligned}
\tag{A.9}
$$

where

$$
\begin{aligned}
&B^{i**}_{T-1} := B^i_{T-1} + F^i_{T-1}C^{j*}_{T-1}, \quad C^{i**}_{T-1} := C^i_{T-1} + F^i_{T-1}B^{j*}_{T-1},\\
&D^{i**}_{T-1} := D^i_{T-1} + F^i_{T-1}D^{j*}_{T-1}, \quad S^{i**}_{T-1} := S^i_{T-1} + F^i_{T-1}S^{j*}_{T-1}, \quad i, j = 1, 2, \quad i \neq j,
\end{aligned}
\tag{A.10}
$$

and for trader $i, j = 1, 2, i \neq j$,

$$
\begin{aligned}
&G^i_{T-1} := \lambda_T + \frac{1}{2}R^i\sigma^2_{\varepsilon_{T-1}} - (\lambda_T - \lambda_{T-1}\alpha^{T-1})C^{j*}_{T-1} - \frac{(B^{i**}_{T-1})^2}{4A^i_{T-1}},\\
&H^i_{T-1} := \mu_{\varepsilon_{T-1}} - (\lambda_T - \lambda_{T-1}\alpha^{T-1})S^{j*}_{T-1} - \frac{B^{i**}_{T-1}S^{i**}_{T-1}}{2A^i_{T-1}},\\
&I^i_{T-1} := (1 - \mathrm{e}^{-\rho}) + (\lambda_T - \lambda_{T-1}\alpha^{T-1})D^{j*}_{T-1} + \frac{B^{i**}_{T-1}D^{i**}_{T-1}}{2A^i_{T-1}},\\
&J^i_{T-1} := \frac{(D^{i**}_{T-1})^2}{4A^i_{T-1}}, \quad L^i_{T-1} := \frac{D^{i**}_{T-1}S^{i**}_{T-1}}{2A^i_{T-1}}, \quad M^i_{T-1} := \frac{(C^{i**}_{T-1})^2}{4A_{T-1}},\\
&N^i_{T-1} := \frac{C^{i**}_{T-1}S^{i**}_{T-1}}{2A^i_{T-1}}, \quad X^i_{T-1} := \frac{C^{i**}_{T-1}D^{i**}_{T-1}}{2A^i_{T-1}},\\
&Y^i_{T-1} := (\lambda_T - \lambda_{T-1}\alpha^{T-1})B^{j*}_{T-1} + \frac{B^{i**}_{T-1}C^{i**}_{T-1}}{2A^i_{T-1}} - \lambda_T, \quad Z^i_{T-1} := \frac{(S^{i**}_{T-1})^2}{4A^i_{T-1}}.
\end{aligned}
\tag{A.11}
$$

Hereafter, we assume that the expected utility payoff functions of the large trader $i = 1, 2$ at time $t + 1$ take the following functional form:

$$
\begin{aligned}
V^i_{t+1}(\pi^{1*}_{t+1}, \pi^{2*}_{t+1})[s_{t+1}] = -\exp\{&-R^i(w^i_{t+1} - p_{t+1}\overline{Q}^i_{t+1} - G^i_{t+1}(\overline{Q}^i_{t+1})^2 - H^i_{t+1}\overline{Q}^i_{t+1}\\
&+ I^i_{t+1}r_{t+1}\overline{Q}^i_{t+1} + J^i_{t+1}r^2_{t+1} + L^i_{t+1}r_{t+1} + M^i_{t+1}(\overline{Q}^j_{t+1})^2\\
&+ N^i_{t+1}\overline{Q}^j_{t+1} + X^i_{t+1}r_{t+1}\overline{Q}^j_{t+1} + Y^i_{t+1}\overline{Q}^i_{t+1}\overline{Q}^j_{t+1} + Z^i_{t+1})\}.
\end{aligned}
\tag{A.12}
$$

Then, $V_t^i(\pi_t^{1*}, \pi_t^{2*})[s_t], i = 1, 2$ become

$$
\begin{aligned}
&V_t^i(\pi_t^{1*}, \pi_t^{2*})[s_t] \\
&= \sup_{q_t^i \in \mathbb{R}} \mathbb{E}[V_{t+1}^i(\pi_{t+1}^{1*}, \pi_{t+1}^{2*})[s_{t+1}] | s_t] \\
&= \sup_{q_t^i \in \mathbb{R}} -\exp\{-R^i[-A_t^i(q_t^i)^2 + \{B_t^i\overline{Q}_t^i + C_t^i\overline{Q}_t^j + D_t^i r_t + F_t^i q_t^j + S_t^i\}q_t^i \\
&\quad + w_t^i - p_t\overline{Q}_t^i - (G_{T+1}^i + \frac{1}{2}R^i\sigma_{\varepsilon_t})(\overline{Q}_t^i)^2 - (H_{t+1}^i + \mu_{\varepsilon_t})\overline{Q}_t^i + \{(1 - \mathrm{e}^{-\rho}) + I_{t+1}^i \mathrm{e}^{-\rho}\}r_t\overline{Q}_t^i \\
&\quad + J_{t+1}^i \mathrm{e}^{-2\rho} r_t^2 + L_{t+1}^i \mathrm{e}^{-\rho} r_t + M_{t+1}^i(\overline{Q}_t^j)^2 + N_{t+1}^i\overline{Q}_t^j + X_{t+1}^i \mathrm{e}^{-\rho} r_t\overline{Q}_t^j + Y_{t+1}\overline{Q}_t^i\overline{Q}_t^j + Z_{t+1}^i \\
&\quad + \{(-\lambda_t\alpha^t + I_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t - Y_{t+1}^i)\overline{Q}_t^i + (-2M_{t+1}^i + Y_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t)\overline{Q}_t^j \\
&\quad + (2J_{t+1}^i \mathrm{e}^{-2\rho}\lambda_t\alpha_t - X_{t+1}^i \mathrm{e}^{-\rho})r_t + (L_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t - N_{t+1}^i)\}q_t^j \\
&\quad + (J_{t+1}^i \mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 + M_{t+1}^i - X_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t)(q_t^j)^2]\},
\end{aligned}
\tag{A.13}
$$

where for $i = 1, 2$

$$
\begin{aligned}
A_t^i &:= (1 - \alpha^t)\lambda_t + G_{t+1}^i + L_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t - J_{t+1}^i \mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 + \frac{1}{2}R^i\sigma_{\varepsilon_t}^2, \\
B_t^i &:= -\lambda_t\alpha^t + 2G_{t+1}^i + I_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t + R^i\sigma_{\varepsilon_t}^2, \\
C_t^i &:= X_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t - Y_{t+1}^i, \\
D_t^i &:= -(1 - \mathrm{e}^{-\rho}) - I_{t+1}^i \mathrm{e}^{-\rho} + 2J_{t+1}^i \mathrm{e}^{-2\rho}\lambda_t\alpha_t, \\
F_t^i &:= -(1 - \alpha^t)\lambda_t - I_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t + 2J_{t+1}^i \mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 - X_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t + Y_{t+1}^i, \\
S_t^i &:= H_{t+1}^i + L_{t+1}^i \mathrm{e}^{-\rho}\lambda_t\alpha_t - \mu_{\varepsilon_t}.
\end{aligned}
\tag{A.14}
$$

Therefore, the execution volume of the large trader $i = 1, 2$ at time $t \in \{T-2, \ldots, 1\}$ attaining the supremum of $V_t^i(\pi_t^1, \pi_t^2)[s_t]$ becomes like Eq. (A.5) by the same derivation methods beforehand:

$$
q_t^{i*}(s_t) = \frac{1}{2A_t^i}(B_t^i\overline{Q}_t^i + C_t^i\overline{Q}_t^j + D_t^i r_t + F_t^i q_t^j + S_t^i), \quad i, j = 1, 2, \quad i \neq j. \tag{A.15}
$$

From Eq. (A.15), we obtain the following execution volume at the Markov perfect equilibrium $q_t^{i*}, i = 1, 2$:

$$
\begin{aligned}
f_t^{i*}(s_t) &= B_t^{i*}\overline{Q}_t^i + C_t^{i*}\overline{Q}_t^j + D_t^{i*} r_t + S_t^{i*} \\
&= a_t^i + b_t^i\overline{Q}_t^i + c_t^i\overline{Q}_t^j + d_t^i r_t, \quad i, j = 1, 2, \quad i \neq j,
\end{aligned}
\tag{A.16}
$$

where for $i, j = 1, 2, i \neq j$,

$$
\begin{aligned}
&\delta_t^i := 2A_t^i - \frac{F_t^i F_t^j}{2A_t^j}, \\
&B_t^{i*} := b_t^i = \frac{1}{\delta_t^i}(B_t^i + \frac{F_t^j C_t^i}{2A_t^i}), \quad C_t^{i*} := c_t^i = \frac{1}{\delta_t^i}(C_t^i + \frac{F_t^j B_t^i}{2A_t^i}), \\
&D_t^{i*} := d_t^i = \frac{1}{\delta_t^i}(D_t^i + \frac{F_t^j D_t^i}{2A_t^i}), \quad S_t^{i*} := a_t^i = \frac{1}{\delta_t^i}(S_t^i + \frac{F_t^j S_t^i}{2A_t^i}),
\end{aligned}
\tag{A.17}
$$

and the expected equilibrium payoff for large trader $i = 1, 2$ at the Markov Perfect equilibrium $(\pi^{1*}, \pi^{2*}) \in \Pi_M^1 \times \Pi_M^2$ become

$$
\begin{aligned}
&V_t^i(\pi_t^{1*}, \pi_t^{2*})[s_t] \\
&= -\exp\{-R^i[w_t^i - p_t\overline{Q}_t^i - (G_{T+1}^i + \frac{1}{2}R_i\sigma_{\varepsilon_t}^2)(\overline{Q}_t^i)^2 - (H_{t+1}^i + \mu_{\varepsilon_t})\overline{Q}_t^i \\
&\quad + \{(1-\mathrm{e}^{-\rho}) + I_{t+1}^i\mathrm{e}^{-\rho}\}r_t\overline{Q}_t^i + J_{t+1}^i\mathrm{e}^{-2\rho}r_t^2 + L_{t+1}^i\mathrm{e}^{-\rho}r_t + M_{t+1}(\overline{Q}_t^j)^2 + N_{t+1}\overline{Q}_t^j \\
&\quad + X_{t+1}^i\mathrm{e}^{-\rho}r_t\overline{Q}_t^j + Y_{t+1}\overline{Q}_t^i\overline{Q}_t^j + Z_{t+1}^i + \{(-\lambda_t\alpha^t + I_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t - Y_{t+1}^i)\overline{Q}_t^i \\
&\quad + (-2M_{t+1}^i + Y_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t)\overline{Q}_t^j + (2J_{t+1}^i\mathrm{e}^{-2\rho}\lambda_t\alpha_t - X_{t+1}^i\mathrm{e}^{-\rho})r_t \\
&\quad + (L_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t - N_{t+1}^i)\}q_t^{j*} + (J_{t+1}^i\mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 + M_{t+1}^i - X_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t)(q_t^{j*})^2 \\
&\quad + \frac{1}{4A_t^i}(B_t^{i**}\overline{Q}_t^i + C_t^{i**}\overline{Q}_t^j + D_t^{i**}r_t + S_t^{i**})^2]\} \\
&= -\exp\{-R^i(w_t^i - p_t\overline{Q}_t^i - G_t(\overline{Q}_t^i)^2 - H_t\overline{Q}_t^i + I_tr_t\overline{Q}_t^i + J_tr_t^2 + L_tr_t \\
&\quad + M_t^i(\overline{Q}_t^j)^2 + N_t^i\overline{Q}_t^j + X_t^ir_t\overline{Q}_t^j + Y_t^i\overline{Q}_t^i\overline{Q}_t^j + Z_t^i)\},
\end{aligned}
\tag{A.18}
$$

where

$$
\begin{aligned}
\psi_t^i &:= -\lambda_t\alpha^t + I_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t - Y_{t+1}^i, \\
\phi_t^i &:= -2M_{t+1}^i + Y_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t, \\
\theta_t^i &:= 2J_{t+1}^i\mathrm{e}^{-2\rho}\lambda_t\alpha_t - X_{t+1}^i\mathrm{e}^{-\rho}, \\
\iota_t^i &:= L_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t - N_{t+1}^i, \\
\nu_t^i &:= J_{t+1}^i\mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 + M_{t+1}^i - X_{t+1}^i\mathrm{e}^{-\rho}\lambda_t\alpha_t,
\end{aligned}
\tag{A.19}
$$

and

$$
\begin{aligned}
&B_t^{i**} := B_t^i + F_t^iC_t^{j*}, \quad C_t^{i**} := C_t^i + F_t^iB_t^{j*}, \\
&D_t^{i**} := D_t^i + F_t^iD_t^{j*}, \quad S_t^{i**} := S_t^i + F_t^iS_t^{j*}, \quad i, j = 1, 2, \quad i \neq j,
\end{aligned}
\tag{A.20}
$$

and

$$
\begin{aligned}
G_t^i &:= G_{t+1}^i + \frac{1}{2} R^i \sigma_\varepsilon^2 - \psi_t^i C_t^{j*} - \nu_t^i (B_t^{j*})^2 - \frac{(B_t^{i**})^2}{4A_t}, \\
H_t^i &:= H_{t+1}^i + \mu_{\epsilon_t} - \psi_t^i S_t^{j*} - \iota_t^i C_t^{j*} - 2\nu_t^i C_t^{j*} S_t^{j*} - \frac{B_t^{i**} S_t^{i**}}{2A_t^i}, \\
I_t^i &:= I_{t+1}^i \mathrm{e}^{-\rho} + (1 - \mathrm{e}^{-\rho}) + \psi_t^i D_t^{j*} + \theta_t^i C_t^{j*} + 2\nu_t^i C_t^{j*} D_t^{j*} + \frac{B_t^{i**} D_t^{i**}}{2A_t^i}, \\
J_t^i &:= J_{t+1}^i \mathrm{e}^{-2\rho} + \theta_t^i D_t^{j*} + \nu_t^i (D_t^{j*})^2 + \frac{(D_t^{i**})^2}{4A_t^i}, \\
L_t^i &:= L_{t+1}^i \mathrm{e}^{-\rho} + \theta_t^i S_t^{j*} + \iota_t^i D_t^{j*} + 2\nu_t^i D_t^{j*} S_t^{j*} + \frac{D_t^{i**} S_t^{i**}}{2A_t^i}, \\
M_t^i &:= M_{t+1}^i + \phi_t^i B_t^{j*} + \nu_t^i (C_t^{j*})^2 + \frac{(C_t^{i**})^2}{4A_t^i}, \\
N_t^i &:= N_{t+1}^i + \phi_t^i S_t^{j*} + \iota_t^i B_t^{j*} + 2\nu_t^i B_t^{j*} S_t^{j*} + \frac{C_t^{i**} S_t^{i**}}{2A_t^i}, \\
X_t^i &:= X_{t+1}^i \mathrm{e}^{-\rho} + \phi_t^i D_t^{j*} + \theta_t^i B_t^{j*} + 2\nu_t^i B_t^{j*} D_t^{j*} + \frac{C_t^{i**} D_t^{i**}}{2A_t^i}, \\
Y_t^i &:= Y_{t+1}^i + \psi_t^i B_t^{j*} + \phi_t^i C_t^{j*} + 2\nu_t^i B_t^{j*} C_t^{j*} + \frac{B_t^{i**} C_t^{i**}}{2A_t^i}, \\
Z_t^i &:= Z_{t+1}^i + \iota_t^i S_t^{j*} + \nu_t^i (S_t^{j*})^2 + \frac{(S_t^{i**})^2}{4A_t^i}.
\end{aligned}
\tag{A.21}
$$

□

Graduate School of Economics, Osaka University
Osaka, 560-0043, Japan
&
Center for Mathematical Modeling and Data Science, Osaka University
Osaka, 560–8531, Japan
E-mail address: Ohnishi@econ.osaka-u.ac.jp

Masamitsu OHNISHI

and

Graduate School of Economics, Osaka University
Osaka, 560-0043, Japan
E-mail address: shimosshi1q94@gmail.com

Makoto SHIMOSHIMIZU