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Diagrams of numerical semigroups whose general members are non-Weierstrass

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Abstract

We construct diagrams consisting of an infinite number of numerical semigroups through dividing by two whose general members are non-Weierstrass where the bottom of the diagram is some Weierstrass numerical semigroup.

1 Introduction

Let \( \mathbb{N}_0 \) be the additive monoid of non-negative integers. A submonoid \( H \) of \( \mathbb{N}_0 \) is called a numerical semigroup if the complement \( \mathbb{N}_0 \setminus H \) is finite. The cardinality of \( \mathbb{N}_0 \setminus H \) is called the genus of \( H \), denoted by \( g(H) \). In this article \( H \) always stands for a numerical semigroup. We set

\[
c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},
\]

which is called the conductor of \( H \). It is known that \( c(H) \leq 2g(H) \). \( H \) is said to be symmetric if \( c(H) = 2g(H) \). \( H \) is said to be quasi-symmetric if \( c(H) = 2g(H) - 1 \). We are interested in the case \( c(H) = 2g(H) - 2 \).

A curve means a complete non-singular irreducible algebraic curve over an algebraically closed field \( k \) of characteristic 0. For a pointed curve \( (C, P) \) we set

\[
H(P) = \{ n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = np \},
\]

where \( k(C) \) is the field of rational functions on \( C \). Then \( H(P) \) is a numerical semigroup of genus \( g(C) \) where \( g(C) \) is the genus of \( C \).

We set \( d_2(H) = \{ h' \in \mathbb{N}_0 \mid 2h' \in H \} \), which is a numerical semigroup. Let \( \pi : \tilde{C} \to C \) be a double covering of a curve with a ramification point \( \tilde{P} \). Then \( d_2(H(\tilde{P})) = H(\pi(\tilde{P})) \).

\( H \) is said to be Weierstrass if there exists a pointed curve \( (C, P) \) with \( H(P) = H \). \( H \) is said to be of double covering type (abbreviated to DC) if there exists a double covering \( \pi : C \to C' \) with a ramification point \( P \) such that \( H = H(P) \). If \( H \) is DC, then both \( H \) and \( d_2(H) \) are Weierstrass. For positive integers \( a_1, \ldots, a_s \) we denote by \( \langle a_1, \ldots, a_s \rangle \) the monoid generated by \( a_1, \ldots, a_s \). For example, \( H = \langle 2, 2g + 1 \rangle \) is DC with \( d_2(H) = \mathbb{N}_0 \).

Indeed, let \( \pi \) be a double covering from a curve of genus \( g \) to the projective line \( \mathbb{P}^1 \) and \( P \) be any ramification point. Then \( H(P) = H \). The following is an open problem:

\[1\) This paper is an extended abstract and the details will appear elsewhere.

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Problem. ([4] and [1]) What is the proportion of non-Weierstrass numerical semigroups in the whole set of numerical semigroups?

Our purpose in this article is to construct diagrams consisting of an infinite number of numerical semigroups through the map $d_2$ whose general members in the diagram are non-Weierstrass where the bottom of the diagram is a Weierstrass numerical semigroup $H$ with $c(H) = 2g(H) - 2$.

2 Towers of symmetric numerical semigroups

We set $m(H) = \min\{h \in H \mid h > 0\}$, which is called the multiplicity of $H$.

Remark 2.1 (i) Let $n$ be an odd integer. Then $2H + n\mathbb{N}_0$ is a numerical semigroup.
(ii) Let $n$ be an odd integer with $n \geq c(H) + m(H) - 1$. Then we have $d_2(2H + n\mathbb{N}_0) = H$.

We have the following result for the above numerical semigroups:

Theorem 2.2 (Komeda-Ohbuchi [5]) Let $n$ be an odd integer with

$$n \geq \max\{c(H) + m(H) - 1, 2g(H) + 1\}.$$

If $H$ is Weierstrass, then $2H + n\mathbb{N}_0$ is DC, hence Weierstrass.

We have towers consisting of symmetric numerical semigroups which are DC.

Theorem 2.3 Let $H_0$ be a symmetric Weierstrass numerical semigroup. For each $i \geq 1$ let us take an odd integer

$$n_i \geq \max\{c(H_{i-1}) + m(H_{i-1}) - 1, 2g(H_{i-1}) + 1\}$$

where we set $H_i = 2H_{i-1} + n_i\mathbb{N}_0$ for $i \geq 1$. Then we have towers of symmetric numerical semigroups which are DC as follows:

$$H_i \downarrow d_2 \quad \text{for } i \geq 1.$$

3 Towers of quasi-symmetric numerical semigroups

Lemma 3.1 ([2]) Let $H$ and $\tilde{H}$ be quasi-symmetric numerical semigroups with $d_2(\tilde{H}) = H$. Then we obtain $g(\tilde{H}) = 2g(H) - 1$.

By the above lemma and Riemann-Hurwitz Formula we get the following:

Theorem 3.2 Let $H$ and $\tilde{H}$ be quasi-symmetric numerical semigroups with $d_2(\tilde{H}) = H$. Then $\tilde{H}$ is not DC.
Proposition 3.3 ([2]) Let $H'$ be a quasi-symmetric numerical semigroup. We set 

$$n = \min\{h' \in H' \mid h' \text{ is odd}\} \text{ and } s_i = \min\{h' \in H' \mid h' \equiv i \mod n\}$$

for all $i = 1, \ldots, n - 1$. We set 

$$\{s_1, \ldots, s_{n-1}\} = \{s^{(1)} < \cdots < s^{(n-1)}\}$$

and 

$$H = \langle n, 2s^{(1)}, \ldots, 2s^{(\frac{n-1}{2})}, 2s^{(\frac{n+1}{2})} - n, \ldots, 2s^{(n-1)} - n \rangle.$$ 

Then $H$ is a quasi-symmetric numerical semigroup of genus $2g(H') - 1$ with $d_2(H) = H'$.

Example. Let $H_0 = \langle 3, 4, 5 \rangle$. For each odd $m \geq 1$ (resp. even $m \geq 2$) we set 

$$H_m = \langle 3, 3m + 2, 3 \cdot 2m + 1 \rangle$$ (resp. $H_m = \langle 3, 3m + 1, 3(2m - 1) + 2 \rangle$).

Then we have towers of quasi-symmetric numerical semigroups which are not DC as follows:

$$H_{i+1} \downarrow d_2 \text{ for } i \geq 0.$$ 

4 Diagrams of numerical semigroups with 

$c(H) = 2g(H) - 2$

We set 

$$PF(H) = \{\gamma \in \mathbb{N}_0 \setminus H \mid \gamma + h \in H, \text{ all } h \in H > 0\},$$

whose elements are called pseudo-Frobenius numbers of $H$. We have $c(H) - 1 \in PF(H)$. We set $t(H) = \#PF(H)$, which is called the type of $H$.

Remark 4.1 We have $c(H) + t(H) \leq 2g(H) + 1$. (For example, see [6].)

$H$ is said to be almost symmetric if the equality $c(H) + t(H) = 2g(H) + 1$ holds.

Remark 4.2 i) $H$ is symmetric if and only if $t(H) = 1$. In this case $H$ is almost symmetric.

ii) If $H$ is quasi-symmetric, then $t(H) = 2$. The converse does not hold. In this case $H$ is also almost symmetric.

iii) If $c(H) = 2g(H) - 2$, then $t(H) = 2$ or 3.

We set $PF^*(H) = PF(H) \setminus \{c(H) - 1\}$.

Proposition 4.3 ([3]) If $H$ is almost symmetric, then we have an automorphism of $PF^*(H)$ sending $\gamma$ to $c(H) - 1 - \gamma$. 
Corollary 4.4 If $c(H) = 2g(H) - 2$ and $t(H) = 3$, we have $PF^*(H) = \{\gamma, 2g(H) - 3 - \gamma\}$ for some $\gamma \in \mathbb{N}_0 \setminus H$.

Example. Let $H = \langle 4, 6, 4l + 1, 4l + 3 \rangle$ for $l \geq 1$. Then we have $c(H) = 4l = 2g(H) - 2$ and $PF^*(H) = \{2, 4l - 1 - 2\}$, hence $t(H) = 3$, i.e., $H$ is almost symmetric.

Example. Let $H = \langle 4, 4l+1, 4(2l-1)+3 \rangle$ for $l \geq 1$. Then we have $c(H) = 12l-4 = 2g(H) - 2$ and $PF^*(H) = \{4 \cdot 2l-2\}$, hence $t(H) = 2$, i.e., $H$ is not almost symmetric.

Remark 4.5 Let $n$ be an odd integer with $n \geq \max\{c(H) + m - 1, 2m\}$. Then we have $g(2H + n\mathbb{N}_0) = 2g(H) + \frac{n-1}{2}$ with $d_2(2H + n\mathbb{N}_0) = H$.

By the definition of $PF^*(H)$ we get the following:

Lemma 4.6 Let $d_2(\tilde{H}) = H$ and $n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h} \text{ is odd}\}$. Then the following are equivalent:

i) $g(\tilde{H}) = 2g(H) + \frac{n-1}{2} - 1$.

ii) $\tilde{H} = 2H + \langle n, n + 2f \rangle$ for some $f \in PF(H)$.

Theorem 4.7 Assume that $c(H) = 2g(H) - 2$ and $t(H) = 3$. Let $PF^*(H) = \{f_1, f_2\}$. We set $\tilde{H}_i = 2H + \langle n, n + 2f \rangle$ for $i = 1, 2$. Then one of the following holds:

i) $H$ is Weierstrass and $\tilde{H}_1, \tilde{H}_2$ are DC.

ii) $H$ is Weierstrass and renumbering 1 and 2 $\tilde{H}_1$ is DC and $\tilde{H}_2$ is not DC.

iii) $H$ is non-Weierstrass.

If $n >> 0$, then both $\tilde{H}_1$ and $\tilde{H}_2$ are non-Weierstrass.

Proof. For i) and ii) see [2]. Applying [7] we get iii).}

For $1 \leq i \leq m(H) - 1$ we define $s_i$ by $\min\{h \in H \mid h \equiv i \mod m(H)\}$. We set $S(H) = \{m(H)\} \cup \{s_i \mid i = 1, \ldots, m(H) - 1\}$, which is called the standard basis for $H$.

Remark 4.8 ([3]) We have $PF(H) = \{s_i - m(H) \mid s_i + s_j \notin S(H) \text{ for all } j\}$.

Theorem 4.9 ([2]) Assume that $c(H) = 2g(H) - 2$. Let $f = s_i - m(H) \in PF^*(H)$. Let $n$ be an odd number with $n \geq 4((2m(H) - 1)s_i - m(H)) + 1 - g(H) + 1$.

We set $\tilde{H} = 2H + \langle n, n + 2f \rangle$. Then we have the following:

i) $c(\tilde{H}) = 2g(\tilde{H}) - 2$ and $t(\tilde{H}) = 3$.

ii) $\tilde{H}$ is not DC. For odd $N \geq n + 2(2g(H) - 3 + m(H))$ we obtain that $\tilde{H} = 2\tilde{H} + \langle N, N + 2(2s_i - 2m(H)) \rangle$ is not DC.
Using the above theorem we get our main result in this article.

**Corollary 4.10** Let $H$ be a numerical semigroup with $c(H) = 2g(H) - 2$. Assume that $m(H)$ is a power of 2. Then we can construct a diagram of numerical semigroups whose general members are non-Weierstrass such that the bottom of the diagram is $H$. Here, general members mean all members in the interior of the diagram except finite ones.

**References**


