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Abstract

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On affine Osserman connections

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Abstract. An affine manifold (M, ∇) is Osserman if the eigenvalues of the affine Jacobi operators vanishe. In this paper, explicit examples of affine Osserman connections on 3 and 4-manifolds are constructed and their applications are given.

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1. Introduction

The theory of affine connections is a classical topic in differential geometry, initially developped to solve pure geometrical problems. It provides an extremely important tool to study geometrical structures on manifolds and, as such, has been applied with great sources in many different settings. For affine connections, a survey of the development of the theory can be found in [19] and references therein. In [13], García-Rio *et al.* introduced the notion of the affine Osserman connections. Affine Osserman connections are well-understood in dimension two. For instance, in [5] and [13], the authors proved, in a different way, that an affine connection is Osserman if and only if its Ricci tensor is skew-symmetric. The situation is however more involved in higher dimensions where the skew-symmetry of the Ricci tensor is a necessary (but not a sufficient) condition for an affine connection to be Osserman. The concept of an affine Osserman connection has become a very active research subject. (See [6, 7, 8] for more details).

The aim of the present paper is to give examples of two families of affine Osserman connections on 3 and 4 dimensional manifolds which are Ricci flat. Our paper is organized as follows. Section 1 introduces this topic. The section 2 contains some definitions and basic results we shall need. In section 3, we study the Osserman condition on a family of affine connection. The last section is devoted to the study of Riemann extension that associates to an affine structure on a manifold a corresponding metric of neutral signature on its cotangent bundle. It plays an important role in various questions involving the spectral geometry of the curvature operator. (See [1, 2] for more informations).

2. Preliminaries

2.1. Affine manifolds

Let M be a m-dimensional smooth manifold and ∇ be an affine connection on M. Let us consider a system of coordinates $(u_1, \dots, u_m)$ in a neighborhood $\mathcal{U}$ of a point p in M. In $\mathcal{U}$, the connection is given by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k, \qquad (2.1)$$

where $\{\partial_i = \frac{\partial}{\partial u_i}\}_{1\leq i\leq m}$ is a basis of the tangent space T_pM and the functions $\Gamma_{ij}^k(i,j,k = 1,\cdots,m)$ are called the coefficients of the affine connection. The pair (M,∇) shall be called *affine manifold*.
We define a few tensor fields associated to a given affine connection ∇. The *torsion tensor field* T, which is of type $(1,2)$, is defined by

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

for any vector fields X and Y on M. The components of the torsion tensor T in local coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

If the torsion tensor of a given affine connection ∇ vanishes, we say that ∇ is torsion-free.
The *curvature tensor field* $\mathcal{R}$, which is of type $(1,3)$, is defined by

$$\mathcal{R}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for any vector fields X, Y and Z on M. The components in local coordinates are

$$\mathcal{R}(\partial_k,\partial_l)\partial_j = \sum_i R_{jkl}^i\partial_i.$$

We shall assume that ∇ is torsion-free. If $\mathcal{R} = 0$ on M, we say that ∇ is *flat affine connection*. It is known that ∇ is flat if and only if around a point, there exists a local coordinates system such that $\Gamma_{ij}^k = 0$ for all i, j and k.
We define the *Ricci tensor Ric*, of type $(0,2)$ by

$$\mathrm{Ric}(Y,Z) = \mathrm{trace}\{X \mapsto \mathcal{R}(X,Y)Z\}.$$

The components in local coordinates are given by

$$\mathrm{Ric}(\partial_j,\partial_k) = \sum_i R_{kij}^i.$$

It is known, in Riemannian geometry, that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor. But, this property is not true for an arbitrary affine connection which is torsion-free. In fact, for property is closely related to the concept of parallel volume element (cf. [19] for more details).
In a 2-dimensional manifold, the curvature tensor $\mathcal{R}$ and the Ricci tensor Ric are related by

$$\mathcal{R}(X,Y)Z = \mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y. \tag{2.2}$$

For $X \in \Gamma(T_pM)$ and $p \in M$, we define the *affine Jacobi operator* $J_{\mathcal{R}}$ with respect to X by $J_{\mathcal{R}}(X) : T_pM \longrightarrow T_pM$ such that

$$J_{\mathcal{R}}(X)Y := \mathcal{R}(Y,X)X. \tag{2.3}$$

for any vector field Y. The affine Jacobi operator satisfies $J_{\mathcal{R}}(X)X = 0$ and $J_{\mathcal{R}}(\alpha X)Y = \alpha^2 J_{\mathcal{R}}(X)Y$, for $\alpha \in \mathbb{R} - \{0\}$ and $X \in T_pM$.

2.2. The Riemann extension construction

Let $N := T^*M$ be the cotangent bundle of an m-dimensional manifold and let $\pi : T^*M \to M$ be the natural projection. A point ξ of the cotangent bundle is represented by an ordered pair (ω, p), where $p = \pi(\xi)$ is a point on M and ω is a 1-form on T_pM. If $u = (u_1,\ldots,u_m)$ are local coordinates on M, let $u' = (u_{1'},\ldots,u_{m'})$ be the associated dual coordinates on the fiber where we expand a 1-form ω as $\omega = u_{i'}du_i$ $(i = 1,\ldots,m; i' = i+m)$; we shall adopt the Einstein convention and sum over repeated indices henceforth.

For each vector field $X = X^i\partial_i$ on M, the evaluation map $\iota X(p,\omega) = \omega(X_p)$ defines on function on N which, in local coordinates is given by

$$\iota X(u_i, u_{i'}) = u_{i'}X^i.$$

Vector fields on N are characterized by their action on function ιX; the complete lift X^C of a vector field X on M to N is characterized by the identity

$$X^C(\iota Z) = \iota[X, Z], \quad \text{for all} \quad Z \in \mathcal{C}^\infty(TM).$$

Moreover, since a $(0, s)$-tensor field on M is characterized by its evaluation on complete lifts of vectors fields on M, for each tensor field T of type $(1, 1)$ on M, we define a 1-form ιT on N which is characterized by the identity

$$\iota T(X^C) = \iota(TX).$$

Definition 2.1. *Let (M, ∇) be an affine manifold of dimension m. The Riemann extension $\bar{g}$ of (M, ∇) is the pseudo-Riemannian metric of neutral signature (m, m) on the cotangent bundle T^*M, which is characterized by the identity*

$$\bar{g}(X^C, Y^C) = -\iota(\nabla_X Y + \nabla_Y X).$$

In the system of induced coordinates $(u_i, u_{i'})$ on T^*M, the Riemann extension takes the form:

$$\bar{g} = \begin{pmatrix} -2u_{k'}\Gamma^k_{ij} & \mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix}, \tag{2.4}$$

with respect to $\{\partial_{u_1}, \ldots, \partial_{u_m}, \partial_{u'_1}, \ldots, \partial_{u'_m}\}$; here the indices i and j range from $1, \ldots, m$, $i' = i + m$, and Γ^k_{ij} are the coefficients of the affine connection ∇ with respect to the coordinates (u_i) on M. More explicitly:

$$\bar{g}(\partial_{u_i}, \partial_{u_j}) = -2u_{k'}\Gamma^k_{ij}, \quad \bar{g}(\partial_{u_i}, \partial_{u'_j}) = \delta^j_i, \quad \bar{g}(\partial_{u'_i}, \partial_{u'_j}) = 0.$$

Let (M, g) be a pseudo-Riemannian manifold. The Riemann extension of the Levi-Civita connection inherits many of the properties of the base manifold. For instance, (M, g) has constant sectional curvature if and only if $(T^*M, \bar{g})$ is locally conformally flat. However, the main applications of the Riemann extensions appear when considering affine connections that are not the Levi-Civita connection of any metric. We refer to Yano and Ishihara [21] for the proof of the following result:

Lemma 2.2. *Let (M, ∇) be an affine manifold. Let $\tilde{R}$ be the curvature operator of the Riemann extension $(T^*M, \bar{g})$ and let $\mathcal{R}$ be the curvature operator of (M, ∇). Then:*

$$\begin{aligned} \tilde{R}^h_{kji} &= \mathcal{R}^h_{kji}, \quad \tilde{R}^{h'}_{kji'} = -\mathcal{R}^i_{kjh}, \quad \tilde{R}^{h'}_{kj'i} = -\mathcal{R}^j_{hik}, \quad \tilde{R}^{h'}_{k'ji} = -\mathcal{R}^k_{hij}, \\ \tilde{R}^{h'}_{kji} &= u_{a'}\Big\{\nabla_{\partial_{u_h}} R^a_{kji} - \nabla_{\partial_{u_i}} \mathcal{R}^a_{kjh} + \Gamma^a_{ht}\mathcal{R}^t_{kji} + \Gamma^a_{kt}\mathcal{R}^t_{ihj} \\ &\quad + \Gamma^a_{jt}\mathcal{R}^t_{hik} + \Gamma^a_{it}\mathcal{R}^t_{kjh}\Big\}, \end{aligned}$$

where $\mathcal{R}^\delta_{\alpha\beta\gamma}$ (respectively $\tilde{R}^\delta_{\alpha\beta\gamma}$) denote the components of $\mathcal{R}$ and $\tilde{R}$.

By Lemma, $(T^*M, \bar{g})$ is locally conformally symmetric if and only if (M, ∇) is locally symmetric. Furthermore, $(T^*M, \bar{g})$ is locally conformally flat if and only if (M, ∇) is projectively flat. Any projectively flat pseudo-Riemannian manifold has constant sectional curvature. However, there are many projectively flat affine connections. We have the following result:

Theorem 2.3. *([13]) Let (M, ∇) be a smooth torsion-free affine manifold. Then the following statements are equivalent:*

1. *(M, ∇) is affine Osserman.*
2. *The Riemann extension $(T^*M, \bar{g})$ of (M, ∇) is a pseudo-Riemannian Osserman manifold.*

3. Affine Osserman manifolds

Let (M, ∇) be a m-dimensional affine manifold, i.e., ∇ is a torsion free connection on the tangent bundle of a smooth manifold M of dimension m. Let $\mathcal{R}$ be the curvature operator and $J_{\mathcal{R}}(X)$ the Jacobi operator with respect to a vector $X \in T_pM$ associated.

Definition 3.1. [14] *One says that an affine manifold (M, ∇) is affine Osserman at $p \in M$ if the characteristic polynomial of $J_{\mathcal{R}}(X)$ is independent of $X \in T_pM$. Also (M, ∇) is called affine Osserman if (M, ∇) is affine Osserman at each $p \in M$.*

Theorem 3.2. [14] *Let (M, ∇) be a m-dimensional affine manifold. Then (M, ∇) is called affine Osserman at $p \in M$ if and only if the characteristic polynomial of $J_{\mathcal{R}}(X)$ is $P_{J_{\mathcal{R}(X)}}[\lambda] = \lambda^m$ for every $X \in T_pM$.*

Corollary 3.3. *We say that (M, ∇) is affine Osserman if $\mathrm{Spect}\{J_{\mathcal{R}}(X)\} = \{0\}$ for any vector X.*

Corollary 3.4. *If (M, ∇) is affine Osserman at $p \in M$ then the Ricci tensor is skew-symmetric at $p \in M$.*

The affine Osserman manifolds arise naturally as generalized affine plane wave manifolds [12] and was a fruitful field of inquiry [9, 10, 11]. Also, affine connections with skew-symmetric Ricci tensor have received attention in the literature [4].

3.1. Example of affine Osserman connection on a 3-manifolds

Let M be a three-dimensional manifold and ∇ a smooth torsion free affine connection. Choose a system (u_1, u_2, u_3) of local coordinates in a domaine $\mathcal{U} \subset M$ such that the affine connection ∇ is determined by the functions $f_1, \ldots, f_6$ given by the formulas

$$\left\{ \begin{array}{lcl} \nabla_{\partial_1}\partial_1 &=& f_1(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_1}\partial_2 &=& f_2(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_1}\partial_3 &=& f_3(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_2}\partial_2 &=& f_4(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_2}\partial_3 &=& f_5(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_3}\partial_3 &=& f_6(u_1, u_2, u_3)\partial_1. \end{array} \right. \tag{3.1}$$

A straighforward calculation from (3.1) shows that the non-zero components of the curvature tensor are given by

$$\begin{array}{lcl} \mathcal{R}(\partial_1, \partial_2)\partial_1 &=& (\partial_1 f_2 - \partial_2 f_1)\partial_1 \\ \mathcal{R}(\partial_1, \partial_2)\partial_2 &=& (f_1 f_4 + \partial_1 f_4 - f_2^2 - \partial_2 f_2)\partial_1 \\ \mathcal{R}(\partial_1, \partial_2)\partial_3 &=& (f_1 f_5 + \partial_1 f_5 - f_2 f_3 - \partial_2 f_3)\partial_1 \\ \mathcal{R}(\partial_1, \partial_3)\partial_1 &=& (\partial_1 f_3 - \partial_3 f_1)\partial_1 \\ \mathcal{R}(\partial_1, \partial_3)\partial_2 &=& (f_1 f_5 + \partial_1 f_5 - f_2 f_3 - \partial_3 f_2)\partial_1 \\ \mathcal{R}(\partial_1, \partial_3)\partial_3 &=& (f_1 f_6 + \partial_1 f_6 - f_3^2 - \partial_3 f_3)\partial_1 \\ \mathcal{R}(\partial_2, \partial_3)\partial_1 &=& (\partial_2 f_3 - \partial_3 f_2)\partial_1 \\ \mathcal{R}(\partial_2, \partial_3)\partial_2 &=& (f_2 f_5 + \partial_2 f_5 - f_3 f_4 - \partial_3 f_4)\partial_1 \\ \mathcal{R}(\partial_2, \partial_3)\partial_3 &=& (f_2 f_6 + \partial_2 f_6 - f_3 f_5 - \partial_3 f_5)\partial_1. \end{array}$$

The non-zero components of the Ricci tensor are given by

$$\begin{array}{rcl}
\mathrm{Ric}(\partial_2,\partial_1) & = & f_1f_4+\partial_1 f_4-f_2^2-\partial_2 f_1; \\
\mathrm{Ric}(\partial_2,\partial_2) & = & f_1f_4+\partial_1 f_4-f_2^2-\partial_2 f_1 \\
\mathrm{Ric}(\partial_2,\partial_3) & = & f_1f_5+\partial_1 f_5-f_2f_3-\partial_2 f_3 \\
\mathrm{Ric}(\partial_3,\partial_1) & = & \partial_1 f_3-\partial_3 f_1 \\
\mathrm{Ric}(\partial_3,\partial_2) & = & f_1f_5+\partial_1 f_5-f_2f_3-\partial_3 f_2 \\
\mathrm{Ric}(\partial_3,\partial_3) & = & f_1f_6+\partial_1 f_6-f_3^2-\partial_3 f_3.
\end{array}$$

The skew-symmetry of Ricci tensor means that, in any local coordinates, we have:

$$\left\{\begin{array}{lcl}
\mathrm{Ric}(\partial_1,\partial_1)=\mathrm{Ric}(\partial_2,\partial_2)=\mathrm{Ric}(\partial_3,\partial_3) & = & 0 \\
\mathrm{Ric}(\partial_1,\partial_2)+Ric(\partial_2,\partial_1) & = & 0 \\
\mathrm{Ric}(\partial_1,\partial_3)+Ric(\partial_3,\partial_1) & = & 0 \\
\mathrm{Ric}(\partial_2,\partial_3)+Ric(\partial_3,\partial_2) & = & 0.
\end{array}\right. \tag{3.2}$$

According (3.1) and (3.2), we have the following:

Proposition 3.5. *The Ricci tensor of the affine connection ∇ defined in (3.1) is skew-symmetric if the functions $f_i, i=1,\ldots,6$ satisfy the following partial differential equations:*

$$\begin{array}{rcl}
\partial_1 f_2-\partial_2 f_1=0 \quad \partial_1 f_3-\partial_3 f_1 & = & 0 \\
\partial_1 f_4-\partial_2 f_2+f_1f_4-f_2^2 & = & 0 \\
\partial_1 f_6-\partial_3 f_3+f_1f_6-f_3^2 & = & 0 \\
2\partial_1 f_5-\partial_2 f_3-\partial_3 f_2+2f_1f_5-2f_2f_3 & = & 0.
\end{array}$$

Proof. It follows from (3.1) and (3.2).

■

Corollary 3.6. [8] *Let ∇ be as (3.1). Assume that $f_2=f_3=f_5=0$, then the Ricci tensor of the affine connection (3.1) is skew-symmetric if and only if the coefficients of the connection (3.1) satisfy*

$$f_1(u_1,u_2,u_3)=f_1(u_1), \quad \partial_1 f_4+f_1f_4=0, \quad \textit{and} \quad \partial_1 f_6+f_1f_6=0.$$

Proposition 3.7. *Let (M,∇) be a 3-dimensional affine manifold with torsion free connection given by (3.1). Then (M,∇) is affine Osserman if and only if the Ricci tensor is skew-symmetric.*

Proof. Since the Ricci tensor of any affine Osserman connection is skew-symmetric. It follows that we have the following necessary conditions for the affine connection (3.1) to be Osserman

$$\partial_1 f_2-\partial_2 f_1=0 \quad \partial_1 f_3-\partial_3 f_1=0 \quad \partial_1 f_4-\partial_2 f_2+f_1f_4-f_2^2=0$$
$$\partial_1 f_6-\partial_3 f_3+f_1f_6-f_3^2=0 \quad 2\partial_1 f_5-\partial_2 f_3-\partial_3 f_2+2f_1f_5-2f_2f_3=0.$$

Then, the matrix associated to the affine Jacobi operator can be expressed, with respect to the coordinate basis, as

$$(J_{\mathcal{R}}(X))=\left(\begin{array}{ccc} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

with

$$\begin{aligned} b &= \alpha_1\alpha_3\Big(2\partial_2 f_3 - \partial_3 f_1 - \partial_1 f_5 - f_1 f_5 + f_2 f_3\Big) \\ &\quad +\alpha_2\alpha_3\Big(\partial_2 f_5 - \partial_3 f_4 + f_2 f_5 - f_3 f_4\Big) \\ &\quad +\alpha_3^2\Big(\partial_2 f_6 - \partial_3 f_5 + f_2 f_6 - f_3 f_5\Big); \\ c &= -\alpha_1\alpha_2\Big(\partial_2 f_3 - 2\partial_3 f_2 + \partial_1 f_5 + f_1 f_5 - f_2 f_3\Big) \\ &\quad -\alpha_2^2\Big(\partial_2 f_5 - \partial_3 f_4 + f_2 f_5 - f_3 f_4\Big) \\ &\quad -\alpha_2\alpha_3\Big(\partial_2 f_6 - \partial_3 f_5 + f_2 f_6 - f_3 f_5\Big). \end{aligned}$$

The characteristic polynomial of the affine Jacobi operator is now seen to be: $P_{J_{\mathcal{R}}(X)}[\lambda] = -\lambda^3$.

■

Example 3.8. *One can construct examples of affine Osserman connections. The following connection on $\mathbb{R}^3$ whose non-zero coefficients of the cofficients are given by $\nabla_{\partial_2}\partial_2 = u_2u_3\partial_1$ and $\nabla_{\partial_3}\partial_3 = (u_2+u_3)\partial_1$ is affine Osserman.*

3.2. Example of affine Osserman connection on a 4-manifolds

In the following M denotes a four-dimensional manifold and ∇ a smooth torsion-free affine connection. Choose a system $\{u_1, u_2, u_3, u_4\}$ of local coordinates in a domaine $\mathcal{U} \subset M$ such that the affine connection ∇ is determined by the functions $f_1, \ldots, f_6$ given by the formulas:

$$\begin{cases} \nabla_{\partial_1}\partial_1 = f_1(u_1,u_2,u_3,u_4)\partial_4 \\ \nabla_{\partial_1}\partial_4 = f_2(u_1,u_2,u_3,u_4)\partial_4 \\ \nabla_{\partial_2}\partial_2 = f_3(u_1,u_2,u_3,u_4)\partial_4 \\ \nabla_{\partial_2}\partial_3 = f_4(u_1,u_2,u_3,u_4)\partial_4 \\ \nabla_{\partial_3}\partial_3 = f_5(u_1,u_2,u_3,u_4)\partial_4 \\ \nabla_{\partial_4}\partial_4 = f_6(u_1,u_2,u_3,u_4)\partial_4. \end{cases} \tag{3.3}$$

The non-zero components of the curvature tensor are given by

$$\begin{aligned} \mathcal{R}(\partial_1,\partial_2)\partial_1 &= -\partial_2 f_1\partial_4, \quad \mathcal{R}(\partial_1,\partial_2)\partial_2 = (\partial_1 f_3 + f_2 f_3)\partial_4; \\ \mathcal{R}(\partial_1,\partial_2)\partial_3 &= (\partial_1 f_4 + f_2 f_4)\partial_4, \quad \mathcal{R}(\partial_1,\partial_2)\partial_4 = -\partial_2 f_2\partial_4; \\ \mathcal{R}(\partial_1,\partial_3)\partial_1 &= -\partial_3 f_1\partial_4, \quad \mathcal{R}(\partial_1,\partial_3)\partial_2 = (\partial_1 f_4 + f_2 f_4)\partial_4; \\ \mathcal{R}(\partial_1,\partial_3)\partial_3 &= (\partial_1 f_5 + f_2 f_5)\partial_4, \quad \mathcal{R}(\partial_1,\partial_3)\partial_4 = -\partial_3 f_2\partial_4; \\ \mathcal{R}(\partial_1,\partial_4)\partial_1 &= (\partial_1 f_2 - \partial_4 f_1 + f_2^2 - f_1 f_6)\partial_4, \quad \mathcal{R}(\partial_2,\partial_4)\partial_1 = \partial_2 f_2\partial_4 \\ \mathcal{R}(\partial_2,\partial_3)\partial_2 &= (\partial_2 f_4 - \partial_3 f_3)\partial_4, \quad \mathcal{R}(\partial_2,\partial_3)\partial_3 = (\partial_2 f_5 - \partial_3 f_4)\partial_4 \\ \mathcal{R}(\partial_1,\partial_4)\partial_4 &= (\partial_1 f_6 - \partial_4 f_2)\partial_4, \quad \mathcal{R}(\partial_2,\partial_4)\partial_2 = (-\partial_4 f_3 - f_3 f_6)\partial_4 \\ \mathcal{R}(\partial_2,\partial_4)\partial_3 &= (-\partial_4 f_4 - f_4 f_6)\partial_4, \quad \mathcal{R}(\partial_2,\partial_4)\partial_4 = \partial_2 f_6\partial_4 \\ \mathcal{R}(\partial_3,\partial_4)\partial_1 &= \partial_3 f_2\partial_4, \quad \mathcal{R}(\partial_3,\partial_4)\partial_2 = (-\partial_4 f_4 - f_4 f_6)\partial_4 \\ \mathcal{R}(\partial_3,\partial_4)\partial_3 &= (-\partial_4 f_5 - f_5 f_6)\partial_4, \quad \mathcal{R}(\partial_3,\partial_4)\partial_4 = \partial_3 f_6\partial_4. \end{aligned}$$

The non-zero components of the Ricci tensor are given by

$$\begin{aligned} \mathrm{Ric}(\partial_1,\partial_1) &= \partial_1 f_2 - \partial_4 f_1 + f_2^2 - f_1 f_6, \quad \mathrm{Ric}(\partial_1,\partial_2) = \partial_2 f_2, \\ \mathrm{Ric}(\partial_1,\partial_3) &= \partial_3 f_2, \quad \mathrm{Ric}(\partial_2,\partial_2) = -\partial_4 f_3 - f_3 f_6, \\ \mathrm{Ric}(\partial_2,\partial_3) &= -\partial_4 f_4 - f_4 f_6, \quad \mathrm{Ric}(\partial_3,\partial_2) = -\partial_4 f_4 - f_4 f_6, \\ \mathrm{Ric}(\partial_3,\partial_3) &= -\partial_4 f_5 - f_5 f_6, \mathrm{Ric}(\partial_4,\partial_1) = \partial_1 f_6 - \partial_4 f_2, \\ \mathrm{Ric}(\partial_4,\partial_2) &= \partial_2 f_6, \quad \mathrm{Ric}(\partial_4,\partial_3) = \partial_3 f_6. \end{aligned}$$

Proposition 3.9. *The Ricci tensor of the affine connection ∇ defined in (3.3) is skew-symmetric if the functions f_2 and f_6 has the form*

$$f_2 = f(u_1, u_4) \quad \text{and} \quad f_6 = f(u_1, u_4)$$

and $f_i, i = 1, \dots, 6$ satisfy the following partial differential equations:

$$\begin{aligned} \partial_1 f_2 - \partial_4 f_1 + f_2^2 - f_1 f_6 = 0 \quad \partial_4 f_3 + f_3 f_6 &= 0 \\ \partial_4 f_5 + f_5 f_6 = 0 \quad \partial_1 f_6 - \partial_4 f_2 = 0 \quad \partial_4 f_4 + f_4 f_6 &= 0. \end{aligned}$$

Now, if $X = \sum_{i=1}^{4} \alpha_i X^i$ is a vector fields on a four-dimensional affine manifold (M, ∇), then the affine Jacobi operator heve the following form:

$$\begin{aligned} J_{\mathcal{R}}(X)\partial_i &= \alpha_1^2 \mathcal{R}(\partial_i, \partial_1)\partial_1 + \alpha_1\alpha_2 \mathcal{R}(\partial_i, \partial_2)\partial_1 + \alpha_1\alpha_3 \mathcal{R}(\partial_i, \partial_3)\partial_1 \\ &+ \alpha_1\alpha_4 \mathcal{R}(\partial_i, \partial_4)\partial_1 + \alpha_1\alpha_2 \mathcal{R}(\partial_i, \partial_1)\partial_2 + \alpha_2^2 \mathcal{R}(\partial_i, \partial_2)\partial_2 \\ &+ \alpha_2\alpha_3 \mathcal{R}(\partial_i, \partial_3)\partial_2 + \alpha_2\alpha_4 \mathcal{R}(\partial_i, \partial_4)\partial_2 + \alpha_1\alpha_3 \mathcal{R}(\partial_i, \partial_1)\partial_3 \\ &+ \alpha_2\alpha_3 \mathcal{R}(\partial_i, \partial_2)\partial_3 + \alpha_3^2 \mathcal{R}(\partial_i, \partial_3)\partial_3 + \alpha_3\alpha_4 \mathcal{R}(\partial_i, \partial_4)\partial_3 \\ &+ \alpha_1\alpha_4 \mathcal{R}(\partial_i, \partial_1)\partial_4 + \alpha_2\alpha_4 \mathcal{R}(\partial_i, \partial_2)\partial_4 + \alpha_3\alpha_4 \mathcal{R}(\partial_i, \partial_3)\partial_4 \\ &+ \alpha_4^2 \mathcal{R}(\partial_i, \partial_4)\partial_4 \end{aligned}$$

and the matrix associated is given by

$$(J_{\mathcal{R}}(X)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= -\alpha_1\alpha_2\partial_2 f_1 - \alpha_1\alpha_3\partial_3 f_1 + \alpha_2^2(\partial_1 f_3 + f_2 f_3) \\ &\quad +2\alpha_2\alpha_3(\partial_1 f_4 + f_2 f_4) + \alpha_3^2(\partial_1 f_5 + f_2 f_5) \\ a_2 &= \alpha_1^2\partial_2 f_1 - \alpha_1\alpha_2(\partial_1 f_3 + f_2 f_3) + \alpha_2\alpha_3(\partial_2 f_4 - \partial_3 f_3) \\ &\quad -\alpha_1\alpha_3(\partial_1 f_4 + f_2 f_4) + \alpha_3^2(\partial_2 f_5 - \partial_3 f_4) \\ a_3 &= \alpha_1^2\partial_3 f_1 - \alpha_1\alpha_2(\partial_1 f_4 + f_2 f_4) - \alpha_2^2(\partial_2 f_4 - \partial_3 f_3) \\ &\quad -\alpha_1\alpha_3(\partial_1 f_5 + f_2 f_5) - \alpha_2\alpha_3(\partial_2 f_5 - \partial_3 f_4) \end{aligned}$$

It is easy to see that the characteristic polynomial of the affine Jacobi operator is $P_{J_{\mathcal{R}}(X)}[\lambda] = \lambda^4$. We have the following result:

Proposition 3.10. *Let (M, ∇) be a 4-dimensional affine manifold with torsion free connection given by (3.3). Then (M, ∇) is affine Osserman if and only if the Ricci tensor is skew-symetric.*

Example 3.11. *The following connection on $\mathbb{R}^4$ whose non-zero coefficients of the cofficients are given by $\nabla_{\partial_1}\partial_4 = (u_1 + u_4)\partial_4$ and $\nabla_{\partial_4}\partial_4 = (u_1 u_4)\partial_4$ is affine Osserman*

4. Pseudo-Riemannian Osserman manifolds

Let (M, g) be a pseudo-Riemannian manifold of signature (p, q) and R be the curvature tensor of the Levi-Civita connection. If (M, g) is Riemannian and if it is flat or it is local rank one symmetric space, then the set of local isometries acts transitively on the unit sphere bundle $S(M, g)$. Conversely, the eigenvalues of the Jacobi operator J_R are constant on the unit sphere bundle $S(M, g)$. Robert Osserman [20] wondered if the converse holds; later authirs called this problem the *Osserman conjecture*. Works of Chi [3], of Gilkey et al. [15] and of Nikolayevsky [16, 17] show that any complete and simply connected Osserman manifold of dimension $m \neq 16$ is a rank-one symmetric space; the 16-dimensional setting is exceptional and the situation is still not clear in that setting although there are some partial result due, again, to Nikolayevsky [18]. In the Lorentzian setting, it is known [14], that a Lorentzian Osserman manifold has constant sectional curvature; the geometry of such manifolds is very special.

Let $S^{\pm}(M,g) := \{X \in TM : g(X,X) = \pm 1\}$ the pseudo-sphere unit bundles. One says that a pseudo-Riemannian manifold (M,g) is Osserman if the eigenvalues of J_R are constant on the pseudo-sphere unit bundles $S^{\pm}(M,g)$. It is known that there exist pseudo-Riemannian Osserman manifolds which are neither flat, nor local rank one symmetric spaces ([14]).

Example 4.1. *Let $M = \mathbb{R}^4$ with usual coordinates $\{u_1, u_2, u_3, u_4\}$. Then*

$$\begin{aligned} g &= u_1u_3du^1 \otimes du^1 + adu^1 \otimes du^2 + bdu^1 \otimes du^3 + adu^2 \otimes du^1 \\ &\quad + u_2u_4du^2 \otimes du^2 + bdu^2 \otimes du^4 + bdu^3 \otimes du^1 + bdu^4 \otimes du^2 \end{aligned}$$

define a pseudo-Riemannian Osserman metric on $\mathbb{R}^4$ of signature $(2,2)$ at the points where $u_1 = 0$ or $u_1u_3^2 - 3bu_3 + bu_1 = 0$.

Next we will use the Riemann extension to exhibit a pseudo-Riemannian Osserman metrics of signatures $(3,3)$ and $(4,4)$.

1. Let (u_1, u_2, u_3) be the local coordinates on a 3-dimensional affine manifold (M, ∇). We expand $\nabla_{\partial_i}\partial_j = \sum_k f_{ij}^k \partial_k$ for $i,j,k = 1,2,3$ to define the Christoffel symbols of ∇. Let $\omega = u_4du_1 + u_5du_2 + u_6du_3 \in T^*M : (u_4, u_5, u_6)$ are the dual fiber coordinates. The Riemann extension is the pseudo-Riemannian metric $\bar{g}$ on the cotangent bundle T^*M of neutral signature $(3,3)$ defined by setting

$$\begin{aligned} \bar{g}(\partial_1, \partial_4) &= \bar{g}(\partial_2, \partial_5) = \bar{g}(\partial_3, \partial_6) = 1, \\ \bar{g}(\partial_1, \partial_1) &= -2u_4f_{11}^1 - 2u_5f_{11}^2 - 2u_6f_{11}^3, \\ \bar{g}(\partial_1, \partial_2) &= -2u_4f_{12}^1 - 2u_5f_{12}^2 - 2u_6f_{12}^3, \\ \bar{g}(\partial_1, \partial_3) &= -2u_4f_{13}^1 - 2u_5f_{13}^2 - 2u_6f_{13}^3, \\ \bar{g}(\partial_2, \partial_2) &= -2u_4f_{22}^1 - 2u_5f_{22}^2 - 2u_6f_{22}^3, \\ \bar{g}(\partial_2, \partial_3) &= -2u_4f_{23}^1 - 2u_5f_{23}^2 - 2u_6f_{23}^3, \\ \bar{g}(\partial_3, \partial_3) &= -2u_4f_{33}^1 - 2u_5f_{33}^2 - 2u_6f_{33}^3. \end{aligned}$$

Let consider the affine Osserman connection given in the example (3.8). Its Riemann extenxion $\bar{g}$ on $\mathbb{R}^6$ is define by:

$$\bar{g} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -u_4u_2u_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2u_4(u_2+u_3) & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

The pseudo-Riemannian metric (4.1) is Osserman of signature $(3,3)$.

2. Let (u_1, u_2, u_3, u_4) be the local coordinates on a 4-dimensional affine manifold (M, ∇). We expand $\nabla_{\partial_i}\partial_j = \sum_k f_{ij}^k \partial_k$ for $i,j,k = 1,2,3,4$ to define the Christoffel symbols of ∇. Let $\omega = u_5du_1 + u_6du_2 + u_7du_3 + u_8du_4 \in T^*M : (u_5, u_6, u_7, u_8)$ are the dual fiber coordinates. The Riemann extension is the pseudo-Riemannian metric $\bar{g}$ on the cotangent bundle T^*M

of neutral signature $(4,4)$ defined by setting

$$\begin{array}{rcl}
\bar{g}(\partial_1,\partial_5) & = & \bar{g}(\partial_2,\partial_6)=\bar{g}(\partial_3,\partial_7)=\bar{g}(\partial_4,\partial_8)=1,\\
\bar{g}(\partial_1,\partial_1) & = & -2u_5f_{11}^1-2u_6f_{11}^2-2u_7f_{11}^3-2u_8f_{11}^4,\\
\bar{g}(\partial_1,\partial_2) & = & -2u_5f_{12}^1-2u_6f_{12}^2-2u_7f_{12}^3-2u_8f_{12}^4,\\
\bar{g}(\partial_1,\partial_2) & = & -2u_5f_{12}^1-2u_6f_{12}^2-2u_7f_{12}^3-2u_8f_{12}^4,\\
\bar{g}(\partial_1,\partial_3) & = & -2u_5f_{13}^1-2u_6f_{13}^2-2u_7f_{13}^3-2u_8f_{13}^4,\\
\bar{g}(\partial_1,\partial_4) & = & -2u_5f_{14}^1-2u_6f_{14}^2-2u_7f_{14}^3-2u_8f_{14}^4,\\
\bar{g}(\partial_2,\partial_2) & = & -2u_5f_{22}^1-2u_6f_{22}^2-2u_7f_{22}^3-2u_8f_{22}^8,\\
\bar{g}(\partial_2,\partial_3) & = & -2u_5f_{23}^1-2u_6f_{23}^2-2u_7f_{23}^3-2u_8f_{23}^4,\\
\bar{g}(\partial_2,\partial_4) & = & -2u_5f_{24}^1-2u_6f_{24}^2-2u_7f_{24}^3-2u_8f_{24}^4,\\
\bar{g}(\partial_3,\partial_3) & = & -2u_5f_{33}^1-2u_6f_{33}^2-2u_7f_{33}^3-2u_8f_{33}^4,\\
\bar{g}(\partial_3,\partial_4) & = & -2u_5f_{34}^1-2u_6f_{34}^2-2u_7f_{34}^3-2u_8f_{34}^4,\\
\bar{g}(\partial_4,\partial_4) & = & -2u_5f_{44}^1-2u_6f_{44}^2-2u_7f_{44}^3-2u_8f_{44}^4.
\end{array}$$

Let consider the affine Osserman connection given in the example (3.11). Its Riemannian extenxion $\bar{g}$ on $\mathbb{R}^8$ is define by:

$$\bar{g}=\begin{pmatrix}
0 & 0 & 0 & -2u_8(u_1+u_4) & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & -2u_8u_1u_4 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix} \qquad (4.2)$$

The metric (4.2) is Osserman of signature $(4,4)$

The affine Osserman connections are of interest, not only in the affine geometry, but also in the study of the pseudo-Riemannian Osserman metrics since they provide some nice examples of Osserman manifolds whose Jacobi operators have non-trivial Jordan normal form and which are not nilpotent. It has long been a task in this field to build examples of Osserman manifolds which not nilpotent and which exhibited non-trivial Jordan normal form. We will refer [1, 2] and references therein for more information.

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