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JANNE GRÖHN, JANNE HEITTOKANGAS, RISTO KORHONEN AND JOUNI RÄTTYÄ (EDITORS)

PROCEEDINGS OF THE SUMMERSCHOOL IN COMPLEX AND HARMONIC ANALYSIS, AND RELATED TOPICS

Dedicated to the memory of Rauno Aulaskari

JANNE GRÖHN, JANNE HEITTOKANGAS, RISTO KORHONEN
AND JOUNI RÄTTYÄ (EDITORS)

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Preface

The Summer School in Complex and Harmonic Analysis, and Related Topics, was held at the Mekrijärvi Research Station of the University of Eastern Finland, from June 14 to 18, 2014, continuing our tradition to organize intensive courses for doctoral and post-doctoral students.

The summer school attracted 27 doctoral, post-doctoral and senior participants, including 12 participants from foreign universities. The key program consisted of 3 short lecture courses, delivered by Artur Nicolau (Universitat Autònoma de Barcelona, Spain), José-Ángel Peláez (University of Málaga, Spain) and Brett D. Wick (Georgia Institute of Technology, Atlanta, USA). These courses have been developed further to be included in this volume. In addition to the three lecture courses, the participants of the summer school presented 10 contributed talks of 40 minutes each during the meeting.

The Midsummer weather was exceptionally chilly during the summer school, we experienced temperatures even some degrees below the zero at night. Low temperatures combined with rain and strong wind at times and clouds of mosquitos did not prevent us to enjoy lake shore sauna, smoke sauna and other activities the Mekrijärvi Research Station had to offer. We are grateful for the staff of the station for the great conditions they offered for our event. Unfortunately, we learned that the station will be shut down in the near future, and hence organizing further meetings at the station will be impossible.

Some months after the summer school we received the sad news that our friend and colleague Professor Emeritus Rauno Aulaskari passed away on December, 2014, due to serious illness. Rauno gave a talk in the summer school, but nobody could imagine that it was going to be his last mathematics talk. This volume is dedicated to the memory of Rauno.

Joensuu, December 4, 2015 *Jouni Rättyä*

We acknowledge financial support from Academy of Finland #268009, Finnish National Doctoral Programme in Mathematics and its Applications, Finnish Academy of Science and Letters, and University of Eastern Finland. Their contributions made organizing this summer school possible.

Lectures given at the summer school

Short courses:

Artur Nicolau *(Universitat Autònoma de Barcelona, Spain)*
Nevanlinna-Pick interpolation problem

José Ángel Peláez *(Universidad de Málaga, Spain)*
Weighted Hardy-Bergman spaces

Brett Wick *(Georgia Institute of Technology, USA)*
Carleson Measures in Spaces of Analytic Functions

Contributed talks:

Rauno Aulaskari *(University of Eastern Finland, Finland)*
Some function classes on Riemann surfaces

Sita Benedict *(University of Jyväskylä, Finland)*
Intrinsic Hardy-Orlicz spaces of conformal mappings

Christoph Böhm *(OTH Regensburg, Germany)*
Loewner evolution in multiply connected domains

Tingbin Cao *(Nanchang University, China)*
Uniqueness problems for meromorphic mappings in one and several complex variables

Julia Koch *(University of Würzburg, Germany)*
Dieudonné's Lemma for schlicht functions

Shamil Makhmutov *(Sultan Qaboos University, Oman)*
On growth of the spherical derivative and the Nevanlinna characteristic

Maria Martin *(University of Eastern Finland, Finland)*
Order of families of harmonic mappings with bounded Schwarzian norm

Jordi Pau *(Universitat de Barcelona, Spain)*
Bounded projections and the reproducing formula on large weighted Bergman spaces

Atte Reijonen *(University of Eastern Finland, Finland)*
On the complexity of finding a necessary and sufficient condition for Blaschke-oscillatory equations

Jarno Talponen *(University of Eastern Finland, Finland)*
ODE representation for varying exponent L^p norms

Participants at the summer school

Rauno Aulaskari	*(University of Eastern Finland, Finland)*
Sita Benedict	*(University of Jyväskylä, Finland)*
Christoph Böhm	*(OTH Regensburg, Germany)*
Tingbin Cao	*(Nanchang University, China)*
Janne Gröhn	*(University of Eastern Finland, Finland)*
Janne Heittokangas	*(University of Eastern Finland, Finland)*
Juha-Matti Huusko	*(University of Eastern Finland, Finland)*
Julia Koch	*(University of Würzburg, Germany)*
Risto Korhonen	*(University of Eastern Finland, Finland)*
Taneli Korhonen	*(University of Eastern Finland, Finland)*
Nan Li	*(University of Eastern Finland, Finland)*
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Noel Merchán Álvarez	*(Universidad de Málaga, Spain)*
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Artur Nicolau	*(Universitat Autónoma de Barcelona, Spain)*
Roc Oliver	*(Universitat de Barcelona, Spain)*
Jordi Pau	*(Universitat de Barcelona, Spain)*
José Ángel Peláez	*(Universidad de Málaga, Spain)*
Robert Rahm	*(Georgia Institute of Technology, USA)*
Atte Reijonen	*(University of Eastern Finland, Finland)*
Jouni Rättyä	*(University of Eastern Finland, Finland)*
Kian Sierra McGettigan	*(Universidad de Málaga, Spain)*
Jarno Talponen	*(University of Eastern Finland, Finland)*
Brett Wick	*(Georgia Institute of Technology, USA)*
Wen Xu	*(University of Eastern Finland, Finland)*

Group photo (taken by Juha-Matti Huusko)

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The Nevanlinna-Pick Interpolation Problem

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Abstract. This paper collects the material of my course at the Summerschool at Mekrijärvi in 2014. Schur's algorithm is applied to parametrize the set of all solutions of an indeterminate Nevanlinna-Pick interpolation problem and to study the corresponding Nevanlinna's coefficients. Next we prove the classical result of R. Nevanlinna on the existence of inner solutions as well as the more recent result of A. Stray on the existence of Blaschke products among the solutions. Finally several refinements of these results in the context of scaled Nevanlinna-Pick problems are presented.

MSC 2010: 30E05, 30J05, 30J10.
Keywords: Blaschke product, Hardy-Sobolev space, inner function, Nevanlinna's parametrization.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. The Nevanlinna-Pick problem can be stated as follows: *Given a sequence of distinct points* $\{z_n\} \subset \mathbb{D}$ *and a sequence of values* $\{w_n\} \subset \mathbb{C}$*, is there an analytic function* $f : \mathbb{D} \to \overline{\mathbb{D}}$ *such that* $f(z_n) = w_n$*,* $n = 1, 2, \ldots$*?* We will discuss three types of questions:

1. When the problem has a solution?

2. How can the set of all solutions be described?

3. How can one find solutions with certain extremal properties?

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First session is devoted to questions 1 and 2. There are two classical approaches due respectively, to Nevanlinna and Pick. We will follow Nevanlinna's ideas which also lead to a solution of 2. See [32] and [33]. At the end of the session we will mention a modern approach to Pick's result which leads to a different set of problems and results. Next sessions are devoted to 3. Second session is devoted to prove Nevanlinna's main result, which states that if the Nevanlinna-Pick problem has more than one solution, then all extremal solutions are inner functions. In the third section we will prove a refinement of this result due to A. Stray which states that actually most extremal solutions are Blaschke products. Last session is devoted to extremal solutions of scaled problems.

The Nevanlinna-Pick problem has been considered in many different spaces and extended in many different directions. In 1968, Adamyan, Arov and Krein extended Nevanlinna's parametrization. See [1] or [19, p.146] or [47]. D. Sarason found deep relations between the Nevanlinna-Pick problem and several results in operator theory, see [42] and [43, p.68]. His work has been extremely influential and has been extended by many authors. See the monography [2]. The Nevanlinna-Pick problem has also been considered in other spaces of analytic functions, see [2], [30], and the monography [44]. In these lectures we will not try to review these results; instead, we will follow a geodesic which will bring us from the classical ideas of Nevanlinna to some modern results, mainly due to Arne Stray.

This paper collects the material of the four lectures I gave at the Summerschool in Mekrijärvi, Finland, in June 2014. It is a pleasure to thank the organizers for their kind invitation and to all participants for the nice atmosphere, modulo mosquitoes, during those days at this wonderful research facility.

2. NEVANLINNA'S AND PICK'S APPROACHES

Nevanlinna published his results a few years later than Pick but was unaware of the latter's work due probably to the poor communication during the First World War. The approaches of Nevanlinna and Pick are quite different. We will follow Nevanlinna's ideas, which are based on Schur's algorithm. This is a beautiful technique which can also be used in other problems such as the Caratheodory problem, where one assigns Taylor coefficients instead of function values.

2.1. Nevanlinna's Approach

We start with some notation. The space of bounded analytic functions in the unit disc is denoted by H^∞. Given $f \in H^\infty$, consider $||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Any function $f \in H^\infty$ has radial limit $f\left(e^{i\theta}\right) = \lim_{r\to 1} f\left(re^{i\theta}\right)$ at almost every point $e^{i\theta}$ of the unit circle and $||f||_\infty = ||f||_{L^\infty(\partial\mathbb{D})}$.

Bilinear transformations $T(z) = \frac{az+b}{cz+d}$, with $ad - bc \neq 0$, will be represented by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z$. The main advantage is the following fact: if $\tilde{T}(z) = \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$, $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$, the composition $T \circ \tilde{T}$ is represented by the product of these two matrices: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$.

We use the notation b_a for the automorphism of the unit disc given by $b_a(z) = \frac{|a|}{a}\frac{a-z}{1-\bar{a}z}$, where $a \in \mathbb{D}\setminus\{0\}$ is fixed. Also, $b_0(z) = z$. We will use the following two elementary facts.

> Fact 1. If $f : \mathbb{D} \to \mathbb{D}$ is analytic and $f(a) = 0$, then $f(z) = b_a(z) f_1(z)$ where $f_1 : \mathbb{D} \to \overline{\mathbb{D}}$ is an analytic function. This follows by considering the analytic function $\frac{f(z)}{b_a(z)} = f_1(z)$ for $z \in \mathbb{D}$ and applying Maximum Modulus Principle.
>
> Fact 2. If $\sum(1 - |a_n|) < \infty$, then $B(z) = \prod b_{a_n}(z)$ converges uniformly on compacts of $\mathbb{D}$. The function $B(z)$ is analytic, $||B||_\infty = \sup_{z\in\mathbb{D}} |B(z)| = 1$ and $B(a_n) = 0, n = 1, 2, \ldots$. $B(z)$ is called the Blaschke product with zeros $\{a_n\}$. See, for instance, [19, p. 51].

We first consider Nevanlinna-Pick problems with finitely many points.

Finite Case. Assume we have a finite set of points $\{z_1, \ldots, z_N\}$ and values $\{w_1, \ldots, w_N\}$; in other words, given $\{z_1, \ldots, z_N\} \subset \mathbb{D}$ and $\{w_1, \ldots, w_N\} \subset \mathbb{C}$, the problem can be stated as follows:

$(*)_N$: Find $f \in H^\infty$, $||f||_\infty \leq 1$ with $f(z_i) = w_i$, $i = 1, \ldots, N$.

We will consider simultaneously questions 1 and 2:

Case $N = 1$, that is, if we have a single point z_1 and a single value w_1. There are three cases:

- If $|w_1| > 1$, we have no solution.

- If $|w_1| = 1$, we have a unique solution.

- If $|w_1| < 1$, we have infinitely many solutions. Moreover, assume f is a solution. Then by Fact 1,

$$\frac{f - w_1}{1 - \overline{w_1} f} = b_{z_1} f_1, \qquad ||f_1||_\infty \leq 1,$$

that is,

$$f = \frac{b_{z_1} f_1 + w_1}{1 + \overline{w_1} b_{z_1} f_1}, \qquad ||f_1||_\infty \leq 1\,.$$

Hence, the set of all solutions is

$$\begin{aligned}&\big\{f \in H^\infty : ||f||_\infty \leq 1, f(z_1) = w_1\big\}\\ &\quad = \left\{f = \frac{b_{z_1} f_1 + w_1}{1 + \overline{w_1} b_{z_1} f_1} : f_1 \in H^\infty, ||f_1||_\infty \leq 1\right\}\\ &\quad = \left\{f = \frac{\begin{pmatrix} b_{z_1} & w_1 \\ \overline{w_1} b_{z_1} & 1 \end{pmatrix}}{\sqrt{1 - |w_1|^2}} f_1 : f_1 \in H^\infty, ||f_1||_\infty \leq 1\right\}.\end{aligned}$$

It will be useful to denote

$$U_1 = \frac{1}{\sqrt{1 - |w_1|^2}} \begin{pmatrix} b_{z_1} & w_1 \\ \overline{w_1} b_{z_1} & 1 \end{pmatrix}.$$

The factor in front of the matrix is chosen so that $\det U_1 = b_{z_1}$.

Case $N > 1$. We will argue inductively: function f is a solution of the Nevanlinna-Pick problem with N points $z_1, \ldots, z_N$ if and only if $f = \frac{b_{z_1} f_1 + w_1}{1 + \overline{w_1} b_{z_1} f_1}$ and $f_1(z_j) = \frac{1}{b_{z_1}(z_j)} \frac{w_j - w_1}{1 - \overline{w_1} w_j}$ for $j = 2, \ldots, N$. Writing $w_2^{(1)} = \frac{1}{b_{z_1}(z_2)} \frac{w_2 - w_1}{1 - \overline{w_1} w_2}$, we have three possibilities:

- If $|w_2^{(1)}| > 1$ we have no solution.

- If $|w_2^{(1)}| = 1$ we have a unique f_1 and therefore a unique solution of the Nevanlinna-Pick problem $(*)_2$ with two points z_1, z_2.

- If $|w_2^{(1)}| < 1$, then the previous argument gives

$$f_1 = \frac{b_{z_2} f_2 + w_2^{(1)}}{1 + \overline{w_2^{(1)}} b_{z_2} f_2} = U_2 f_2$$

for some $f_2 \in H^\infty$ with $||f_2||_\infty \leq 1$, where

$$U_2 = \frac{1}{\sqrt{1-|w_2^{(1)}|^2}} \begin{pmatrix} b_{z_2} & w_2^{(1)} \\ \overline{w_2^{(1)}} b_{z_2} & 1 \end{pmatrix}.$$

Then $f = U_1 U_2(f_2),\ f_2 \in H^\infty,\ ||f_2||_\infty \leq 1.$

Iterating this procedure we have that the problem $(*)_N$ has more than one solution if and only if $|w_i^{(i-1)}| < 1,\ i = 1, \ldots, N$. If $|w_i^{(i-1)}| > 1$ for some i, then the problem has no solution, while if $|w_i^{(i-1)}| = 1$ and $|w_k^{(k-1)}| < 1$ for $k \leq i$, then the problem

$$(*)_i : \text{Find } f \in H^\infty,\ ||f||_\infty \leq 1,\ f(z_k) = w_k,\ k = 1, \ldots, i.$$

has a unique solution.

We will restrict attention to Nevanlinna-Pick problems with more than one solution. If the problem $(*)_N$ has more than one solution, the set of all solutions to problem $(*)_N$ is given by $f = U_1 \cdots U_N(f_N)$, where $f_N \in H^\infty,\ ||f_N||_\infty \leq 1$ and

$$U_i = \frac{1}{\sqrt{1-|w_i^{(i-1)}|^2}} \begin{pmatrix} b_{z_i} & w_i^{(i-1)} \\ \overline{w_i^{(i-1)}} b_{z_i} & 1 \end{pmatrix}.$$

In other words, denoting $\left(\begin{smallmatrix} P_N & Q_N \\ R_N & S_N \end{smallmatrix}\right) = U_1 \cdots U_N$, we have

$$f = \begin{pmatrix} P_N & Q_N \\ R_N & S_N \end{pmatrix} f_N.$$

The functions P_N, Q_N, R_N, S_N are called Nevanlinna coefficients. Let us mention some of their properties:

- P_N, Q_N, R_N, S_N are rational functions with poles contained in the set $\{\frac{1}{\overline{z_i}} : i = 1, \ldots, N\}$. This is clear because the components of the matrices U_i satisfy it.

- $P_N S_N - Q_N R_N = B_N$, the Blaschke product with zeros $z_1, \ldots, z_N$. This is clear because $\det U_j = b_{z_j},\ j = 1, \ldots, N$.

- Consider $\Delta_N(z) = \{f(z) : f \text{ solves problem } (*)_N\}$. Then $\Delta_N(z)$

is a Euclidean disc of center $c_N(z)$ and radius $\rho_N(z)$ given by

$$c_N(z) = \frac{P_N(z)\overline{\left(-\frac{R_N}{S_N}(z)\right)} + Q_N(z)}{R_N(z)\overline{\left(-\frac{R_N}{S_N}(z)\right)} + S_N(z)},$$

$$\rho_N(z) = \frac{|B_N(z)|}{|S_N(z)|^2 - |R_N(z)|^2}.$$

Let's prove this third property:

Proof. Fix $z \in \overline{\mathbb{D}}$. Since $|w_i^{(i-1)}| < 1$, the map $U_{i,z} = U_i : \mathbb{D} \to \mathbb{D}$ defined by

$$U_i(w) = \frac{b_{z_i}(z)w + w_i^{(i-1)}}{1 + \overline{w_i^{(i-1)}} b_{z_i}(z)w}$$

maps the unit disc into itself and if $|z| = 1$, U_i is onto. The same holds for $\left(\begin{smallmatrix} P_N & Q_N \\ R_N & S_N \end{smallmatrix}\right) = U_1 \cdots U_N$, that is, fixed $z \in \overline{\mathbb{D}}$, the map

$$\begin{aligned} T_{N,z} &: \mathbb{D} \to \mathbb{D} \\ &w \mapsto \frac{P_N(z)w + Q_N(z)}{R_N(z)w + S_N(z)} \end{aligned}$$

is into and if $|z| = 1$ it is onto. Consider $\Delta_N(z) = \{f(z) : f$ solves problem $(*)_N\}$. Then,

$$\Delta_N(z) = \left\{ \frac{P_N(z)w + Q_N(z)}{R_N(z)w + S_N(z)} : w \in \overline{\mathbb{D}} \right\} = T_{N,z}(\overline{\mathbb{D}}).$$

Hence $\Delta_N(z)$ is a disc. Since $T_{N,z}\left(-\frac{S_N}{R_N}(z)\right) = \infty$, by reflection, $T_{N,z}$ maps the point $\overline{-\frac{R_N}{S_N}(z)}$ to the center of $\Delta_N(z)$. Since

$$\rho_N(z) = \left| T_{N,z}(\mathrm{e}^{i\theta}) - T_{N,z}\left(-\frac{\overline{R_N(z)}}{S_N}(z) \right) \right|$$

for any $e^{i\theta} \in \partial\,\mathbb{D}$, a calculation shows that the radius $\rho_N(z)$ of $\Delta_N(z)$ is

$$\rho_N(z) = \frac{|B_N(z)|}{|S_N(z)|^2 - |R_N(z)|^2}.$$

$\square$

- On $\partial\,\mathbb{D}$, the following identities hold: $|S_N|^2 - |R_N|^2 = 1$, $P_N\overline{R_N} = Q_N\overline{S_N}$, $P_N = B_N\overline{S_N}$, $Q_N = B_N\overline{R_N}$.

Proof. If $|z| = 1$, $\Delta_N(z) = \overline{\mathbb{D}}$, hence $\rho_N(z) = 1$, that is,

$$|S_N(z)|^2 - |R_N(z)|^2 = 1, \quad C_N(z) = 0,$$

$P_N(z)\overline{R_N(z)} = Q_N(z)\overline{S_N(z)}$. Since $P_N(z)S_N(z) - Q_N(z)R_N(z) = B_N(z)$, we have that

$$P_N(z)|S_N(z)|^2 - Q_N(z)\overline{S_N(z)}R_N(z) = B_N(z)\overline{S_N(z)},$$

and since $Q_N(z)\overline{S_N(z)} = P_N(z)\overline{R_N(z)}$ we deduce that

$$P_N(z)|S_N(z)|^2 - P_N(z)|R_N(z)|^2 = B_N(z)\overline{S_N(z)}$$

and we obtain that $P_N(z) = B_N(z)\overline{S_N(z)}$. A similar argument proves last identity. □

- For $z \in \mathbb{D}$, the following identities hold:

$$\begin{gathered}
|S_N(z)| \geq 1, \\
P_N(z) = B_N(z)\overline{S_N(1/\overline{z})}, \\
Q_N(z) = B_N(z)\overline{R_N(1/\overline{z})}, \\
\max\{|P_N(z)|, |Q_N(z)|, |R_N(z)|\} \leq |S_N(z)|.
\end{gathered}$$

Proof. Since $|S_N(z)|^2 - |R_N(z)|^2 \geq 0$ we deduce that $S_N(z) \neq 0$ because otherwise $R_N(z) = 0$ and then $P_N(z) = Q_N(z) = 0$, but $B_N = P_N S_N - Q_N R_N$, which has no double zeros. Hence S_N does not vanish at $\mathbb{D}$, so $1/S_N \in H^\infty$ is continuous in $\overline{\mathbb{D}}$ and $|1/S_N| \leq 1$ on $\partial\mathbb{D}$, because on $\partial\mathbb{D}$ we have $|S_N|^2 = 1 + |R_N|^2 \geq 1$. By the maximum principle we deduce that $|1/S_N| \leq 1$ on $\mathbb{D}$. The formula $P_N(z) = B_N(z)\overline{S_N(1/\overline{z})}$ follows from $P_N = B_N\overline{S_N}$ on $\partial\mathbb{D}$ by analytic continuation. The formula $Q_N(z) = B_N(z)\overline{R_N(1/\overline{z})}$ follows similarly. Since $|P_N| = |S_N|$ and $|Q_N| = |R_N| < |S_N|$ on $\partial\mathbb{D}$ and S_N has no zeros on $\mathbb{D}$, we deduce that

$$\max\{|P_N(z)|, |Q_N(z)|, |R_N(z)|\} \leq |S_N(z)|$$

for all $z \in \mathbb{D}$. □

Infinite Case. Assume we have an infinite sequence of points $\{z_i\}$ and values $\{w_i\}$ and consider the Nevanlinna-Pick problem

$(*)$ *Find* $f \in H^\infty$, $||f||_\infty \leq 1$, $f(z_i) = w_i$, $i = 1, 2, \ldots$.

Consider the Nevanlinna-Pick problem with the first N points. We know that if the problem has more than one solution, then all solutions can be parametrized as

$$\left\{f \in H^\infty : ||f||_\infty \leq 1,\, f(z_i) = w_i,\, i = 1, \ldots, N\right\}$$
$$= \left\{f = \frac{P_N\varphi + Q_N}{R_N\varphi + S_N} : \varphi \in H^\infty,\, ||\varphi||_\infty \leq 1\right\}.$$

We can assume $R_N(0) = 0$. This is just a normalization one can achieve replacing φ in Nevanlinna's formula by

$$\frac{\varphi - \alpha}{1 - \overline{\alpha}\varphi}, \quad \alpha = \overline{\left(\frac{R_N(0)}{S_N(0)}\right)}.$$

In order to consider Nevanlinna's parametrization of the set of solutions in the infinite case we need to make sure that the infinite case problem has more than one solution. One of the nice features of Schur's algorithm is that it is reversible. In our approach, given the values $\{w_i\}$, we have obtained the points $\{w_i^{(i-1)}\}$. Conversely, given points $\{w_i^{(i-1)} : i = 1, \ldots, N\} \subset \mathbb{D}$, one can obtain the values $\{w_i : i = 1, \ldots, N\}$ and consider the correspondent Nevanlinna-Pick problem. We shall mention the following fact, proved by Denjoy: *The Nevanlinna-Pick problem with infinite points has more than one solution if and only if*

$$\sum_i \frac{1 - |z_i|}{1 - \left|w_i^{(i-1)}\right|} < \infty,$$

see [52, p. 300]. Nonetheless, we will not use (and we will not prove) this result. The first main Theorem is the following.

Theorem 2.1. (Nevanlinna, 1919) *Assume the Nevanlinna-Pick problem*

$(*)$ *Find* $f \in H^\infty$, $||f||_\infty \leq 1$, $f(z_n) = w_n$, $n = 1, 2, \ldots$

has more than one solution. Then the set of all solutions can be *parametrized as*

$$\begin{aligned}\big\{f\in H^\infty\,:\,\|f\|_\infty\le 1,\ f(z_n)=w_n,\ n=1,2,\dots\big\}\\ =\left\{f=\frac{P\varphi+Q}{R\varphi+S}\,:\,\varphi\in H^\infty,\ \|\varphi\|_\infty\le 1\right\},\end{aligned}\tag{2.1}$$

where P,Q,R,S are analytic functions in $\mathbb{D}$ which satisfy

(i) *Let B be the Blaschke product with zeros $\{z_n\}$. Then, we have $PS-QR=B$.*

(ii) *The Nevanlinna's coefficients P,S,Q,R belong to the Nevanlinna class $N(\mathbb{D})$.*[2]

(iii) *The set*

$$\Delta(z)=\big\{f(z):f\ \textit{solves}\ (*)\big\}=\left\{\frac{P(z)w+Q(z)}{R(z)w+S(z)}\,:\,w\in\overline{\mathbb{D}}\right\}$$

is a Euclidean disc of center $c(z)$ and radius $\rho(z)$, given by

$$c(z)=\frac{P(z)\overline{\big(-\frac{R}{S}(z)\big)}+Q(z)}{R(z)\overline{\big(-\frac{R}{S}(z)\big)}+S(z)},\qquad \rho(z)=\frac{|B(z)|}{|S(z)|^2-|R(z)|^2}.$$

(iv) *At almost every point of $\partial\mathbb{D}$ we have $|S|^2-|R|^2=1$, $P=B\overline{S}$, $Q=B\overline{R}$ and $P\overline{R}-Q\overline{S}=0$.*

(v) *For all $z\in\mathbb{D}$ we have $\max\{|P(z)|,|Q(z)|,|R(z)|\}\le|S(z)|$. Moreover, S is an outer function and $|S(z)|\ge 1$.*

Proof. Consider the truncated Nevanlinna-Pick problem $(*)_N$ and the corresponding Nevanlinna's coefficients P_N,Q_N,R_N,S_N. Since $R_N(0)=0$, by Schwarz's lemma, we have $\big|\frac{R_N(z)}{S_N(z)}\big|\le|z|$. Hence,

$$\frac{|B_N(z)|}{|S_N(z)|^2(1-|z|^2)}\ge\rho_N(z)=\frac{|B_N(z)|}{|S_N(z)|^2-|R_N(z)|^2}\ge\frac{|B_N(z)|}{|S_N(z)|^2}.$$

Since the problem has more than one solution, there exists $z_0\in\mathbb{D}$ such that $\lim_{N\to\infty}\rho_N(z_0)\neq 0$. Hence, $|S_N(z_0)|\not\to\infty$. Since $\log|S_N|$

[2]Moreover, A. Stray proved that P,S,Q and R are meromorphic in $\mathbb{C}\backslash\overline{\{1/\overline{z_j}\}}$. See [46].

are positive harmonic functions, considering a subsequence if necessary, Harnack's principle gives that $\{S_N\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Since $\max\{|P_N|,|Q_N|,|R_N|\} \leq |S_N|$, we deduce that there exist subsequences $P_{N_k}, Q_{N_k}, R_{N_k}, S_{N_k}$ which converge uniformly on compacts of $\mathbb{D}$. The limit functions are called P, Q, R and S. Then, the parametrization (2.1) follows from the finite case. Moreover,

$$\Delta(z) = \left\{ \frac{P(z)w + Q(z)}{R(z)w + S(z)} : w \in \overline{\mathbb{D}} \right\}$$

and property (iii) follows as in the finite case. Also, since $P_N S_N - Q_N R_N = B_N$ and $\max\{|P_N|,|Q_N|,|R_N|\} \leq |S_N|$ on $\mathbb{D}$, we deduce (i) and $\max\{|P|,|Q|,|R|\} \leq |S|$ on $\mathbb{D}$. Since $|S(z)| \geq 1$ for all $z \in \mathbb{D}$, we have $1/S \in H^\infty$ and then $S \in N(\mathbb{D})$. Since S has no zeros and $\max\{|P|,|Q|,|R|\} \leq |S|$, we deduce that $P, Q, R \in N(\mathbb{D})$.

Property (iv) will be proven as a consequence of a theorem of Nevanlinna which will be the main topic of the next session. It remains to prove that S is outer. A function $g \in H^\infty$, $||g||_\infty \leq 1$ is an extreme point of the unit ball of H^∞ if it can not be written as $g = (g_1 + g_2)/2$ where $g_i \in H^\infty$ and $||g_i||_\infty \leq 1$, $i = 1, 2$. A result by de Leeuw-Rudin, see [27], tells us that $g \in H^\infty$, $||g||_\infty \leq 1$ is an extreme point of the unit ball of H^∞ if and only if

$$\int_0^{2\pi} \log \frac{1}{1 - |g(\mathrm{e}^{i\theta})|} \, \mathrm{d}\theta = +\infty.$$

This is related to the Nevanlinna-Pick problem because the set of solutions is clearly convex. Hence, if there are two solutions to $(*)$, there exists a solution $f_0 \in H^\infty$, $||f_0||_\infty \leq 1$ such that

$$\int_0^{2\pi} \log \frac{1}{1 - |f_0(\mathrm{e}^{i\theta})|} \, \mathrm{d}\theta < +\infty.$$

Write $f_0 = \frac{P\varphi_0 + Q}{R\varphi_0 + S}$ for some $\varphi_0 \in H^\infty$, $||\varphi_0||_\infty \leq 1$. Consider $f_1 = f_0 + BE$ where E is the outer function whose boundary values have modulus $1 - |f_0|$, that is,

$$E(z) = \exp\left(\int_0^{2\pi} \frac{\mathrm{e}^{i\theta} + z}{e^{i\theta} - z} \log\left(1 - \left|f_0(\mathrm{e}^{i\theta})\right|\right) \mathrm{d}\theta \right), \quad z \in \mathbb{D}.$$

Then f_1 is a solution of $(*)$. Hence, $f_1 = \frac{P\varphi_1 + Q}{R\varphi_1 + S}$ for some $\varphi_1 \in H^\infty$,

$||\varphi_1||_\infty \leq 1$. Therefore,

$$BE = f_1 - f_0 = \frac{P\varphi_1 + Q}{R\varphi_1 + S} - \frac{P\varphi_0 + Q}{R\varphi_0 + S} = \frac{B}{S^2} \frac{\varphi_1 - \varphi_0}{\left(\frac{R}{S}\varphi_1 + 1\right)\left(\frac{R}{S}\varphi_0 + 1\right)}.$$

We know $|S| \geq 1$ on $\mathbb{D}$. Assume $1/S$ has a singular inner factor. Then, since $1 + \frac{R}{S}\varphi_i$, $i = 0, 1$, are outer because they have positive real part, we deduce that BE would be divisible by a singular inner function, which leads to contradiction. □

In classical moment problems one tries to find positive measures in a half line with prescribed moments. It is worth mentioning that under suitable conditions, one can parametrize all solutions of the problem by a formula which is analogue to Nevanlinna's parametrization. See [29].

2.2. Pick's Approach

In this section, we shall describe an idea due to Sarason which leads to Pick's classical result on the existence of solutions to the Nevanlinna-Pick problems, see [42]. A different proof can be found in [19, p. 7].

Theorem 2.2. (Pick, 1916) *The Nevanlinna-Pick problem* $(*)$ *has a solution if and only if the matrices*

$$\left(\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right)_{i,j=1,\ldots,N}$$

are positive semidefinite for any $N = 1, 2, \ldots$.

Proof. By normal families it is enough to prove the result for Nevanlinna-Pick problem with finitely many points. We will only prove the necessity. Let H be a Hilbert space of analytic functions in $\mathbb{D}$. For instance, H could be the Hardy space

$$H = \mathbb{H}^2 = \Bigg\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} : \\ ||f||_2^2 = \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \,\mathrm{d}t < \infty \Bigg\}$$

and

$$\langle f, g \rangle_H = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\overline{g(e^{it})} \,\mathrm{d}t.$$

Let M_H be its multiplier space, that is, $M_H = \{\varphi : \mathbb{D} \to \mathbb{C}$ analytic : $\varphi f \in H$ for any $f \in H\}$. The norm in M_H is given by

$$||\varphi||_{M_H} = \sup_{f \in H \setminus \{0\}} \frac{||\varphi f||_H}{||f||_H}.$$

For instance, $M_{\mathbb{H}^2} = \mathbb{H}^\infty$ and $||\varphi||_{M_{\mathbb{H}^2}} = ||\varphi||_\infty$. Assume that the evaluation at a point $z \in \mathbb{D}$ given by

$$\begin{aligned} H &\to \mathbb{C} \\ f &\mapsto f(z) \end{aligned}$$

is continuous. Then, there exists $k_z \in H$, called reproducing kernel, such that $f(z) = \langle f, k_z \rangle_H$ for any $f \in H$. For instance, if $f \in H^2$, by Cauchy's formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - z\mathrm{e}^{-i\theta}} \, \mathrm{d}\theta = \langle f, k_z \rangle,$$

where

$$k_z(\mathrm{e}^{i\theta}) = \frac{1}{1 - \overline{z}\mathrm{e}^{i\theta}}.$$

The main idea is the following. Pick $\varphi \in M_H$ and consider the multiplication operator $M_\varphi : H \to H$ defined by $M_\varphi(f) = \varphi f$. Then, the adjoint operator M_φ^* satisfies $M_\varphi^*(k_z) = \overline{\varphi(z)} k_z$ because

$$\langle f, M_\varphi^*(k_z) \rangle = \langle M_\varphi(f), k_z \rangle = \langle \varphi f, k_z \rangle = \varphi(z) \langle f, k_z \rangle = \langle f, \overline{\varphi(z)} k_z \rangle$$

for any $f \in H$. Hence, if $||\varphi||_{M_H} \leq 1$, then $||M_\varphi^*(\sum \lambda_i k_{z_i})||_H \leq ||\sum \lambda_i k_{z_i}||_H$, that is,

$$\begin{aligned} \sum_{i,j} \lambda_i \overline{\lambda_j} \, \overline{\varphi(z_i)} \varphi(z_j) k_{z_i}(z_j) &= \Big|\Big| \sum \lambda_i \overline{\varphi(z_i)} k_{z_i} \Big|\Big|_H^2 \leq \Big|\Big| \sum \lambda_i k_{z_i} \Big|\Big|_H^2 \\ &= \Big\langle \sum \lambda_i k_{z_i}, \sum \lambda_j k_{z_j} \Big\rangle = \sum_{i,j} \lambda_i \overline{\lambda_j} \, k_{z_i}(z_j). \end{aligned}$$

In the case $H = \mathbb{H}^2$, $M_H = \mathbb{H}^\infty$ and $k_{z_i}(z) = 1/(1 - \overline{z_i} z)$, if φ is a solution to the Nevanlinna-Pick problem $(*)$, then

$$\sum \lambda_i \overline{\lambda_j} \overline{w_i} w_j \frac{1}{1 - \overline{z_i} z_j} \leq \sum \lambda_i \overline{\lambda_j} \frac{1}{1 - \overline{z_i} z_j},$$

that is, the matrix

$$\left(\frac{1-\overline{w_i}w_j}{1-\overline{z_i}z_j}\right)_{i,j=1,\dots,N},$$

is positive semidefinite. This proves the necessity in Pick's result. □

A Hilbert space H of analytic functions in the disc with reproducing kernel k_z has the Pick property if for any sequence $\{w_j\} \subset \mathbb{D}$ such that the matrices $((1-w_i\overline{w_j})k_{z_i}(z_j))_{i,j=1,\dots,N}$ are positive semidefinite for any N, there exists $\varphi \in M_H$, $||\varphi||_{M_H} \leq 1$ such that $\varphi(z_i) = w_i$, $i = 1, 2, \dots$. Pick's theorem tells us that $H = \mathbb{H}^2$ has the Pick's property. Agler proved that the Dirichlet space has the Pick property. The Bergman space does not have Pick's property. Pick's property is closely related to many other important notions as interpolating sequences and Carleson Measures. See the books by Seip [44] and by Agler and McCarthy, [2].

3. EXTREMAL SOLUTIONS

Given a Nevanlinna-Pick problem with more than one solution, Theorem 2.1 provides a parametrization of the set of all solutions. If one chooses φ to be a unimodular constant $\lambda \in \partial\,\mathbb{D}$, the corresponding solution $\frac{P\lambda+Q}{R\lambda+S}$ is called an extremal solution. This section is devoted to present Nevanlinna's classical result and its refinement due to Stray.

3.1. Extremal Solutions for Finite Problems

In this section we will show the following elementary fact. If we have a Nevanlinna-Pick problem with N points and with more than one solution, then, for any $\lambda \in \partial\,\mathbb{D}$, $\frac{P_N\lambda+Q_N}{R_N\lambda+S_N}$ is a Blaschke product of degree less or equal to N. This is clear because $\frac{P_N\lambda+Q_N}{R_N\lambda+S_N} = U_1 \cdots U_N(\lambda)$ and $U_i(f)$ is a Blaschke product with i zeros whenever f is a Blaschke product with $i-1$ zeros.

An inner function is a bounded analytic function in $\mathbb{D}$ whose radial limits are of modulus one at almost every point of the unit circle. If I is an inner function, then $\frac{P_NI+Q_N}{R_NI+S_N}$ is also inner because on $\partial\,\mathbb{D}$ we have $P_N = B_N\overline{S_N}$, $Q_N = B_N\overline{R_N}$, and hence

$$\frac{P_N I + Q_N}{R_N I + S_N} = B_N \frac{\overline{S_N} I + \overline{R_N}}{R_N I + S_N},$$

which is unimodular at each point where $|I| = 1$.

3.2. Extremal Solutions for Infinite Problems

We now state the second main result by Nevanlinna:

Theorem 3.1. (Nevanlinna, 1929) *Given a Nevanlinna-Pick problem with more than one solution, consider its Nevanlinna's coefficients P, Q, R, S. Then for any $\lambda \in \partial\mathbb{D}$, the function $I_\lambda = \frac{P\lambda+Q}{R\lambda+S}$ is inner.*

Proof. Fix $\lambda \in \partial\mathbb{D}$ and $I = \frac{P\lambda+Q}{R\lambda+S}$. Since I solves the problem with finitely many points, for any $N = 1, 2, \ldots$, we have

$$I = \frac{P_N \varphi_N + Q_N}{R_N \varphi_N + S_N}$$

for some $\varphi_N \in H^\infty$, $||\varphi_N||_\infty \leq 1$. Taking convenient subsequences, we may assure that $P_{N_j}, Q_{N_j}, R_{N_j}, S_{N_j} \longrightarrow P, Q, R, S$ and $\varphi_{N_j} \longrightarrow \varphi$ uniformly on compacts of $\mathbb{D}$. Hence,

$$I = \frac{P_{N_j}\varphi_{N_j} + Q_{N_j}}{R_{N_j}\varphi_{N_j} + S_{N_j}} \longrightarrow \frac{P\varphi + Q}{R\varphi + S}.$$

Thus, $\varphi \equiv \lambda$, that is, $\varphi_{N_j} \longrightarrow \lambda$ uniformly on compacts of $\mathbb{D}$. Assume I is not inner, that is, there exists $K \subset \partial\mathbb{D}$ with $|K| > 0$ and $|I| \leq m < 1$ on K, where $|K|$ denotes the Lebesgue measure of K. Recall that if $|z| = 1$, the mapping $T_{N,z}$ is onto from $\mathbb{D}$ to $\mathbb{D}$, hence it preserves the pseudohyperbolic distances, that is, $\rho(a, b) = \rho(T_{N,z}(a), T_{N,z}(b))$. Here, $\rho(z, w) = \left|\frac{z-w}{1-\overline{w}z}\right|$ for $z, w \in \mathbb{D}$. Using that $T_{N,z}(\varphi_N) = I$, $T_{N,z}(0) = Q_N/S_N$, at almost every point of K, one has that

$$|\varphi_{N_j}| = \rho(\varphi_{N_j}, 0) = \rho\left(I, \frac{Q_{N_j}}{S_{N_j}}\right) \leq \frac{|I| + \left|\frac{Q_{N_j}}{S_{N_j}}\right|}{1 + \left|\frac{Q_{N_j}}{S_{N_j}}\right| |I|} \leq \frac{m + \left|\frac{Q_{N_j}}{S_{N_j}}\right|}{1 + \left|\frac{Q_{N_j}}{S_{N_j}}\right| m}.$$

Since

$$C_0 \geq \log |S_{N_j}(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| S_{N_j}(\mathrm{e}^{i\theta}) \right| \mathrm{d}\theta$$
$$\geq \frac{1}{2\pi} \int_K \log \left| S_{N_j}(\mathrm{e}^{i\theta}) \right| \mathrm{d}\theta,$$

there exists $K_1 \subset K$ with $|K_1| \geq |K|/2$ such that $|S_{N_j}| \leq C_1 = C_1(K)$ on K_1. Hence,

$$\frac{|Q_{N_j}|^2}{|S_{N_j}|^2} = 1 - \frac{1}{|S_{N_j}|^2} \leq 1 - \frac{1}{C_1^2} = C_2^2 \text{ on } K_1,$$

and we deduce

$$|\varphi_{N_j}| \leq \frac{m + C_2}{1 + C_2 m} = m_1 < 1 \text{ on } K_1.$$

Then,

$$|\varphi_{N_j}(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \varphi_{N_j}(\mathrm{e}^{i\theta}) \right| \mathrm{d}\theta \leq m_1 |K_1| + (1 - |K_1|) < 1,$$

which leads to contradiction. □

Corollary 3.2. *If a Nevanlinna-Pick problem has more than one solution, then it has an inner solution.*

We can now deduce the identities stated in property (iv) of Theorem 2.1. See [48].

Corollary 3.3. *At almost every point on* $\partial\,\mathbb{D}$ *we have* $|S|^2 - |R|^2 = 1$, $P = B\overline{S}$, $Q = B\overline{R}$ *and* $P\overline{R} = Q\overline{S}$. *Moreover,* $\rho\left(r\mathrm{e}^{i\theta}\right) \to 1$ *as* $r \to 1$ *at almost every point* $\mathrm{e}^{i\theta} \in \partial\,\mathbb{D}$. □

Proof. Fix $\mathrm{e}^{i\theta} \in \partial\,\mathbb{D}$ such that $\left| I_\lambda \left(\mathrm{e}^{i\theta}\right) \right| = 1$ for three different values of $\lambda \in \partial\,\mathbb{D}$. Assume also $P\left(\mathrm{e}^{i\theta}\right)$, $Q\left(\mathrm{e}^{i\theta}\right)$, $R\left(\mathrm{e}^{i\theta}\right)$, $S\left(\mathrm{e}^{i\theta}\right)$ exist and are finite. We have that

$$T(\mathrm{e}^{i\theta}) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}, \quad w \mapsto \frac{P\left(\mathrm{e}^{i\theta}\right) w + Q\left(\mathrm{e}^{i\theta}\right)}{R\left(\mathrm{e}^{i\theta}\right) w + S\left(\mathrm{e}^{i\theta}\right)},$$

maps $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. Since the center of $T(\mathbb{D})$ is the origin and the radius is 1, we have $P(\mathrm{e}^{i\theta})\overline{R\left(\mathrm{e}^{i\theta}\right)} - Q(\mathrm{e}^{i\theta})\overline{S\left(\mathrm{e}^{i\theta}\right)} = 0$, $\left|S(\mathrm{e}^{i\theta})\right|^2 - \left|R(\mathrm{e}^{i\theta})\right|^2 = 1$.

Hence, $\rho\left(re^{i\theta}\right) \to 1$ as $r \to 1$. Since $PS - QR = B$, we deduce $P\overline{R}S - Q|R|^2 = B\overline{R}$. Thus, $Q|S|^2 - Q|R|^2 = B\overline{R}$ and we deduce $Q = B\overline{R}$ on $\partial\mathbb{D}$. A similar argument shows that $P = B\overline{S}$. □

Arguing as in the finite case one can also deduce the following corollary:

Corollary 3.4. *Let I be an inner function. Then $\frac{PI+Q}{RI+S}$ is also inner.*

3.3. Blaschke Products Among Extremal Solutions

Let us first recall the following classical description of Blaschke products among the set of inner functions:

Lemma 3.5. *(Frostman) Let I be an inner function. Then I is a Blaschke product if and only if*

$$\lim_{r\to 1}\int_0^{2\pi} \log\left|I(re^{i\theta})\right|^{-1} \mathrm{d}\theta = 0.$$

Proof. Assume $I = BS$, where S is a non trivial singular inner function. Then, $|I| \leq |S|$ and

$$\int_0^{2\pi} \log\left|I(re^{i\theta})\right|^{-1} \mathrm{d}\theta \geq \int_0^{2\pi} \log\left|S(re^{i\theta})\right|^{-1} \mathrm{d}\theta = 2\pi \log|S(0)|^{-1}.$$

Conversely, assume I is a Blaschke product with zeros $\{z_n\}$. Then,

$$\begin{aligned}\int_0^{2\pi} \log\left|I(re^{i\theta})\right|^{-1} \mathrm{d}\theta &= \sum_n \int_0^{2\pi} \log\left|\frac{1-\overline{z_n}re^{i\theta}}{re^{i\theta}-z_n}\right| \mathrm{d}\theta \\ &= -\sum_n \int_0^{2\pi} \log\left|re^{i\theta}-z_n\right| \mathrm{d}\theta.\end{aligned}$$

Now,

$$\int_0^{2\pi} \log\left|re^{i\theta}-z_n\right| \mathrm{d}\theta = \begin{cases} 2\pi\log|z_n|, & |z_n| \geq r, \\ 2\pi\log r, & |z_n| < r. \end{cases}$$

Hence,

$$\int_0^{2\pi} \log\left|I(re^{i\theta})\right|^{-1} \mathrm{d}\theta = \sum_{|z_n|<r} \log(r) + \sum_{|z_n|\geq r} \log|z_n|,$$

which tends to 0 by the Blaschke condition, $\sum(1-|z_n|) < \infty$. □

A compact set $K \subset \mathbb{C}$ has positive logarithmic capacity if there exists a probability measure μ supported on K such that the logarithmic potential

$$u(z) = \int_K \log \frac{1}{|z-w|} \, \mathrm{d}\mu(w)$$

is uniformly bounded. A countable set has logarithmic capacity zero while a rectifiable curve has positive logarithmic capacity. Sets of logarithmic capacity zero are small in terms of size; for instance, they have Hausdorff dimension zero. Let I be an inner function. A classical result by Frostman says that for all $\alpha \in \mathbb{D}$, except possibly for a set of logarithmic capacity zero, the function $\frac{I-\alpha}{1-\overline{\alpha}I}$ is a Blaschke product. See [18] or [19, p.75]. We now state the main result of this section:

Theorem 3.6. (A. Stray, 1988, [49]) *Assume the Nevanlinna-Pick problem* $(*)$ *has more than one solution. Then, for all* $\lambda \in \partial\mathbb{D}$, *except possibly for a set of logarithmic capacity zero, the function* $I_\lambda = \frac{P\lambda+Q}{R\lambda+S}$ *is a Blaschke product.*

Proof. Let

$$E = \left\{ \lambda \in \partial\mathbb{D} : I_\lambda = \frac{P\lambda+Q}{R\lambda+S} \text{ is not a Blaschke product} \right\}.$$

We want to show that the logarithmic capacity of E is 0. Let μ be a probability measure supported in E with

$$\sup_{z\in\mathbb{C}} \left| \int_E \log|z-w|^{-1} \, \mathrm{d}\mu(w) \right| = M < \infty.$$

We want to show that $\mu(E) = 0$. By Lemma 3.5, it is enough to show that

$$\int_E \lim_{r\to 1} \int_0^{2\pi} \log \left| I_\lambda(r\mathrm{e}^{i\theta}) \right|^{-1} \mathrm{d}\theta \, \mathrm{d}\mu(\lambda) = 0.$$

We know that $\rho\left(r\mathrm{e}^{i\theta}\right) \to 1$ at almost every $\mathrm{e}^{i\theta}$, hence, given $\varepsilon > 0$ and $\eta > 0$ there exists $K \subset \partial\mathbb{D}$, $|K| \geq 2\pi - \varepsilon$ with $\rho(r\mathrm{e}^{i\theta}) \geq 1-\eta$ for $\mathrm{e}^{i\theta} \in K$ if $1-r$ is sufficiently small. Since $I_\lambda\left(r\mathrm{e}^{i\theta}\right)$ is a boundary point of $\Delta\left(r\mathrm{e}^{i\theta}\right)$, we deduce that $\left|I_\lambda\left(r\mathrm{e}^{i\theta}\right)\right| \geq 1-2\eta$ for $\mathrm{e}^{i\theta} \in K$ and

$\int_K \log\left|I_\lambda\left(r\mathrm{e}^{i\theta}\right)\right|^{-1}\,\mathrm{d}\theta \leq C\eta$ for any $\lambda\in\partial\,\mathbb{D}$ if $1-r$ is sufficiently small. So, it is enough to show

$$\int\limits_E \lim_{r\to 1}\int\limits_{\partial\,\mathbb{D}\setminus K}\log\left|I_\lambda(r\mathrm{e}^{i\theta})\right|^{-1}\mathrm{d}\theta\,\mathrm{d}\mu(\lambda)=0.$$

Using that S is outer, one can show that

$$\int\limits_{\partial\,\mathbb{D}\setminus K}\log\left|S(r\mathrm{e}^{i\theta})\right|\mathrm{d}\theta\longrightarrow\int\limits_{\partial\,\mathbb{D}\setminus K}\log\left|S(\mathrm{e}^{i\theta})\right|\mathrm{d}\theta,\quad r\to 1.$$

Now, by Fatou's lemma and Fubini, one has

$$\begin{aligned}&\int\limits_E \lim_{r\to 1}\int\limits_{\partial\,\mathbb{D}\setminus K}\log\left|I_\lambda(r\mathrm{e}^{i\theta})\right|^{-1}\mathrm{d}\theta\,\mathrm{d}\mu(\lambda)\\&\qquad\leq\liminf_{r\to 1}\int\limits_{\partial\,\mathbb{D}\setminus K}\int\limits_E\log\left|\frac{\frac{P}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda+\frac{Q}{S}\left(r\mathrm{e}^{i\theta}\right)}{\frac{R}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda+1}\right|^{-1}\mathrm{d}\mu(\lambda)\mathrm{d}\theta.\end{aligned}$$

Since

$$\sup_{z\in\mathbb{C}}\left|\int_{\partial\,\mathbb{D}}\log|z-\lambda|^{-1}\,\mathrm{d}\mu(\lambda)\right|=M<\infty,$$

we have

$$\int\limits_E\log\left|\frac{R}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda+1\right|\mathrm{d}\mu(\lambda)\leq M$$

and

$$\begin{aligned}&\int\limits_E\log\left|\frac{P}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda+\frac{Q}{S}\left(r\mathrm{e}^{i\theta}\right)\right|^{-1}\mathrm{d}\mu(\lambda)\\&\qquad\leq-\log\left(\max\left(\left|\frac{P}{S}\left(r\mathrm{e}^{i\theta}\right)\right|,\left|\frac{Q}{S}\left(r\mathrm{e}^{i\theta}\right)\right|\right)\right)+M.\end{aligned}$$

Since $\frac{P}{S}-\frac{Q}{S}\frac{R}{S}=\frac{B}{S^2}$, we deduce that

$$\max\left(\left|\frac{P}{S}\left(r\mathrm{e}^{i\theta}\right)\right|,\left|\frac{Q}{S}\left(r\mathrm{e}^{i\theta}\right)\right|\right)\geq\frac{1}{2}\frac{\left|B\left(r\mathrm{e}^{i\theta}\right)\right|}{\left|S\left(r\mathrm{e}^{i\theta}\right)\right|^2}.$$

Hence,

$$\int_E \lim_{r\to 1} \int_{\partial\mathbb{D}\setminus K} \log\left|I_\lambda(r\mathrm{e}^{i\theta})\right|^{-1} \mathrm{d}\theta\,\mathrm{d}\mu(\lambda)$$

$$\leq CM|\partial\mathbb{D}\setminus K| + C\liminf_{r\to 1}\left(\int_{\partial\mathbb{D}\setminus K} \log\left|B(r\mathrm{e}^{i\theta})\right|^{-1}\mathrm{d}\theta\right.$$

$$\left. + \int_{\partial\mathbb{D}\setminus K} \log\left|S(r\mathrm{e}^{i\theta})\right|^{2}\mathrm{d}\theta\right).$$

The first integral tends to zero as $r \to 1$ because of Lemma 3.5 and the second tends to $\int_{\partial\mathbb{D}\setminus K} \log\left|S(\mathrm{e}^{i\theta})\right|^{2}\mathrm{d}\theta$, which is arbitrarily small if $|\partial\mathbb{D}\setminus K|$ is small. □

We now mention an application of last theorem. Let $K \subset \partial\mathbb{D}$ be a compact set of zero length and let $\varphi : K \to \overline{\mathbb{D}}$ be a continuous function. In [34] it is proved that there exists a Blaschke product I such that for any $e^{i\theta} \in K$, one has $\lim_{r\to 1} I(re^{i\theta}) = \varphi(e^{i\theta})$. The proof proceeds as follows. One first finds a non extremal function f_0 of the unit ball of H^∞ whose radial limit at each point of $e^{i\theta} \in K$ is $\varphi(e^{i\theta})$. R. Berman constructed a Blaschke product B with zeros $\{z_n\}$ such that the radial limit of B vanishes at each point of K. One considers the Nevanlinna-Pick problem with points $\{z_n\}$ and values $\{f_0(z_n)\}$. Then applying Theorem 4, one can choose I to be a convenient extremal solution.

4. SCALED NEVANLINNA-PICK PROBLEMS

A. Stray has found relations between the classical Nevanlinna-Pick problem and more modern topics in function theory, see [46], [47] and [48]. We will show that a certain refinement of the Corona Theorem provides convenient estimates of the radius of $\Delta(z)$.

4.1. The Corona Theorem

The Corona Theorem appeared as part of an effort to understand the Banach algebra properties of H^∞. It is actually equivalent to the non existence of a corona in the maximal ideal space M, that is, that $\mathbb{D}$

is dense in M. The techniques introduced by Carleson in his solution had a huge impact in both Complex and Harmonic Analysis.

Theorem 4.1. (Carleson, 1962, [10] or [19, Chapter VIII]). *Let $f_1, \dots, f_n \in H^\infty$ such that*

$$\inf_{z\in\mathbb{D}} \sum_{j=1}^{n} |f_j(z)| \geq \delta > 0.$$

Then there exist $g_1, \dots, g_n \in H^\infty$ with $\sum_{j=1}^{n} f_j g_j \equiv 1$.

This famous result has been extremely influential. For instance, in his proof, Carleson invented a geometric construction known as the Corona construction that has led to many deep results in the theory of H^∞ as well as in harmonic analysis and many other areas. Another simpler proof based on Littlewood-Paley integrals was obtained by T. Wolff in the eighties, see [19, p.315]. Among other important concepts, Carleson introduced the notion of what we know today as Carleson measure (for the Hardy space). It is not known if the Corona Theorem holds for any domain in the complex plane. It is also open in the unit ball of $\mathbb{C}^N$, $N > 1$ or in the polydisc. On the negative direction, it is known that the Corona Theorem fails in Riemann surfaces. The proof by Carleson is quite technical but contains also the notion of Carleson contour which will appear later, so let us describe it. In general, the level set of a bounded analytic function may be unrectifiable. Actually, P. Jones constructed $f \in H^\infty$, $||f||_\infty \leq 1$ such that for any $\varepsilon \in (0,1)$ the level set $\{z \in \mathbb{D} : |f(z)| = \varepsilon\}$ has infinite lenght; see [24]. Carleson constructed a variant of a level set which is rectifiable.

Lemma 4.2. (Carleson, [10], [19, p.333]). *Let $f \in H^\infty$, $||f||_\infty = 1$ and $0 < \varepsilon < 1$. Then, there exists $\delta = \delta(\varepsilon) > 0$ and $\Gamma = \Gamma(\varepsilon) = \cup_j \Gamma_j$, where $\Gamma_j = \Gamma_j(\varepsilon)$ are piecewise $\mathcal{C}^1$ closed curves with interior $\mathring{\Gamma}_j$ such that*

(a) $|f(z)| \geq \varepsilon$ *if* $z \in \mathbb{D} \setminus \cup \mathring{\Gamma}_j$.

(b) $|f(z)| \leq \delta$ *if* $z \in \cup \mathring{\Gamma}_j$.

(c) *Lenght on Γ is a Carleson measure, that is, there exists $C = C(\varepsilon) > 0$ such that* $\operatorname{length}(D \cap \Gamma) \leq Cr$ *for any disc D of radius r.*

We will use the following refinement of the Corona Theorem:

Theorem 4.3. (P. Jones, [25]) *Let* $f_1, f_2 \in H^\infty$, $||f_i||_\infty \le 1$, $i = 1, 2$. *Assume* $1/2 > \eta > 0$ *satisfies*

$$\inf_{z \in \mathbb{D}} \Big(|f_1(z)| + |f_2(z)| \Big) \ge 1 - \eta.$$

Then there exist $g_1, g_2 \in H^\infty$ *with* $f_1 g_1 + f_2 g_2 \equiv 1$ *and*

$$\sup_{z \in \mathbb{D}} \Big(|f_1(z) g_1(z)| + |f_2(z) g_2(z)| \Big) \le 1 + \frac{A}{\log(1/\eta)},$$

where A *is an absolute constant.*

4.2. The Radius of a Scaled Problem

A Nevanlinna-Pick problem is called scaled if it has a solution f_0 such that $||f_0||_\infty < 1$. The crucial idea in this section is due to A. Stray, see [50], who, using the result of P. Jones stated above, observed that one can estimate the radius of $\Delta(z)$ of an scaled problem.

Lemma 4.4. ([50]) *Assume* $(*)$ *is an scaled Nevanlinna-Pick problem. Then,* $\rho(z) \to 1$ *as* $|B(z)| \to 1$.

Proof. Take $\varepsilon > 0$ small, to be fixed later. Fix $z \in \mathbb{D}$ with $|B(z)| \ge 1 - \varepsilon$. Consider the functions $B(w)$ and $\tau_z(w) = \frac{w-z}{1-\overline{z}w}$. Then, since by Schwarz's lemma

$$\left| \frac{B(w) - B(z)}{1 - \overline{B(z)} B(w)} \right| \le \left| \frac{w - z}{1 - \overline{z} w} \right| = |\tau_z(w)|,$$

we deduce that $|B(w)| + |\tau_z(w)| \ge 1 - C(\varepsilon)$, where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, by Theorem 4.3, there exist $g_1, g_2 \in H^\infty$ such that $B g_1 + \tau_z g_2 \equiv 1$ and $|B(w) g_1(w)| + |\tau_z(w) g_2(w)| \le 1 + A(\varepsilon)$ for any $w \in \mathbb{D}$. Here, $A(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let f_0 be a solution of $(*)$ with $||f_0||_\infty < 1$. Then, for any $s \in \overline{\mathbb{D}}$, the function $f_s = f_0 \tau_z g_2 + \frac{s B g_1}{1 + A(\varepsilon)}$ is a solution of $(*)$ if $\varepsilon > 0$ is chosen small enough so that $||f_0||_\infty \le 1/(1 + A(\varepsilon))$. Now,

$$f_s(z) = \frac{s}{1 + A(\varepsilon)} B(z) g_1(z) = \frac{s}{1 + A(\varepsilon)}.$$

Hence, $\Delta(z)$ contains the disc $\{w \in \mathbb{C} : |w| \le 1/(1 + A(\varepsilon))\}$. Since

$\varepsilon > 0$ can be taken arbitrarily small, this finishes the proof of the lemma. $\square$

Lemma 4.4 was used in [36] to study the Nevanlinna coefficients of a scaled problem.

In the last lecture we will discuss the following question: *If we know an additional information of the sequence of points* $\{z_n\}$*, what can be deduced about the behaviour of the extremal solutions of the Nevanlinna-Pick problem* $(*)$*?* Let us first consider several classes of inner functions. For $0 < \alpha < 1$, the class $\mathcal{B}_\alpha$ consists of the Blaschke products B for which its zero sequence satisfies

$$\sum_{z\,:\,B(z)=0} (1-|z|)^{1-\alpha} < \infty.$$

The following result can be found in the dissertation of L. Carleson, [9]. Let B be a Blaschke product. Then, $B \in \mathcal{B}_\alpha$ if and only if

$$\int_{\mathbb{D}} \frac{\log |B(z)|^{-1}}{(1-|z|^2)^{1+\alpha}} \,\mathrm{d}A(z) < \infty. \tag{4.1}$$

This fact follows easily applying second Green's formula to the functions $\log |B(z)|^{-1}$ and $(1-|z|^2)^{1-\alpha}$. Our second class is defined as follows. An inner function I is in (the Hardy-Sobolev space) $H^{1,\alpha}$ if I' is in the Hardy space H^α, that is,

$$\sup_{r<1} \int_0^{2\pi} \left|I'(r\mathrm{e}^{i\theta})\right|^\alpha \mathrm{d}\theta < \infty. \tag{4.2}$$

Let $\mathrm{d}A(z)$ be the two dimensional Lebesgue measure. For $\frac{1}{2} < \alpha < 1$ and $1 \le p \le 2$, it is well known that $I \in H^{1,\alpha}$ if and only if

$$\int_{\mathbb{D}} |I'(z)|^p (1-|z|)^{p-\alpha-1} \,\mathrm{d}A(z) < \infty \tag{4.3}$$

See Theorem 13 of [14]. So, in this sense, inner functions do not distinguish between these Hardy-Sobolev and Besov spaces. The class of inner functions in $H^{1,\alpha}$ have been extensively studied by many authors but there is still no description of the inner functions I in $H^{1,\alpha}$ in terms of the geometry of its zero set and the behaviour of its associated singular measure. In 1973, Protas proved that for $\frac{1}{2} < \alpha < 1$, $\mathcal{B}_\alpha \subset H^{1,\alpha}$, see [39]. The converse is not true, but Ahern proved that if an inner function $I \in H^{1,\alpha}$ then there exists $w \in \mathbb{D}$ such that $\frac{I-w}{1-\overline{w}I} \in \mathcal{B}_\alpha$ (see [3]). The paper [3] contains also many interesting related results. There is also a beautiful description of inner functions I in $H^{1,\alpha}$, $\frac{1}{2} < \alpha < 1$, in terms of Carleson contours given by Cohn, see [12], which reads as follows: *Fix $\frac{1}{2} < \alpha < 1$. An inner function I is in $H^{1,\alpha}$ if and only if*

$$\int_\Gamma \frac{|\mathrm{d}z|}{(1-|z|)^\alpha} < \infty,$$

where Γ is a Carleson contour of I.

Smoothness properties of inner functions have attracted the attention of many researchers. See [4], [5], [6], [7], [8], [13], [15], [16], [17], [20], [21], [23], [37], [38], [40], [41], [51], and the monography [28]. Observe that in the case $\alpha = 1$, either condition (4.2) or (4.3) implies that I is a finite Blaschke product. For instance, if (4.2) holds, then I extends continuously to the closed unit disc and hence it is a finite Blaschke product. If (4.3) holds and $p = 2$, condition (4.3) tells that the area of the image $I(\mathbb{D})$, counting multiplicities, must be finite. Then, I is a finite Blaschke product. The general case $1 < p < \infty$, $p \neq 2$ was considered in [26]. So, in the case $\alpha = 1$, it is natural to consider weak spaces. An inner function I is in the class $H^{1,1}_\infty$ if there exists a constant $C > 0$ such that for any $0 < r < 1$ and any $\lambda > 0$ one has

$$\left|\left\{\mathrm{e}^{i\theta} : |I'(r\mathrm{e}^{i\theta})| > \lambda\right\}\right| \leq \frac{C}{\lambda}.$$

Here, $|E|$ denotes the length of the measurable set $E \subset \partial\,\mathbb{D}$. It is well known that this last condition holds if and only if the non-tangential maximal function $M(I')$ of I' satisfies the weak estimate

$$\left|\left\{\mathrm{e}^{i\theta} : M(I')(\mathrm{e}^{i\theta}) > \lambda\right\}\right| \leq \frac{C_1}{\lambda}$$

for any $\lambda > 0$. For $1 < p < \infty$ consider the measure $\mathrm{d}\mu_p(z) = (1-|z|)^{p-2}\,\mathrm{d}A(z)$. The weak analogue of condition (4.3) would read as follows: *An inner function I is in the weak Besov space B^p_∞ if there exists a constant $C > 0$ such that for any $\lambda > 0$ one has*

$$\mu_p\left(\left\{z \in \mathbb{D} : |I'(z)| > \lambda\right\}\right) \leq \frac{C}{\lambda^p}.$$

In the papers [11] and [22], written in collaboration with J. Cima and J. Gröhn, it was proved that for any $1 < p < \infty$, $I \in H^{1,1}_{\infty}$ if and only if $I \in B^p_{\infty}$, and this holds if and only if I is a Blaschke product for which there exists a constant $C = C(I) > 0$ such that for any $n = 1, 2, \ldots$ one has

$$\#\left\{z : I(z) = 0,\, 2^{-n-1} < 1 - |z| \leq 2^{-n}\right\} \leq C. \tag{4.4}$$

It is easy to show that the sequence of zeros of I satisfies condition (4.4) if and only if it is the union of finitely many sequences which approach the unit circle exponentially fast. Let us denote $\mathcal{B}_1$ the class of Blaschke products satisfying (4.4). It is obvious that $\mathcal{B}_1 \subset \mathcal{B}_\alpha$ for any $0 < \alpha < 1$. We can now state the result on the Nevanlinna-Pick problem.

Theorem 4.5. ([31]) *Let* $(*)$ *be a scaled Nevanlinna-Pick problem and let* B *be the Blaschke product with zeros* $\{z_n\}$*. Let* I_λ, $\lambda \in \partial\mathbb{D}$, *be the extremal solutions of* $(*)$.

(a) *Fix* $0 < \alpha < 1$ *and assume* $B \in H^{1,\alpha}$*. Then,* $I_\lambda \in H^{1,\alpha}$ *for any* $\lambda \in \partial\mathbb{D}$.

(b) *Assume* $B \in \mathcal{B}_1$*. Then,* $I_\lambda \in \mathcal{B}_1$ *for any* $\lambda \in \partial\mathbb{D}$.

(c) *Fix* $0 < \alpha < 1$*. Assume*

$$\sum_n (1 - |z_n|)^{1-\alpha} \log\left(\frac{1}{1 - |z_n|}\right) < \infty.$$

Then, for all $\lambda \in \partial\mathbb{D}$, *except possibly for a set of logarithmic capacity zero,* $I_\lambda \in \mathcal{B}_\alpha$.

We shall not prove this result but we will make a few comments about it. We do not know if the assumption that $(*)$ be scaled is essential. The main obstacle is that there is no analogue to Lemma 4.4 for non-scaled problems, see [35]. We do not know whether condition (c) holds under the weaker assumption that $B \in \mathcal{B}_\alpha$. The essential tool in the proof of (a) and (b) is Lemma 4.4, where the assumption $(*)$ scaled is used. The proof of (c) uses the description (4.1) and arguments similar to the proof of Stray's theorem. We state an easy consequence Theorem 4.5.

Corollary 4.6. *Let $\{z_n\}$ be a Blaschke sequence and let B be the corresponding Blaschke product. Let $\{w_n\}$ be a bounded sequence of complex numbers such that*

$$M = \sup \left\{ ||f||_\infty \, : \, f \in H^\infty, \, f(z_n) = w_n, \, n = 1, 2, \dots \right\} < \infty.$$

Fix $\varepsilon > 0$.

(a) *There exists a Blaschke product I with $(M+\varepsilon)I(z_n) = w_n$, $n = 1, 2, \dots$.*

(b) *Fix $0 < \alpha < 1$. Assume $B \in H^{1,\alpha}$. Then, there exists a Blaschke product $I \in H^{1,\alpha}$ with $(M+\varepsilon)I(z_n) = w_n$, $n = 1, 2, \dots$.*

(c) *Assume $B \in \mathcal{B}_1$. Then, there exists a Blaschke product $I \in \mathcal{B}_1$ with $(M+\varepsilon)I(z_n) = w_n$, $n = 1, 2, \dots$.*

(d) *Fix $0 < \alpha < 1$. Assume $\sum (1 - |z_n|)^{1-\alpha} |\log(1 - |z_n|)| < \infty$, then there exists $I \in \mathcal{B}_\alpha$ with $(M+\varepsilon)I(z_n) = w_n$, $n = 1, 2, \dots$.*

This result follows from Theorems 3.6 and 4.5 because for $\varepsilon > 0$ the Nevanlinna-Pick problem

$$(*) \text{ Find } f \in H^\infty, \, ||f||_\infty \le 1, \, f(z_n) = \tfrac{w_n}{M+\varepsilon}, \, n = 1, 2, \dots$$

is scaled. It is worth mentioning that the result does not hold when $\varepsilon = 0$, see [45].

We finally state an open question which could have applications: *Let δ_z be the Dirac mass at the point $z \in \mathbb{D}$. Assume $\sum (1 - |z_n|)\delta_{z_n}$ is a Carleson measure (for the Hardy space H^2). Is it then true that there exists $\lambda \in \partial\,\mathbb{D}$ such that*

$$\sum_{z \,:\, I_\lambda(z) = 0} (1 - |z|)\delta_z$$

is a Carleson measure?

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Small Weighted Bergman Spaces

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Abstract. This paper is based on the course "Weighted Hardy-Bergman spaces" I delivered in the Summer School "Complex and Harmonic Analysis and Related Topics" at the Mekrijärvi research station of University of Eastern Finland, June 2014. The main purpose of this survey is to present recent progress on the theory of Bergman spaces A^p_ω, induced by radial weights ω satisfying the doubling property $\int_r^1 \omega(s)\,ds \le C \int_{\frac{1+r}{2}}^1 \omega(s)\,ds$.

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Keywords: Bergman space, Carleson measure, composition operator, doubling weight, factorization, integral operator, maximal function, regular weight.

1. INTRODUCTION

Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. For $f \in \mathcal{H}(\mathbb{D})$ and $0 < r < 1$, set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \le \infty$, the Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty$. A nonnegative integrable function ω on the unit disc $\mathbb{D}$ is called a weight. It is radial if $\omega(z) = \omega(|z|)$

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for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight ω, the weighted Bergman space A^p_ω is the space of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. That is, $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$ where L^p_ω is the corresponding weighted Lebesgue space. As usual, we write A^p_α for the standard weighted Bergman space induced by the radial weight $(1-|z|^2)^\alpha$, where $-1 < \alpha < \infty$ [26, 33, 63]. We denote $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha\, dA(z)$ and $\omega(E) = \int_E \omega(z)\, dA(z)$ for short. We recall that the Bloch space $\mathcal{B}$ [7] consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) + |f(0)| < \infty.$$

The Carleson square $S(I)$ based on an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{it} \in \mathbb{D} : e^{it} \in I,\ 1-|I| \le r < 1\}$, where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D}\setminus\{0\}$ the interval $I_a = \left\{e^{i\theta} : |\arg(ae^{-i\theta})| \le \frac{1-|a|}{2}\right\}$, and denote $S(a) = S(I_a)$.

The theory of standard Bergman spaces A^p_α has evolved enormously throughout the last decades, although important problems such as a description of zero sets or a characterization of invariant subspaces remain open, see [26, 33, 63] for details.

With respect to a general weighted Bergman space A^p_ω, a fact which affects the way of approaching a good number of problems is whether or not ω is radial. Roughly speaking, we can say that the theory of weighted Bergman spaces A^p_ω induced by non-radial weights is at early stages and essential facts are unknown. For instance, if ω is a radial weight, one can easily prove that polynomials are dense in A^p_ω, but this does not remain true for a general weight. For example, the weight

$$\omega(z) = |S(z)|^2 = \left|\exp\left(-\frac{1+z}{1-z}\right)\right|^2 = \exp\left(-\frac{1-|z|^2}{|1-z|^2}\right)$$

satisfies that polynomials are not dense in A^2_ω [26, p. 138]. Concerning embeddings, the sharp inequality $M_p(r,f) \lesssim \|f\|_{\mathcal{B}}\left(\log\frac{e}{1-r}\right)^{p/2}$ and known results on lacunary series [16], show that $\mathcal{B} \subset A^p_\omega$ if and only if $\int_0^1 \omega(r)\left(\log\frac{e}{1-r}\right)^{p/2} dr < \infty$, whenever ω is a radial weight.

These observations lead us to the following open questions;

1. Which are those weights such that the polynomials are dense in A^p_ω?

2. Which are those weights such that $\mathcal{B} \subset A^p_\omega$?

Despite these and other obstacles, some progress has been achieved on the theory of weighed Bergman spaces A^p_ω induced by non-radial weights [3, 8, 9, 12, 49].

As for the Bergman spaces A^p_ω induced by radial weights it is worth noticing that some advances have been obtained on Bergman spaces A^p_ω, in the case when ω belongs to certain classes of radial weights, see [26, 33, 49, 63] and the references therein. However, many questions such that the existence of a (strong or weak) factorization of A^p_ω-functions or the boundedness of the Bergman projection P_ω on A^p_ω [51], are not understood yet. In this paper, we will be specially concerned to the theory of Bergman spaces A^p_ω induced by radial weights ω such that

$$\int_r^1 \omega(s)\,ds \le C \int_{\frac{1+r}{2}}^1 \omega(s)\,ds.$$

We shall write $\widehat{\mathcal{D}}$ for this class of radial weights. A primary motivation for this study is the so called "transition phenomena" from the standard Bergman spaces A^p_α to the Hardy space H^p. That is, in many respects the Hardy space H^p is the limit of A^p_α, as $\alpha \to -1$, but it is a very rough estimate since most of the finer function-theoretic properties of the classical weighted Bergman space A^p_α are not carried over to the Hardy space H^p. Plenty of results in [49–51] show that spaces A^p_ω induced by rapidly increasing weights (Section 2 below for a definition), lie "closer" to H^p than any A^p_α. Here we will present some of them. Moreover, many tools used in the theory of the classical Bergman spaces fail to work in A^p_ω, $\omega \in \widehat{\mathcal{D}}$, so frequently we have to employ appropriate techniques for A^p_ω, $\omega \in \widehat{\mathcal{D}}$, which usually work on standard Bergman spaces and even on Hardy spaces.

The paper is organized as follows; Section 2 contains the definition of classes of radial weights that are considered in these notes, shows relations between them, and contains several descriptions of the class $\widehat{\mathcal{D}}$. In Section 3 we characterize q-Carleson measures for A^p_ω, $\omega \in \widehat{\mathcal{D}}$. This result has been recently proved in [50]. For the range $q \ge p$, we offer a different proof from that in [50]. Here we follow ideas from [49, Chapter 2] and in particular we prove the pointwise estimate

$$\begin{aligned}|f(z)|^\alpha &\le C(\alpha,\omega) \sup_{I:\, z\in S(I)} \frac{1}{\omega\left(S(I)\right)} \int_{S(I)} |f(\xi)|^\alpha \omega(\xi)\, dA(\xi)\\ &= CM_\omega(|f|^\alpha)(z)\end{aligned}$$

for any $f \in \mathcal{H}(\mathbb{D})$, $\alpha > 0$, $\omega \in \widehat{\mathcal{D}}$ and $z \in \mathbb{D}$. We also show some equivalent norms on A^p_ω and a description of q-Carleson measures for A^p_ω in the case $q < p$. Most of these last results are presented without a detailed proof. Section 4 contains the main result in [49, Chapter 3]. There, by using a probabilistic method introduced by Horowitz [36], we prove that if ω is a weight (not necessarily radial) such that

$$\omega(z) \asymp \omega(\zeta), \quad z \in \Delta(\zeta, r), \quad \zeta \in \mathbb{D},$$

where $\Delta(\zeta, r)$ denotes a pseudohyperbolic disc, and polynomials are dense in A^p_ω, then each $f \in A^p_\omega$ can be represented in the form $f = f_1 \cdot f_2$, where $f_1 \in A^{p_1}_\omega$, $f_2 \in A^{p_2}_\omega$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and the following norm estimates hold

$$\|f_1\|^p_{A^{p_1}_\omega} \cdot \|f_2\|^p_{A^{p_2}_\omega} \le \frac{p}{p_1}\|f_1\|^{p_1}_{A^{p_1}_\omega} + \frac{p}{p_2}\|f_2\|^{p_2}_{A^{p_2}_\omega} \le C(p_1,p_2,\omega)\|f\|^p_{A^p_\omega}.$$

In Section 5, by mimicking the corresponding proofs in [49, Section 3.2], we prove that whenever $\omega \in \widehat{\mathcal{D}}$, the union of two A^p_ω-zero sets is not an A^p_ω-zero set.

In Section 6 we characterize those analytic symbols g on $\mathbb{D}$ such that the integral operator $T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\, d\zeta$ is bounded from A^p_ω into A^q_ω, where $0 < p, q < \infty$. Finally, in Section 7 we deal with composition operators $C_\varphi(f) = f\circ\varphi$, where $f \in \mathcal{H}(\mathbb{D})$ and φ is an analytic self-map φ of $\mathbb{D}$. We recall a recent description [52] of bounded and compact composition operators, from A^p_ω into A^q_v, when $\omega \in \widehat{\mathcal{D}}$ and v a radial weight. In the case $q < p$, Theorem 7.1 (below) gives a characterization of bounded (and compact) composition operators that differs from the one in the existing literature [62] in the classical case $C_\varphi : A^p_\alpha \to A^q_\beta$. Here we extend this last result in order to describe bounded (and compact) composition operators from A^p_ω into A^q_v, where ω is a regular weight (see Section 2 below for a definition) and v a radial weight. As far as we know, this result is new.

Throughout these notes, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

2. RADIAL WEIGHTS. PRELIMINARY RESULTS

We recall that $\widehat{\mathcal{D}}$ is the class of radial weights such that $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\,ds$ is doubling, that is, there exists $C = C(\omega) \geq 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. We call a radial weight ω regular, denoted by $\omega \in \mathcal{R}$, if $\omega \in \widehat{\mathcal{D}}$ and $\omega(r)$ behaves as its integral average over $(r, 1)$, that is,

$$\omega(r) \asymp \frac{\int_r^1 \omega(s)\,ds}{1-r}, \quad 0 \leq r < 1.$$

As to concrete examples, we mention that every standard weight as well as those given in [6, (4.4)–(4.6)] are regular. It is clear that $\omega \in \mathcal{R}$ if and only if for each $s \in [0, 1)$ there exists a constant $C = C(s, \omega) > 1$ such that

$$C^{-1}\omega(t) \leq \omega(r) \leq C\omega(t), \quad 0 \leq r \leq t \leq r + s(1-r) < 1, \tag{1}$$

and

$$\frac{\int_r^1 \omega(s)\,ds}{1-r} \lesssim \omega(r), \quad 0 \leq r < 1. \tag{2}$$

The definition of regular weights used here is slightly more general than that in [49], but the main properties are essentially the same by Lemma 2.1 below and [49, Chapter 1].

A radial continuous weight ω is called rapidly increasing, denoted by $\omega \in \mathcal{I}$, if

$$\lim_{r\to 1^-} \frac{\int_r^1 \omega(s)\,ds}{\omega(r)(1-r)} = \infty.$$

It follows from [49, Lemma 1.1] that $\mathcal{I} \subset \widehat{\mathcal{D}}$. Typical examples of rapidly increasing weights are

$$v_\alpha(r) = \left((1-r)\left(\log\frac{e}{1-r}\right)^\alpha\right)^{-1}, \quad 1 < \alpha < \infty.$$

Despite their name, rapidly increasing weights may admit a strong oscillatory behavior. Indeed, the weight

$$\omega(r) = \left|\sin\left(\log \frac{1}{1-r}\right)\right| v_\alpha(r) + 1, \quad 1 < \alpha < \infty,$$

belongs to $\mathcal{I}$ but it does not satisfy (1) [49, p. 7]. Due to this fact, occasionally we consider the class $\widetilde{\mathcal{I}}$ of those weights $\omega \in \mathcal{I}$ satisfying (1).

A radial continuous weight ω is called rapidly decreasing if

$$\lim_{r\to 1^-} \frac{\int_r^1 \omega(s)\,ds}{\omega(r)(1-r)} = 0.$$

The exponential type weights $\omega_{\gamma,\alpha}(r) = (1-r)^\gamma \exp\left(\frac{-c}{(1-r)^\alpha}\right)$, $\gamma \geq 0$, $\alpha, c > 0$, are rapidly decreasing. It is worth mentioning that the pseudohyperbolic metric is not the right one to describe problems on A^p_ω in this case. Roughly speaking, the substitute of a pseudohyperbolic disc of center z and radius $r < 1$ is constructed by writing $\omega = e^{-\varphi}$, where $\Delta\varphi > 0$, and considering the disc $D\big(z, c/\sqrt{\Delta\varphi(z)}\,\big)$.

The weighted Bergman spaces A^p_ω induced by rapidly decreasing weights are similar, but not identical, to weighted Fock spaces [44]. See [8, 9, 22, 23, 45, 46, 58] for progress on the theory of these spaces. For further information on any of these classes, see [49, Chapter 1] and the references therein.

The main aim of this section is to obtain different characterizations and properties of the classes of weights $\widehat{\mathcal{D}}$ and $\mathcal{R}$. We shall go further and in the next result (and only there in these notes) ω is assumed to be a finite positive Borel measure on $[0, 1)$ and $\widehat{\omega}(z) = \int_{|z|}^1 d\omega(t)$ for all $z \in \mathbb{D}$. If there exists $C = C(\omega) > 0$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $r \in [0, 1)$, we denote $\omega \in \widehat{\mathcal{D}}$. We write $d(\omega \otimes m)(z) = d\theta\, r d\omega(r)/\pi$ for $z = re^{i\theta} \in \mathbb{D}$, and

$$\omega_x = \int_0^1 r^x\, d\omega(r), \quad x > -1.$$

For each $K > 1$, let $\rho_n = \rho_n(\omega, K)$ be the sequence defined by $\widehat{\omega}(\rho_n) = \widehat{\omega}(0)K^{-n}$.

The following characterizations of the class $\widehat{\mathcal{D}}$ will be frequently used from here on.

Lemma 2.1. *Let ω be a finite positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) \geq 1$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(iii) *There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that*

$$\int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma} d\omega(s) \leq C\widehat{\omega}(t), \quad 0 \leq t < 1;$$

(iv) *There exist constants $C_0 = C_0(\omega) > 0$ and $C = C(\omega) > 0$ such that*

$$\widehat{\omega}(0) \leq C_0 \widehat{\omega}\left(\frac{1}{2}\right) \tag{3}$$

and

$$\int_0^t s^{\frac{1}{1-t}}\, d\omega(s) \leq C\widehat{\omega}(t), \quad 0 \leq t < 1; \tag{4}$$

(v) *There exist constants $C_0 = C_0(\omega) > 0$ and $C = C(\omega) > 0$ such that (3) holds and*

$$\widehat{\omega}(r) \leq C r^{-\frac{1}{1-t}} \widehat{\omega}(t), \quad 0 \leq r \leq t < 1; \tag{5}$$

(vi) *Condition (3) and the asymptotic equality*

$$\int_0^1 s^x\, d\omega(s) \asymp \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty), \tag{6}$$

are valid;

(vii) *There exists $\lambda = \lambda(\omega) \geq 0$ such that*

$$\int_{\mathbb{D}} \frac{d(\omega \otimes m)(z)}{|1 - \overline{\zeta} z|^{\lambda+1}} \asymp \frac{\widehat{\omega}(\zeta)}{(1 - |\zeta|)^{\lambda}}, \quad \zeta \in \mathbb{D};$$

(viii) *Conditions* (3) *and* $\omega^\star(z) \asymp \widehat{\omega}(z)(1-|z|)$ *as* $|z| \geq \frac{1}{2}$, *hold. Here and on the following*

$$\omega^\star(z) = \int_{|z|}^1 \log \frac{s}{|z|} s\, d\omega(s), \quad z \in \mathbb{D} \setminus \{0\};$$

(ix) *Condition* (3) *holds and there exists* $C = C(\omega) > 0$ *such that* $\omega_n \leq C\omega_{2n}$ *for all* $n \in \mathbb{N}$*;*

(x) *Condition* (3) *holds and there exist* $C = C(\omega) > 0$ *and* $\eta = \eta(\omega) > 0$ *such that*

$$\omega_x \leq C\left(\frac{y}{x}\right)^\eta \omega_y, \quad 0 < x \leq y < \infty;$$

(xi) *There exist* $K = K(\omega) > 1$ *and* $C = C(\omega, K) > 1$ *such that* $1 - \rho_n(\omega, K) \geq C(1 - \rho_{n+1}(\omega, K))$ *for all* $n \in \mathbb{N} \cup \{0\}$*.*

Moreover, if $\omega \in \widehat{\mathcal{D}}$*, there exists* $C = C(\omega) > 0$ *such that*

$$\int_0^r \frac{dt}{\widehat{\omega}(t)(1-t)} \geq \frac{C}{\widehat{\omega}(r)}, \quad r \in \big[1/2, 1\big).$$

Before presenting the proof of Lemma 2.1, let us observe that condition (3) holds for any weight (absolutely continuous measure) such that $\omega > 0$ on an interval contained in $[1/2, 1)$, so it is not a real restriction for an admissible weight but a consequence of working in the general setting of positive Borel measures.

Proof of Lemma 2.1. We will prove (i)$\Leftrightarrow$(ii), (i)$\Leftrightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i), (iv)$\Leftrightarrow$(vi), (iii)$\Rightarrow$(vii)$\Rightarrow$(i)$\Leftrightarrow$(viii), and since (i) and (vi) together imply (ix), finally (ix)$\Rightarrow$(vi), (ix)$\Leftrightarrow$(x), and (ii)$\Leftrightarrow$(xi).

Let $\omega \in \widehat{\mathcal{D}}$. If $0 \leq r \leq t < 1$ and $r_n = 1 - 2^{-n}$ for all $n \in \mathbb{N} \cup \{0\}$, then there exist k and m such that $r_k \leq r < r_{k+1}$ and $r_m \leq t < r_{m+1}$. Therefore

$$\begin{aligned}
\widehat{\omega}(r) &\leq \widehat{\omega}(r_k) \leq C\widehat{\omega}(r_{k+1}) \leq \cdots \leq C^{m-k+1}\widehat{\omega}(r_{m+1}) \\
&\leq C^{m-k+1}\widehat{\omega}(t) = C^2 2^{(m-k-1)\log_2 C}\widehat{\omega}(t) \\
&\leq C^2 \left(\frac{1-r}{1-t}\right)^{\log_2 C} \widehat{\omega}(t), \quad 0 \leq r \leq t < 1,
\end{aligned}$$

and hence (ii) is satisfied. Since the choice $t = (1+r)/2$ in (ii) gives

$\widehat{\omega}(r) \leq C2^{\beta}\widehat{\omega}(\frac{1+r}{2})$ for all $r \in [0,1)$, we have shown that $\omega \in \widehat{\mathcal{D}}$ if and only if (ii) is satisfied.

Let $\omega \in \widehat{\mathcal{D}}$. If $0 \leq t < 1$ and $r_n = 1 - 2^{-n}$ for all $n \in \mathbb{N} \cup \{0\}$, then there exists m such that $r_m \leq t < r_{m+1}$. Therefore

$$
\begin{aligned}
&\int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma} d\omega(s) \\
&\quad \leq \int_0^{r_{m+1}} \left(\frac{1-t}{1-s}\right)^{\gamma} d\omega(s) = \sum_{n=0}^{m} \int_{r_n}^{r_{n+1}} \left(\frac{1-t}{1-s}\right)^{\gamma} d\omega(s) \\
&\quad \leq \sum_{n=0}^{m} \left(\frac{1-r_m}{1-r_{n+1}}\right)^{\gamma} \left(\widehat{\omega}(r_n) - \widehat{\omega}(r_{n+1})\right) \\
&\quad \leq \sum_{n=0}^{m} \frac{C}{2^{\gamma(m-n-1)}} \widehat{\omega}(r_{n+1}) \leq \widehat{\omega}(r_{m+1}) 2^{2\gamma} \sum_{n=0}^{m} \left(\frac{C}{2^{\gamma}}\right)^{m-n+1} \\
&\quad \leq \widehat{\omega}(t) 2^{2\gamma} \sum_{j=1}^{\infty} \left(\frac{C}{2^{\gamma}}\right)^{j},
\end{aligned}
$$

and we deduce (iii) for $\gamma = \gamma(\omega) > \frac{\log C}{\log 2}$. Conversely, if (iii) is satisfied and $0 \leq r \leq t < 1$, then

$$
\begin{aligned}
C\widehat{\omega}(t) &\geq \int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma} d\omega(s) \\
&= (1-t)^{\gamma} \int_0^t \left(\int_0^s \gamma(1-x)^{-\gamma-1}\, dx + 1\right) d\omega(s) \\
&= (1-t)^{\gamma}\gamma \int_0^t (1-x)^{-\gamma-1} \int_x^t d\omega(s)\, dx + (1-t)^{\gamma} \int_0^t d\omega(s) \\
&= (1-t)^{\gamma}\gamma \int_0^t (1-x)^{-\gamma-1} \left(\widehat{\omega}(x) - \widehat{\omega}(t)\right) dx \\
&\qquad + (1-t)^{\gamma} \int_0^t d\omega(s) \\
&\geq (1-t)^{\gamma}\gamma \int_0^r (1-x)^{-\gamma-1} \left(\widehat{\omega}(x) - \widehat{\omega}(t)\right) dx \\
&\qquad + (1-t)^{\gamma} \int_0^t d\omega(s)
\end{aligned}
$$

$$\begin{aligned}
&\geq (1-t)^\gamma \gamma \widehat{\omega}(r) \int_0^r (1-x)^{-\gamma-1}\,dx \\
&\qquad - (1-t)^\gamma \widehat{\omega}(t)\gamma \int_0^t (1-x)^{-\gamma-1}\,dx + (1-t)^\gamma \int_0^t \, d\omega(s) \\
&= \left(\frac{1-t}{1-r}\right)^\gamma \widehat{\omega}(r) - (1-t)^\gamma \widehat{\omega}(r) \\
&\qquad - \widehat{\omega}(t) + (1-t)^\gamma \widehat{\omega}(t) + (1-t)^\gamma \int_0^t \, d\omega(s) \\
&= \left(\frac{1-t}{1-r}\right)^\gamma \widehat{\omega}(r) - \widehat{\omega}(t) + (1-t)^\gamma (\widehat{\omega}(0) - \widehat{\omega}(r)) \\
&\geq \left(\frac{1-t}{1-r}\right)^\gamma \widehat{\omega}(r) - \widehat{\omega}(t), \quad 0 \leq r \leq t < 1.
\end{aligned}$$

Therefore (ii), and thus also (i), is valid.

The proof of [49, Lemma 1.3] shows that (iii) implies (iv). We include a proof for the sake of completeness. Condition (3) follows trivially from (i). A simple calculation shows that for all $s \in (0,1)$ and $x > 1$,

$$s^{x-1}(1-s)^\gamma \leq \left(\frac{x-1}{x-1+\gamma}\right)^{x-1} \left(\frac{\gamma}{x-1+\gamma}\right)^\gamma \leq \left(\frac{\gamma}{x-1+\gamma}\right)^\gamma.$$

Therefore (iii), with $t = 1 - \frac{1}{x}$, yields

$$\begin{aligned}
\int_0^{1-\frac{1}{x}} s^x \omega(s)\,ds &\leq \left(\frac{\gamma x}{x-1+\gamma}\right)^\gamma \int_0^{1-\frac{1}{x}} \frac{\omega(s)}{x^\gamma (1-s)^\gamma} s\,ds \\
&\lesssim \int_{1-\frac{1}{x}}^1 \omega(s)\,ds, \quad x > 1,
\end{aligned}$$

which gives (4). On the other hand, if (iv) is satisfied and $0 \leq r \leq t < 1$, then

$$\begin{aligned}
C\widehat{\omega}(t) &\geq \int_0^t s^{\frac{1}{1-t}} \omega(s)\,ds = \int_0^t \frac{x^{\frac{t}{1-t}}}{1-t} \int_x^t d\omega(s)\,dx \\
&= \int_0^t \frac{x^{\frac{t}{1-t}}}{1-t} \left(\widehat{\omega}(x) - \widehat{\omega}(t)\right)\,dx \\
&= \int_0^t \frac{x^{\frac{t}{1-t}}}{1-t} \widehat{\omega}(x)\,dx - \widehat{\omega}(t) \int_0^t \frac{x^{\frac{t}{1-t}}}{1-t}\,dx
\end{aligned}$$

$$\geq \int_0^r \frac{x^{\frac{t}{1-t}}}{1-t}\widehat{\omega}(x)\,dx - \widehat{\omega}(t)\int_0^t \frac{x^{\frac{t}{1-t}}}{1-t}\,dx$$
$$\geq \widehat{\omega}(r)\int_0^r \frac{x^{\frac{t}{1-t}}}{1-t}\,dx - \widehat{\omega}(t)\int_0^t \frac{x^{\frac{t}{1-t}}}{1-t}\,dx = \widehat{\omega}(r)r^{\frac{1}{1-t}} - \widehat{\omega}(t)t^{\frac{1}{1-t}},$$

and thus

$$r^{\frac{1}{1-t}}\widehat{\omega}(r) \leq \left(C + t^{\frac{1}{1-t}}\right)\widehat{\omega}(t), \quad 0 \leq r \leq t < 1,$$

which is (5). Now, by choosing $t = (1+r)/2$, (5) implies

$$\widehat{\omega}(r) \leq A^{-1}r^{\frac{2}{1-r}}\widehat{\omega}(r) \leq A^{-1}(C+1)\widehat{\omega}\left(\frac{1+r}{2}\right), \quad \frac{1}{2} \leq r < 1, \quad (7)$$

where $A = \min_{r\in[1/2,1)} r^{\frac{2}{1-r}} > 0$. Now, by combining (3) and (7) we deduce

$$\widehat{\omega}(s) \leq \widehat{\omega}(0) \leq C_1\widehat{\omega}\left(\frac{1}{2}\right) \lesssim \widehat{\omega}\left(\frac{3}{4}\right) \leq \widehat{\omega}\left(\frac{1+s}{2}\right), \quad 0 \leq s \leq \frac{1}{2},$$

which together with (7) gives $\omega \in \widehat{\mathcal{D}}$.

By integrating only from 0 to $1 - 1/x$ on the left of (6), we see that (vi)$\Rightarrow$(iv). Conversely, (iv) implies

$$\widehat{\omega}\left(1-\frac{1}{x}\right) \leq \widehat{\omega}(0) \leq C_1\widehat{\omega}\left(\frac{1}{2}\right) \leq 4C_1\int_{\frac{1}{2}}^1 s^2 d\omega(s) \leq 4C_1\int_0^1 s^2 d\omega(s)$$
$$\leq 4C_1\left(\int_0^{1-\frac{1}{x}} s^x\,d\omega(s) + \int_{1-\frac{1}{x}}^1 s^x\,d\omega(s)\right)$$
$$\lesssim \widehat{\omega}\left(1-\frac{1}{x}\right), \quad 1 \leq x \leq 2,$$

which gives (6) for $1 \leq x \leq 2$. Moreover, (iv) implies

$$\widehat{\omega}\left(1-\frac{1}{x}\right) \asymp \int_{1-\frac{1}{x}}^1 s^x\,d\omega(s) \leq \int_0^1 s^x\,d\omega(s)$$
$$= \int_0^{1-\frac{1}{x}} s^x\,d\omega(s) + \int_{1-\frac{1}{x}}^1 s^x\,d\omega(s)$$
$$\lesssim \widehat{\omega}\left(1-\frac{1}{x}\right) + \widehat{\omega}\left(1-\frac{1}{x}\right) \asymp \widehat{\omega}\left(1-\frac{1}{x}\right), \quad 2 \leq x < \infty,$$

and thus (vi) is satisfied.

Now, let us see (iii) implies (vii). If $|\zeta| \le \frac{1}{2}$, (vii) is equivalent to

$$\widehat{\omega}(0) \lesssim d(\omega \otimes m)(\mathbb{D}) = \int_0^1 s\, d\omega(s) \lesssim \widehat{\omega}(1/2), \tag{8}$$

which clearly follows from (i). Moreover,

$$\begin{aligned}\int_{\mathbb{D}} \frac{d(\omega \otimes m)(z)}{|1-\overline{\zeta}z|^{\lambda+1}} &\asymp \int_0^1 \frac{s\, d\omega(s)}{(1-|\zeta|s)^\lambda} = \left(\int_0^{|\zeta|} + \int_{|\zeta|}^1\right) \frac{s\, d\omega(s)}{(1-|\zeta|s)^\lambda} \\ &\asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda} + \int_0^{|\zeta|} \frac{s\, d\omega(s)}{(1-|\zeta|s)^\lambda}, \quad |\zeta| \ge \frac{1}{2},\end{aligned}$$

so by using (iii)

$$\begin{aligned}\frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda} &\le \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda} + \int_0^{|\zeta|} \frac{s\, d\omega(s)}{(1-|\zeta|s)^\lambda} \\ &\le \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda} + \int_0^{|\zeta|} \frac{d\omega(s)}{(1-s)^\lambda} \lesssim \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^\lambda}, \quad |\zeta| \ge \frac{1}{2},\end{aligned}$$

and hence (iii)$\Rightarrow$(vii). Assuming (vii), in particular we have (8), which implies

$$\widehat{\omega}(x) \le \widehat{\omega}(0) \asymp \widehat{\omega}(1/2) \le 2\int_{1/2}^1 s\, d\omega(s) \le 2\int_x^1 s\, d\omega(s), \quad 0 \le x \le \frac{1}{2}.$$

So

$$\widehat{\omega}(x) \asymp \int_x^1 s\, d\omega(s) = \widehat{\omega_1}(x), \quad 0 \le x < 1. \tag{9}$$

Moreover, for $0 < r \le t \in [1/2, 1)$, (vii) yields

$$\begin{aligned}\frac{\widehat{\omega}(t)}{(1-t)^\lambda} &\gtrsim \int_0^t \frac{s d\omega(s)}{(1-ts)^\lambda} = \int_0^t \left(\int_0^s \frac{\lambda t}{(1-tx)^{\lambda+1}} dx + 1\right) s d\omega(s) \\ &= \int_0^t \frac{\lambda t}{(1-tx)^{\lambda+1}} \left(\widehat{\omega_1}(x) - \widehat{\omega_1}(t)\right) dx + \int_0^t s d\omega(s) \\ &= \int_0^t \frac{\lambda t}{(1-tx)^{\lambda+1}} \widehat{\omega_1}(x)\, dx - \widehat{\omega_1}(t) \int_0^t \frac{\lambda t}{(1-tx)^{\lambda+1}}\, dx \\ &\quad + \int_0^t s d\omega(s)\end{aligned}$$

$$\begin{aligned}&\geq \widehat{\omega_1}(r)\int_0^r \frac{\lambda t}{(1-tx)^{\lambda+1}}\,dx-\frac{\widehat{\omega_1}(t)}{(1-t^2)^\lambda}+\widehat{\omega_1}(0)\\&\geq \widehat{\omega_1}(r)\frac{1}{(1-tr)^\lambda}-\frac{\widehat{\omega_1}(t)}{(1-t)^\lambda},\end{aligned}$$

and thus bearing in mind (9)

$$\widehat{\omega}(r)\lesssim \frac{(1-tr)^\lambda}{(1-t)^\lambda}\widehat{\omega}(t),\quad 0<r\leq t\in\big[1/2,1\big).$$

By choosing $t=(1+r)/2$ we deduce $\omega\in\widehat{\mathcal{D}}$.

The inequalities $1-t\leq -\log t\leq (1-t)/t$ show that $\omega^\star(r)\asymp \int_r^1(s-r)\,d\omega(s)$ for $r\geq\frac12$, and hence $\omega^\star(r)\lesssim\widehat{\omega}(r)(1-r)$ for all $r\geq\frac12$ and any ω. Moreover, if $\omega\in\widehat{\mathcal{D}}$, then

$$\omega^\star(r)\gtrsim\int_{\frac{1+r}{2}}^1(s-r)\,d\omega(s)\geq\left(\frac{1+r}{2}-r\right)\widehat{\omega}\left(\frac{1+r}{2}\right)\asymp\widehat{\omega}(r)(1-r),$$

and thus (i)⇒(viii). Conversely, assume that there exists a constant $C=C(\omega)>0$ such that

$$\widehat{\omega}(r)(1-r)\leq C\int_r^1(s-r)\,d\omega(s),\quad \frac12\leq r<1,$$

and let $r_p=\frac{p+r}{p+1}$, where $p>0$. Then

$$\begin{aligned}\widehat{\omega}(r)(1-r)&\leq C\int_r^{r_p}(s-r)\,d\omega(s)+C\int_{r_p}^1(s-r)\,d\omega(s)\\&\leq C\widehat{\omega}(r)\,(r_p-r)+C(1-r)\widehat{\omega}(r_p),\end{aligned}$$

and hence

$$\widehat{\omega}(r)\leq\frac{C(p+1)}{1+p-Cp}\widehat{\omega}(r_p),\quad\frac12\leq r<1.$$

If $C<2$ we may take $p=1$ and deduce $\omega\in\widehat{\mathcal{D}}$. For otherwise, fix $p>0$ sufficiently small and use the argument employed in the proof of (i)⇒(ii) together with $1-r_p=(1-r)/(1+p)\asymp 1-r$ to obtain

$$\widehat{\omega}(r)\lesssim\widehat{\omega}\left(\frac{1+r}{2}\right),\quad\frac12\leq r<1.$$

This together with (3), gives $\omega\in\widehat{\mathcal{D}}$. Thus (viii)⇒(i).

It is clear that (i) and (vi) together imply (ix). Conversely, assume (ix) is satisfied. Let $A = \sup_n \left(1 - \frac{1}{n+1}\right)^n$ and fix k large enough such that $C^k A^{2^k} < 1$. Then

$$\begin{aligned}\omega_n \le C\omega_{2n} \le C^k \omega_{2^k n} &= C^k \left(\int_0^{1-\frac{1}{n+1}} + \int_{1-\frac{1}{n+1}}^1 \right) r^{2^k n}\, d\omega(r) \\ &\le C^k A^{2^k} \omega_n + C^k \widehat{\omega}\left(1 - \frac{1}{n+1}\right), \quad n \in \mathbb{N},\end{aligned}$$

and hence

$$\omega_n \le \frac{C^k}{1 - C^k A^{2^k}} \widehat{\omega}\left(1 - \frac{1}{n+1}\right).$$

So, if $n \le x < n+1$, we deduce

$$\int_0^1 s^x\, d\omega(s) \le \omega_n \lesssim \widehat{\omega}\left(1 - \frac{1}{n+1}\right) \le \widehat{\omega}\left(1 - \frac{1}{x}\right),$$

and (vi) follows.

Assume now (ix) and let $1 \le x \le y < \infty$. Then there exist $n, m \in \mathbb{N} \cup \{0\}$ such that $n \le x \le n+1$ and $2^m n \le y \le 2^{m+1} n$. Then (ix) gives

$$\begin{aligned}\omega_x \le \omega_n \le C^{m+1} \omega_{2^{m+1} n} &\le 2^{(m+1)\log_2 C} \omega_y \\ &\le \left(\frac{2y}{n+1} \frac{n+1}{n} \right)^{\log_2 C} \omega_y \le C^2 \left(\frac{y}{x}\right)^{\log_2 C} \omega_y,\end{aligned}$$

and (x) follows. The choice $y = 2n = 2x$ gives (x)$\Rightarrow$(ix).

Assume there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $1 - \rho_n \ge C(1 - \rho_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Let $0 \le r \le t < 1$ and fix $n, k \in \mathbb{N} \cup \{0\}$ such that $\rho_n \le r < \rho_{n+1}$ and $\rho_k \le t < \rho_{k+1}$. Then

$$\begin{aligned}1 - r \ge 1 - \rho_{n+1} &\ge C(1 - \rho_{n+2}) \ge \cdots \ge C^{k-n-1}(1 - \rho_k) \\ &\ge C^{-2} \left(\frac{K^{-n}}{K^{-(k+1)}} \right)^{\log_K C} (1 - t) \ge C^{-2} \left(\frac{\widehat{\omega}(r)}{\widehat{\omega}(t)} \right)^{\log_K C} (1 - t),\end{aligned}$$

and hence

$$\widehat{\omega}(r) \le C^{\frac{2}{\log_K C}} \left(\frac{1 - r}{1 - t} \right)^{\frac{1}{\log_K C}} \widehat{\omega}(t), \quad 0 \le r \le t < 1,$$

and thus (ii) is satisfied. Conversely, by choosing $t = \rho_{n+1}$ and $r = \rho_n$

in (ii), we deduce $1-\rho_{n+1}\le\left(\frac{C}{K}\right)^{1/\beta}(1-\rho_n)$, and (xi) follows by choosing $K>C$.

Moreover, if $\omega\in\widehat{\mathcal{D}}$, there exists $C=C(\omega)>0$ such that

$$\begin{aligned}\int_0^r\frac{dt}{\widehat{\omega}(t)(1-t)}&\ge\int_{2r-1}^r\frac{dt}{\widehat{\omega}(t)(1-t)}\ge\frac{1}{\widehat{\omega}(2r-1)}\log 2\\&\ge\frac{\log 2}{C\widehat{\omega}(r)},\quad r\in\big[1/2,1\big).\end{aligned}$$

The proof of the lemma is now complete. □

Let $1<p_0,p_0'<\infty$ such that $1/p_0+1/p_0'=1$, and let $\eta>-1$. A weight $\omega:\mathbb{D}\to(0,\infty)$ satisfies the *Bekollé-Bonami* $B_{p_0}(\eta)$-*condition*, denoted by $\omega\in B_{p_0}(\eta)$, if there exists a constant $C=C(p_0,\eta,\omega)>0$ such that

$$\begin{aligned}&\left(\int_{S(I)}\omega(z)(1-|z|)^\eta\,dA(z)\right)\left(\int_{S(I)}\omega(z)^{\frac{-p_0'}{p_0}}(1-|z|)^\eta\,dA(z)\right)^{\frac{p_0}{p_0'}}\\&\qquad\le C|I|^{(2+\eta)p_0}\end{aligned}$$

for every interval $I\subset\mathbb{T}$. Bekollé and Bonami introduced these weights in [11, 12], and showed that $\frac{\omega(z)}{(1-|z|)^\eta}\in B_{p_0}(\eta)$ if and only if the Bergman projection

$$P_\eta(f)(z)=(\eta+1)\int_{\mathbb{D}}\frac{f(\xi)}{(1-\bar{\xi}z)^{2+\eta}}(1-|\xi|^2)^\eta\,dA(\xi)$$

is bounded from $L^{p_0}_\omega$ to $A^{p_0}_\omega$ [12].

The next lemma shows that a radial weight ω that satisfies (1) is regular if and only if it is a Bekollé-Bonami weight. Moreover, Part (iii) quantifies in a certain sense the self-improving integrability of radial weights.

Lemma 2.2.

(i) *If $\omega\in\mathcal{R}$, then for each $p_0>1$ there exists $\eta=\eta(p_0,\omega)>-1$ such that $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$.*

(ii) *If ω is a radial weight such that (1) is satisfied and $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$ for some $p_0>0$ and $\eta>-1$, then $\omega\in\mathcal{R}$.*

(iii) *For each radial weight* ω *and* $0<\alpha<1$, *define*

$$\widetilde{\omega}(r)=\left(\int_r^1 \omega(s)\,ds\right)^{-\alpha}\omega(r),\quad 0\le r<1.$$

Then $\widetilde{\omega}$ *is also a weight and*

$$\frac{\int_r^1 \widetilde{\omega}(s)\,ds}{(1-r)\widetilde{\omega}(r)}=\frac{1}{1-\alpha}\frac{\int_r^1 \omega(s)\,ds}{(1-r)\omega(r)},\quad 0\le r<1.$$

Proof. (i) Since each regular weight is radial, it suffices to show that there exists a constant $C=C(p,\eta,\omega)>0$ such that

$$\left(\int_{1-|I|}^1 \omega(t)\,dt\right)\left(\int_{1-|I|}^1 \omega(t)^{\frac{-p_0'}{p_0}}(1-t)^{p_0'\eta}\,dt\right)^{\frac{p_0}{p_0'}}\le C|I|^{(1+\eta)p_0}\quad (10)$$

for every interval $I\subset\mathbb{T}$. To prove (10), set

$$s_0=1-|I|,\quad s_{n+1}=s_n+s(1-s_n),$$

where $s\in(0,1)$ is fixed. Take p_0 and η such that $\eta>\frac{\log C}{p_0\log\frac{1}{1-s}}>0$, where the constant $C=C(s,\omega)>1$ is from (1). Then (1) yields

$$\begin{aligned}\int_{1-|I|}^1 \omega(t)^{\frac{-p_0'}{p_0}}(1-t)^{p_0'\eta}\,dt &\le \sum_{n=0}^{\infty}(1-s_n)^{p_0'\eta}\int_{s_n}^{s_{n+1}}\omega(t)^{\frac{-p_0'}{p_0}}\,dt\\ &\le C^{\frac{p_0'}{p_0}}\sum_{n=0}^{\infty}(1-s_n)^{p_0'\eta+1}\omega(s_n)^{\frac{-p_0'}{p_0}}\\ &\le |I|^{p_0'\eta+1}\omega(1-|I|)^{\frac{-p_0'}{p_0}}\\ &\qquad\cdot\sum_{n=0}^{\infty}(1-s)^{n(p_0'\eta+1)}C^{(n+1)\frac{p_0'}{p_0}}\\ &=C(p_0,\eta,s,\omega)|I|^{p_0'\eta+1}\omega(1-|I|)^{\frac{-p_0'}{p_0}},\end{aligned}$$

which together with (2) gives (10).

(ii) The asymptotic inequality $\frac{\int_r^1\omega(s)\,ds}{\omega(r)}\lesssim(1-r)$ follows by (10) and further appropriately modifying the argument in the proof of (i). Since the assumption (1) gives $\frac{\int_r^1\omega(s)\,ds}{\omega(r)}\gtrsim(1-r)$, we deduce $\omega\in\mathcal{R}$.

(iii) If $0 \leq r < t < 1$, then an integration by parts yields

$$\int_r^t \frac{\omega(s)}{\left(\int_s^1 \omega(v)\,dv\right)^\alpha}\,ds = \left(\int_r^1 \omega(v)\,dv\right)^{1-\alpha} - \left(\int_t^1 \omega(v)\,dv\right)^{1-\alpha} + \alpha \int_r^t \frac{\omega(s)}{\left(\int_s^1 \omega(v)\,dv\right)^\alpha}\,ds,$$

from which the assertion follows by letting $t \to 1^-$. □

3. CARLESON MEASURES

For a given Banach space (or a complete metric space) X of analytic functions on $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a *q-Carleson measure for X* if the identity operator $I_d : X \to L^q(\mu)$ is bounded. We shall obtain a description of q-Carleson measures for the weighted Bergman space A^p_ω, $\omega \in \widehat{\mathcal{D}}$. We shall offer a detailed proof for the case $q \geq p$ which differs from that in [50] and follows the lines of [49, Chapter 2].

3.1. Test functions and the weighted maximal function

The next result follows from Lemma 2.1(vii) and its proof.

Lemma 3.1. *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Then there is $\lambda_0(\omega)$ such that for any $\lambda \geq \lambda_0$ and each $a \in \mathbb{D}$ the function $F_{a,p}(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\frac{\lambda+1}{p}}$ is analytic in $\mathbb{D}$ and satisfies*

$$|F_{a,p}(z)| \asymp 1, \quad z \in S(a), \quad a \in \mathbb{D}, \tag{11}$$

and

$$\|F_{a,p}\|^p_{A^p_\omega} \asymp \omega\left(S(a)\right), \quad a \in \mathbb{D}. \tag{12}$$

It is known that q-Carleson measures for $\omega \in \mathcal{R}$ can be characterized either in terms of Carleson squares or pseudohyperbolic discs [21]. However, this is no longer true when $\omega \in \widehat{\mathcal{D}}$. So, we shall use tools from harmonic analysis.

Let us consider the maximal function

$$M_\omega(\varphi)(z) = \sup_{I:\, z\in S(I)} \frac{1}{\omega\left(S(I)\right)} \int_{S(I)} |\varphi(\xi)|\omega(\xi)\,dA(\xi), \quad z \in \mathbb{D},$$

introduced by Hörmander [34]. Here we must require $\varphi \in L^1_\omega$ and that $\varphi(re^{i\theta})$ is 2π-periodic with respect to θ for all $r \in (0,1)$. The function $M_\omega(\varphi)$ plays a role on A^p_ω similar to that of the Hardy-Littlewood maximal function on the Hardy space H^p.

Now, we are going to get a pointwise control of $|f|$ in terms of $M_\omega(|f|)$.

Lemma 3.2. *Let $0 < s < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Then there exists a constant $C = C(s,\omega) > 0$ such that*

$$|f(z)|^s \le C M_\omega(f^s)(z), \quad z \in \mathbb{D}, \tag{13}$$

for all $f \in \mathcal{H}(\mathbb{D})$.

Proof. Let $\omega \in \widehat{\mathcal{D}}$ and let $C = C(\omega) \ge 1$ and $\beta = \beta(\omega) > 0$ be those of Lemma 2.1(ii). Write $s = \alpha\gamma$, where $\gamma > \beta + 1 + \log_2 C > 1$. It suffices to prove the assertion for the points $re^{i\theta} \in \mathbb{D}$ with $r > \frac{1}{2}$. If $r < \rho < 1$, then using that $|f|^\alpha$ is subharmonic and Hölder's inequality

$$\begin{aligned}
|f(re^{i\theta})|^\alpha &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-(\frac{r}{\rho})^2}{|1-\frac{r}{\rho}e^{it}|^2} |f(\rho e^{i(t+\theta)})|^\alpha \, dt \\
&\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(1-(\frac{r}{\rho})^2\right)^{\gamma-1}}{|1-\frac{r}{\rho}e^{it}|^\gamma} |f(\rho e^{i(t+\theta)})|^{\alpha\gamma} \, dt \right)^{1/\gamma} \\
&\quad \cdot \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(1-(\frac{r}{\rho})^2\right)^{\gamma'-1}}{|1-\frac{r}{\rho}e^{it}|^{\gamma'}} \, dt \right)^{1/\gamma'},
\end{aligned}$$

that is

$$\begin{aligned}
|f(re^{i\theta})|^s &\le C(\omega,s)\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(1-(\frac{r}{\rho})^2\right)^{\gamma-1}}{|1-\frac{r}{\rho}e^{it}|^\gamma} |f(\rho e^{i(t+\theta)})|^s \, dt \\
&= C(\omega,s)\int_{-\pi}^{\pi} P_\gamma\left(\frac{r}{\rho}, t\right) |f(\rho e^{i(t+\theta)})|^s \, dt,
\end{aligned}$$

where

$$P_\gamma(r,t) = \frac{1}{2\pi}\frac{(1-r^2)^{\gamma-1}}{|1-re^{it}|^\gamma}, \quad 0 < r < 1.$$

Set $t_n = 2^{n-1}(1-r)$ and $J_n = [-t_n, t_n]$ for $n = 0, 1, \ldots, N+1$, where N is the largest natural number such that $t_N < \frac{1}{2}$. Further, set $G_0 = J_0$, $G_n = J_n \setminus J_{n-1}$ for $n = 1, \ldots, N$, and $G_{N+1} = [-\pi, \pi] \setminus J_N$. Then

$$\begin{aligned}
|f(re^{i\theta})|^s &\leq \sum_{n=0}^{N+1} \int_{G_n} P_\gamma\left(\frac{r}{\rho}, t\right) |f(\rho e^{i(t+\theta)})|^s dt \\
&\leq \sum_{n=0}^{N+1} P_\gamma\left(\frac{r}{\rho}, t_{n-1}\right) \int_{G_n} |f(\rho e^{i(t+\theta)})|^s dt \\
&\lesssim \frac{1}{1-\frac{r}{\rho}} \sum_{n=0}^{N+1} 2^{-n\gamma} \int_{G_n} |f(\rho e^{i(t+\theta)})|^s dt,
\end{aligned}$$

and therefore

$$\begin{aligned}
&|f(re^{i\theta})|^s (1-r) \int_{(1+r)/2}^{1} \omega(\rho)\rho\, d\rho \\
&\quad \leq 2 \int_r^1 |f(re^{i\theta})|^s (\rho - r)\omega(\rho)\rho\, d\rho \\
&\quad \lesssim \sum_{n=0}^{N+1} 2^{-n\gamma} \int_r^1 \int_{G_n} \left| f\left(\rho e^{i(t+\theta)}\right)\right|^s dt\, \omega(\rho)\rho^2\, d\rho.
\end{aligned}$$

It follows that

$$\begin{aligned}
|f(re^{i\theta})|^s &\lesssim \sum_{n=0}^{N} 2^{-n(\gamma-1)} \frac{\int_r^1 \int_{-t_n}^{t_n} \left| f\left(\rho e^{i(t+\theta)}\right)\right|^s dt\, \omega(\rho)\rho\, d\rho}{\int_{-t_n}^{t_n} \int_{(1+r)/2}^{1} \omega(\rho)\rho\, d\rho\, dt} \\
&\quad + 2^{-N(\gamma-1)} \frac{\int_r^1 \int_{-\pi}^{\pi} \left| f\left(\rho e^{it}\right)\right|^s dt \omega(\rho)\rho\, d\rho}{\int_{-\pi}^{\pi} \int_{(1+r)/2}^{1} \omega(\rho)\rho d\rho\, dt} \\
&\lesssim \sum_{n=0}^{N} 2^{-n(\gamma-1)} \frac{\int_{1-t_{n+1}}^1 \int_{-t_n}^{t_n} \left| f\left(\rho e^{i(t+\theta)}\right)\right|^s dt\, \omega(\rho)\rho\, d\rho}{\int_{-t_n}^{t_n} \int_{(1+r)/2}^{1} \omega(\rho)\rho\, d\rho\, dt} \\
&\quad + 2^{-N(\gamma-1)} \frac{\int_0^1 \int_{-\pi}^{\pi} \left| f\left(\rho e^{it}\right)\right|^s dt \omega(\rho)\rho\, d\rho}{\int_{-\pi}^{\pi} \int_{(1+r)/2}^{1} \omega(\rho)\rho d\rho\, dt},
\end{aligned}$$

where the last step is a consequence of the inequalities $0 < 1 - t_{n+1} \leq r$. Denoting the interval centered at $e^{i\theta}$ and of the same length as J_n by $J_n(\theta)$, and applying Lemma 2.1(ii), to the denominators, we obtain

$$
\begin{aligned}
|f(re^{i\theta})|^s &\lesssim \sum_{n=0}^{N} C^n 2^{-n(\gamma-1-\beta)} \frac{\int_{S(J_n(\theta))} |f(z)|^s \,\omega(z)\, dA(z)}{\omega(S(J_n(\theta)))} \\
&\quad + C^N 2^{-N(\gamma-1-\beta)} \frac{\int_{\mathbb{D}} |f(z)|^s \omega(z)\, dA(z)}{\omega(\mathbb{D})} \\
&\lesssim \left(\sum_{n=0}^{\infty} 2^{-n(\gamma-1-\beta-\log_2 C)} \right) M_\omega(|f|^s)(re^{i\theta}) \\
&\lesssim M_\omega(|f|^s)(re^{i\theta}),
\end{aligned}
$$

where in the last inequality we have used the election of γ. This finishes the proof. □

3.2. Carleson measures. Case $\mathbf{0 < p \leq q < \infty}$.

Next, we prove our main result in this section, by combining a weak $(1,1)$ inequality for the maximal function with the pointwise estimate (13).

Theorem 3.3. *Let $0 < p \leq q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and let μ be a positive Borel measure on $\mathbb{D}$. Then μ is a q-Carleson measure for A^p_ω if and only if*

$$
\sup_{I \subset \mathbb{T}} \frac{\mu\left(S(I)\right)}{\left(\omega\left(S(I)\right)\right)^{\frac{q}{p}}} < \infty. \tag{14}
$$

Moreover, if μ is a q-Carleson measure for A^p_ω, then the identity operator $I_d : A^p_\omega \to L^q(\mu)$ satisfies

$$
\|I_d\|^q_{\left(A^p_\omega, L^q(\mu)\right)} \asymp \sup_{I \subset \mathbb{T}} \frac{\mu\left(S(I)\right)}{\left(\omega\left(S(I)\right)\right)^{\frac{q}{p}}}.
$$

Proof. Let $0 < p \leq q < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and assume first that μ is a q-Carleson measure for A^p_ω. Consider the test functions $F_{a,p}$ defined in Lemma 3.1. Then the assumption together with relations (11) and (12) yield

$$
\begin{aligned}
\mu(S(a)) &\lesssim \int_{S(a)} |F_{a,p}(z)|^q \, d\mu(z) \leq \int_{\mathbb{D}} |F_{a,p}(z)|^q \, d\mu(z) \\
&\lesssim \|F_{a,p}\|^q_{A^p_\omega} \lesssim \omega\left(S(a)\right)^{\frac{q}{p}}
\end{aligned}
$$

for all $a \in \mathbb{D}$, and thus μ satisfies (14).

Conversely, let μ be a positive Borel measure on $\mathbb{D}$ such that (14) is satisfied. We begin with proving that there exists a constant $K = K(p,q,\omega) > 0$ such that the L^1_ω-weak type inequality

$$\mu\left(E_s\right) \le K s^{-\frac{q}{p}} \|\varphi\|_{L^1_\omega}^{\frac{q}{p}}, \quad E_s = \big\{z \in \mathbb{D} : M_\omega(\varphi)(z) > s\big\}, \tag{15}$$

is valid for all $\varphi \in L^1_\omega$ and $0 < s < \infty$.

If $E_s = \emptyset$, then (15) is clearly satisfied. If $E_s \neq \emptyset$, then recall that $I_z = \big\{e^{i\theta} : |\arg(ze^{-i\theta})| < (1-|z|)/2\big\}$ and $S(z) = S(I_z)$, and define for each $\varepsilon > 0$ the sets

$$A_s^\varepsilon = \left\{z \in \mathbb{D} : \int_{S(I_z)} |\varphi(\xi)|\omega(\xi)\, dA(\xi) > s\left(\varepsilon + \omega(S(z))\right)\right\}$$

and

$$B_s^\varepsilon = \big\{z \in \mathbb{D} :\, I_z \subset I_u \text{ for some } u \in A_s^\varepsilon\big\}.$$

The sets B_s^ε expand as $\varepsilon \to 0^+$, and

$$E_s = \big\{z \in \mathbb{D} : M_\omega(\varphi)(z) > s\big\} = \bigcup_{\varepsilon>0} B_s^\varepsilon,$$

so

$$\mu(E_s) = \lim_{\varepsilon\to 0^+} \mu(B_s^\varepsilon). \tag{16}$$

We notice that for each $\varepsilon > 0$ and $s > 0$ there are finitely many points $z_n \in A_s^\varepsilon$ such that the arcs I_{z_n} are disjoint. Namely, if there were infinitely many points $z_n \in A_s^\varepsilon$ with this property, then the definition of A_s^ε would yield

$$s\sum_n [\varepsilon + \omega(S(z))] \le \sum_n \int_{S(I_{z_n})} |\varphi(\xi)|\omega(\xi)\, dA(\xi) \le \|\varphi\|_{L^1_\omega}, \tag{17}$$

and therefore $\infty = s\sum_n \varepsilon \le \|\varphi\|_{L^1_\omega}$, which is impossible because $\varphi \in L^1_\omega$.

We now use Covering lemma [25, p. 161] to find $z_1, \dots, z_m \in A_s^\varepsilon$ such that the arcs I_{z_n} are disjoint and

$$A_s^\varepsilon \subset \bigcup_{n=1}^m \{z : I_z \subset J_{z_n}\},$$

where J_z is the arc centered at the same point as I_z and of length $5|I_z|$.

It follows easily that

$$B_s^\varepsilon \subset \bigcup_{n=1}^{m} \{z : I_z \subset J_{z_n}\}. \tag{18}$$

But now the assumption (14) and the hypothesis $\omega \in \widehat{\mathcal{D}}$ give

$$\begin{aligned}\mu\left(\{z : I_z \subset J_{z_n}\}\right) &= \mu\left(\{z : S(z) \subset S(J_{z_n})\}\right) \le \mu\left(S(J_{z_n})\right) \\ &\lesssim \left(\omega\left(S(J_{z_n})\right)\right)^{\frac{q}{p}} \lesssim \left(\omega\left(S(z_n)\right)\right)^{\frac{q}{p}}, \quad n = 1, \dots, m.\end{aligned}$$

This combined with (18) and (17) yields

$$\mu(B_s^\varepsilon) \lesssim \sum_{n=1}^{m} \left(\omega\left(S(z_n)\right)\right)^{\frac{q}{p}} \le \left(\sum_{n=1}^{m} \omega\left(S(z_n)\right)\right)^{\frac{q}{p}} \le s^{-\frac{q}{p}} \|\varphi\|_{L^1_\omega}^{\frac{q}{p}},$$

which together with (16) gives (15) for some $K = K(p, q, \omega)$.

We will now use Lemma 3.2 and (15) to show that μ is a q-Carleson measure for A^p_ω. To do this, fix $\alpha > 1/p$ and let $f \in A^p_\omega$. For $s > 0$, let $|f|^{1/\alpha} = \psi_{\frac{1}{\alpha},s} + \chi_{\frac{1}{\alpha},s}$, where

$$\psi_{\frac{1}{\alpha},s}(z) = \begin{cases} |f(z)|^{\frac{1}{\alpha}}, & \text{if } |f(z)|^{\frac{1}{\alpha}} > s/(2K), \\ 0, & \text{otherwise,} \end{cases}$$

and K is the constant in (15), chosen such that $K \ge 1$. Since $p > \frac{1}{\alpha}$, the function $\psi_{\frac{1}{\alpha},s}$ belongs to L^1_ω for all $s > 0$. Moreover,

$$M_\omega(|f|^{\frac{1}{\alpha}}) \le M_\omega(\psi_{\frac{1}{\alpha},s}) + M_\omega(\chi_{\frac{1}{\alpha},s}) \le M_\omega(\psi_{\frac{1}{\alpha},s}) + \frac{s}{2K},$$

and therefore

$$\left\{z \in \mathbb{D} : M_\omega(|f|^{\frac{1}{\alpha}})(z) > s\right\} \subset \left\{z \in \mathbb{D} : M_\omega(\psi_{\frac{1}{\alpha},s})(z) > s/2\right\}. \tag{19}$$

Using Lemma 3.2, the inclusion (19), (15) and Minkowski's inequality in continuous form (Fubini in the case $q = p$), we finally deduce

$$\begin{aligned}&\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \\ &\quad\lesssim \int_{\mathbb{D}} \left(M_\omega(|f|^{\frac{1}{\alpha}})(z)\right)^{q\alpha} \, d\mu(z) \\ &\quad= q\alpha \int_0^\infty s^{q\alpha - 1} \mu\left(\left\{z \in \mathbb{D} : M_\omega(|f|^{\frac{1}{\alpha}})(z) > s\right\}\right) \, ds\end{aligned}$$

$$
\begin{aligned}
&\leq q\alpha \int_0^\infty s^{q\alpha-1}\mu\left(\left\{z\in\mathbb{D}: M_\omega(\psi_{\frac{1}{\alpha},s})(z)>s/2\right\}\right)ds\\
&\lesssim \int_0^\infty s^{q\alpha-1-\frac{q}{p}}\|\psi_{\frac{1}{\alpha},s}\|_{L^1_\omega}^{\frac{q}{p}}\,ds\\
&= \int_0^\infty s^{q\alpha-1-\frac{q}{p}}\left(\int_{\{z:\,|f(z)|^{1/\alpha}>\frac{s}{2K}\}}|f(z)|^{\frac{1}{\alpha}}\omega(z)\,dA(z)\right)^{\frac{q}{p}}ds\\
&\leq \left(\int_{\mathbb{D}}|f(z)|^{\frac{1}{\alpha}}\omega(z)\left(\int_0^{2K|f(z)|^{\frac{1}{\alpha}}}s^{q\alpha-1-\frac{q}{p}}\,ds\right)^{\frac{p}{q}}dA(z)\right)^{\frac{q}{p}}\\
&\lesssim \left(\int_{\mathbb{D}}|f(z)|^p\omega(z)\,dA(z)\right)^{\frac{q}{p}}.
\end{aligned}
$$

Therefore μ is a q-Carleson measure for A^p_ω, and the proof of Theorem 3.3(i) is complete. □

The next useful result follows from the proof of Theorem 3.3.

Theorem 3.4. *Let $0<p\leq q<\infty$ and $0<\alpha<\infty$ such that $p\alpha>1$. Let $\omega\in\widehat{\mathcal{D}}$, and let μ be a positive Borel measure on $\mathbb{D}$. Then $[M_\omega((\cdot)^{\frac{1}{\alpha}})]^\alpha : L^p_\omega\to L^q(\mu)$ is bounded if and only if μ satisfies* (14). *Moreover,*

$$
\|[M_\omega((\cdot)^{\frac{1}{\alpha}})]^\alpha\|^q_{\left(L^p_\omega,L^q(\mu)\right)}\asymp \sup_{I\subset\mathbb{T}}\frac{\mu\left(S(I)\right)}{\left(\omega\left(S(I)\right)\right)^{\frac{q}{p}}}.
$$

Before presenting a description of q-Carleson measures for A^p_ω, where $\omega\in\widehat{\mathcal{D}}$ and $q<p$, we shall obtain several equivalent A^p_ω-norms which are useful to study this problem and some other questions throughout the manuscript.

3.3. Equivalent norms on A^p_ω

A description of A^p_ω in terms of the maximal function follows from Lemma 3.2 and Theorem 3.4.

Corollary 3.5. *Let $0<p<\infty$ and $0<\alpha<\infty$ such that $p\alpha>1$. Let $\omega\in\widehat{\mathcal{D}}$. Then,*

$$
\|f\|^p_{A^p_\omega}\asymp\|[M_\omega((f)^{\frac{1}{\alpha}})]^\alpha\|^p_{L^p_\omega},\quad f\in\mathcal{H}(\mathbb{D}).
$$

It is well-known that a choice of an appropriate norm is often a key step when solving a problem on a space of analytic functions. For instance, in the study of the integration operator

$$T_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \quad z \in \mathbb{D}, \quad g \in \mathcal{H}(\mathbb{D}),$$

one wants to get rid of the integral symbol, so one looks for norms in terms of the first derivative. The first known result in this area was proved by Hardy and Littlewood for the standard weights [63].

Theorem A. *If* $0 < p < \infty$ *and* $\alpha > -1$*, then*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha \, dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p+\alpha} \, dA(z)$$

for all $f \in \mathcal{H}(\mathbb{D})$*.*

Later, this Littlewood-Paley type formula was extended to the following class of weights [48], which includes any differentiable decreasing weight and all the standard ones. See also [5, 23, 61] for previous and further results. The distortion function of a radial weight ω is

$$\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(s)\, ds, \quad 0 \le r < 1.$$

It was introduced by Siskakis [61].

Theorem 3.6. *Let* $0 < p < \infty$ *and let* ω *be a differentiable radial weight. If*

$$\sup_{0<r<1} \frac{\omega'(r)}{\omega^2(r)} \int_r^1 \omega(s)\, ds < \infty,$$

then

$$\|f\|_{A^p_\omega}^p \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \psi_\omega^p(|z|) \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}).$$

See also [3] for a Littlewood-Paley type formula for $\|\cdot\|_{A^p_\omega}$-norm, where ω is a Bekollé-Bonami weight. However, an analogue of Theorem 3.6 does not exist if $\omega \in \widehat{\mathcal{D}}$ and $p \neq 2$.

Proposition 3.7. *Let $p \neq 2$. Then there exists $\omega \in \widehat{\mathcal{D}}$ such that, for any function $\varphi : [0,1) \to (0,\infty)$, the relation*

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f'(z)|^p \varphi(|z|)^p \omega(z)\, dA(z) + |f(0)|^p \tag{20}$$

can not be valid for all $f \in \mathcal{H}(\mathbb{D})$.

Proof. Let first $p > 2$ and consider the weight function $v_\alpha(r) = (1-r)^{-1}\big(\log \frac{e}{1-r}\big)^{-\alpha}$, where $\alpha > 1$ is fixed such that $2 < 2(\alpha - 1) \leq p$. Assume on the contrary to the assertion that (20) is satisfied for all $f \in \mathcal{H}(\mathbb{D})$. Applying this relation to the function $h_n(z) = z^n$, we obtain

$$\int_0^1 r^{np} v_\alpha(r)\, dr \asymp n^p \int_0^1 r^{(n-1)p} \varphi(r)^p v_\alpha(r)\, dr, \quad n \in \mathbb{N}. \tag{21}$$

Consider now the lacunary series $h(z) = \sum_{k=0}^\infty z^{2^k}$. It is easy to see that

$$M_p(r,h) \asymp \left(\log \frac{1}{1-r}\right)^{1/2}, \quad M_p(r,h') \asymp \frac{1}{1-r}, \quad 0 \leq r < 1. \tag{22}$$

By combining the relations (21), (22) and

$$\left(\frac{1}{1-r^p}\right)^p \asymp \sum_{n=1}^\infty n^{p-1} r^{(n-1)p}, \; \log \frac{1}{1-r^p} \asymp \sum_{n=1}^\infty n^{-1} r^{np}, \quad 0 \leq r < 1,$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |h'(z)|^p \varphi(z)^p v_\alpha(z)\, dA(z) \\
&\quad \asymp \int_0^1 \left(\frac{1}{1-r^p}\right)^p \varphi(r)^p v_\alpha(r)\, dr \\
&\quad \asymp \int_0^1 \left(\sum_{n=1}^\infty n^{p-1} r^{(n-1)p}\right) \varphi(r)^p v_\alpha(r)\, dr \\
&\quad \asymp \sum_{n=1}^\infty n^{p-1} \int_0^1 r^{(n-1)p} \varphi(r)^p v_\alpha(r)\, dr \asymp \sum_{n=1}^\infty n^{-1} \int_0^1 r^{np} v_\alpha(r)\, dr \\
&\quad \asymp \int_0^1 \left(\sum_{n=1}^\infty n^{-1} r^{np}\right) v_\alpha(r)\, dr \asymp \int_0^1 \log \frac{1}{1-r^p}\, v_\alpha(r)\, dr,
\end{aligned}$$

where the last integral is convergent because $\alpha > 2$. However,

$$\|h\|_{A^p_{v_\alpha}}^p \asymp \int_0^1 \left(\log \frac{1}{1-r}\right)^{p/2} v_\alpha(r)\,dr = \infty,$$

since $p \geq 2(\alpha - 1)$, and therefore (20) fails for $h \in \mathcal{H}(\mathbb{D})$. This is the desired contradiction.

If $0 < p < 2$, we again consider v_α, where α is chosen such that $p < 2(\alpha - 1) \leq 2$, and use an analogous reasoning to that above to prove the assertion. Details are omitted. □

Because of the above result we look for other equivalent norms to $\|\cdot\|_{A^p_\omega}$ in terms (or involving) the derivative. In fact, applying the Hardy-Stein-Spencer identity [28]

$$\|f\|_{H^p}^p = \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|}\,dA(z) + |f(0)|^p,$$

to the dilated functions $f_r(z) = f(rz)$, $0 < r < 1$, and integrating with respect to $r\omega(r)\,dr$ we obtain such equivalent norm.

Theorem 3.8. *Let $0 < p < \infty$, $n \in \mathbb{N}$ and $f \in \mathcal{H}(\mathbb{D})$, and let ω be a radial weight. Then*

$$\|f\|_{A^p_\omega}^p = p^2 \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \omega^\star(z)\,dA(z) + \omega(\mathbb{D})|f(0)|^p, \qquad (23)$$

where

$$\omega^\star(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s\,ds, \qquad z \in \mathbb{D} \setminus \{0\}.$$

In particular,

$$\|f\|_{A^2_\omega}^2 = 4\|f'\|_{A^2_{\omega^\star}}^2 + \omega(\mathbb{D})|f(0)|^2. \qquad (24)$$

Fefferman and Stein [27] obtained the following extension of the classical Littlewood-Paley formula for H^2

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{T}} \left(\int_{\Gamma(e^{i\theta})} |f'(z)|^2\,dA(z)\right)^{p/2} d\theta + |f(0)|^p,$$

where

$$\Gamma(e^{i\theta}) = \left\{z \in \mathbb{D} : |\theta - \arg z| < \frac{1}{2}(1 - |z|)\right\}, \quad u = e^{i\theta} \in \mathbb{T}.$$

Usually the function $e^{i\theta} \mapsto \left(\int_{\Gamma(e^{i\theta})} |f'(z)|^2 \, dA(z) \right)^{1/2}$ is called the square (Lusin) area function.

In order to get an extension of this result to weighted Bergman spaces, we need to define tangential lens type regions

$$\Gamma(u) = \left\{ z \in \mathbb{D} : \, |\theta - \arg z| < \frac{1}{2}\left(1 - \frac{|z|}{r}\right) \right\}, \quad u = re^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\},$$

induced by points in $\mathbb{D}$, and the tents

$$T(z) = \left\{ u \in \mathbb{D} : \, z \in \Gamma(u) \right\}, \quad z \in \mathbb{D},$$

which are closely interrelated. By the same method used in the proof of Theorem 3.8, we get the following result.

Theorem 3.9. *Let $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$, and let ω be a radial weight. Then*

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |f'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, dA(u) + |f(0)|^p, \qquad (25)$$

where the constants of comparison depend only on p and ω.

It is worth mentioning that $\omega^\star$ is smoother than ω. In fact,

$$\omega(T(z)) \asymp \omega^\star(z), \quad |z| \geq \frac{1}{2}.$$

So, bearing in mind Lemma 2.1,

$$\omega^\star(z) \asymp \omega\left(T(z)\right) \asymp \omega\left(S(z)\right), \quad z \in \mathbb{D}, \quad \omega \in \widehat{\mathcal{D}}. \qquad (26)$$

Before ending this section, for a function f defined in $\mathbb{D}$, we consider the non-tangential maximal function of f in the (punctured) unit disc by

$$N(f)(u) = \sup_{z \in \Gamma(u)} |f(z)|, \quad u \in \mathbb{D} \setminus \{0\}.$$

Lemma 3.10. *Let $0 < p < \infty$ and let ω be a radial weight. Then there exists a constant $C > 0$ such that*

$$\|f\|_{A^p_\omega}^p \leq \|N(f)\|_{L^p_\omega}^p \leq C\|f\|_{A^p_\omega}^p$$

for all $f \in \mathcal{H}(\mathbb{D})$.

A proof can be obtained by dilating and integrating the well-known inequality [28, Theorem 3.1 on p. 57]

$$\|f^\star\|^p_{L^p(\mathbb{T})} \le C\|f\|^p_{H^p}$$

respect to ω. Here, and on the sequel, $f^\star(\zeta) = \sup_{z\in\Gamma(\zeta)}|f(z)|$ for $\zeta \in \mathbb{T}$.

3.4. Carleson measures. Case $0 < q < p < \infty$.

For several classes of weights, q-Carleson measures for A^p_ω [21, 42, 46] have been described, in the triangular case $p > q$, by using an atomic decomposition theorem in the sense of standard Bergman spaces [55, Theorem 2.2]. However, this approach does not seem to be adequate for the class $\widehat{\mathcal{D}}$. A sufficient condition can be easily obtained.

Proposition 3.11. *Let $0 < q < p < \infty$, ω a radial weight and μ be a positive Borel measure on $\mathbb{D}$. If*

$$B_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L^{\frac{p}{p-q}}_\omega$, then μ is a q-Carleson measure for A^p_ω.

Proof. Fubini's theorem, Hölder's inequality and Lemma 3.10 yield

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^q\, d\mu(z) &= \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} \frac{|f(z)|^q\, d\mu(z)}{\omega(T(z))}\right)\omega(\zeta)\, dA(\zeta)\\
&\le \int_{\mathbb{D}} (N(f)(\zeta))^q B_\mu(\zeta)\omega(\zeta)\, dA(\zeta)\\
&\le \|N(f)\|^q_{L^p_\omega}\|B_\mu\|_{L_\omega^{\frac{p}{p-q}}} \asymp \|f\|^q_{A^p_\omega}\|B_\mu\|_{L_\omega^{\frac{p}{p-q}}},
\end{aligned}$$

for all $f \in A^p_\omega$. □

It turns out that the reverse of the above result is true [50, Theorem 1] for $\omega \in \widehat{\mathcal{D}}$. However, its proof its much more involved. As in the case $q \ge p$, methods from harmonic analysis are the appropriate ones. To some extent this is natural because the weighted Bergman space A^p_ω induced by $\omega \in \widehat{\mathcal{D}}$ may lie essentially much closer to the Hardy space H^p than any standard Bergman space A^p_α [49]. Luecking [41] employed the theory of tent spaces, introduced by Coifman, Meyer and Stein [18] and further considered by Cohn and Verbitsky [17], to study the analogue problem for Hardy spaces. In [50], an analogue of this theory for Bergman spaces is built and it is a key ingredient in the proof of the following result.

Theorem 3.12. *Let $0 < q < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent:*

(i) *μ is a q-Carleson measure for A^p_ω;*

(ii) *The function*

$$B_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D} \setminus \{0\},$$

belongs to $L_\omega^{\frac{p}{p-q}}$;

(iii) $M_\omega(\mu)(z) = \sup_{z \in S(a)} \frac{\mu(S(a))}{(\omega(S(a)))^\alpha} \in L_\omega^{\frac{p}{p-q}}$.

4. FACTORIZATION OF FUNCTIONS IN A^P_ω

Factorization theorems in spaces of analytic functions are related with plenty of issues such as zero sets, dual spaces, Hankel operators or integral operators. We remind the reader of the following well-known factorization of H^p-functions [25].

Theorem B. *If $f \not\equiv 0$, $f \in H^p$, then $f = B \cdot g$ where B is the Blaschke product of zeros of f and g does not vanish on $\mathbb{D}$. Moreover, $\|f\|_{H^p} = \|g\|_{H^p}$. In particular, $f \not\equiv 0$, $f \in H^1$, can be written as $f = f_1 \cdot f_2$ where f_2 does not vanish on $\mathbb{D}$. Moreover, $\|f\|_{H^1} = \|f_j\|_{H^2}$, $j = 1, 2$.*

Because of the following result, Theorem B does not remain true for standard Bergman spaces A^p_α [35].

Theorem C. *Let $0 < p < q < \infty$. Then there exists an A^p zero set which is not an A^q zero set. In particular, it is not possible to represent an arbitrary A^1 function as the product of two functions in A^2, one of them nonvanishing.*

Some years later, a weak factorization result was obtained in the context of Hardy spaces in several variables [19].

Theorem D. *If $f \in A^1$ function, then*

$$f = \sum_{j=1}^{\infty} F_j G_j$$

and $\sum_{j=1}^{\infty} \|F_j\|_{A^2} \|G_j\|_{A^2} \leq C\|f\|_{A^1}$.

Essentially at the same time, Horowitz [36] improved this result, obtaining a strong factorization of A^p_α-functions.

Theorem E. *Assume that* $0 < p < \infty$, $\alpha > -1$ *and* $p^{-1} = p_1^{-1} + p_2^{-1}$. *If* $f \in A^p_\alpha$, *then there exist* $f_1 \in A^{p_1}_\alpha$ *and* $f_2 \in A^{p_2}_\alpha$ *such that* $f = f_1 \cdot f_2$ *and*

$$\|f_1\|^p_{A^{p_1}_\alpha} \cdot \|f_2\|^p_{A^{p_2}_\alpha} \leq C\|f\|^p_{A^p_\alpha}$$

for some constant $C = C(p_1, p_2, \alpha) > 0$.

Motivated by the study of integral operators, we are interested in finding out a large class of weights ω which allow a (strong) factorization of A^p_ω-functions.

Throughout these notes, we shall use the following notation. For $a \in \mathbb{D}$, define $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$. The pseudohyperbolic distance from z to w is defined by $\varrho(z, w) = |\varphi_z(w)|$, and the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $r \in (0, 1)$ is denoted by $\Delta(a, r) = \{z : \varrho(a, z) < r\}$.

A careful inspection of Horowitz's techniques lead us to consider the following class of weights. A weight ω (not necessarily radial neither continuous) is called invariant, $\omega \in \mathcal{I}nv$, if for each $r \in (0, 1)$ there exists a constant $C = C(r) \geq 1$ such that

$$C^{-1}\omega(a) \leq \omega(z) \leq C\omega(a), \qquad z \in \Delta(a, r). \tag{27}$$

We note that a radial weight ω belongs to $\mathcal{I}nv$ if and only if ω does not have zeros and ω satisfies the property (1). Therefore, $\mathcal{R} \cup \widetilde{\mathcal{I}} \subset \mathcal{I}nv$. Moreover, by using results in [3] it is not difficult to prove that a differentiable weight ω is invariant whenever

$$|\nabla\omega(z)|(1 - |z|^2) \leq C\omega(z), \quad z \in \mathbb{D}.$$

The following result is based on the additivity of the hyperbolic distance on geodesics.

Lemma 4.1. *If $\omega \in \mathcal{I}nv$, then there exists a function $C : \mathbb{D} \to [1, \infty)$ such that*

$$\omega(u) \leq C(z)\omega(\varphi_u(z)), \quad u, z \in \mathbb{D}, \tag{28}$$

and

$$\int_{\mathbb{D}} \log C(z)\, dA(z) < \infty. \tag{29}$$

Conversely, if ω is a weight having no zeros, satisfying (28) and the function C is uniformly bounded in compact subsets of $\mathbb{D}$, then $\omega \in \mathcal{I}nv$.

Proof. Let first $\omega \in \mathcal{I}nv$. Then there exists a constant $C \geq 1$ such that

$$C^{-1}\omega(a) \leq \omega(z) \leq C\omega(a), \quad z \in \Delta_h(a, 1). \tag{30}$$

For each $z, u \in \mathbb{D}$, the hyperbolic distance between u and $\varphi_u(z)$ is

$$\varrho_h(u, \varphi_u(z)) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

By the additivity of the hyperbolic distance on the geodesic joining u and $\varphi_u(z)$, and (30) we deduce

$$\omega(u) \leq C^{E(\varrho_h(u,\varphi_u(z)))+1}\omega(\varphi_u(z)) \leq C \left(\frac{1 + |z|}{1 - |z|}\right)^{\frac{\log C}{2}} \omega(\varphi_u(z)),$$

where $E(x)$ is the integer such that $E(x) \leq x < E(x) + 1$. It follows that (28) and (29) are satisfied.

Conversely, let ω be a weight satisfying (28) such that the function C is uniformly bounded in compact subsets of $\mathbb{D}$. Then, for each $r \in (0, 1)$, there exists a constant $C = C(r) > 0$ such that $\omega(u) \leq C(r)\omega(z)$ whenever $|\varphi_u(z)| < r$. Thus $\omega \in \mathcal{I}nv$. $\square$

The next result plays an important role in the proof of our factorization theorem. The proof is technical, see [49, Lemma 3.3].

Lemma 4.2. *Let $0 < p < q < \infty$ and $\omega \in \mathcal{I}nv$. Let $\{z_k\}$ be the zero set of $f \in A^p_\omega$, and let*

$$g(z) = |f(z)|^p \prod_k \frac{1 - \frac{p}{q} + \frac{p}{q}|\varphi_{z_k}(z)|^q}{|\varphi_{z_k}(z)|^p}.$$

Then there exists a constant $C = C(p, q, \omega) > 0$ *such that*

$$\|g\|_{L^1_\omega} \leq C\|f\|^p_{A^p_\omega}.$$

Moreover, the constant C *has the following properties:*

(i) *If* $0 < p < q \leq 2$*, then* $C = C(\omega)$*, that is,* C *is independent of* p *and* q*.*

(ii) *If* $2 < q < \infty$ *and* $\frac{q}{p} \geq 1 + \epsilon > 1$*, then* $C = C_1 q e^{C_1 q}$*, where* $C_1 = C_1(\epsilon, \omega)$*.*

Now, we prove our main result in this section.

Theorem 4.3. *Let* $0 < p < \infty$ *and* $\omega \in \mathcal{I}nv$ *such that the polynomials are dense in* A^p_ω*. Let* $f \in A^p_\omega$*, and let* $0 < p_1, p_2 < \infty$ *such that* $p^{-1} = p_1^{-1} + p_2^{-1}$*. Then there exist* $f_1 \in A^{p_1}_\omega$ *and* $f_2 \in A^{p_2}_\omega$ *such that* $f = f_1 \cdot f_2$ *and*

$$\|f_1\|^p_{A^{p_1}_\omega} \cdot \|f_2\|^p_{A^{p_2}_\omega} \leq \frac{p}{p_1}\|f_1\|^{p_1}_{A^{p_1}_\omega} + \frac{p}{p_2}\|f_2\|^{p_2}_{A^{p_2}_\omega} \leq C\|f\|^p_{A^p_\omega} \tag{31}$$

for some constant $C = C(p_1, p_2, \omega) > 0$*.*

Proof. Let $0 < p < \infty$ and $\omega \in \mathcal{I}nv$ such that the polynomials are dense in A^p_ω, and let $f \in A^p_\omega$. Assume first that f has finitely many zeros only. Such functions are of the form $f = gB$, where $g \in A^p_\omega$ has no zeros and B is a finite Blaschke product. Let $z_1, \dots, z_m$ be the zeros of f so that $B = \prod_{k=1}^m B_k$, where $B_k = \frac{z_k}{|z_k|}\varphi_{z_k}$. Write $B = B^{(1)} \cdot B^{(2)}$, where the factors $B^{(1)}$ and $B^{(2)}$ are random subproducts of $B_0, B_1, \dots, B_m$, where $B_0 \equiv 1$. Setting $f_j = \left(\frac{f}{B}\right)^{p/p_j} B^{(j)}$, we have $f = f_1 \cdot f_2$. We now choose $B^{(j)}$ probabilistically. For a given $j \in \{1, 2\}$, the factor $B^{(j)}$ will contain each B_k with the probability p/p_j. The obtained m random variables are independent, so the expected value of $|f_j(z)|^{p_j}$ is

$$\begin{aligned} E(|f_j(z)|^{p_j}) &= \left|\frac{f(z)}{B(z)}\right|^p \prod_{k=1}^m \left(1 - \frac{p}{p_j} + \frac{p}{p_j}|\varphi_{z_k}(z)|^{p_j}\right) \\ &= |f(z)|^p \prod_{k=1}^m \frac{\left(1 - \frac{p}{p_j}\right) + \frac{p}{p_j}|\varphi_{z_k}(z)|^{p_j}}{|\varphi_{z_k}(z)|^p} \end{aligned} \tag{32}$$

for all $z \in \mathbb{D}$ and $j \in \{1,2\}$. Now, bearing in mind (32) and Lemma 4.2, we find a constant $C_1 = C_1(p,p_1,\omega) > 0$ such that

$$\begin{aligned}\|E(f_1^{p_1})\|_{L^1_\omega} &= \int_{\mathbb{D}} \left[|f(z)|^p \prod_{k=1}^m \frac{\left(1-\frac{p}{p_1}\right)+\frac{p}{p_1}|\varphi_{z_k}(z)|^{p_1}}{|\varphi_{z_k}(z)|^p} \right] \omega(z)dA(z) \\ &= \int_{\mathbb{D}} \left[|f(z)|^p \prod_{k=1}^m \frac{\frac{p}{p_2}+\left(1-\frac{p}{p_2}\right)|\varphi_{z_k}(z)|^{p_1}}{|\varphi_{z_k}(z)|^p} \right] \omega(z)dA(z) \\ &\leq C_1 \|f\|^p_{A^p_\omega}.\end{aligned}$$

Analogously, by (32) and Lemma 4.2 there exists a constant $C_2 = C_2(p,p_2,\omega) > 0$ such that

$$\begin{aligned}\|E(f_2^{p_2})\|_{L^1_\omega} &= \int_{\mathbb{D}} \left[|f(z)|^p \prod_{k=1}^m \frac{\left(1-\frac{p}{p_2}\right)+\frac{p}{p_2}|\varphi_{z_k}(z)|^{p_2}}{|\varphi_{z_k}(z)|^p} \right] \omega(z)dA(z) \\ &\leq C_2 \|f\|^p_{A^p_\omega}.\end{aligned}$$

By combining the two previous inequalities, we obtain

$$\begin{aligned}&\left\| E\left(\frac{p}{p_1} f_1^{p_1}\right) \right\|_{L^1_\omega} + \left\| E\left(\frac{p}{p_2} f_2^{p_2}\right) \right\|_{L^1_\omega} \\ &\qquad \leq \left(\frac{p}{p_1}C_1 + \frac{p}{p_2}C_2\right) \|f\|^p_{A^p_\omega}.\end{aligned} \tag{33}$$

On the other hand,

$$\begin{aligned}&\left\| E\left(\frac{p}{p_1} f_1^{p_1}\right) \right\|_{L^1_\omega} + \left\| E\left(\frac{p}{p_2} f_2^{p_2}\right) \right\|_{L^1_\omega} \\ &\quad = \frac{p}{p_1} \int_{\mathbb{D}} \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left(\frac{p}{p_2} + \left(1-\frac{p}{p_2}\right) |\varphi_{z_k}(z)|^{p_1} \right) \omega(z)dA(z) \\ &\qquad + \frac{p}{p_2} \int_{\mathbb{D}} \left| \frac{f(z)}{B(z)} \right|^p \prod_{k=1}^m \left(\left(1-\frac{p}{p_2}\right) + \frac{p}{p_2}|\varphi_{z_k}(z)|^{p_2} \right) \omega(z)dA(z) \\ &\quad = \int_{\mathbb{D}} I(z)\omega(z)\, dA(z),\end{aligned} \tag{34}$$

where

$$I(z) = \left|\frac{f(z)}{B(z)}\right|^p \left[\frac{p}{p_1} \cdot \prod_{k=1}^{m} \left(\frac{p}{p_2} + \left(1 - \frac{p}{p_2}\right) |\varphi_{z_k}(z)|^{p_1}\right) \right.$$
$$\left. + \frac{p}{p_2} \cdot \prod_{k=1}^{m} \left(\left(1 - \frac{p}{p_2}\right) + \frac{p}{p_2}|\varphi_{z_k}(z)|^{p_2}\right)\right].$$

It is clear that the m zeros of f must be distributed to the factors f_1 and f_2, so if f_1 has n zeros, then f_2 has the remaining $(m-n)$ zeros. Therefore

$$I(z) = \sum_{f_{l_1} \cdot f_{l_2} = f} \left(\left(1 - \frac{p}{p_2}\right)^n \left(\frac{p}{p_2}\right)^{m-n} \left[\frac{p}{p_1}|f_{l_1}(z)|^{p_1} + \frac{p}{p_2}|f_{l_2}(z)|^{p_2}\right]\right). \tag{35}$$

This sum consists of 2^m addends, f_{l_1} contains $\left(\frac{f}{B}\right)^{p/p_1}$ and n zeros of f, and f_{l_2} contains $\left(\frac{f}{B}\right)^{p/p_2}$ and the remaining $(m-n)$ zeros of f, and thus $f = f_{l_1} \cdot f_{l_2}$. Further, for a fixed $n = 0, 1, \dots, m$, there are $\binom{m}{n}$ ways to choose f_{l_1} (once f_{l_1} is chosen, f_{l_2} is determined). Consequently,

$$\begin{aligned}\sum_{f_{l_1} \cdot f_{l_2} = f} &\left(1 - \frac{p}{p_2}\right)^n \left(\frac{p}{p_2}\right)^{m-n} \\ &= \sum_{n=0}^{m} \binom{m}{n} \left(1 - \frac{p}{p_2}\right)^n \left(\frac{p}{p_2}\right)^{m-n} = 1.\end{aligned} \tag{36}$$

Now, by joining (33), (34) and (35), we deduce

$$\begin{aligned}\sum_{f_{l_1} \cdot f_{l_2} = f} &\left(1 - \frac{p}{p_2}\right)^n \left(\frac{p}{p_2}\right)^{m-n} \left[\frac{p}{p_1}\|f_{l_1}\|^{p_1}_{A^{p_1}_\omega} + \frac{p}{p_2}\|f_{l_2}\|^{p_2}_{A^{p_2}_\omega}\right] \\ &\le \left(\frac{p}{p_1}C_1 + \frac{p}{p_2}C_2\right) \|f\|^p_{A^p_\omega}.\end{aligned}$$

This together with (36) shows that there must exist a concrete factorization $f = f_1 \cdot f_2$ such that

$$\frac{p}{p_1}\|f_1\|^{p_1}_{A^{p_1}_\omega} + \frac{p}{p_2}\|f_2\|^{p_2}_{A^{p_2}_\omega} \le C(p_1, p_2, \omega)\|f\|^p_{A^p_\omega}.$$

By combining this with the inequality

$$x^{\alpha} \cdot y^{\beta} \leq \alpha x + \beta y, \quad x, y \geq 0, \quad \alpha + \beta = 1,$$

we finally obtain (31) under the hypotheses that f has finitely many zeros only.

To deal with the general case, we first prove that every norm-bounded family in A^p_ω is a normal family of analytic functions. If $f \in A^p_\omega$, then

$$\begin{aligned} \|f\|^p_{A^p_\omega} &\geq \int_{D(0,\frac{1+\rho}{2})\setminus D(0,\rho)} |f(z)|^p \omega(z)\, dA(z) \\ &\gtrsim M^p_p(\rho, f) \left(\min_{|z| \leq \frac{1+\rho}{2}} \omega(z) \right), \quad 0 \leq \rho < 1, \end{aligned}$$

from which the well-known relation

$$M_\infty(r, f) \lesssim M_p\left(\frac{1+r}{2}, f\right)(1-r)^{-1/p}, \quad 0 \leq r < 1,$$

yields

$$M^p_\infty(r, f) \lesssim \frac{\|f\|^p_{A^p_\omega}}{(1-r)\left(\min_{|z|\leq \frac{3+r}{4}} \omega(z)\right)}, \quad 0 \leq r < 1.$$

Therefore every norm-bounded family in A^p_ω is a normal family of analytic functions by Montel's theorem.

Finally, assume that $f \in A^p_\omega$ has infinitely many zeros. Since polynomials are dense in A^p_ω by the assumption, we can choose a sequence f_l of functions with finitely many zeros that converges to f in norm, and then, by the previous argument, we can factorize each $f_l = f_{l,1} \cdot f_{l,2}$ as earlier. Now, since every norm-bounded family in A^p_ω is a normal family of analytic functions, by passing to subsequences of $\{f_{l,j}\}$ with respect to l if necessary, we have $f_{l,j} \to f_j$, where the functions f_j form the desired bounded factorization $f = f_1 \cdot f_2$ satisfying (31). This finishes the proof. □

At first glance the next result might seem a bit artificial. However, it turns out to be a key ingredient in the proof of Proposition 6.7 (below) where we get the uniform boundedness of a certain family of integral operators, which is usually established by using interpolation theorems.

Corollary 4.4. *Let $0 < p < 2$ and $\omega \in \mathcal{I}nv$ such that the polynomials are dense in A^p_ω. Let $0 < p_1 \leq 2 < p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $p_2 \geq 2p$. If $f \in A^p_\omega$, then there exist $f_1 \in A^{p_1}_\omega$ and $f_2 \in A^{p_2}_\omega$ such that $f = f_1 \cdot f_2$ and*

$$\|f_1\|_{A^{p_1}_\omega} \cdot \|f_2\|_{A^{p_2}_\omega} \leq C\|f\|_{A^p_\omega}$$

for some constant $C = C(p_1, \omega) > 0$.

It can be proved mimicking the proof of of Theorem 4.3, but paying special attention to the constants coming from Lemma 4.2, see [49, Corollary 3.4] for details.

Before ending this section, let us observe that there are non-radial weights satisfying the hypotheses of our factorization result for A^p_ω.

Lemma 4.5. *Let f be a non-vanishing univalent function in $\mathbb{D}$, $0 < \gamma < 1$ and $\omega = |f|^\gamma$. Then the polynomials are dense in A^p_ω for all $p \geq 1$.*

Proof. Since f is univalent and zero-free, so is $1/f$, and hence both f and $1/f$ belong to A^p for all $0 < p < 1$. By choosing $\delta > 0$ such that $\gamma(1+\delta) < 1$ we deduce that both ω and $\frac{1}{\omega}$ belong to $L^{1+\delta}$. Therefore the polynomials are dense in A^p_ω by [31, Theorem 2]. □

Finally, let us consider the class of weights that appears in a paper by Abkar [1] concerning norm approximation by polynomials in weighted Bergman spaces. A function u defined on $\mathbb{D}$ is said to be *superbiharmonic* if $\Delta^2 u \geq 0$, where Δ stands for the Laplace operator

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

in the complex plane $\mathbb{C}$. The superbihamonic weights play an essential role in the study of invariant subspaces of the Bergman space A^p.

Theorem F. *Let ω be a superbiharmonic weight such that*

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \omega(r\zeta)\, dm(\zeta) = 0.$$

Then the polynomials are dense in A^p_ω.

The proof of Theorem F relies on showing that these type of weights ω satisfy

$$\omega(z) \leq C(\omega)\omega(rz), \quad r_0 \leq r < 1, \quad r_0 \in (0,1),$$

which asserts that polynomials are dense on A^p_ω. In [49, Lemma 1.11] it is proved the following.

Lemma 4.6. *Every superbihamonic weight that satisfies*

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \omega(r\zeta)\, dm(\zeta) = 0,$$

is invariant and the polynomials are dense in A^p_ω.

5. ZERO SETS

For a given space X of analytic functions in $\mathbb{D}$, a sequence $\{z_k\}$ is called an X-zero set, if there exists a function f in X such that f vanishes precisely on the points $\{z_k\}$ and nowhere else. A sequence $\{z_k\}$ is a H^p-zero set if and only if satisfies the Blaschke condition $\sum_k (1-|z_k|) < \infty$.

5.1. The Bergman-Nevanlinna class

Using Lemma 2.1, Jensen's formula and the elementary factors from the classical Weierstrass factorization for the theory of entire functions, it can be proved the following [49, Proposition 3.16]. The weighted Bergman-Nevanlinna class consists of those analytic functions in $\mathbb{D}$ for which

$$\int_{\mathbb{D}} \log^+ |f(z)| \omega(z)\, dA(z) < \infty.$$

Theorem 5.1. *Let $\omega \in \widehat{\mathcal{D}}$. Then $\{z_k\}$ is a zero set of the Bergman-Nevanlinna class if and only if*

$$\sum_k [(1-|z_k|)\widehat{\omega}(z_k)] = \sum_k \left[(1-|z_k|) \int_{|z_k|}^1 \omega(s)\, ds\right] < \infty.$$

As far as we know, it is still an open problem to find a complete description of zero sets of functions in the Bergman spaces $A^p = A^p_0$, but the gap between the known necessary and sufficient conditions is very small. We refer to [26, Chapter 4], [33, Chapter 4] and [39, 43, 56, 57]. The analogous question is also unsolved for classical Dirichlet spaces $\mathcal{D}^2_\alpha$, $0 \le \alpha < 1$, of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{D}^2_\alpha} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|)^\alpha \, dA(z) < \infty.$$

The most important results are the ones given by Carleson in [14], [15], and by Shapiro and Shields in [59]. Some progress was achieved in [47].

5.2. A^p_ω zeros sets

Our results on zeros set of A^p_ω follow the line of those due to Horowitz [35, 37, 38]. Roughly speaking we will study basic properties of unions, subsets and the dependence on p of the zero sets of functions in A^p_ω. By using ideas and estimates obtained in the proof of Theorem 4.3 we get our first result in this section, see [49, Theorem 3.5].

Theorem 5.2. *Let $0 < p < \infty$ and $\omega \in \mathcal{I}nv$. Let $\{z_k\}$ be an arbitrary subset of the zero set of $f \in A^p_\omega$, and let*

$$H(z) = \prod_k B_k(z)\big(2 - B_k(z)\big), \quad B_k = \frac{z_k}{|z_k|}\varphi_{z_k},$$

with the convention $z_k/|z_k| = 1$ if $z_k = 0$. Then there exists a constant $C = C(\omega) > 0$ such that $\|f/H\|^p_{A^p_\omega} \le C\|f\|^p_{A^p_\omega}$. In particular, each subset of an A^p_ω-zero set is an A^p_ω-zero set.

Now we turn to work with radial weights. The first of them will be used to show that A^p_ω-zero sets depend on p.

Theorem 5.3. *Let $0 < p < \infty$ and let ω be a radial weight. Let $f \in A^p_\omega$, $f(0) \ne 0$, and let $\{z_k\}$ be its zero sequence repeated according to multiplicity and ordered by increasing moduli. Then*

$$\prod_{k=1}^n \frac{1}{|z_k|} = \mathrm{o}\left(\left(\int_{1-\frac{1}{n}}^1 \omega(r)\, dr\right)^{-\frac{1}{p}}\right), \quad n \to \infty. \tag{37}$$

Proof. Let $f \in A^p_\omega$ and $f(0) \neq 0$. By multiplying Jensen's formula

$$\log|f(0)| + \sum_{k=1}^{n} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|d\theta, \quad 0 < r < 1,$$

by p, and applying the arithmetic-geometric mean inequality, we obtain

$$|f(0)|^p \prod_{k=1}^{n} \frac{r^p}{|z_k|^p} \leq M_p^p(r, f) \tag{38}$$

for all $0 < r < 1$ and $n \in \mathbb{N}$. Moreover,

$$\lim_{r \to 1^-} M_p^p(r, f) \int_r^1 \omega(s)\, ds \leq \lim_{r \to 1^-} \int_r^1 M_p^p(s, f)\omega(s)\, ds = 0,$$

so taking $r = 1 - 1/n$ in (38), we deduce

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \lesssim M_p\left(1 - \frac{1}{n}, f\right) = \mathrm{o}\left(\left(\int_{1-\frac{1}{n}}^1 \omega(r)\, dr\right)^{-\frac{1}{p}}\right), \quad n \to \infty,$$

as desired. □

The next result shows that condition (37) is a sharp necessary condition for $\{z_k\}$ to be an A^p_ω-zero set.

Theorem 5.4. *Let $0 < q < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Then there exists $f \in \bigcap_{p<q} A^p_\omega$ such that its zero sequence $\{z_k\}$, repeated according to multiplicity and ordered by increasing moduli, does not satisfy* (37) *with $p = q$. In particular, there is a $\cap_{p<q} A^p_\omega$-zero set which is not an A^q_ω-zero set.*

Proof. The proof uses ideas from [30, Theorem 3], see also [37, 38]. Define

$$f(z) = \prod_{k=1}^{\infty} F_k(z), \quad z \in \mathbb{D}, \tag{39}$$

where

$$F_k(z) = \frac{1 + a_k z^{2^k}}{1 + a_k^{-1} z^{2^k}}, \quad z \in \mathbb{D}, \quad k \in \mathbb{N},$$

and

$$a_k = \left(\frac{\int_{1-2^{-k}}^1 \omega(s)\, ds}{\int_{1-2^{-(k+1)}}^1 \omega(s)\, ds}\right)^{1/q}, \quad k \in \mathbb{N}.$$

By Lemma 2.1 there exists a constant $C_1 = C_1(q,\omega) > 0$ such that

$$1 < a_k \le C_1 < \infty, \quad k \in \mathbb{N}. \tag{40}$$

Therefore $\limsup_{k\to\infty}(a_k - a_k^{-1})^{2^{-k}} \le \limsup_{k\to\infty} a_k^{2^{-k}} = 1$, and hence the product in (39) defines an analytic function in $\mathbb{D}$. The zero set of f is the union of the zero sets of the functions F_k, so f has exactly 2^k simple zeros on the circle $\left\{z : |z| = a_k^{-2^{-k}}\right\}$ for each $k \in \mathbb{N}$. Let $\{z_j\}_{j=1}^{\infty}$ be the sequence of zeros of f ordered by increasing moduli, and denote $N_n = 2 + 2^2 + \cdots + 2^n$. Then $2^n \le N_n \le 2^{n+1}$, and hence

$$\prod_{k=1}^{N_n} \frac{1}{|z_k|} \ge \prod_{k=1}^{n} a_k = \left(\frac{\int_{\frac{1}{2}}^1 \omega(s)\,ds}{\int_{1-2^{-(n+1)}}^1 \omega(s)\,ds}\right)^{1/q} \ge \left(\frac{\int_{\frac{1}{2}}^1 \omega(s)\,ds}{\int_{1-\frac{1}{N_n}}^1 \omega(s)\,ds}\right)^{1/q}.$$

It follows that $\{z_j\}_{j=1}^{\infty}$ does not satisfy (37), and thus $\{z_j\}_{j=1}^{\infty}$ is not an A^q_ω-zero set by Theorem 5.3.

We turn to prove that the function f defined in (39) belongs to A^p_ω for all $p \in (0,q)$. Set $r_n = e^{-2^{-n}}$ for $n \in \mathbb{N}$, and observe that

$$|f(z)| = \left|\prod_{k=1}^{n} a_k \frac{a_k^{-1} + z^{2^k}}{1 + a_k^{-1} z^{2^k}}\right| \left|\prod_{j=1}^{\infty} \frac{1 + a_{n+j} z^{2^{n+j}}}{1 + a_{n+j}^{-1} z^{2^{n+j}}}\right|. \tag{41}$$

The function $h_1(x) = \frac{\alpha + x}{1+\alpha x}$ is increasing on $[0,1)$ for each $\alpha \in [0,1)$, and therefore

$$\begin{aligned}\left|\frac{1 + a_{n+j} z^{2^{n+j}}}{1 + a_{n+j}^{-1} z^{2^{n+j}}}\right| &= a_{n+j}\left|\frac{a_{n+j}^{-1} + z^{2^{n+j}}}{1 + a_{n+j}^{-1} z^{2^{n+j}}}\right| \le a_{n+j}\frac{a_{n+j}^{-1} + |z|^{2^{n+j}}}{1 + a_{n+j}^{-1} |z|^{2^{n+j}}} \\ &\le \frac{1 + a_{n+j}\left(\frac{1}{e}\right)^{2^j}}{1 + a_{n+j}^{-1}\left(\frac{1}{e}\right)^{2^j}}, \quad |z| \le r_n, \quad j, n \in \mathbb{N}.\end{aligned} \tag{42}$$

Since $h_2(x) = \frac{1+x\alpha}{1+x^{-1}\alpha}$ is increasing on $(0,\infty)$ for each $\alpha \in (0,\infty)$, (40) and (42) yield

$$\begin{aligned}\left|\prod_{j=1}^{\infty} \frac{1 + a_{n+j} z^{2^{n+j}}}{1 + a_{n+j}^{-1} z^{2^{n+j}}}\right| &\le \prod_{j=1}^{\infty} \frac{1 + a_{n+j}\left(\frac{1}{e}\right)^{2^j}}{1 + a_{n+j}^{-1}\left(\frac{1}{e}\right)^{2^j}} \le \prod_{j=1}^{\infty} \frac{1 + C_1\left(\frac{1}{e}\right)^{2^j}}{1 + C_1^{-1}\left(\frac{1}{e}\right)^{2^j}} \\ &= C_2 < \infty,\end{aligned} \tag{43}$$

whenever $|z| \le r_n$ and $n \in \mathbb{N}$. So, by using (41), (43), Lemma 2.1

and the inequality $e^{-x} \geq 1 - x$, $x \geq 0$, we obtain

$$\begin{aligned} |f(z)| &\leq C_2 \prod_{k=1}^{n} a_k \lesssim \left(\frac{1}{\int_{1-2^{-(n+1)}}^{1} \omega(s)\, ds} \right)^{1/q} \lesssim \left(\frac{1}{\int_{1-2^{-n}}^{1} \omega(s)\, ds} \right)^{1/q} \\ &\leq \left(\frac{1}{\int_{r_n}^{1} \omega(s)\, ds} \right)^{1/q}, \quad |z| \leq r_n, \quad n \in \mathbb{N}. \end{aligned} \tag{44}$$

Let now $|z| \geq 1/\sqrt{e}$ be given and fix $n \in \mathbb{N}$ such that $r_n \leq |z| < r_{n+1}$. Then (44), the inequality $1 - x \leq e^{-x} \leq 1 - \frac{x}{2}$, $x \in [0, 1]$, and Lemma 2.1 give

$$\begin{aligned} |f(z)| \leq M_\infty(r_{n+1}, f) &\lesssim \left(\frac{1}{\int_{r_{n+1}}^{1} \omega(s)\, ds} \right)^{1/q} \\ &\leq \left(\frac{1}{\int_{1-2^{-(n+2)}}^{1} \omega(s)\, ds} \right)^{1/q} \lesssim \left(\frac{1}{\int_{1-2^{-n}}^{1} \omega(s)\, ds} \right)^{1/q} \\ &\leq \left(\frac{1}{\int_{r_n}^{1} \omega(s)\, ds} \right)^{1/q} \leq \left(\frac{1}{\int_{|z|}^{1} \omega(s)\, ds} \right)^{1/q}, \end{aligned}$$

and hence

$$M_\infty(r, f) \lesssim \left(\frac{1}{\int_{r}^{1} \omega(s)\, ds} \right)^{1/q}, \quad 0 < r < 1.$$

This and the identity $\psi_{\widetilde{\omega}}(r) = \frac{1}{1-\alpha} \psi_\omega(r)$ of Lemma 2.2(iii), with $\alpha = p/q < 1$ and $r = 0$, yield

$$\begin{aligned} \|f\|_{A^p_\omega}^p &\lesssim \int_0^1 \frac{\omega(r)\, dr}{\left(\int_r^1 \omega(s)\, ds \right)^{p/q}} = \int_0^1 \widetilde{\omega}(r)\, dr \\ &= \frac{q}{q-p} \left(\int_0^1 \omega(s)\, ds \right)^{\frac{q-p}{q}} < \infty. \end{aligned}$$

This finishes the proof. □

The proof of the above result implies that the union of two A^p_ω-zero sets is not an A^p_ω-zero set. Going further, we obtain the following result.

Corollary 5.5. *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Then the union of two A^p_ω-zero sets is an $A^{p/2}_\omega$-zero set. However, there are two $\bigcap_{p<q} A^p_\omega$-zero sets such that their union is not an $A^{q/2}_\omega$-zero set.*

Since the angular distribution of zeros plays a role in a description of the zero sets of functions in the classical weighted Bergman space A^p_α, it is natural to expect that the same happens also in A^p_ω, when $\omega \in \widehat{\mathcal{D}}$. However, we do not venture into generalizing the theory, developed among others by Korenblum [39], Hedenmalm [32] and Seip [56, 57], and based on the use of densities defined in terms of partial Blaschke sums, Stolz star domains and Beurling-Carleson characteristic of the corresponding boundary set.

6. INTEGRAL OPERATORS

The main aim of this section is to characterize those symbols $g \in \mathcal{H}(\mathbb{D})$ such that the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

is bounded or compact from A^p_ω to A^q_ω, when $\omega \in \widehat{\mathcal{D}}$. The choice $g(z) = z$ gives the usual Volterra operator and the Cesàro operator is obtained when $g(z) = -\log(1-z)$. The bilinear operator $(f, g) \to \int f\, g'$ was introduced by A. Calderón in harmonic analysis in the 60's for his research on commutators of singular integral operators [13] which leads to the study of "paraproducts". Regarding the complex function theory, Pommerenke considered the operator T_g [53] to study the space $BMOA$ proving that $T_g : H^2 \to H^2$ is bounded if and only $g \in$ BMOA. We recall that BMOA consists of functions in the Hardy space H^1 that have *bounded mean oscillation* on the boundary $\mathbb{T}$ [10, 29]. We will use the norm given by

$$\|g\|^2_{\mathrm{BMOA}} = \sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2 (1-|z|^2)\, dA(z)}{1-|a|} + |g(0)|^2.$$

Later, Aleman and Cima [2] proved that $T_g : H^p \to H^p$ is bounded if and only if $g \in$ BMOA. The analogue holds for A^p_ω, $\omega \in \mathcal{R}$, if and only if $g \in \mathcal{B}$ [6]. Recently, the spectrum of T_g has been studied on the Hardy space H^p [4] and on the classical weighted Bergman space A^p_α [3]. The following family of spaces of analytic functions will appear in the description of those symbols g such that $T_g: A^p_\omega \to A^q_\omega$ is bounded.

6.1. Non-conformally invariant spaces

We say that $g \in \mathcal{H}(\mathbb{D})$ belongs to $\mathcal{C}^{q,p}(\omega^\star)$, $0 < p, q < \infty$, if the measure $|g'(z)|^2\omega^\star(z)\,dA(z)$ is a q-Carleson measure for A^p_ω. If $q \geq p$ and $\omega \in \widehat{\mathcal{D}}$, then Theorem 3.3 shows that these spaces only depend on the quotient $\frac{q}{p}$. Consequently, for $q \geq p$ and $\omega \in \widehat{\mathcal{D}}$, we simply write $\mathcal{C}^{q/p}(\omega^\star)$ instead of $\mathcal{C}^{q,p}(\omega^\star)$. Thus, if $\alpha \geq 1$ and $\omega \in \widehat{\mathcal{D}}$, then $\mathcal{C}^\alpha(\omega^\star)$ consists of those $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|^2_{\mathcal{C}^\alpha(\omega^\star)} = |g(0)|^2 + \sup_{I \subset \mathbb{T}} \frac{\int_{S(I)} |g'(z)|^2\omega^\star(z)\,dA(z)}{(\omega\,(S(I)))^\alpha} < \infty. \tag{45}$$

Unlike $\mathcal{B}$, the space $\mathcal{C}^1(\omega^\star)$ can not be described by a simple growth condition on the maximum modulus of g' if $\omega \in \widehat{\mathcal{D}}$. This follows by Proposition 6.1 (below) and the fact that $\log(1-z) \in A^p_\omega$ for all $\omega \in \widehat{\mathcal{D}}$.

The spaces BMOA and $\mathcal{B}$ are conformally invariant. This property has been used, among other things, in describing those symbols $g \in \mathcal{H}(\mathbb{D})$ for which T_g is bounded on H^p or A^p_α. However, the space $\mathcal{C}^1(\omega^\star)$ is not necessarily conformally invariant, and therefore different techniques must be employed in the case of A^p_ω with $\omega \in \widehat{\mathcal{D}}$.

Recall that $h: [0,1) \to (0,\infty)$ is essentially increasing on $[0,1)$ if there exists a constant $C > 0$ such that $h(r) \leq Ch(t)$ for all $0 \leq r \leq t < 1$.

Proposition 6.1.

(A) *If* $\omega \in \widehat{\mathcal{D}}$, *then* $\mathcal{C}^1(\omega^\star) \subset \bigcap_{0<p<\infty} A^p_\omega$.

(B) *If* $\omega \in \widehat{\mathcal{D}}$, *then* BMOA $\subset \mathcal{C}^1(\omega^\star) \subset \mathcal{B}$.

(C) *If* $\omega \in \mathcal{R}$, *then* $\mathcal{C}^1(\omega^\star) = \mathcal{B}$.

(D) *If* $\omega \in \mathcal{I}$, *then* $\mathcal{C}^1(\omega^\star) \subsetneq \mathcal{B}$.

(E) *If* $\omega \in \mathcal{I}$ *and both* $\omega(r)$ *and* $\frac{\psi_\omega(r)}{1-r}$ *are essentially increasing on* $[0,1)$, *then* BMOA $\subsetneq \mathcal{C}^1(\omega^\star)$.

Proof. (A). Let $g \in \mathcal{C}^1(\omega^\star)$. By Theorem 3.3, $|g'(z)|^2\omega^\star(z)\,dA(z)$ is a p-Carleson measure for A^p_ω for all $0 < p < \infty$. In particular, $|g'(z)|^2\omega^\star(z)\,dA(z)$ is a finite measure and hence $g \in A^2_\omega$ by (24). Therefore (23) yields

$$\begin{aligned}\|g\|^4_{A^4_\omega} &= 4^2 \int_{\mathbb{D}} |g(z)|^2|g'(z)|^2\omega^\star(z)\,dA(z) + |g(0)|^4\\ &\lesssim \|g\|^2_{A^2_\omega} + |g(0)|^4,\end{aligned}$$

and thus $g \in A^4_\omega$. Continuing in this fashion, we deduce $g \in A^{2n}_\omega$ for all $n \in \mathbb{N}$, and the assertion follows.

(B). If $g \in \mathrm{BMOA}$, then $|g'(z)|^2 \log\frac{1}{|z|}\,dA(z)$ is a classical Carleson measure [28] (or [29, Section 8]), that is,

$$\sup_{I\subset\mathbb{T}} \frac{\int_{S(I)} |g'(z)|^2 \log\frac{1}{|z|}\,dA(z)}{|I|} < \infty.$$

Therefore

$$\begin{aligned}\int_{S(I)} |g'(z)|^2\omega^\star(z)\,dA(z) &\le \int_{S(I)} |g'(z)|^2 \log\frac{1}{|z|}\left(\int_{|z|}^1 \omega(s)s\,ds\right) dA(z)\\ &\le \left(\int_{1-|I|}^1 \omega(s)s\,ds\right)\int_{S(I)} |g'(z)|^2 \log\frac{1}{|z|}\,dA(z)\\ &\lesssim \left(\int_{1-|I|}^1 \omega(s)s\,ds\right)|I| \asymp \omega\left(S(I)\right),\end{aligned}$$

which together with Theorem 3.3 gives $g \in \mathcal{C}^1(\omega^\star)$ for all $\omega \in \widehat{\mathcal{D}}$.

Let now $g \in \mathcal{C}^1(\omega^\star)$ with $\omega \in \widehat{\mathcal{D}}$. It is well known that $g \in \mathcal{H}(\mathbb{D})$ is a Bloch function if and only if

$$\int_{S(I)} |g'(z)|^2(1-|z|^2)^\gamma\,dA(z) \lesssim |I|^\gamma, \quad I \subset \mathbb{T},$$

for some (equivalently for all) $\gamma > 1$. Fix $\beta = \beta(\omega) > 0$ and $C = C(\beta,\omega) > 0$ as in Lemma 2.1(ii). Then (26) and Lemma 2.1(ii) yield

$$\begin{aligned}\int_{S(I)} |g'(z)|^2(1-|z|)^{\beta+1}\,dA(z) &= \int_{S(I)} |g'(z)|^2\omega^\star(z)\frac{(1-|z|)^{\beta+1}}{\omega^\star(z)}\,dA(z)\\ &\asymp \int_{S(I)} |g'(z)|^2\omega^\star(z)\frac{(1-|z|)^\beta}{\int_{|z|}^1 \omega(s)s\,ds}\,dA(z)\end{aligned}$$

$$\begin{aligned}&\lesssim \frac{|I|^\beta}{\int_{1-|I|}^1 \omega(s)s\,ds}\int_{S(I)}|g'(z)|^2\omega^\star(z)\,dA(z)\\&\lesssim |I|^{\beta+1},\quad |I|\le\frac{1}{2},\end{aligned}$$

and so $g\in\mathcal{B}$.

(C). By Part (B) it suffices to show that $\mathcal{B}\subset\mathcal{C}^1(\omega^\star)$ for $\omega\in\mathcal{R}$. To see this, let $g\in\mathcal{B}$ and $\omega\in\mathcal{R}$. Let us consider the weight $\tilde{\omega}(r)=\frac{\widehat{\omega}(r)}{1-r}$. Since $\omega\in\mathcal{R}$, $\tilde{\omega}(r)$ is a continuous weight such that

$$C_1\le\frac{\psi_{\tilde{\omega}}(r)}{1-r}\le C_2,\quad 0<r<1.$$

A calculation shows that

$$\tilde{h}(r)=\frac{\int_r^1\tilde{\omega}(s)\,ds}{(1-r)^\alpha},\quad \alpha=\frac{1}{C_2},$$

is decreasing on $[0,1)$. So,

$$h(r)=\frac{\int_r^1\omega(s)\,ds}{(1-r)^\alpha}$$

is essentially decreasing on $[0,1)$. This together with (26) gives

$$\begin{aligned}&\int_{S(I)}|g'(z)|^2\omega^\star(z)\,dA(z)\\&\quad=\int_{S(I)}|g'(z)|^2\frac{\omega^\star(z)}{(1-|z|)^{\alpha+1}}(1-|z|)^{\alpha+1}\,dA(z)\\&\quad\asymp\int_{S(I)}|g'(z)|^2\frac{\int_{|z|}^1\omega(s)s\,ds}{(1-|z|)^\alpha}(1-|z|)^{\alpha+1}\,dA(z)\\&\quad\lesssim\frac{\int_{1-|I|}^1\omega(s)\,ds}{|I|^\alpha}\int_{S(I)}|g'(z)|^2(1-|z|)^{\alpha+1}\,dA(z)\\&\quad\lesssim\omega\left(S(I)\right),\quad |I|\le\frac{1}{2},\end{aligned}$$

and therefore $g\in\mathcal{C}^1(\omega^\star)$.

(D). Let $\omega\in\mathcal{I}$, and assume on the contrary to the assertion that $\mathcal{B}\subset\mathcal{C}^1(\omega^\star)$. Ramey and Ullrich [54, Proposition 5.4] constructed $g_1,g_2\in\mathcal{B}$ such that $|g_1'(z)|+|g_2'(z)|\ge(1-|z|)^{-1}$ for all $z\in\mathbb{D}$. Since

$g_1, g_2 \in \mathcal{C}^1(\omega^\star)$ by the antithesis, (26) yields

$$
\begin{aligned}
\|f\|_{A^2_\omega}^2 &\gtrsim \int_{\mathbb{D}} |f(z)|^2 \left(|g_1'(z)|^2 + |g_2'(z)|^2\right) \omega^\star(z)\, dA(z) \\
&\ge \frac{1}{2} \int_{\mathbb{D}} |f(z)|^2 \left(|g_1'(z)| + |g_2'(z)|\right)^2 \omega^\star(z)\, dA(z) \\
&\ge \frac{1}{2} \int_{\mathbb{D}} |f(z)|^2 \frac{\omega^\star(z)}{(1-|z|)^2}\, dA(z) \asymp \int_{\mathbb{D}} |f(z)|^2 \frac{\int_{|z|}^1 \omega(s)\, ds}{(1-|z|)}\, dA(z) \\
&= \int_{\mathbb{D}} |f(z)|^2 \frac{\psi_\omega(|z|)}{1-|z|}\, \omega(z)\, dA(z) \qquad (46)
\end{aligned}
$$

for all $f \in \mathcal{H}(\mathbb{D})$. If $\int_{\mathbb{D}} \frac{\psi_\omega(|z|)}{1-|z|}\, \omega(z)\, dA(z) = \infty$, we choose $f \equiv 1$ to obtain a contradiction. Assume now that $\int_{\mathbb{D}} \frac{\psi_\omega(|z|)}{1-|z|}\, \omega(z)\, dA(z) < \infty$, and replace f in (46) by the test function $F_{a,2}$ from Lemma 3.1. Then (12) and Lemma 2.1 yield

$$
\omega^\star(a) \gtrsim \int_0^1 \frac{(1-|a|)^{\gamma+1}}{(1-|a|r)^\gamma} \frac{\psi_\omega(r)}{1-r}\, \omega(r)\, dr \gtrsim (1-|a|) \int_{|a|}^1 \frac{\psi_\omega(r)}{1-r}\, \omega(r)\, dr,
$$

and hence

$$
\int_{|a|}^1 \frac{\psi_\omega(r)}{1-r}\, \omega(r)\, dr \lesssim \int_{|a|}^1 \omega(r)\, dr, \quad a \in \mathbb{D}.
$$

By letting $|a| \to 1^-$, Bernouilli-l'Hôpital theorem and the assumption $\omega \in \mathcal{I}$ yield a contradiction.

(E) Recall that BMOA $\subset \mathcal{C}^1(\omega^\star)$ by Part (B). See [49, Proposition 5.2] for the remaining inclusion. In fact, there is constructed a lacunary series $g \in \mathcal{C}^1(\omega^\star) \setminus H^2$. □

Proposition 6.2. *Let $\omega \in \mathcal{I}$ such that both $\omega(r)$ and $\frac{\psi_\omega(r)}{1-r}$ are essentially increasing on $[0,1)$, and*

$$
\int_r^1 \omega(s) s\, ds \lesssim \int_{\frac{2r}{1+r^2}}^1 \omega(s) s\, ds, \quad 0 \le r < 1. \qquad (47)
$$

Then $\mathcal{C}^1(\omega^\star)$ is not conformally invariant.

Proof. Let $\omega \in \mathcal{I}$ be as in the assumptions. An standard calculation and Lemma 2.1 gives that $g \in \mathcal{C}^1(\omega^\star)$ if and only if

$$\sup_{b\in\mathbb{D}} \frac{(1-|b|)^2}{\omega(S(b))} \int_{\mathbb{D}} \frac{|g'(z)|^2}{|1-\overline{b}z|^2}\omega(S(z))\,dA(z) < \infty.$$

Let $g \in \mathcal{C}^1(\omega^\star) \setminus H^2$ be the function constructed in the proof of Proposition 6.1(E). Then

$$\begin{aligned}
\sup_{b\in\mathbb{D}} & \frac{(1-|b|)^2}{\omega(S(b))} \int_{\mathbb{D}} \frac{|(g\circ\varphi_a)'(z)|^2}{|1-\overline{b}z|^2}\omega(S(z))\,dA(z) \\
&\geq \frac{(1-|a|)^2}{\omega(S(a))} \int_{\mathbb{D}} \frac{|g'(\zeta)|^2}{|1-\overline{a}\varphi_a(\zeta)|^2}\omega(S(\varphi_a(\zeta)))\,dA(\zeta) \qquad (48)\\
&\geq \int_{D(0,|a|)} |g'(\zeta)|^2(1-|\zeta|)\left(\frac{\omega(S(\varphi_a(\zeta)))}{\omega(S(a))}\frac{|1-\overline{a}\zeta|^2}{1-|\zeta|}\right)dA(\zeta),
\end{aligned}$$

where

$$\begin{aligned}
\frac{\omega(S(\varphi_a(\zeta)))}{\omega(S(a))}\frac{|1-\overline{a}\zeta|^2}{1-|\zeta|} &= \frac{(1-|\varphi_a(\zeta)|)\int_{|\varphi_a(\zeta)|}^1 \omega(s)s\,ds}{(1-|a|)\int_{|a|}^1 \omega(s)s\,ds}\frac{|1-\overline{a}\zeta|^2}{1-|\zeta|} \\
&\gtrsim \frac{\int_{\frac{2|a|}{1+|a|^2}}^1 \omega(s)s\,ds}{\int_{|a|}^1 \omega(s)s\,ds} \gtrsim 1, \quad |\zeta| \leq |a|,
\end{aligned}$$

by Lemma 2.1 and (47). Since $g \notin H^2$, the assertion follows by letting $|a| \to 1^-$ in (48). □

6.2. Boundedness of the integral operator. Case $\mathbf{q} = \mathbf{p}$

We shall use the following preliminary result.

Lemma 6.3. *Let $0 < p, q < \infty$ and $\omega \in \widehat{\mathcal{D}}$. If $T_g : A^p_\omega \to A^q_\omega$ is bounded, then*

$$M_\infty(r, g') \lesssim \frac{(\omega^\star(r))^{\frac{1}{p}-\frac{1}{q}}}{1-r}, \quad 0 < r < 1.$$

Proof. Let $0 < p, q < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and assume that $T_g : A^p_\omega \to A^q_\omega$ is bounded. Consider the functions

$$f_{a,p}(z) = \frac{(1-|a|)^{\frac{\gamma+1}{p}}}{(1-\overline{a}z)^{\frac{\gamma+1}{p}}\,\omega\left(S(a)\right)^{\frac{1}{p}}}, \quad a \in \mathbb{D}.$$

By Lemma 3.1 there is $\gamma > 0$ such that $\sup_{a\in\mathbb{D}} \|f_{a,p}\|_{A^p_\omega} < \infty$. Since

$$\|h\|^q_{A^q_\omega} \ge \int_{\mathbb{D}\setminus D(0,r)} |h(z)|^q \omega(z)\,dA(z) \gtrsim M_q^q(r,h) \int_r^1 \omega(s)\,ds, \quad r \ge \frac{1}{2},$$

for all $h \in A^q_\omega$, we obtain

$$\begin{aligned} M_q^q(r, T_g(f_{a,p})) &\lesssim \frac{\|T_g(f_{a,p})\|^q_{A^q_\omega}}{\int_r^1 \omega(s)\,ds} \le \frac{\|T_g\|^q_{(A^p_\omega, A^q_\omega)} \cdot \left(\sup_{a\in\mathbb{D}} \|f_{a,p}\|^q_{A^p_\omega}\right)}{\int_r^1 \omega(s)\,ds} \\ &\lesssim \frac{1}{\int_r^1 \omega(s)\,ds}, \quad r \ge \frac{1}{2}, \end{aligned}$$

for all $a \in \mathbb{D}$. This together with the well-known relations

$$M_\infty(r,f) \lesssim \frac{M_q(\rho,f)}{(1-r)^{1/q}}, \quad M_q(r,f') \lesssim \frac{M_q(\rho,f)}{1-r}, \quad \rho = \frac{1+r}{2},$$

Lemma 2.1 and (26) yield

$$\begin{aligned} |g'(a)| &\asymp (\omega^\star(a))^{\frac{1}{p}} |T_g(f_{a,p})'(a)| \lesssim (\omega^\star(a))^{\frac{1}{p}} \frac{M_q\left(\frac{1+|a|}{2}, (T_g(f_{a,p}))'\right)}{(1-|a|)^{\frac{1}{q}}} \\ &\lesssim (\omega^\star(a))^{\frac{1}{p}} \frac{M_q((3+|a|)/4, T_g(f_{a,p}))}{(1-|a|)^{1+\frac{1}{q}}} \asymp \frac{(\omega^\star(a))^{\frac{1}{p}-\frac{1}{q}}}{1-|a|}, \ \ |a| \ge \frac{1}{2}. \end{aligned}$$

The assertion follows from this inequality. $\square$

Theorem 6.4. *If $\omega \in \widehat{\mathcal{D}}$, $g \in \mathcal{H}(\mathbb{D})$ and $0 < p < \infty$, then $T_g : A^p_\omega \to A^p_\omega$ is bounded if and only if $g \in \mathcal{C}^1(\omega^\star)$.*

Proof. If $p = 2$ the equivalence follows from Theorem 3.8, the definition of $\mathcal{C}^1(\omega^\star)$ and (45). The rest of the proof is divided in four cases.

Let $p > 2$ and assume that $g \in \mathcal{C}^1(\omega^\star)$. Since $L^{p/2}_\omega \backsimeq \left(L^{\frac{p}{p-2}}_\omega\right)^\star$, Theorem 3.9 shows that $T_g : A^p_\omega \to A^p_\omega$ is bounded if and only if

$$\begin{aligned} &\left|\int_{\mathbb{D}} h(u) \left(\int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2\,dA(z)\right) \omega(u)\,dA(u)\right| \\ &\qquad \lesssim \|h\|_{L^{\frac{p}{p-2}}_\omega} \|f\|^2_{A^p_\omega}, \quad h \in L^{\frac{p}{p-2}}_\omega. \end{aligned}$$

Bearing in mind Theorems 3.3 and 3.4,

$$\begin{aligned}
&\left|\int_{\mathbb{D}} h(u)\left(\int_{\Gamma(u)}|f(z)|^2|g'(z)|^2\,dA(z)\right)\omega(u)\,dA(u)\right|\\
&\quad\le \int_{\mathbb{D}}|f(z)|^2|g'(z)|^2\left(\int_{T(z)}|h(u)|\omega(u)\,dA(u)\right)\,dA(z)\\
&\quad\asymp \int_{\mathbb{D}}|f(z)|^2|g'(z)|^2\omega^\star(z)\left(\frac{1}{\omega\left(S(z)\right)}\int_{T(z)}|h(u)|\omega(u)\,dA(u)\right)dA(z)\\
&\quad\lesssim \int_{\mathbb{D}}|f(z)|^2|g'(z)|^2\omega^\star(z)M_\omega(|h|)(z)\,dA(z)\\
&\quad\lesssim \left(\int_{\mathbb{D}}|f(z)|^p|g'(z)|^2\omega^\star(z)\,dA(z)\right)^{\frac{2}{p}}\\
&\qquad\cdot\left(\int_{\mathbb{D}}\left(M_\omega(|h|)(z)\right)^{\frac{p}{p-2}}|g'(z)|^2\omega^\star(z)\,dA(z)\right)^{\frac{p-2}{p}}\\
&\quad\lesssim \|f\|^2_{A^p_\omega}\|h\|_{L^{\frac{p}{p-2}}_\omega},
\end{aligned}$$

so $T_g : A^p_\omega \to A^p_\omega$ is bounded.

Reciprocally, let $p>2$ and assume that $T_g : A^p_\omega \to A^p_\omega$ is bounded. By Theorem 3.9 this is equivalent to

$$\begin{aligned}
\|T_g(f)\|^p_{A^p_\omega} &\asymp \int_{\mathbb{D}}\left(\int_{\Gamma(u)}|f(z)|^2|g'(z)|^2\,dA(z)\right)^{\frac{p}{2}}\omega(u)\,dA(u)\\
&\lesssim \|f\|^p_{A^p_\omega}
\end{aligned}$$

for all $f\in A^p_\omega$. By using this together with (26), Fubini's theorem, Hölder's inequality and Lemma 3.10, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}}|f(z)|^p|g'(z)|^2\omega^\star(z)\,dA(z)\\
&\qquad\asymp \int_{\mathbb{D}}|f(z)|^p|g'(z)|^2\omega(T(z))\,dA(z)\\
&\qquad= \int_{\mathbb{D}}\int_{\Gamma(u)}|f(z)|^p|g'(z)|^2dA(z)\omega(u)\,dA(u)\\
&\qquad\le \int_{\mathbb{D}}N(f)(u)^{p-2}\int_{\Gamma(u)}|f(z)|^2|g'(z)|^2\,dA(z)\omega(u)\,dA(u)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_{\mathbb{D}} N(f)(u)^p \omega(u)\,dA(u)\right)^{\frac{p-2}{p}}\\
&\quad\cdot\left(\int_{\mathbb{D}}\left(\int_{\Gamma(u)}|f(z)|^2|g'(z)|^2 dA(z)\right)^{\frac{p}{2}}\omega(u)\,dA(u)\right)^{\frac{2}{p}}\\
&\lesssim \|f\|_{A^p_\omega}^p
\end{aligned}
$$

for all $f\in A^p_\omega$. Therefore $|g'(z)|^2\omega^\star(z)dA(z)$ is a p-Carleson measure for A^p_ω, and thus $g\in\mathcal{C}^1(\omega^\star)$ by the definition.

Let now $0<p<2$, and assume that $g\in\mathcal{C}^1(\omega^\star)$. Then

$$
\begin{aligned}
\|T_g(f)\|_{A^p_\omega}^p &\asymp \int_{\mathbb{D}}\left(\int_{\Gamma(u)}|f(z)|^2|g'(z)|^2\,dA(z)\right)^{\frac{p}{2}}\omega(u)\,dA(u)\\
&\leq \int_{\mathbb{D}} N(f)(u)^{\frac{p(2-p)}{2}}\left(\int_{\Gamma(u)}|f(z)|^p|g'(z)|^2\,dA(z)\right)^{\frac{p}{2}}\omega(u)\,dA(u)\\
&\leq \left(\int_{\mathbb{D}} N(f)(u)^p\omega(u)\,dA(u)\right)^{\frac{2-p}{2}}\\
&\quad\cdot\left(\int_{\mathbb{D}}\int_{\Gamma(u)}|f(z)|^p|g'(z)|^2\,dA(z)\omega(u)\,dA(u)\right)^{\frac{p}{2}}\\
&\lesssim \|f\|_{A^p_\omega}^{\frac{p(2-p)}{2}}\left(\int_{\mathbb{D}}|f(z)|^p|g'(z)|^2\omega(T(z))\,dA(z)\right)^{\frac{p}{2}}\\
&\asymp \|f\|_{A^p_\omega}^{\frac{p(2-p)}{2}}\left(\int_{\mathbb{D}}|f(z)|^p|g'(z)|^2\omega^\star(z)\,dA(z)\right)^{\frac{p}{2}}\lesssim \|f\|_{A^p_\omega}^p.
\end{aligned}
$$

Let now $0<p<2$, and assume that $T_g: A^p_\omega\to A^p_\omega$ is bounded. Then Lemma 6.3 and its proof imply $g\in\mathcal{B}$ and

$$
\|g\|_{\mathcal{B}}\lesssim\|T_g\|. \tag{49}
$$

Choose $\gamma>0$ large enough, and consider the functions

$$
F_{a,p}=\left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\frac{\gamma+1}{p}}
$$

of Lemma 3.1. Let $1<\alpha,\beta<\infty$ such that $\beta/\alpha=p/2<1$, and let α' and β' be the conjugate indexes of α and β. Then (26), Fubini's theorem, Hölder's inequality, (11) and (25) yield

$$\begin{aligned}
&\int_{S(a)} |g'(z)|^2\omega^\star(z)\,dA(z)\\
&\quad\asymp \int_{\mathbb{D}}\left(\int_{S(a)\cap\Gamma(u)} |g'(z)|^2|F_{a,p}(z)|^2\,dA(z)\right)^{\frac{1}{\alpha}+\frac{1}{\alpha'}}\omega(u)\,dA(u)\\
&\quad\le \left(\int_{\mathbb{D}}\left(\int_{\Gamma(u)} |g'(z)|^2|F_{a,p}(z)|^2\,dA(z)\right)^{\frac{\beta}{\alpha}}\omega(u)\,dA(u)\right)^{\frac{1}{\beta}}\\
&\qquad\cdot\left(\int_{\mathbb{D}}\left(\int_{\Gamma(u)\cap S(a)} |g'(z)|^2\,dA(z)\right)^{\frac{\beta'}{\alpha'}}\omega(u)\,dA(u)\right)^{\frac{1}{\beta'}}\\
&\quad\asymp \|T_g(F_{a,p})\|_{A^p_\omega}^{\frac{p}{\beta}}\|S_g(\chi_{S(a)})\|_{L^{\frac{\beta'}{\alpha'}}_\omega}^{\frac{1}{\alpha'}},\quad a\in\mathbb{D}, \qquad (50)
\end{aligned}$$

where

$$S_g(\varphi)(u)=\int_{\Gamma(u)}|\varphi(z)|^2|g'(z)|^2\,dA(z),\quad u\in\mathbb{D}\setminus\{0\},$$

for any bounded function φ on $\mathbb{D}$. Since $\beta/\alpha = p/2 < 1$, we have $\frac{\beta'}{\alpha'}>1$ with the conjugate exponent $\left(\frac{\beta'}{\alpha'}\right)'=\frac{\beta(\alpha-1)}{\alpha-\beta}>1$. Therefore

$$\|S_g(\chi_{S(a)})\|_{L^{\frac{\beta'}{\alpha'}}_\omega}=\sup_{\|h\|_{L_\omega^{\frac{\beta(\alpha-1)}{\alpha-\beta}}}\le 1}\left|\int_{\mathbb{D}}h(u)S_g(\chi_{S(a)})(u)\omega(u)\,dA(u)\right|.$$

By using Fubini's theorem, estimate (26), Hölder's inequality and Theorem 3.4, we deduce

$$\begin{aligned}
&\left|\int_{\mathbb{D}}h(u)S_g(\chi_{S(a)})(u)\omega(u)\,dA(u)\right|\\
&\quad\le\int_{\mathbb{D}}|h(u)|\int_{\Gamma(u)\cap S(a)}|g'(z)|^2\,dA(z)\,\omega(u)\,dA(u)\\
&\quad\lesssim\int_{S(a)}M_\omega(|h|)(z)|g'(z)|^2\omega^\star(z)\,dA(z)\\
&\quad\le\left(\int_{S(a)}|g'(z)|^2\omega^\star(z)\,dA(z)\right)^{\frac{\alpha'}{\beta'}}\\
&\qquad\cdot\left(\int_{\mathbb{D}}M_\omega(|h|)^{\left(\frac{\beta'}{\alpha'}\right)'}|g'(z)|^2\omega^\star(z)\,dA(z)\right)^{1-\frac{\alpha'}{\beta'}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\int_{S(a)} |g'(z)|^2 \omega^\star(z)\, dA(z)\right)^{\frac{\alpha'}{\beta'}} \\
&\quad \cdot \left(\sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))}\right)^{1-\frac{\alpha'}{\beta'}} \|h\|_{L_\omega^{\left(\frac{\beta'}{\alpha'}\right)'}}.
\end{aligned} \tag{51}
$$

By replacing $g(z)$ by $g_r(z) = g(rz)$, $0 < r < 1$, and combining (50)–(51), we obtain

$$
\begin{aligned}
&\int_{S(a)} |g_r'(z)|^2 \omega^\star(z)\, dA(z) \\
&\quad \lesssim \|T_{g_r}(F_{a,p})\|_{A^p_\omega}^{\frac{p}{\beta}} \left(\int_{S(a)} |g_r'(z)|^2 \omega^\star(z)\, dA(z)\right)^{\frac{1}{\beta'}} \\
&\qquad \cdot \left(\sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g_r'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))}\right)^{\frac{1}{\alpha'}\left(1-\frac{\alpha'}{\beta'}\right)}.
\end{aligned}
$$

We now claim that there exists a constant $C = C(\omega) > 0$ such that

$$
\sup_{0<r<1} \|T_{g_r}(F_{a,p})\|_{A^p_\omega}^p \le C\|T_g\|_{A^p_\omega}^p \omega(S(a)), \quad a \in \mathbb{D}. \tag{52}
$$

Taking this for granted for a moment, we deduce

$$
\begin{aligned}
&\left(\frac{\int_{S(a)} |g_r'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))}\right)^{\frac{1}{\beta}} \\
&\quad \lesssim \|T_g\|_{A^p_\omega}^{\frac{p}{\beta}} \left(\sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g_r'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))}\right)^{\frac{1}{\alpha'}\left(1-\frac{\alpha'}{\beta'}\right)}
\end{aligned}
$$

for all $0 < r < 1$ and $a \in \mathbb{D}$. This yields

$$
\frac{\int_{S(a)} |g_r'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))} \lesssim \|T_g\|^2, \quad a \in \mathbb{D},
$$

and so

$$
\begin{aligned}
\sup_{a\in\mathbb{D}} \frac{\int_{S(a)}|g(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))} &\le \sup_{a\in\mathbb{D}} \liminf_{r\to 1^-} \left(\frac{\int_{S(a)}|g_r'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S(a))}\right) \\
&\lesssim \|T_g\|^2
\end{aligned}
$$

by Fatou's lemma. Therefore $g \in \mathcal{C}^1(\omega^\star)$ by Theorem 3.3.

It remains to prove (52). To do this, let $a \in \mathbb{D}$. If $|a| \le r_0$, where $r_0 \in (0,1)$ is fixed, then the inequality in (52) follows by Theorem 3.9, the change of variable $rz = \zeta$, the fact

$$\Gamma(ru) \subset \Gamma(u), \quad 0 < r < 1, \tag{53}$$

and the assumption that $T_g : A^p_\omega \to A^p_\omega$ is bounded. If $a \in \mathbb{D}$ is close to the boundary, we consider two separate cases.

Let first $\frac{1}{2} < |a| \le \frac{1}{2-r}$. Then

$$|1 - \overline{a}z| \le \left|1 - \overline{a}\frac{z}{r}\right| + \frac{1-r}{2-r} \le 2\left|1 - \overline{a}\frac{z}{r}\right|, \quad |z| \le r.$$

Therefore Theorem 3.9, (53) and (12) yield

$$\begin{aligned}
\|T_{g_r}(F_{a,p})\|^p_{A^p_\omega} &\asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} r^2 |g'(rz)|^2 \, |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, du \\
&= \int_{\mathbb{D}} \left(\int_{\Gamma(ru)} |g'(z)|^2 \left| F_{a,p}\left(\frac{z}{r}\right)\right|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, du \\
&\le 2^{\gamma+1} \int_{\mathbb{D}} \left(\int_{\Gamma(ru)} |g'(z)|^2 \, |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, du \\
&\le 2^{\gamma+1} \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |g'(z)|^2 \, |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, du \\
&\asymp \|T_g(F_{a,p})\|^p_{A^p_\omega} \lesssim \|T_g\|^p_{A^p_\omega} \omega(S(a))
\end{aligned}$$

and hence

$$\|T_{g_r}(F_{a,p})\|^p_{A^p_\omega} \lesssim \|T_g\|^p_{A^p_\omega} \omega(S(a)), \quad \frac{1}{2} < |a| \le \frac{1}{2-r}. \tag{54}$$

Let now $|a| > \max\{\frac{1}{2-r}, \frac{1}{2}\}$. Then, by Theorem 3.9, (49) and Lemma 3.1, we deduce

$$\begin{aligned}
\|T_{g_r}(F_{a,p})\|^p_{A^p_\omega} &\asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} r^2 |g'(rz)|^2 \, |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, du \\
&\le M_\infty(r, g')^p \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |F_{a,p}(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \omega(u) \, du
\end{aligned}$$

$$
\begin{aligned}
&\lesssim M_\infty\left(2-\frac{1}{|a|},g'\right)^p(1-|a|)^p\left\|\left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\frac{\gamma+1}{p}-1}\right\|^p_{A^p_\omega}\\
&\lesssim \|g\|^p_{\mathcal{B}}\,\omega(S(a))\lesssim \|T_g\|^p_{A^p_\omega}\,\omega(S(a))
\end{aligned}
$$

for $\gamma>0$ large enough. This together with (54) gives (52). The proof of (i) is now complete. □

6.3. Boundedness of the integral operator. Case $\mathbf{q \geq p}$.

If $\alpha>1$ and $\omega\in\widehat{\mathcal{D}}$, $g\in\mathcal{C}^\alpha(\omega^\star)$ if and only if [49, Proposition 4.7]

$$M_\infty(r,g')\lesssim\frac{(\omega^\star(r))^{\frac{\alpha-1}{2}}}{1-r},\quad 0<r<1.$$

So using analogous ideas to those employed in the proof of Theorem 6.4 we can prove the following.

Theorem 6.5. *Let $0<p<q<\infty$, $\omega\in\widehat{\mathcal{D}}$ and $g\in\mathcal{H}(\mathbb{D})$.*

(i) *If $0<p<q$ and $\frac{1}{p}-\frac{1}{q}<1$, then the following conditions are equivalent:*

(a) *$T_g: A^p_\omega\to A^q_\omega$ is bounded;*

(b) $M_\infty(r,g')\lesssim\dfrac{(\omega^\star(r))^{\frac{1}{p}-\frac{1}{q}}}{1-r},\quad r\to 1^-$;

(c) $\displaystyle\sup_{I\subset\mathbb{T}}\frac{\int_{S(I)}|g'(z)|^2\omega^\star(z)\,dA(z)}{(\omega\left(S(I)\right))^\alpha}<\infty.$

(ii) *If $\frac{1}{p}-\frac{1}{q}\geq 1$, then $T_g: A^p_\omega\to A^q_\omega$ is bounded if and only if g is constant.*

6.4. Boundedness of the integral operator. Case $\mathbf{0<q<p}$.

We shall use Corollary 4.4 on factorization of A^p_ω-functions in order to study the remaining case.

Theorem 6.6. *If $0<q<p<\infty$, $g\in\mathcal{H}(\mathbb{D})$ and $\omega\in\widetilde{\mathcal{I}}\cup\mathcal{R}$, $T_g: A^p_\omega\to A^q_\omega$ is bounded if and only if $g\in A^s_\omega$, where $\frac{1}{s}=\frac{1}{q}-\frac{1}{p}$.*

Proof. The sufficiency can be proved arguing as in Proposition 3.11 and it is valid for any radial weight ω. Let first $g \in A^s_\omega$, where $s = \frac{pq}{p-q}$. Then Theorem 3.9, Hölder's inequality and Lemma 3.10 yield

$$\begin{aligned}
\|T_g(f)\|^q_{A^q_\omega} &\asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |f(z)|^2 |g'(z)|^2 \, dA(z) \right)^{\frac{q}{2}} \omega(u)\, dA(u) \\
&\le \int_{\mathbb{D}} (N(f)(u))^q \left(\int_{\Gamma(u)} |g'(z)|^2 \, dA(z) \right)^{\frac{q}{2}} \omega(u)\, dA(u) \\
&\le \left(\int_{\mathbb{D}} (N(f)(u))^p \omega(u)\, dA(u) \right)^{\frac{q}{p}} \\
&\quad \cdot \left(\int_{\mathbb{D}} \left(\int_{\Gamma(u)} |g'(z)|^2 \, dA(z) \right)^{\frac{pq}{2(p-q)}} \omega(u)\, dA(u) \right)^{\frac{p-q}{p}} \\
&\le C_1^{q/p} C_2(p,q,\omega) \|f\|^q_{A^p_\omega} \|g\|^q_{A^s_\omega}. \qquad (55)
\end{aligned}$$

Thus $T_g : A^p_\omega \to A^q_\omega$ is bounded.

In order to prove the converse we we will use ideas from [2, p. 170–171], where T_g acting on Hardy spaces is studied. We begin with the following result whose proof relies on Corollary 4.4.

Proposition 6.7. *Let* $0 < q < p < \infty$ *and* $\omega \in \widetilde{\mathcal{I}} \cup \mathcal{R}$, *and let* $T_g : A^p_\omega \to A^q_\omega$ *be bounded. Then* $T_g : A^{\hat{p}}_\omega \to A^{\hat{q}}_\omega$ *is bounded for any* $\hat{p} < p$ *and* $\hat{q} < q$ *with* $\frac{1}{\hat{q}} - \frac{1}{\hat{p}} = \frac{1}{q} - \frac{1}{p}$. *Further, if* $0 < p \le 2$, *then there exists* $C = C(p,q,\omega) > 0$ *such that*

$$\limsup_{\hat{p} \to p^-} \|T_g\|_{\left(A^{\hat{p}}_\omega, A^{\hat{q}}_\omega\right)} \le C \|T_g\|_{\left(A^p_\omega, A^q_\omega\right)}. \qquad (56)$$

Proof. Theorem 4.3 shows that for any $f \in A^{\hat{p}}_\omega$, there exist $f_1 \in A^p_\omega$ and $f_2 \in A^{\hat{p}p/(p-\hat{p})}_\omega$ such that

$$f = f_1 f_2 \quad \text{and} \quad \|f_1\|_{A^p_\omega} \cdot \|f_2\|_{A^{\frac{\hat{p}p}{p-\hat{p}}}_\omega} \le C_3 \|f\|_{A^{\hat{p}}_\omega} \qquad (57)$$

for some constant $C_3 = C_3(p,\hat{p},\omega) > 0$. We observe that $T_g(f) = T_F(f_2)$, where $F = T_g(f_1)$. Since $T_g : A^p_\omega \to A^q_\omega$ is bounded,

$$\|F\|_{A^q_\omega} = \|T_g(f_1)\|_{A^q_\omega} \le \|T_g\|_{\left(A^p_\omega, A^q_\omega\right)} \|f_1\|_{A^p_\omega} < \infty, \qquad (58)$$

and hence $F \in A^q_\omega$. Then (55) and the identity $\frac{1}{q} = \frac{1}{\hat{q}} - \frac{p-\hat{p}}{\hat{p}p}$ yield

$$\|T_g(f)\|_{A^{\hat{q}}_\omega} = \|T_F(f_2)\|_{A^{\hat{q}}_\omega} \le C_1^{\frac{1}{\hat{p}}-\frac{1}{p}} C_2 \|f_2\|_{A_\omega^{\frac{\hat{p}p}{p-\hat{p}}}} \|F\|_{A^q_\omega},$$

where $C_2 = C_2(q,\omega) > 0$. This together with (57) and (58) gives

$$\begin{aligned}\|T_g(f)\|_{A^{\hat{q}}_\omega} &\le C_1^{\frac{1}{\hat{p}}-\frac{1}{p}} C_2 \|T_g\|_{\left(A^p_\omega, A^q_\omega\right)} \|f_1\|_{A^p_\omega} \cdot \|f_2\|_{A_\omega^{\frac{\hat{p}p}{p-\hat{p}}}} \\ &\le C_1^{\frac{1}{\hat{p}}-\frac{1}{p}} C_2\, C_3\, \|T_g\|_{\left(A^p_\omega, A^q_\omega\right)} \|f\|_{A^{\hat{p}}_\omega}.\end{aligned} \tag{59}$$

Therefore $T_g : A^{\hat{p}}_\omega \to A^{\hat{q}}_\omega$ is bounded.

To prove (56), let $0 < p \le 2$ and let $0 < \hat{p} < 2$ be close enough to p such that

$$\min\left\{\frac{p}{p-\hat{p}}, \frac{\hat{p}p}{p-\hat{p}}\right\} > 2.$$

If $f \in A^{\hat{p}}_\omega$, then Corollary 4.4 shows that (57) holds with $C_3 = C_3(p,\omega)$. Therefore the reasoning in the previous paragraph and (59) give (56). □

With this result in hand, we are ready to prove that $g \in A^s_\omega$ whenever $T_g : A^p_\omega \to A^q_\omega$ is bounded and $1/s = 1/q - 1/p$. Let $0 < q < p < \infty$ and $\omega \in \widetilde{\mathcal{I}} \cup \mathcal{R}$, and let $T_g : A^p_\omega \to A^q_\omega$ be bounded. By the first part of Proposition 6.7, we may assume that $p \le 2$. We may also assume, without loss of generality, that $g(0) = 0$. Define $t^* = \sup\{t : g \in A^t_\omega\}$. Since the constant function 1 belongs to A^p_ω, we have $g = T_g(1) \in A^q_\omega$, and hence $t^* \ge q > 0$. Fix a positive integer m such that $\frac{t^*}{m} < p$. For each $t < t^*$, set $\hat{p} = \hat{p}(t) = \frac{t}{m}$, and define $\hat{q} = \hat{q}(t)$ by the equation $1/s = 1/\hat{q} - 1/\hat{p}$. Then $\hat{p} < p$, $\hat{q} < q$ and $T_g : A^{\hat{p}}_\omega \to A^{\hat{q}}_\omega$ is bounded by Proposition 6.7. Since $g^m = g^{\frac{t}{\hat{p}}} \in A^{\hat{p}}_\omega$, we have $g^{m+1} = (m+1)T_g(g^m) \in A^{\hat{q}}_\omega$ and

$$\|g^{m+1}\|_{A^{\hat{q}}_\omega} \le (m+1)\|T_g\|_{(A^{\hat{p}}_\omega, A^{\hat{q}}_\omega)} \|g^m\|_{A^{\hat{p}}_\omega},$$

that is,

$$\|g\|^{m+1}_{A^{(m+1)\hat{q}}_\omega} \le (m+1)\|T_g\|_{(A^{\hat{p}}_\omega, A^{\hat{q}}_\omega)} \|g\|^m_{A^t_\omega}. \tag{60}$$

Suppose first that for some $t < t^*$, we have

$$t \ge (m+1)\hat{q} = \left(\frac{t}{\hat{p}} + 1\right)\hat{q} = \hat{q} + t\left(1 - \frac{\hat{q}}{s}\right).$$

Then $s \le t < t^*$, and the result follows from the definition of t^*. It remains to consider the case in which $t < (m+1)\hat{q}$ for all $t < t^*$. By Hölder's inequality, $\|g\|^m_{A^t_\omega} \le C_1(m,\omega)\|g\|^m_{A^{(m+1)\hat{q}}_\omega}$. This and (60) yield

$$\|g\|_{A^{(m+1)\hat{q}}_\omega} \le C_2(m,\omega)\|T_g\|_{(A^{\hat{p}}_\omega, A^{\hat{q}}_\omega)}, \tag{61}$$

where $C_2(m,\omega) = C_1(m,\omega)(m+1)$. Now, as t increases to t^*, $\hat{p}$ increases to $\frac{t^*}{m}$ and $\hat{q}$ increases to $\frac{t^* s}{t^*+ms}$, so by (61) and (56) we deduce

$$\begin{aligned}\|g\|_{A_\omega^{\frac{(m+1)t^*s}{t^*+ms}}} &\le \limsup_{t\to t^*}\|g\|_{A^{(m+1)\hat{q}}_\omega} \le C_2(m,\omega)\limsup_{\hat{p}\to p^-}\|T_g\|_{(A^{\hat{p}}_\omega, A^{\hat{q}}_\omega)}\\ &\le C(p,q,m,\omega)\|T_g\|_{(A^p_\omega, A^q_\omega)} < \infty.\end{aligned}$$

The definition of t^* implies $\frac{(m+1)t^*s}{t^*+ms} \le t^*$, and so $t^* \ge s$. This finishes the proof of Theorem 6.6. $\square$

The main results of this section are gathered here.

Theorem 6.8. *Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $g \in \mathcal{H}(\mathbb{D})$.*

(i) *The following conditions are equivalent:*

  (ai) $T_g : A^p_\omega \to A^p_\omega$ *is bounded;*

  (bi) $g \in \mathcal{C}^1(\omega^\star)$.

(ii) *If $0 < p < q$ and $\frac{1}{p} - \frac{1}{q} < 1$, then the following conditions are equivalent:*

  (aii) $T_g : A^p_\omega \to A^q_\omega$ *is bounded;*

  (bii) $M_\infty(r, g') \lesssim \dfrac{(\omega^\star(r))^{\frac{1}{p}-\frac{1}{q}}}{1-r}, \quad r \to 1^-;$

  (cii) $g \in \mathcal{C}^{q/p}(\omega^\star)$.

(iii) *If $\frac{1}{p} - \frac{1}{q} \ge 1$, then $T_g : A^p_\omega \to A^q_\omega$ is bounded if and only if g is constant.*

(iv) *If $0 < q < p < \infty$ and $\omega \in \widetilde{\mathcal{I}} \cup \mathcal{R}$, then the following conditions are equivalent:*

  (aiv) $T_g : A^p_\omega \to A^q_\omega$ *is bounded;*

  (biv) $g \in A^s_\omega$, *where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.*

7. COMPOSITION OPERATORS

Each analytic self-map φ of $\mathbb{D}$ induces the composition operator $C_\varphi(f) = f \circ \varphi$ acting on $\mathcal{H}(\mathbb{D})$. With regard to the theory of composition operators, we refer to [24, 60, 63].

Let $\zeta \in \varphi^{-1}(z)$ denote the set of the points $\{\zeta_n\}$ in $\mathbb{D}$, organized by increasing moduli, such that $\varphi(\zeta_n) = z$ for all n, with each point repeated according to its multiplicity. For a radial weight ω and an analytic self-map φ of $\mathbb{D}$ we define the generalized Nevanlinna counting function as

$$N_{\varphi,\omega^\star}(z) = \sum_{\zeta\in\varphi^{-1}(z),} \omega^\star(z), \quad z \in \mathbb{D}\setminus\{\varphi(0)\}.$$

Using the characterization of the q-Carleson measures for A^p_ω provided in Theorem 3.3, Theorem 3.12 and a description of bounded differentiation operators from A^p_ω to L^q_μ [50], it has recently been proved the following result [52].

Theorem 7.1. *Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and v be a radial weight, and let φ be an analytic self-map of $\mathbb{D}$.*

(a) *If $p > q$, then the following assertions are equivalent:*

  (i) $C_\varphi : A^p_\omega \to A^q_v$ *is bounded;*

  (ii) $C_\varphi : A^p_\omega \to A^q_v$ *is compact;*

  (iii) $N\left(\frac{N_{\varphi,v^\star}}{\omega^\star}\right) \in L^{\frac{p}{p-q}}_\omega$.

(b) *If $q \geq p$, then $C_\varphi : A^p_\omega \to A^q_v$ is bounded if and only if*

$$\limsup_{|z|\to1^-} \frac{N_{\varphi,v^\star}(z)}{\omega^\star(z)^{\frac{q}{p}}} < \infty.$$

(c) *If $q \geq p$, then $C_\varphi : A^p_\omega \to A^q_v$ is compact if and only if*

$$\lim_{|z|\to1^-} \frac{N_{\varphi,v^\star}(z)}{\omega^\star(z)^{\frac{q}{p}}} = 0.$$

We observe that condition (iii) in the classical case $C_\varphi : A^p_\alpha \to A^q_\beta$ gives a characterization of bounded (and compact) operators that differs from the one in the existing literature [62]. Here we shall prove an extension of this last result to the class of regular weights.

Theorem 7.2. *Let $0 < q < p < \infty$, $\omega \in \mathcal{R}$ and v be a radial weight, and let φ be an analytic self-map of $\mathbb{D}$. Then the following assertions are equivalent:*

(i) $C_\varphi : A^p_\omega \to A^q_v$ *is bounded;*

(ii) $C_\varphi : A^p_\omega \to A^q_v$ *is compact;*

(iii) *The function*

$$z \mapsto \frac{N_{\varphi,v^\star}(z)}{\omega^\star(z)}$$

belongs to $L_\omega^{\frac{p}{p-q}}$*;*

(iv) *The function*

$$z \mapsto \frac{\int_{\Delta(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\, dA(\zeta)}{(1-|z|)^2}$$

belongs to $L_\omega^{\frac{p}{p-q}}$ *for some (equivalently for all) fixed* $r \in (0,1)$.

7.1. Preliminary results

A key result in the proof of Theorem 7.2 is the local good behavior of the generalized Nevanlinna counting function [52, Lemma 14].

Lemma 7.3. *Let φ be an analytic self-map of $\mathbb{D}$ and v a radial weight. Then $N_{\varphi,v^\star}$ is subharmonic on $\mathbb{D} \setminus \{\varphi(0)\}$.*

Next, using the subharmonicity of $|f|^p$, the definition of the class $\mathcal{I}nv$ and the fact that $\inf_{z\in K} \omega(z) > 0$ for any compact subset $K \subset \mathbb{D}$, it can be deduced the following.

Lemma 7.4. *Let $0 < p < \infty$ and $\omega \in \mathcal{I}nv$. Then the norm convergence in A^p_ω implies the uniform convergence on compact subsets of $\mathbb{D}$.*

We shall use the following result on composition operators acting on weighted Bergman spaces induced by weights that are not necessarily radial.

Proposition 7.5. *Let $0 < q, p < \infty$, $\omega \in \mathcal{I}nv$ such that the polynomials are dense in A^p_ω, and v be a weight. If $n \in \mathbb{N}$, then the following assertions are valid:*

(i) *$C_\varphi : A^p_\omega \to A^q_v$ is bounded if and only if $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is bounded. Moreover,*

$$\|C_\varphi\|_{(A^{np}_\omega, A^{nq}_v)} \asymp \|C_\varphi\|^{1/n}_{(A^p_\omega, A^q_v)}.$$

(ii) *If the norm convergence in A^q_v implies the uniform convergence on compact subsets of $\mathbb{D}$, then $C_\varphi : A^p_\omega \to A^q_v$ is compact if and only if $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is compact.*

Proof. (i) Let first $C_\varphi : A^p_\omega \to A^q_v$ be bounded and $f \in A^{np}_\omega$. Then $f^n \in A^p_\omega$ and

$$\begin{aligned}
\|C_\varphi(f)\|^{nq}_{A^{nq}_v} &= \int_{\mathbb{D}} |f \circ \varphi(z)|^{nq} v(z)\, dA(z) \\
&= \int_{\mathbb{D}} |f^n \circ \varphi(z)|^{q} v(z)\, dA(z) = \|C_\varphi(f^n)\|^q_{A^q_v} \\
&\le \|C_\varphi\|^q_{(A^p_\omega, A^q_v)} \|f^n\|^q_{A^p_\omega} = \|C_\varphi\|^q_{(A^p_\omega, A^q_v)} \|f\|^{nq}_{A^{np}_\omega},
\end{aligned}$$

so $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is bounded and $\|C_\varphi\|_{(A^{np}_\omega, A^{nq}_v)} \le \|C_\varphi\|^{1/n}_{(A^p_\omega, A^q_v)}$.

Conversely, let $C_\varphi : A^{np}_\omega \to A^{nq}_v$ be bounded and $f \in A^p_\omega$. Now n applications of Theorem 4.3 show that f can be represented in the form $f = \prod_{k=1}^n f_k$, where each $f_k \in A^{np}_\omega$ and

$$\prod_{k=1}^n \|f_k\|_{A^{np}_\omega} \le C(n, p, \omega) \|f\|_{A^p_\omega}.$$

Therefore Hölder's inequality gives

$$\begin{aligned}
\|C_\varphi(f)\|^q_{A^q_v} &= \int_{\mathbb{D}} \left| \left(\left(\prod_{k=1}^n f_k \right) \circ \varphi \right)(z) \right|^q v(z)\, dA(z) \\
&= \int_{\mathbb{D}} \prod_{k=1}^n |\left(f_k \circ \varphi\right)(z)|^q v(z)\, dA(z) \le \prod_{k=1}^n \|C_\varphi(f_k)\|^q_{A^{nq}_v} \\
&\le \|C_\varphi\|^{nq}_{(A^{np}_\omega, A^{nq}_v)} \prod_{k=1}^n \|f_k\|^q_{A^{np}_\omega} \\
&\le C(n, p, q, \omega) \|C_\varphi\|^{nq}_{(A^{np}_\omega, A^{nq}_v)} \|f\|^q_{A^p_\omega}.
\end{aligned}$$

So $C_\varphi : A^p_\omega \to A^q_v$ is bounded and $\|C_\varphi\|^{1/n}_{(A^p_\omega, A^q_v)} \lesssim \|C_\varphi\|_{(A^{np}_\omega, A^{nq}_v)}$.

(ii) We may assume that φ is not constant. Let first $C_\varphi : A^p_\omega \to A^q_v$ be compact. To see that also $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is compact, take $\{f_j\} \subset A^{np}_\omega$ such that $\sup_j \|f_j\|_{A^{np}_\omega} < \infty$. Since $\omega \in \mathcal{I}nv$ by the assumption, Lemma 7.4 and Montel's theorem imply the existence of a subsequence $\{f_{j_k}\}$ such that f_{j_k} converges uniformly on compact subsets of $\mathbb{D}$ to some $f \in \mathcal{H}(\mathbb{D})$, and further $f \in A^{np}_\omega$ by Fatou's lemma. Therefore the sequence $\{g_k\} = \{f_{j_k} - f\}$ converges uniformly to 0 on compact subsets of $\mathbb{D}$ and $\sup_k \|g_k\|_{A^{np}_\omega} < \infty$. Hence $\{g^n_k\}$ converges uniformly to 0 compact subsets of $\mathbb{D}$ and $\sup_k \|g^n_k\|_{A^p_\omega} < \infty$. Now, since $C_\varphi : A^p_\omega \to A^q_v$ is compact there is a subsequence $\{g^n_{k_m}\}$ and $G \in \mathcal{H}(\mathbb{D})$ such that

$$\|C_\varphi(g^n_{k_m} - G)\|^q_{A^q_v} \to 0, \quad m \to \infty. \tag{62}$$

Now, by the hypotheses on v, $g^n_{k_m} \circ \varphi - G \circ \varphi$ converges uniformly to 0 on compact subsets of $\mathbb{D}$, and since φ is not constant, this and the uniform convergence of $\{g^n_k\}$ to zero imply $G \equiv 0$. So, by (62),

$$\|C_\varphi(g_{k_m})\|^{nq}_{A^{nq}_v} \to 0, \quad m \to \infty,$$

and hence $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is compact.

Conversely, let $C_\varphi : A^{np}_\omega \to A^{nq}_v$ be compact and take $\{f_j\} \subset A^p_\omega$ such that $\sup_j \|f_j\|_{A^p_\omega} < \infty$. As earlier, since $\omega \in \mathcal{I}nv$, we may use Lemma 7.4 and Montel's theorem to find a subsequence $\{f_{j_k}\}$ such that f_{j_k} converges uniformly on compact subsets of $\mathbb{D}$ to some $f \in \mathcal{H}(\mathbb{D})$, that in fact belongs to A^p_ω by Fatou's lemma. Therefore $\{g_k\} = \{f_{j_k} - f\}$ convergence uniformly to 0 on compact subsets of $\mathbb{D}$ and $\sup_k \|g_k\|_{A^p_\omega} < \infty$. By n applications of [49, Theorem 3.1], each function g_k can be factorized to $g_k = \prod_{m=1}^n g_{k,m}$, where each $g_{k,m} \in A^{np}_\omega$ and

$$\prod_{m=1}^{n} \|g_{k,m}\|_{A^{np}_\omega} \le C(n,p,\omega)\|g_k\|_{A^p_\omega}.$$

Since $\sup_k \|g_k\|_{A^p_\omega} < \infty$, this implies $\sup_k \|g_{k,m}\|_{A^{np}_\omega} < \infty$ for all $m = 1, 2, \dots, n$. Using that $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is bounded, we get functions $G_1, \dots, G_m \in A^{np}_\omega$ and subsequences $\{g_{k_l,m}\}$ such that

$$\|g_{k_l,m} \circ \varphi - G_m\|_{A^q_v} \to 0, \quad l \to \infty, \quad m = 1, 2, \dots, n. \tag{63}$$

Since the norm convergence in A^q_v implies the uniform convergence on compact subsets of $\mathbb{D}$, and φ is not constant, the uniform convergence of g_k to zero and (63) imply that at least one of the functions $G_1, \dots, G_m$ must be identically zero. Without loss of generality, we may assume that $G_1 \equiv 0$. Then, by Hölder's inequality, we deduce

$$\begin{aligned}
\|C_\varphi(g_{kl})\|^q_{A^q_v} &= \int_{\mathbb{D}} \left| \left(\left(\prod_{m=1}^n g_{k_l,m} \right) \circ \varphi \right)(z) \right|^q v(z)\, dA(z) \\
&\le \prod_{k=1}^n \|C_\varphi(g_{k_l,m})\|^q_{A^{nq}_v} \\
&\le \|C_\varphi(g_{k_l,1})\|^q_{A^{nq}_v} \|C_\varphi\|^{(n-1)q}_{(A^{np}_\omega, A^{nq}_v)} \prod_{k=2}^n \|g_{k_l,m}\|^q_{A^p_\omega} \\
&\le C(n,p,q,\omega) \|C_\varphi(g_{k_l,1}) - G_1\|^q_{A^{nq}_v},
\end{aligned}$$

which together with (63) finishes the proof. □

We also need an atomic decomposition of A^p_ω-functions, $\omega \in \mathcal{R}$. Recall that $A = \{z_k\}_{k=0}^\infty \subset \mathbb{D}$ is uniformly discrete if it is separated in the hyperbolic metric, it is an ε-net if $\mathbb{D} = \bigcup_{k=0}^\infty \Delta(z_k, \varepsilon)$, and finally, it is a δ-lattice if it is a 5δ-net and uniformly discrete with constant $\gamma = \delta/5$.

Proposition 7.6. *Let $1 < p < \infty$, $\omega \in \mathcal{R}$ and $\{z_k\}_{k=1}^\infty \subset \mathbb{D} \setminus \{0\}$ be an ε-net. Then the following assertions hold:*

(i) *If $f \in A^p_\omega$, then there exist $\{c_j\}_{j=1}^\infty \in l^p$ and $M = M(\omega) > 0$ such that*

$$f(z) = \sum_{j=1}^\infty \frac{c_j}{(\omega(\Delta(z_j, \epsilon)))^{1/p}} \left(\frac{1 - |z_j|^2}{1 - \overline{z}_j z} \right)^M \tag{64}$$

and $\|\{c_j\}_{j=1}^\infty\|_{l^p} \lesssim \|f\|_{A^p_\omega}$.

(ii) *If $\{c_j\}_{j=1}^\infty \in l^p$, then there exists $M = M(\omega) > 0$ such that the function defined by the infinite sum in (64) converges uniformly on compact subsets of $\mathbb{D}$ to an analytic function $f \in A^p_\omega$ and $\|f\|_{A^p_\omega} \lesssim \|\{c_j\}_{j=1}^\infty\|_{l^p}$.*

Proof. (i) Let $1 < p < \infty$, $\omega \in \mathcal{R}$ and $f \in A^p_\omega$. Then $\omega \in B_p(\eta)$ for each $\eta = \eta(p,\omega)$ large enough by Lemma 2.2. That is, $\frac{\omega(z)}{(1-|z|)^\eta}$

satisfies [40, (4.2)] with $\beta = 0$, $\gamma = \left(1+\frac{p}{p'}\right)\eta$ and $\alpha = \eta$. Consequently, [40, Theorem 4.1] implies the existence of $\{c_j\}_{j=1}^{\infty} \in l^p$ such that

$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{\left(\omega\left(\Delta(z_j,\epsilon)\right)\right)^{1/p}} \left(\frac{1-|z_j|^2}{1-\overline{z}_j z}\right)^{\eta+2}$$

and $\|\{c_j\}_{j=1}^{\infty}\|_{l^p} \lesssim \|f\|_{A^p_\omega}$. Hence (i) is proved with $M = \eta + 2$.

(ii) Let $\{c_j\}_{j=1}^{\infty} \in l^p$ be given. By the proof of (i) we know that $\omega \in B_p(\eta)$ for each η large enough. The assertion follows by [40, Theorem 4.1]. □

An atomic-decomposition for A^p_ω-functions, $0 < p \leq 1$ and $\omega \in \mathcal{R}$, can also be obtained by using the results by Constantin [20] (see also [3, Theorem 2.2]). However, we do not get into this question for a matter of simplicity and because we are able to prove Theorem 7.2 just by using Proposition 7.6.

7.2. Proof of Theorem 7.2

We shall prove (iv)⇒(iii)⇒(i)⇒(iv)⇒(ii)⇒(i).

(iv)⇒(iii). If $0 < r < 1$ is fixed, then $N_{\varphi,\omega^\star}$ is subharmonic in each pseudohyperbolic disc that is sufficiently close to the boundary by Lemma 7.3. The implication follows by this fact because $\omega^\star$ is essentially constant on pseudohyperbolic discs.

(iii)⇒(i). Let first $2 \leq q < \infty$, and let $0 < r < 1$ be fixed. Then the function $|f|^{q-2}|f'|^2$ is subharmonic. By using this and arguing as in the proof of [62, Lemma 2.4], we deduce

$$\begin{aligned} |f(\zeta)|^{q-2}|f'(\zeta)|^2 &\lesssim \frac{1}{(1-|\zeta|)^2} \int_{\Delta(\zeta,r^2)} |f(z)|^{q-2}|f'(z)|^2\,dA(z) \\ &\lesssim \frac{1}{(1-|\zeta|)^4} \int_{\Delta(\zeta,r)} |f(z)|^q\,dA(z), \quad \zeta \in \mathbb{D}. \end{aligned} \tag{65}$$

Now Theorem 3.8, a change of variable, (65) and Fubini's theorem give

$$\begin{aligned} \|C_\varphi(f)\|^q_{A^q_v} &\asymp \int_{\mathbb{D}} |f(\varphi(z))|^{q-2}|f'(\varphi(z))|^2|\varphi'(z)|^2 v^\star(z)\,dA(z) \\ &\quad + v(\mathbb{D})|f(\varphi(0))|^q \\ &= \int_{\mathbb{D}} |f(\zeta)|^{q-2}|f'(\zeta)|^2 N_{\varphi,v^\star}(\zeta)\,dA(\zeta) + v(\mathbb{D})|f(\varphi(0))|^q \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{D}} \left(\frac{1}{(1-|\zeta|)^4} \int_{\Delta(\zeta,r)} |f(z)|^q \, dA(z) \right) N_{\varphi,v^\star}(\zeta)\, dA(\zeta) \\
&\qquad + v(\mathbb{D}) |f(\varphi(0))|^q \\
&\asymp \int_{\mathbb{D}} \left(\frac{1}{(1-|z|)^2} \int_{\Delta(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{(1-|\zeta|)^2} \, dA(\zeta) \right) |f(z)|^q \, dA(z) \\
&\qquad + v(\mathbb{D}) |f(\varphi(0))|^q.
\end{aligned}
\tag{66}
$$

Let $M[f]$ denote the Hardy-Littlewood maximal function defined by

$$
M[f](z) = \sup_{\delta>0} \frac{1}{A(\Delta(z,\delta))} \int_{\Delta(z,\delta)} |f(\zeta)| \, dA(\zeta), \quad z \in \mathbb{D},
$$

for each $f \in L^1$. The maximal function $M[f]$ is bounded on L^p when $p > 1$. Therefore (66), the assumption $\omega \in \mathcal{R}$, Hölder's inequality and (26) yield

$$
\begin{aligned}
\|C_\varphi(f)\|_{A^q_v}^q &\lesssim \int_{\mathbb{D}} \frac{1}{\omega(z)^{\frac{p-q}{p}}} \left(\frac{1}{(1-|z|)^2} \int_{\Delta(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{(1-|\zeta|)^2 \omega^{q/p}(\zeta)} \, dA(\zeta) \right) \\
&\qquad \cdot |f(z)|^q \omega(z) \, dA(z) + v(\mathbb{D}) |f(\varphi(0))|^q \\
&\lesssim \int_{\mathbb{D}} \frac{1}{\omega(z)^{\frac{p-q}{p}}} M\left[\frac{N_{\varphi,v^\star}}{(1-|\zeta|)^2 \omega^{q/p}} \right](z) |f(z)|^q \omega(z) \, dA(z) \\
&\qquad + v(\mathbb{D}) |f(\varphi(0))|^q \\
&\lesssim \|f\|_{A^p_\omega}^q \left\| M\left[\frac{N_{\varphi,v^\star}}{(1-|\zeta|)^2 \omega^{q/p}} \right] \right\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} \\
&\lesssim \|f\|_{A^p_\omega}^q \left\| \frac{N_{\varphi,v^\star}}{(1-|\zeta|)^2 \omega^{q/p}} \right\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} \\
&\asymp \|f\|_{A^p_\omega}^q \left\| \frac{N_{\varphi,v^\star}}{\omega^\star} \right\|_{L_\omega^{\frac{p}{p-q}}}^{\frac{p}{p-q}} \lesssim \|f\|_{A^p_\omega}^q.
\end{aligned}
$$

Thus $C_\varphi : A^p_\omega \to A^q_v$ is bounded provided $q \geq 2$. If $0 < q < 2$, we may choose $n \in \mathbb{N}$ such that $nq \geq 2$. Then $C_\varphi : A^{np}_\omega \to A^{nq}_v$ is bounded by the previous argument and so is $C_\varphi : A^p_\omega \to A^q_v$ by Proposition 7.5.

(i)$\Rightarrow$(iv). By Proposition 7.5(i) we may assume that $q \geq 2$. Next, bearing in mind Proposition 7.6, we pick up an ϵ-net $\{z_k\}_{k=0}^\infty \subset \mathbb{D} \setminus \{0\}$ and $M = M(\omega) > 0$ large enough such that for any $\{c_j\}_{j=1}^\infty \in l^p$ the function f defined by (64) converges uniformly on compact subsets of $\mathbb{D}$ to an analytic function $f \in A^p_\omega$ such that $\|f\|_{A^p_\omega} \lesssim \|\{c_j\}_{j=1}^\infty\|_{l^p}$.

For simplicity, let us write $h_j(z) = (\omega(\Delta(z_j,\epsilon)))^{-1/p}\left(\frac{1-|z_j|^2}{1-\overline{z}_j z}\right)^M$. Let us consider the classical Rademacher functions $\{r_j(t)\}$ and set $f_t(z) = \sum_{j=1}^\infty r_j(t)c_j h_j(z)$. Since $C_\varphi : A^p_\omega \to A^q_v$ is bounded by the assumption,

$$\|C_\varphi(f_t)\|^q_{A^q_v} \le \|C_\varphi\|^q_{(A^p_\omega, A^q_v)}\|f_t\|^q_{A^p_\omega} \lesssim \|C_\varphi\|^q_{(A^p_\omega, A^q_v)}\|\{c_j\}_{j=1}^\infty\|^q_{l^p},$$

from which an integration with respect to t gives

$$\sum_{j=1}^\infty |c_j|^q\|C_\varphi(h_j)\|^q_{A^q_v} \le \left\|\left(\sum_{j=1}^\infty |c_j|^2|C_\varphi(h_j)|^2\right)^{1/2}\right\|^q_{L^q_v} \lesssim \|\{c_j\}_{j=1}^\infty\|^q_{l^p},$$

where in the first inequality we used the hypothesis $q \ge 2$. Therefore Theorem 3.8 and a change of variable give

$$\sum_{j=1}^\infty |c_j|^q \int_{\mathbb{D}} |h_j(\zeta)|^{q-2}|h_j'(\zeta)|^2 N_{\varphi,v^\star}(\zeta)\,dA(\zeta) \lesssim \|\{c_j\}_{j=1}^\infty\|^q_{l^p}.$$

Now, a calculation shows that

$$\sum_{j=1}^\infty \frac{|c_j|^q}{(\omega(\Delta(z_j,\epsilon)))^{q/p}(1-|z_j|^2)^2}\int_{\Delta(z_j,r)} N_{\varphi,v^\star}(\zeta)\,dA(\zeta) \lesssim \|\{c_j\}_{j=1}^\infty\|^q_{l^p}$$

for any fixed $0<r<1$, and so the sequence

$$\left\{\frac{\int_{\Delta(z_j,r)} N_{\varphi,v^\star}(\zeta)\,dA(\zeta)}{(\omega(\Delta(z_j,\epsilon)))^{q/p}(1-|z_j|^2)^2}\right\}_{j=1}^\infty$$

belongs to $(l^{p/q})^\star$. Since $\omega \in \mathcal{R}$, this is equivalent to

$$\sum_{j=1}^\infty \left(\frac{\int_{\Delta(z_j,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\,dA(\zeta)}{(1-|z_j|^2)^2}\right)^{\frac{p}{p-q}} \omega(\Delta(z_j,\epsilon)) < \infty.$$

Finally, for a given $0<s<1$, by choosing $0<\varepsilon<s<r<1$ appropriately, we deduce

$$\begin{aligned}&\int_{\mathbb{D}}\left(\frac{1}{(1-|z|)^2}\int_{\Delta(z,s)}\frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\,dA(\zeta)\right)^{\frac{p}{p-q}}\omega(z)\,dA(z)\\ &\qquad\le \sum_{j=1}^\infty \int_{\Delta(z_j,\varepsilon)}\left(\frac{1}{(1-|z|)^2}\int_{\Delta(z,s)}\frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\,dA(\zeta)\right)^{\frac{p}{p-q}}\omega(z)\,dA(z)\end{aligned}$$

$$\lesssim \sum_{j=1}^{\infty} \left(\frac{\int_{\Delta(z_j,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\, dA(\zeta)}{(1-|z_j|^2)^2} \right)^{\frac{p}{p-q}} \omega\left(\Delta(z_j,\epsilon)\right) < \infty,$$

and thus (iv) is satisfied.

Since trivially (ii)⇒(i), it suffices to show (iv)⇒(ii) to complete the proof. By Proposition 7.5(ii) we may assume that $q \geq 2$. Since $\omega \in \mathcal{R}$, it is enough to prove that for each $\{f_n\} \in A^p_\omega$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ and $K = \sup_n \|f_n\|_{A^p_\omega} < \infty$ we have

$$\lim_{n\to\infty} \|C_\varphi f_n\|_{A^q_v} = 0. \tag{67}$$

To see this, let $\varepsilon > 0$. Choose r_0 such that $\varphi(0) \in D(0, r_0)$ and

$$\left(\int_{r_0<|z|<1} \left(\frac{1}{(1-|z|)^2} \int_{\Delta(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)} \right)^{\frac{p}{p-q}} \omega(z)\, dA(z) \right)^{(p-q)/p} < \varepsilon.$$

Further, let $n_0 \in \mathbb{N}$ such that $|f_n(z)| < \varepsilon^{1/q}$ for all $n \geq n_0$ and $z \in \overline{D(0,r_0)}$. Then (66) shows that for all $n \geq n_0$ we have

$$\begin{aligned}
\|C_\varphi(f_n)\|^q_{A^q_v} &\lesssim \int_{\mathbb{D}} \left(\frac{1}{(1-|z|)^2} \int_{D(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{(1-|\zeta|)^2}\, dA(\zeta) \right) |f_n(z)|^q\, dA(z) \\
&\quad + v(\mathbb{D})|f(\varphi(0))|^q \\
&\lesssim \int_{\mathbb{D}} \left(\frac{1}{(1-|z|)^2} \int_{D(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\, dA(\zeta) \right) |f_n(z)|^q \omega(z)\, dA(z) \\
&\quad + \varepsilon v(\mathbb{D}) \\
&\lesssim \varepsilon \int_{D(0,r_0)} \left(\frac{1}{(1-|z|)^2} \int_{D(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\, dA(\zeta) \right) \omega(z)\, dA(z) \\
&\quad + \|f_n\|^q_{A^p_\omega} \cdot I + \varepsilon v(\mathbb{D}) \\
&\lesssim \varepsilon,
\end{aligned}$$

where

$$I = \left(\int_{r_0<|z|<1} \left(\frac{1}{(1-|z|)^2} \int_{D(z,r)} \frac{N_{\varphi,v^\star}(\zeta)}{\omega^\star(\zeta)}\, dA(\zeta) \right)^{\frac{p}{p-q}} \omega(z)\, dA(z) \right)^{\frac{p-q}{p}}.$$

This gives (67) and thus completes the proof. □

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Carleson Measures in Spaces of Analytic Functions

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Abstract. These are the course notes generated for a series of lectures to be given at *The Summer School in Complex and Harmonic Analysis and Related Topics* at the University of Eastern Finland, Mekrijärvi, June 14-18, 2014. The goal of the lectures and the lecture notes is to introduce the participants to Carleson measures for spaces of analytic functions. For a domain $\Omega \subset \mathbb{C}^n$ and a reproducing kernel Hilbert space of analytic functions $\mathcal{H}$ we will be interested in studying the *Carleson measures* for the space $\mathcal{H}$. Namely, we will be interested in a characterization of the measures μ for which we have

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq \|\mu\|^2_{CM(\mathcal{H})} \|f\|^2_{\mathcal{H}} \quad \forall f \in \mathcal{H}.$$

The characterization we will seek is in terms of geometric quantities related to the measure. In particular, we will be concerned with cases where it is possible to obtain a characterization in terms of testing the above estimate on simpler classes of functions. We will also be interested in equivalent ways to control the constant $\|\mu\|_{CM(\mathcal{H})}$.

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1. OVERVIEW

These notes are meant to be largely self-contained, but at times out of necessity we will simply refer to known facts and point the interested reader to a reference where this is explained further. Not all topics from these notes will be covered during the lectures, but in the interest of completeness, are provided for the reader to see the connections between these topics and other areas of mathematics. However, it should be mentioned that these notes will work best when coupled with the lectures presented. Every effort has been made to reduce and minimize typographical errors; though with probability one I am sure that some still exist in the file and so I urge the reader to exercise caution while working through the notes.

In this series of lectures we will be interested in special measures for a class of reproducing kernel Hilbert spaces of analytic functions. Let Ω be an open set in $\mathbb{C}^n$ and let $\mathcal{H}$ be a Hilbert function space of analytic functions over Ω with reproducing kernel K_λ. Namely, for any $\lambda \in \Omega$ and for all $f \in \mathcal{H}$ we have that:

$$f(\lambda) = \langle f, K_\lambda \rangle_{\mathcal{H}} .$$

An equivalent formulation of the above, by the Riesz Representation Theorem, is that the linear functional of point evaluation at λ is bounded. The main examples we will be interested in these lectures are the Hardy space on the unit disc (or unit ball) and the Besov-Sobolev spaces of analytic functions on the unit ball in $\mathbb{C}^n$. Each of these spaces will be introduced in the context at hand, and basic properties associated with the spaces will be outlined in the appropriate section. For more detailed information about these spaces, the interested reader will need to consult the references pointed to at the end of each section.

Our main object of interest will be *Carleson Measures* for the space $\mathcal{H}$. More precisely, we are interested in the following objects:

Definition 1.1. ($\mathcal{H}$-Carleson Measure) A non-negative measure μ on Ω is $\mathcal{H}$-Carleson if and only if

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq C^2 \|f\|_{\mathcal{H}}^2 \quad \forall f \in \mathcal{H}.$$

The norm of the Carleson measure μ will be the best constant C appearing above and below will be denoted $\|\mu\|_{CM(\mathcal{H})}$. An alternative way to view these objects, is that we are asking for the measures μ so that $\mathcal{H}$ embeds continuously in $L^2(\Omega\ \mu)$, i.e. $\iota : \mathcal{H} \to L^2(\Omega; \mu)$ is bounded. In particular, our main goal in these talks will be to:

Question 1.2. Give a 'geometric' and 'testable' characterization of the $\mathcal{H}$-Carleson measures.

It turns out we always have some necessary conditions that give good insight into the problem at hand. Let k_λ denote the normalized reproducing kernel for the space $\mathcal{H}$:

$$k_\lambda(z) = \frac{K_\lambda(z)}{\|K_\lambda\|_{\mathcal{H}}}.$$

Testing on the reproducing kernel k_λ we always have the necessary, in general geometric, condition for the measure μ to be Carleson:

$$\sup_{\lambda\in\Omega} \int_\Omega |k_\lambda(z)|^2\, d\mu(z) \leq \|\mu\|^2_{CM(\mathcal{H})}\,.$$

In the cases of interest it is possible to identify a point $\lambda \in \Omega$ with an open set, I_λ on the boundary of Ω. In this setting, the necessary 'geometric' condition will be:

$$\mu\left(T(I_\lambda)\right) \lesssim \|K_\lambda\|^{-2}_{\mathcal{H}}\,.$$

Here $T(I_\lambda)$ is the 'tent' over the set I_λ in the boundary $\partial\Omega$. In some examples of the spaces we will study this simple necessary condition is also sufficient to characterize the Carleson measures in a geometric fashion. However, for other spaces this is no longer the case and much more must be done to understand the measures. In these lectures we will see situations where both the "easy" case and the "hard" case arise. The complete answer to the geometric characterization we will obtain will be intimately connected to the structure of the reproducing kernel for the space $\mathcal{H}$.

Nevertheless, Carleson measures are fundamental objects in complex analysis and harmonic analysis and having a usable characterization of the measures in question is valuable. We list some representative examples of where they play a decisive role:

- Bessel Sequences/Interpolating Sequences/Riesz Sequences: Given $\Lambda = \{\lambda_j\}_{j=1}^{\infty} \subset \Omega$ determine functional analytic basis properties for the set $\{k_{\lambda_j}\}_{j=1}^{\infty}$ in terms of the set Λ. For example, in many cases:

  - $\{k_{\lambda_j}\}_{j=1}^{\infty}$ is a Bessel sequence if and only if μ_Λ is $\mathcal{H}$-Carleson;
  - $\{k_{\lambda_j}\}_{j=1}^{\infty}$ is a Riesz sequence if and only if μ_Λ is $\mathcal{H}$-Carleson and separated.

  Here $\mu_\Lambda = \sum_{j=1}^{\infty} \left\| K_{\lambda_j} \right\|_{\mathcal{H}}^{-2} \delta_{\lambda_j}$. The constants that appear in characterizing the Bessel/Riesz sequences are related to $\|\mu_\Lambda\|_{CM(\mathcal{H})}$.

- Multipliers of $\mathcal{H}$: Characterize the pointwise multipliers for $\mathcal{H}$. In many situations we have that:

  $$\begin{aligned} \text{Multi}(\mathcal{H}) &= H^\infty \cap CM(\mathcal{H}); \\ \|\varphi\|_{\text{Multi}(\mathcal{H})} &\approx \|\varphi\|_{H^\infty(\Omega)} + \|\mu_\varphi\|_{CM(\mathcal{H})}. \end{aligned}$$

  Here $CM(\mathcal{H})$ is the collection of Carleson measures for the space $\mathcal{H}$.

- Commutator/Bilinear Forms/Hankel Form Estimates: Given b, define $T_b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by $T_b(f, g) = \langle fg, b\rangle_{\mathcal{H}}$. Then one typically has:

  $$\|T_b\|_{\mathcal{H}\times\mathcal{H}\to\mathbb{C}} \approx \|\mu_b\|_{CM(\mathcal{H})}$$

  where μ_b is a Carleson measure "built" from the function b.

We refer the reader to the following references where phenomena such as this are explored (this is by no means exhaustive and instead is meant to serve as an introduction to the countless papers and results that illustrate and explore these concepts in much greater detail). Many of these papers have influenced the author's thoughts on these problems in some form or fashion.

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2. CARLESON MEASURES FOR THE HARDY SPACE ON THE DISC

In this section we will focus on the Hardy space $H^2(\mathbb{D})$ and their Carleson measures. The unit disc $\mathbb{D}$ is the set of points in the complex plane of modulus strictly less than 1,

$$\mathbb{D} = \left\{z \in \mathbb{C} : |z| < 1\right\}.$$

The boundary of the disc $\mathbb{D}$ is denoted by $\mathbb{T}$ and is the set

$$\mathbb{T} = \left\{z \in \mathbb{C} : |z| = 1\right\}.$$

In these notes, we will primarily focus on holomorphic functions, and so $\mathrm{Hol}(\mathbb{D})$ denotes the collection of analytic functions on $\mathbb{D}$.

2.1. Basic Definitions for the Hardy Space $H^2(\mathbb{D})$

This section will serve as a "crash course" on the necessary theory for the Hardy space and to motivate the general problem we will be interested in throughout the series of lectures. Much more detail and additional useful facts can be found in the references at the end of this section. Some general facts about reproducing kernel Hilbert spaces will also be collected along the way.

We now introduce the space $H^2(\mathbb{D})$. Let $dm = \frac{1}{2\pi}d\theta$ (normalized Lebesgue measure on $\mathbb{T}$) and $f \in \mathrm{Hol}(\mathbb{D})$, then we say that $f \in H^2(\mathbb{D})$ if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) \equiv \|f\|^2_{H^2(\mathbb{D})} < \infty.$$

We will see that with this norm the space $H^2(\mathbb{D})$ is a Hilbert space of analytic functions. We now show other norms that can be used to study the functions in $H^2(\mathbb{D})$. First, recall that the Fourier transform of a function $f \in L^2(\mathbb{T})$ is given by

$$\hat{f}(n) = \int_{\mathbb{T}} f(e^{i\theta})e^{-in\theta}dm(\theta).$$

A simple computation shows

$$\int_{\mathbb{T}} e^{i(n-k)\theta}dm(\theta) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

Using this, we see that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that

$$\begin{aligned} \|f\|^2_{H^2(\mathbb{D})} &= \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) = \sup_{0<r<1} \int_{\mathbb{T}} \left| \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right|^2 dm(\theta) \\ &= \sup_{0<r<1} \sum_{n,m=0}^{\infty} a_n \overline{a_k} r^n r^k \int_{\mathbb{T}} e^{i(n-k)\theta} dm(\theta) = \sum_{n=0}^{\infty} |a_n|^2. \end{aligned}$$

Note that this says it is possible to study the behavior of the functions in $H^2(\mathbb{D})$ via their Fourier coefficients. For $0 < r < 1$ and $z \in \mathbb{D}$ let $f_r(z) = f(rz)$. Then the computations done above allow us to easily prove the following proposition.

Proposition 2.1. *Suppose that $f \in H^2(\mathbb{D})$. Then the sequence $\{f_r\}$ is Cauchy in $L^2(\mathbb{T})$.*

Proof. Using the computations from above, and obvious modifications, we see that

$$\begin{aligned} \|f_r - f_s\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} \left| \sum_{n=0}^{\infty} a_n (r^n - s^n) e^{in\theta} \right|^2 dm(\theta) \\ &= \sum_{n=0}^{\infty} |(r^n - s^n)|^2 |a_n|^2. \end{aligned}$$

But, as $r, s \to 1$ and by the dominated convergence theorem, since $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, we can conclude that this last summand goes to zero. $\square$

Since $L^2(\mathbb{T})$ is a complete space, then we have an element $f^* \in L^2(\mathbb{T})$ given by $f^* = \lim_{r\to 1} f_r$ also in $L^2(\mathbb{T})$. Since $f^* \in L^2(\mathbb{T})$

we can compute the Fourier coefficients to be

$$\begin{aligned}\widehat{f^*}(n) &= \int_{\mathbb{T}} f^*(e^{i\theta})e^{-in\theta}dm(\theta)\\ &= \lim_{r\to 1}\int_{\mathbb{T}} f_r(e^{i\theta})e^{-in\theta}dm(\theta) = \begin{cases} a_n, & n \geq 0\\ 0, & n < 0.\end{cases}\end{aligned}$$

Collecting all we have done above (essentially) proves the following result.

Proposition 2.2. *Suppose that $f \in H^2(\mathbb{D})$ and denote $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ then*

$$\|f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 = \|f^*\|^2_{L^2(\mathbb{T})}.$$

The only fact that remains to complete the proof of this proposition is to observe that by Parseval's Theorem,

$$\sum_{n=0}^{\infty} |a_n|^2 = \|f^*\|^2_{L^2(\mathbb{T})}.$$

An immediate corollary, obtained by dualization, is that the inner product on $H^2(\mathbb{D})$ will satisfy

$$\langle f, g\rangle_{H^2(\mathbb{D})} = \int_{\mathbb{T}} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}dm(\theta) = \sum_{n=0}^{\infty} a_n\overline{b_n}$$

where we have $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

2.1.1. The Reproducing Kernel for $H^2(\mathbb{D})$

The Hardy space $H^2(\mathbb{D})$ also has an additional property of being a reproducing kernel Hilbert space. Namely, for each point $z \in \mathbb{D}$ there is a, non-zero, function $K_z \in H^2(\mathbb{D})$ such that

$$\langle f, K_z\rangle_{H^2(\mathbb{D})} = f(z).$$

We now turn to determining this function. However, first we prove that point evaluations are in fact bounded.

Proposition 2.3. *Let $z \in \mathbb{D}$, then*

$$|f(z)| \leq \|f\|_{H^2(\mathbb{D})} \frac{1}{\sqrt{1-|z|^2}}.$$

Proof. Via taking limits in Cauchy's formula we have

$$\begin{aligned} f(z) &= \lim_{r\to 1} f_r(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(e^{i\theta})}{e^{i\theta}-z} ie^{i\theta}d\theta \\ &= \int_{\mathbb{T}} \frac{f^*(e^{i\theta})}{1-ze^{-i\theta}}dm(\theta). \end{aligned}$$

A simple computation shows

$$\int_{\mathbb{T}} \frac{1}{|1-ze^{-i\theta}|^2}dm(\theta) = \frac{1}{1-|z|^2}.$$

This then implies, by an application of Cauchy-Schwarz and the above computation, that:

$$|f(z)| \leq \|f^*\|_{L^2(\mathbb{T})} \frac{1}{\sqrt{1-|z|^2}} = \|f\|_{H^2(\mathbb{D})} \frac{1}{\sqrt{1-|z|^2}}.$$

We now present an alternate proof of this. Expanding f in a power series we have

$$\begin{aligned} |f(z)| &= \left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \sum_{n=0}^{\infty} |a_n z^n| \leq \|f\|_{H^2(\mathbb{D})} \left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{1/2} \\ &= \|f\|_{H^2(\mathbb{D})} \frac{1}{\sqrt{1-|z|^2}}. \end{aligned}$$ $\square$

With this proposition proved, we see that pointwise evaluation of functions $f \in H^2(\mathbb{D})$ is a bounded operator. So, by the Riesz representation theorem, we know that there is a *unique* function $K_z \in H^2(\mathbb{D})$ such that

$$f(z) = \langle f, K_z \rangle_{H^2(\mathbb{D})}.$$

Using complex analysis, Cauchy's Theorem, one can immediately see that $K_z(\xi) = 1/(1-\overline{z}\xi)$. But, we want to give an alternate, algorithmic way to compute the answer.

To do so, we collect a general fact about reproducing kernel Hilbert spaces of analytic functions on a domain Ω. Good references for the facts presented here are [1,9,10]. For each point $\lambda \in \Omega$ we will assume that point evaluation of the functions $f \in \mathcal{H}$ is a continuous operation, namely $f \mapsto f(\lambda)$ is bounded. Therefore, again by the Riesz representation theorem, we have a function $K_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, K_\lambda \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}.$$

This vector K_λ is called the reproducing kernel for the space $\mathcal{H}$. A simple application of the reproducing property of the kernel yields the following property for the kernel:

$$K_\lambda(\xi) = \langle K_\lambda, K_\xi \rangle_{\mathcal{H}}.$$

This function is called the kernel function for the Hilbert space $\mathcal{H}$. A simple fact is the following.

Proposition 2.4. *Let $\mathcal{H}$ be a Hilbert function space on Ω and let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$. Then*

$$K_\lambda(\xi) = \sum_i \overline{e_i(\lambda)} e_i(\xi).$$

The proof of this fact is just an application of Parseval's Identity in a general Hilbert space and two applications of the reproducing kernel property for $\mathcal{H}$.

We now specialize back to $H^2(\mathbb{D})$. By simple computation one can show that the functions $\{z^k\}_{k=0}^{\infty}$ are an orthonormal system of functions in $H^2(\mathbb{D})$. Thus, we have that

$$K_\lambda(\xi) = \sum_{k=0}^{\infty} \overline{\lambda}^k \xi^k = \frac{1}{1 - \overline{\lambda}\xi}$$

is the reproducing kernel for the Hardy space $H^2(\mathbb{D})$ (as expected by Cauchy's formula!). In particular, we easily observe that

$$\|K_z\|_{H^2} = \frac{1}{\sqrt{1 - |z|^2}}$$

$$K_\lambda(\xi) = \langle K_\lambda, K_\xi \rangle_{H^2(\mathbb{D})} = \frac{1}{1 - \overline{\lambda}\xi}.$$

2.2. Carleson Measures for $H^2(\mathbb{D})$

Our goal in this section is to prove the following important theorem about certain measures for $H^2(\mathbb{D})$. We say that a non-negative Borel measure μ on $\mathbb{D}$ is an $H^2(\mathbb{D})$-Carleson measure if

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq \|\mu\|^2_{CM(H^2)} \|f\|^2_{H^2(\mathbb{D})} \quad \forall\, f \in H^2(\mathbb{D}).$$

Observe that this is saying that $H^2(\mathbb{D})$ continuously embeds into $L^2(\mathbb{D};\mu)$. The main goal in this section is to provide a geometric characterization of the $H^2(\mathbb{D})$-Carleson measures. A similar and related question is to study the *harmonic* embeddings, which we state now. The proofs for harmonic and analytic are of course closely related. We will be interested in the measures μ such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq \|\mu\|^2_{CM(L^2)} \|f\|^2_{L^2(\mathbb{T})} \quad \forall\, f \in L^2(\mathbb{T})$$

where $f(z)$ denotes the Poisson extension of the function. The connection between these measures and those for $H^2(\mathbb{D})$ is now more clear. The theorem of interest is the following.

Theorem 2.5. (Carleson Embedding Theorem) *Let μ be a non-negative measure in $\mathbb{D}$. Then the following are equivalent:*

(i) *The embedding operator $\mathcal{J} : L^2(\mathbb{T}) \to L^2(\mathbb{D},\mu)$, with $\mathcal{J}(f)(z) = f(z)$, is bounded.*

(ii) *$C(\mu)^2 \equiv \sup_{z\in\mathbb{D}} \|\mathcal{J}k_z\|^2_{L^2(\mu)} = \sup_{z\in\mathbb{D}} \|P_z\|_{L^1(\mu)} < \infty$, where $k_z(\xi) = \frac{(1-|z|^2)^{1/2}}{(1-\xi\overline{z})}$, is the normalized reproducing kernel for the Hardy space $H^2(\mathbb{D})$ and $P_z(\xi) = \frac{1-|z|^2}{|1-\bar{\xi}z|^2}$ is the Poisson kernel.*

(iii) *$I(\mu) = \sup\left\{\frac{1}{r}\,\mu(\mathbb{D}\cap Q(\xi,r)) : r > 0, \xi \in \mathbb{T}\right\} < \infty$, where $Q(\xi,r)$ is a ball measured with respect to the non-isotropic metric $d(z,w) = |1-\overline{z}w|^{1/2}$ associated to $\mathbb{D}$.*

Moreover, the following inequalities hold with absolute constants

$$C(\mu) \leq \|\mathcal{J}\| \lesssim C(\mu) \quad \textit{and} \quad I(\mu) \lesssim C(\mu)^2 \lesssim I(\mu).$$

This means we have that $\|\mu\|_{CM(L^2)} = \|\mathcal{J}\| \approx C(\mu) \approx \sqrt{I(\mu)}$.

2.2.1. Examples of Carleson Measures

We now want to collect a couple of different families of Carleson measures that frequently appear. The first arises through a well-known lemma due to Uchiyama.

Lemma 2.6. (Uchiyama's Lemma) *Let φ be a non-negative, bounded, subharmonic function. Then for any $f \in H^2(\mathbb{D})$:*

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\,|f(z)|^2\,d\mu(z) \leq e\|\varphi\|_\infty \|f\|^2_{H^2(\mathbb{D})}.$$

Here $d\mu(z) = \frac{2}{\pi}\log\frac{1}{|z|}dA(z)$ and $\widetilde{\Delta} = \frac{1}{4}\Delta = \partial\overline{\partial}$.

Proof. Because of homogeneity, we can assume without loss of generality that $\|\varphi\|_\infty = 1$. Direct computation shows that

$$\widetilde{\Delta}\left(e^{\varphi}\,|f|^2\right) = e^{\varphi}\widetilde{\Delta}\varphi\,|f|^2 + e^{\varphi}\,|\partial\varphi f + \partial f|^2 \geq \widetilde{\Delta}\varphi\,|f|^2\,.$$

Then Green's formula implies

$$\begin{aligned}
\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\ |f(z)|^2\ d\mu(z) &\leq \int_{\mathbb{D}} \widetilde{\Delta}\left(e^{\varphi}\,|f|^2\right)(z)\,d\mu(z) \\
&= \int_{\mathbb{T}} e^{\varphi(\xi)}\,|f(\xi)|^2\ dm(\xi) - e^{\varphi(0)}\,|f(0)|^2 \\
&\leq e\int_{\mathbb{T}} |f(\xi)|^2\ dm(\xi) \\
&= e\ \|f\|^2_{H^2(\mathbb{D})}\,.
\end{aligned}$$

□

For the next class of examples, we recall that a function $\varphi \in \mathrm{BMO}(\mathbb{T})$ if

$$\|\varphi\|^2_{BMO} = \sup_{z\in\mathbb{D}} |\varphi|^2(z) - |\varphi(z)|^2 < \infty,$$

where $\varphi(z)$ denotes the harmonic extension of φ to $\mathbb{D}$, and $|\varphi|^2(z)$ denotes the harmonic extension of $|\varphi(\xi)|^2$. This is called the Garsia norm of the function and is one of many useful norms on this space. Note that the expression on the right-hand side of the definition of BMO is always non-negative since we are integrating against a probability measure and by a simple application of Cauchy-Schwarz.

The following identity is a simple computation:

$$\int_{\mathbb{T}} |\varphi(\xi) - \varphi(z)|^2 P_z(\xi) dm(\xi) = |\varphi|^2(z) - |\varphi(z)|^2 . \tag{2.1}$$

If we apply the conformally invariant version of Green's Theorem to the left-hand side of (2.1) then we obtain

$$|\varphi|^2(z) - |\varphi(z)|^2 = \int_{\mathbb{D}} |\nabla\varphi(w)|^2 \log\left|\frac{1-\overline{z}w}{w-z}\right| dA(w).$$

Using the relationship $\log\frac{1}{t} \approx 1-t$ and the identity

$$1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} = \left|\frac{z-w}{1-\overline{z}w}\right|^2$$

we see that

$$\begin{aligned}
&\int_{\mathbb{D}} |\nabla\varphi(w)|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w)\\
&\quad \leq |\varphi|^2(z) - |\varphi(z)|^2\\
&\quad \lesssim \int_{\mathbb{D}} |\nabla\varphi(w)|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w).
\end{aligned}$$

The expressions on the extremes are now simply testing a measure against the normalized reproducing kernels. These computations then, essentially, prove the following

Lemma 2.7. *Suppose that* $\varphi \in \mathrm{BMO}(\mathbb{T})$. *Then*

$$|\nabla\varphi(w)|^2 (1-|w|^2) dA(w)$$

is an $H^2(\mathbb{D})$ *Carleson measure with Carleson measure norm controlled by a constant times* $\|\varphi\|_{\mathrm{BMO}}$.

In fact this lemma gives a characterization of BMO in terms of Carleson measures. For more precise details we refer the reader to the excellent monograph [2].

2.2.2. Carleson Measures via Reproducing Kernels: Proof 1

We give a proof of Theorem 2.5 that uses the reproducing kernel directly. This proof can be found as an exercise in [7], and the exposition below is largely expansion of the details in the exercise mentioned. There are proofs that avoid the use of the kernel and connect the problem to the maximal function. For those see [2, 11].

Equivalence between (i) $\Longleftrightarrow$ (ii)

One direction of this equivalence is immediate, namely (i)$\Rightarrow$(ii). If we know that the embedding operator $\mathcal{J}$ is bounded (and for the rest of the proof we will use $\|\mathcal{J}\|$ as $\|\mu\|_{CM(L^2)}$ since they are the same!), then we have

$$\|\mathcal{J}k_z\|_{L^2(\mu)} \leq \|\mathcal{J}\| \, \|k_z\|_{L^2(\mathbb{T})} = \|\mathcal{J}\|$$

since the reproducing kernel k_z is normalized to have $L^2(\mathbb{T})$ norm one. This also proves that $\|\mu\|_{CM} \leq \|\mathcal{J}\|$. Finally, we should indicate why the equality

$$\sup_{z\in\mathbb{D}} \|\mathcal{J}k_z\|^2_{L^2(\mu)} = \sup_{z\in\mathbb{D}} \|P_z\|_{L^1(\mu)}$$

holds. Since $k_z \in H^2(\mathbb{D})$ then the Poisson kernel P_w reproduces the function value at w, namely

$$k_z(w) = \int_{\mathbb{T}} k_z(\xi) P_w(\xi) dm(\xi).$$

This then implies that $\mathcal{J}(k_z)(w) = k_z(w)$, and using this we have

$$\begin{aligned}
\|\mathcal{J}k_z\|^2_{L^2(\mu)} &= \int_{\mathbb{D}} |\mathcal{J}(k_z)(w)|^2 \, d\mu(w) = \int_{\mathbb{D}} |k_z(w)|^2 \, d\mu(w) \\
&= \int_{\mathbb{D}} \left| \frac{(1-|z|^2)^{1/2}}{(1-\overline{z}w)} \right|^2 d\mu(w) = \int_{\mathbb{D}} P_z(w) d\mu(w).
\end{aligned}$$

Which shows that

$$\|\mathcal{J}k_z\|^2_{L^2(\mu)} = \|P_z\|_{L^1(\mu)},$$

and then the suprema over $z \in \mathbb{D}$ are of course equal.

It only remains to prove that (ii)$\Rightarrow$(i). To prove this direction it is enough to show that the formal adjoint $\mathcal{J}^* : L^2(\mathbb{D}, \mu) \to L^2(\mathbb{T})$ with

$$\mathcal{J}^*(f)(\xi) = \int_{\mathbb{D}} P_z(\xi) f(z) d\mu(z),$$

is bounded. With this in mind we have,

$$\begin{aligned}\|\mathcal{J}^*(f)\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} |\mathcal{J}^*(f)(\xi)|^2\, dm(\xi) = \int_{\mathbb{T}} \mathcal{J}^*(f)(\xi)\overline{\mathcal{J}^*(f)(\xi)} dm(\xi) \\ &= \int_{\mathbb{T}} \Big(\int_{\mathbb{D}} P_z(\xi) f(z) d\mu(z) \Big)\Big(\int_{\mathbb{D}} P_{z'}(\xi) \overline{f(z')} d\mu(z') \Big)\, dm(\xi) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\mathbb{T}} P_z(\xi) P_{z'}(\xi) dm(\xi) \right) f(z) \overline{f(z')} d\mu(z) d\mu(z').\end{aligned}$$

The following lemma will be used to prove that $\mathcal{J}^*$ is bounded.

Lemma 2.8. (Vinogradov-Senichkin Test, [7, p. 151]) *Let $\mathcal{Z}$ be a measurable space and k a non-negative measurable function on $\mathcal{Z} \times \mathcal{Z}$. If*

$$\int_{\mathcal{Z}} k(s,t) k(s,x) ds \leq C(k(t,x) + k(x,t))$$

for a.e. $(t,x) \in \mathcal{Z} \times \mathcal{Z}$, then

$$Q \equiv \iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t) g(s) g(t) ds dt \leq 2C$$

for any non-negative function g with $\|g\|_{L^2(\mathcal{Z})} \leq 1$ and $Q < \infty$.

The proof of this is a good exercise for the reader (it is very straightforward once you have the initial idea). We want to apply the Vinogradov-Senichkin Test, so we define the integral operator $T_k : L^2(\mathbb{D}, \mu) \to L^2(\mathbb{D}, \mu)$ with kernel $k(z', z) = P_z(z')$ by

$$T_k(g)(z) \equiv \int_{\mathbb{D}} k(z', z) g(z') d\mu(z') \quad \forall\, g \in L^2(\mathbb{D}, \mu).$$

Then we have that

$$\begin{aligned}\|\mathcal{J}^*(f)\|^2_{L^2(\mathbb{T})} &\leq \int_{\mathbb{D}} \int_{\mathbb{D}} P_z(z') \left|f(z)\right| \left|f(z')\right| d\mu(z) d\mu(z') \\ &= \langle T_k \left|f\right|, \left|f\right| \rangle_{L^2(\mathbb{D};\mu)} .\end{aligned}$$

Now, we make the following claim.

Proposition 2.9. *Let $z, z', w \in \mathbb{D}$ then*

$$P_z(z') P_w(z') \leq 8 \left(P_z(w) P_w(z') + P_w(z) P_z(z') \right).$$

Proof. Begin by noting that the following inequality holds

$$a^{-1} \equiv |1 - z\overline{w}|^{1/2} \leq \left|1 - z\overline{z'}\right|^{1/2} + \left|1 - z'\overline{w}\right|^{1/2} \equiv b^{-1} + c^{-1}.$$

This fact is simply the triangle inequality for the metric $\rho(z, w) = |1 - z\overline{w}|^{1/2}$, see [12]. Alternatively, one may simply use the facts that $|z - w| \leq |1 - \overline{z}w|$ and the trivial inequality that $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + a^{\frac{1}{2}}$ for $a, b > 0$. Using this inequality we have that $bc \leq a(b+c)$, which in turn implies that $b^4c^4 \leq a^4(b+c)^4 \leq 2^3a^4(b^4+c^4)$. Using this inequality, but with the appropriate substitutions for a,b, and c and using the numerators for $P_z(z')$ and $P_w(z')$ we find that

$$\begin{aligned}
&P_z(z')P_w(z') \\
&\quad \leq \frac{2^3}{|1 - z\overline{w}|^2}(1 - |z|^2)(1 - |w|^2)\left(\frac{1}{\left|1 - z\overline{z'}\right|^2} + \frac{1}{|1 - z'\overline{w}|^2}\right) \\
&\quad = 2^3\left[\frac{(1 - |z|^2)}{|1 - z\overline{w}|^2}\frac{(1 - |w|^2)}{\left|1 - z\overline{z'}\right|^2} + \frac{(1 - |z|^2)}{|1 - z'\overline{w}|^2}\frac{(1 - |w|^2)}{|1 - z\overline{w}|^2}\right] \\
&\quad = 2^3(P_w(z)P_z(z') + P_z(w)P_w(z')).
\end{aligned}$$

□

Since we want to apply the Vinogradov-Senichkin Test we need to know that the kernel $k(z', z) = P_z(z')$ satisfies the hypothesis of the lemma. To this end we need to estimate

$$\int_{\mathbb{D}} k(z', z)k(z', w)d\mu(z').$$

Using the estimate we found for the product of two Poisson kernels, we have

$$\begin{aligned}
&\int_{\mathbb{D}} k(z', z)k(z', w)d\mu(z') \\
&\qquad = \int_{\mathbb{D}} P_z(z')P_w(z')d\mu(z') \\
&\qquad \leq 2^3\int_{\mathbb{D}} P_z(w)P_w(z') + P_w(z)P_z(z')d\mu(z') \\
&\qquad = 2^3\left[P_z(w)\int_{\mathbb{D}} P_w(z')d\mu(z') + P_w(z)\int_{\mathbb{D}} P_z(z')d\mu(z')\right] \\
&\qquad \leq 2^3C(\mu)^2(P_z(w) + P_w(z)).
\end{aligned}$$

The last inequality holds because we are trying to prove that (ii)⇒(i). So the kernel satisfies the hypotheses of the Vinogradov-Senichkin Test. Thus, we have

$$\begin{aligned}\|\mathcal{J}^*(f)\|^2_{L^2(\mathbb{T})} &\leq \langle T_k\,|f|\,,|f|\rangle_{L^2(\mathbb{D};\mu)}\\ &\leq 2\cdot(2^3C(\mu)^2)\,\|f\|^2_{L^2(\mu)} = 2^4C(\mu)^2\,\|f\|^2_{L^2(\mu)}\,.\end{aligned}$$

A duality argument then gives that $\|\mathcal{J}\| \leq 4C(\mu)$, proving that $\mathcal{J}$ is bounded and giving the relationship between $\|\mathcal{J}\|$ and $C(\mu)$.

Equivalence between (ii) $\Longleftrightarrow$ (iii)

To finish the proof of this theorem we only need to dispose of the final equivalence. Again, there is an easy implication and harder implication. We begin by showing that (ii)⇒(iii). Under the hypothesis of (ii) we have

$$C(\mu)^2 \geq \int_{\mathbb{D}} P_z(z')d\mu(z') = \int_{\mathbb{D}} \frac{(1-|z|^2)}{\left|1-z\overline{z'}\right|^2}d\mu(z').$$

Now take $\xi\in\mathbb{T}$ and $0<r<2$, and set $z=(1-r/2)\xi$ and set $Q(\xi,r^{1/2})=\{z\in\mathbb{D}:|1-z\overline{\xi}|^{1/2}<r^{1/2}\}$ (the ball of radius $r^{1/2}$ measured with respect to the metric $\rho(z,w)=|1-z\overline{w}|^{1/2}$). Equivalently, this is the set $\{z\in\mathbb{D}:|z-\xi|<r\}$. A simple calculation shows that $z\in Q(\xi,r^{1/2})$. Then we have

$$\begin{aligned}C(\mu)^2 &\geq \int_{\mathbb{D}} \frac{(1-|z|^2)}{\left|1-z\overline{z'}\right|^2}d\mu(z') \geq \int_{\mathbb{D}\cap Q(\xi,r)} \frac{(1-|z|^2)}{\left|1-z\overline{z'}\right|^2}d\mu(z')\\ &\geq 16^{-1}r^{-2}(1-|z|)\int_{\mathbb{D}\cap Q(\xi,r^{\frac{1}{2}})} d\mu(z')\\ &= 16^{-1}r^{-2}(1-|z|)\,\mu\Big(\mathbb{D}\cap Q(\xi,r^{\frac{1}{2}})\Big)\,.\end{aligned}$$

The last inequality follows since $z\in\mathbb{D}$ and we are considering $z,z'\in Q(\xi,r^{1/2})$. Because if we have two points $z,z'\in Q(\xi,r^{1/2})$ then by the triangle inequality for the non-isotropic metric we have

$$\left|1-z\overline{z'}\right|^{1/2} \leq \left|1-z\overline{\xi}\right|^{1/2} + \left|1-z'\overline{\xi}\right|^{1/2} \leq 2r^{\frac{1}{2}}.$$

Squaring this last inequality gives $|1-z\overline{z'}|\leq 4r$. This then gives $|1-z\overline{z'}|^{-2}\geq 2^{-4}r^{-2}$. We also have that $1-|z|=r/2$. Combining these estimates gives,

$$\begin{aligned} C(\mu)^2 &\geq 16^{-1} r^{-2} (1 - |z|)\, \mu\Big(\mathbb{D} \cap Q(\xi, r^{\frac{1}{2}})\Big) \\ &\geq 16^{-1} 2^{-1} r r^{-2}\, \mu\Big(\mathbb{D} \cap Q(\xi, r^{\frac{1}{2}})\Big). \end{aligned}$$

Taking the supremum over r then gives

$$32^{-1} I(\mu) \leq C(\mu)^2.$$

It only remains to prove that (iii)$\Rightarrow$(ii). We will break this part up into two different cases. First, consider the case where $|z| \leq \frac{3}{4}$. Now we have the following inequality holding for the Poisson kernel,

$$P_z(\xi) = \frac{(1 - |z|^2)}{\left|1 - z\overline{\xi}\right|^2} \leq \frac{(1 - |z|^2)}{(1 - |z|)^2} = \frac{1 + |z|}{1 - |z|} \leq \frac{2}{1 - |z|}.$$

Then if $|z| \leq \frac{3}{4}$ we have that

$$\int_{\mathbb{D}} P_z(w) d\mu(w) \leq \frac{2}{1/4}\, \mu(\mathbb{D}) = 8\, \mu\big(\mathbb{D} \cap Q(\xi, 2)\big) \leq 8 I(\mu).$$

So we only need to deal with the case when $|z| > \frac{3}{4}$. Let $\tilde{z} = \frac{z}{|z|}$ and define the following sets

$$Q_k \equiv \mathbb{D} \cap Q\big(\tilde{z}, 2^{k+1}(1 - |z|^2)\big) \qquad \forall\, k \in \mathbb{N}.$$

Then for $w \in Q_{k+1} \setminus Q_k$ we have

$$\left|1 - w\overline{\tilde{z}}\right| \geq 2^{k+1}(1 - |z|^2).$$

By the triangle inequality for the non-isotropic metric we have

$$\begin{aligned} \left|1 - w\overline{\tilde{z}}\right|^{1/2} &\leq |1 - w\overline{z}|^{1/2} + \left|1 - z\overline{\tilde{z}}\right|^{1/2} \\ &= |1 - w\overline{z}|^{1/2} + (1 - |z|)^{1/2} \\ &\leq |1 - w\overline{z}|^{1/2} + (1 - |z|^2)^{1/2}, \end{aligned}$$

with the last inequality following since $z \in \mathbb{D}$. Squaring this last inequality gives,

$$\left|1 - w\overline{\tilde{z}}\right| \leq 2\left(|1 - w\overline{z}| + (1 - |z|^2)\right),$$

which can be used to conclude

$$\begin{aligned}|1-w\overline{z}| &\geq 2^{-1}\left|1-w\overline{\tilde{z}}\right| - (1-|z|^2)\\ &\geq 2^k(1-|z|^2) - (1-|z|^2)\\ &\geq 2^{k-1}(1-|z|^2).\end{aligned}$$

But this last inequality implies that

$$|1-w\overline{z}|^{-2} \leq 2^{-2(k-1)}(1-|z|^2)^{-2}$$

when $w \in Q_{k+1} \setminus Q_k$. Using this we find:

$$\begin{aligned}\int_{\mathbb{D}} P_z(w)d\mu(w) &= \int_{Q_1} P_z(w)d\mu(w) + \sum_{k=1}^{\infty}\int_{Q_{k+1}\setminus Q_k} P_z(w)d\mu(w)\\ &\leq \int_{Q_1} \frac{2^2}{(1-|z|^2)}d\mu(w)\\ &\quad + \sum_{k=1}^{\infty}\int_{Q_{k+1}\setminus Q_k} \frac{(1-|z|^2)}{4^{k-1}(1-|z|^2)^2}d\mu(w)\\ &\leq 4\frac{\mu(Q_1)}{(1-|z|^2)} + \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k-1}\frac{\mu(Q_{k+1})}{(1-|z|^2)}.\end{aligned}$$

Now we need to recall how $I(\mu)$ was defined and how each of the Q_k was defined. Doing this we have

$$\begin{aligned}&\int_{\mathbb{D}} P_z(w)d\mu(w)\\ &\quad\leq 16\frac{\mu(Q_1)}{(2^2(1-|z|^2))} + \sum_{k=1}^{\infty}(4^{-1})^{k-1}(2^{k+2})\frac{\mu(Q_{k+1})}{(2^{k+2}(1-|z|^2))}\\ &\quad\leq 16I(\mu) + 2^4 I(\mu)\sum_{k=1}^{\infty}2^{-k} \leq 2^5 I(\mu).\end{aligned}$$

Combining the estimates if $|z| \leq \frac{3}{4}$ and if $|z| > \frac{3}{4}$, we have that

$$\int_{\mathbb{D}} P_z(w)d\mu(w) \leq 2^5 I(\mu),$$

then taking the supremum over $z \in \mathbb{D}$ then proves the theorem. $\square$

2.2.3. Carleson Measures via Reproducing Kernels: Proof 2

For this proof we follow the results in [8]. This proof strategy will achieve the best known estimate for the Carleson Embedding Theorem in the case of analytic embeddings. As a reminder, the careful reader will note, Theorem 2.5 applies to *harmonic* embeddings.

Suppose the measure μ satisfies the assumption of Theorem 2.5. Define the Carleson potential

$$\varphi(z) \equiv -\int_{\mathbb{D}} |k_z(\lambda)|^2 \, d\mu(\lambda) = -\int_{\mathbb{D}} P_z(\lambda) d\mu(\lambda),$$

where k_z is the (normalized) reproducing kernel and $P_z(\lambda) = |k_z(\lambda)|^2$ is the Poisson kernel at z. By homogeneity we can assume without loss of generality that $\sup_{z\in \operatorname{supp}\mu} \varphi(z) = -1$. In particular, $-1 \leq \varphi(z) \leq 0$ for $z \in \operatorname{supp}\mu$.

We next compute the Laplacian of the function $\varphi(z)$. Using the fact that for an analytic function f we have $\Delta|f|^2 = \partial\bar{\partial}|f|^2 = 4|f'|^2$ one readily computes

$$\Delta_z P_z(\lambda) = 4\,\frac{|\lambda|^2 - 1}{\left|1 - \bar{\lambda} z\right|^4},$$

(here Δ_z stands for the Laplacian in the variable z). This clearly implies that φ is subharmonic and that

$$\Delta\varphi(z) = 4\int_{\mathbb{D}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda} z|^4} d\mu(\lambda).$$

Applying the proof of Uchiyama's Lemma, Lemma 2.6, we get

$$\int_{\mathbb{D}} |f(z)|^2 \, d\nu(z) \leq \|f\|^2_{H^2(\mathbb{D})},$$

with $d\nu(z) \equiv e^{\varphi(z)}\Delta\varphi(z) \log\frac{1}{|z|} dA(z)$. We will now prove the estimate

$$\int_{\mathbb{D}} |f(\lambda)|^2 \, d\mu(\lambda) \leq 2e\int_{\mathbb{D}} |f(z)|^2 \, d\nu(z) \tag{2.2}$$

which will immediately imply the theorem. First note that

$$\int_{\mathbb{D}} |f(z)|^2 \, d\nu(z) = \frac{4}{2\pi}\int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2 \, e^{\varphi(z)} \frac{1 - |\lambda|^2}{\left|1 - \bar{\lambda} z\right|^4} \log\frac{1}{|z|} dA(z) d\mu(\lambda).$$

Using the estimate $\frac{1}{2}(1-|z|^2) \le \log\frac{1}{|z|}$ we have

$$\int_{\mathbb{D}} |f(z)|^2 \, d\nu(z) \ge \frac{1}{\pi} \int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2 e^{\varphi(z)} \frac{(1-|\lambda|^2)(1-|z|^2)}{\left|1-\overline{\lambda}z\right|^4} dA(z) d\mu(\lambda).$$

Remark 2.10. If we did not care about the constant then the result would be proved. Here is why. In the disc centered at λ of radius $\frac{\delta}{10} > 0$ where $\delta = \operatorname{dist}(\lambda, \mathbb{T})$, call it $D(\lambda, \delta)$, we have that

$$\frac{1-|\lambda|^2}{\left|1-\overline{\lambda}z\right|^4}(1-|z|^2) \approx \frac{1}{\delta^2}.$$

Using the subharmonicity of $e^{\varphi}|f|^2$ and the trivial fact that the volume of $D(\lambda, \delta) \approx \delta^2$ we get

$$e^{\varphi(\lambda)} \, |f(\lambda)|^2 \lesssim \int_{D(\lambda,\delta)} e^{\varphi(z)} \, |f(z)|^2 \, \frac{(1-|\lambda|^2)(1-|z|^2)}{\left|1-\overline{\lambda}z\right|^4} dA(z).$$

Increasing the domain of integration to the whole disc $\mathbb{D}$ clearly does not spoil the inequality, and integrating both sides with respect to $d\mu(\lambda)$ we obtain

$$\int_{\mathbb{D}} e^{\varphi(\lambda)} \, |f(\lambda)|^2 \, d\mu(\lambda) \lesssim \int_{\mathbb{D}} |f(z)|^2 \, d\nu(z) \lesssim \|f\|^2_{H^2(\mathbb{D})}$$

which proves the theorem (without constants). △

Here is how to obtain the sharper estimate. We focus on the inner integral and will prove the inequality

$$\begin{aligned} \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, e^{\varphi(z)} \frac{(1-|\lambda|^2)(1-|z|^2)}{\left|1-\overline{\lambda}z\right|^4} dA(z) \\ \ge \frac{1}{2} e^{\varphi(\lambda)} \, |f(\lambda)|^2 \quad \forall \, \lambda \in \operatorname{supp} \mu, \end{aligned} \tag{2.3}$$

which after integration with respect to $d\mu(\lambda)$ gives (2.2).

Let $w = b_\lambda(z) \equiv \frac{\lambda - z}{1-\overline{\lambda}z}$ denote a conformal change of variables (note that $z = b_\lambda(w)$). A simple computation shows that

$$dA(w) = \left(\frac{1-|\lambda|^2}{\left|1-\overline{\lambda}z\right|^2}\right)^2 dA(z).$$

If we let $\tilde{g}(w) \equiv g \circ b_\lambda(w)$ then the above integral can be recognized as

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^2\, e^{\varphi(z)}\frac{(1-|\lambda|^2)(1-|z|^2)}{\left|1-\overline{\lambda}z\right|^4}dA(z)\\&\quad=\frac{1}{\pi}\int_{\mathbb{D}}e^{\tilde{\varphi}(w)}\left|\tilde{f}(w)\right|^2\frac{1-|w|^2}{\left|1-\overline{\lambda}w\right|^2}dA(w).\end{aligned}$$

In this reduction we have used the algebraic identity that for b_λ defined above,

$$1-|z|^2=\frac{(1-|\lambda|^2)(1-|w|^2)}{\left|1-\overline{\lambda}w\right|^2}.$$

Continuing the estimate we have

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathbb{D}}e^{\tilde{\varphi}(w)}\left|\tilde{f}(w)\right|^2\frac{1-|w|^2}{\left|1-\overline{\lambda}w\right|^2}dA(w)\\&\quad=\frac{1}{\pi}\int_{\mathbb{D}}e^{\tilde{\varphi}(w)}\left|\frac{\tilde{f}(w)}{1-\overline{\lambda}w}\right|^2(1-|w|^2)dA(w).\end{aligned}$$

The function $\frac{\tilde{f}(w)}{1-\overline{\lambda}w}$ is analytic and $\tilde{\varphi}$ is subharmonic, so the function

$$u(w)=e^{\tilde{\varphi}(w)}\left|\frac{\tilde{f}(w)}{1-\overline{\lambda}w}\right|^2$$

is subharmonic. Integrating in polar coordinates and using the mean value property for subharmonic functions we get

$$\begin{aligned}\int_{\mathbb{D}}u(w)(1-|w|^2)dA(w)&=\int_0^1(1-r^2)r\int_0^{2\pi}u(r\theta)d\theta dr\\&\geq 2\pi u(0)\int_0^1(1-r^2)rdr=\frac{\pi}{2}u(0).\end{aligned}$$

Gathering all together we find

$$\frac{1}{\pi}\int_{\mathbb{D}}e^{\tilde{\varphi}(w)}\left|\tilde{f}(w)\right|^2\frac{1-|w|^2}{\left|1-\overline{\lambda}w\right|^2}dA(w)\geq\frac{1}{2}e^{\tilde{\varphi}(0)}\left|\tilde{f}(0)\right|^2=\frac{1}{2}e^{\varphi(\lambda)}\left|f(\lambda)\right|^2,$$

which is equivalent to (2.3). This finally shows that for a Carleson measure μ on $\mathbb{D}$ we have

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq 2e \, \|\varphi\|_\infty \, \|f\|^2_{H^2(\mathbb{D})} = 2eC(\mu)^2 \, \|f\|^2_{H^2(\mathbb{D})}$$

proving Theorem 2.5 for $H^2(\mathbb{D})$ functions.

2.3. Concluding Remarks

There is actually a third proof of this result where one uses the connection with the Poisson kernel and the maximal function, the interested reader can find this proof in [2, 11]. For those interested in analytic functions on the unit ball in $\mathbb{C}^n$, the proofs given in this section carry over and generalize with appropriate modifications. The story of the Carleson measures for the Hardy space of the polydisc is much more complicated, [3–5].

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3. CARLESON MEASURES FOR BESOV-SOBOLEV SPACES

3.1. Besov-Sobolev Spaces on the Unit Ball

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$. The Besov-Sobolev space $B_p^\sigma(\mathbb{B}_n)$ consists of all $f \in \mathrm{Hol}(\mathbb{B}_n)$ such that

$$\|f\|_{B_p^\sigma(\mathbb{B}_n)} \equiv \sum_{k=0}^{m-1} \left|\nabla^k f(0)\right| + \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} < \infty,$$

for some $m > \frac{n}{p} - \sigma$. Above, $d\lambda_n(z) \equiv (1-|z|^2)^{-n-1} dv(z)$, the hyperbolic measure on the unit ball $\mathbb{B}_n$. It turns out the right side is finite for some $m > \frac{n}{p} - \sigma$ if and only if it is finite for all $m > \frac{n}{p} - \sigma$. The choice of derivative is also largely unimportant when computing these norms (again up to equivalence). The interested reader can find the full details in [3].

Specializing to $p = 2$ yields a Hilbert space of analytic functions with the following Hilbert space inner product

$$\langle f, g\rangle_{B_2^\sigma(\mathbb{B}_n)} = \sum_{k=0}^{m-1} f^{(k)}(0)\,\overline{g^{(k)}(0)} + \int_{\mathbb{B}_n} \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) \overline{\left(1-|z|^2\right)^{m+\sigma} g^{(m)}(z)}\, d\lambda_n(z)\,.$$

Various choices of σ give important examples of classical analytic function spaces:

- $\sigma = 0$: Dirichlet Space;
- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
- $\sigma = \frac{n}{2}$: Classical Hardy Space;
- $\sigma > \frac{n}{2}$: Bergman Spaces.

The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces. Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $K_\lambda^\sigma \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, K_\lambda^\sigma \rangle_{B_2^\sigma}.$$

Easy computations show that the kernel function K_λ is given by:

$$K_\lambda^\sigma(z) \equiv \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

- $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $K_\lambda^{\frac{1}{2}}(z) = \frac{1}{1-\overline{\lambda}z}$;
- $\sigma = \frac{n}{2}$: Classical Hardy Space; $K_\lambda^{\frac{n}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^n}$;
- $\sigma = \frac{n+1}{2}$: Bergman Space; $K_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$.

In this section, for vectors $z, w \in \mathbb{C}^n$ we will frequently let $zw = \sum_{j=1}^n z_j w_j$, and so we have that $|z|^2 = \sum_{j=1}^n z_j \overline{z}_j$. Expressions should be clear from context and confusion with the one-variable expression should not present a problem.

3.1.1. Besov-Sobolev Spaces on the Unit Disc

We now (briefly) study the Besov-Sobolev spaces on the unit disc $\mathbb{D}$ (partly to ground things back in reality). Because we are in the unit disc ($n = 1$) it suffices to take only one (usual) derivative when defining these spaces. Fix a parameter $0 \leq \sigma \leq \frac{1}{2}$ and define the Besov-Sobolev space $B_2^\sigma(\mathbb{D})$ as the collection of analytic functions on the disc such that

$$\|f\|_{B_2^\sigma(\mathbb{D})}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{2\sigma} dA(z) < \infty.$$

It is a standard computation to show that when $\sigma = 1/2$ that $H^2(\mathbb{D}) = B_2^{1/2}(\mathbb{D})$, with equivalent norms (the proof of this fact is essentially contained in the section on Carleson measures for Hardy spaces). When $\sigma = 0$, then we are looking at the functions that are analytic on $\mathbb{D}$ such that its derivative is square integrable, nothing other than the Dirichlet space. For those who are interested in computing the norm in terms of Fourier coefficients, it is an easy exercise to show that an equivalent norm on the space $B_2^\sigma(\mathbb{D})$ is given by

$$\sum_{n=0}^{\infty} n^{1-2\sigma} |a_n|^2$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

3.1.2. The Carleson Measure Problem for Besov-Sobolev Spaces

We are interested in obtaining a characterization of the measures on $\mathbb{B}_n$ for which the following inequality holds:

$$\int_{\mathbb{B}_n} |f(z)|^2 \, d\mu(z) \leq \|\mu\|^2_{CM(B_2^\sigma(\mathbb{B}_n))} \|f\|_{B_2^\sigma(\mathbb{B}_n)} .$$

The story changes as the value of σ changes.

When $\frac{n}{2} \leq \sigma$, then the space $B_2^\sigma(\mathbb{B}_n)$ are Hardy or weighted Bergman spaces. The Carleson measures for the Hardy space were discussed in the earlier section of these lecture notes, and can be characterized by testing on the reproducing kernels for the space in question, equivalently, they can be characterized by a simple geometric condition. In particular, we have that a measure μ is Carleson if and only if either of the following conditions hold:

$$\sup_{\lambda \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\left(1-|\lambda|^2\right)^{2\sigma}}{\left|1-\overline{\lambda} z\right|^{4\sigma}} d\mu(z) \leq C^2_{\text{Rep}};$$
$$\mu\left(B(\xi, r)\right) \leq C_{\text{Geo}} r^{2\sigma} \quad \forall \xi \in \partial\mathbb{B}_n, r > 0.$$

Moreover, the norm of the Carleson measure is comparable to C_{Rep} and to $C^{1/2}_{\text{Geo}}$.

On the other hand, when $0 \leq \sigma \leq \frac{1}{2}$ then a geometric characterization of Carleson measures is also known. These characterizations are much more subtle and require substantial work to obtain them.

The challenge in this range of σ is that the norm of the functions are determined by derivative conditions on the function. However, there is some positivity that can be exploited in this range that allows for one to obtain a characterization. When $n = 1$, we point the interested reader toward the paper [9]. In this context, the characterization is of the type:

$$\mu\left(T(\Omega)\right) \lesssim \operatorname{cap}_\sigma(\Omega) \quad \forall\,\text{open}\,\Omega \subset \mathbb{T}$$

where $\operatorname{cap}_\sigma$ is the capacity of the set as measured relative to the space in question. Roughly speaking,

$$\operatorname{cap}_\sigma(E) = \inf\left\{\|f\|^2_{L^2(\mathbb{T})} : f \geq 0,\, k_\sigma * f \geq 1 \text{ on } E\right\}$$

where $k_\sigma(\theta) = |\theta|^{-\frac{1}{2}-\sigma}$. See [9] for precise definitions. The key point to make is that the computation of capacity is solving a certain extremal problem, which is in general difficult to do. Nevertheless, this quantity actually characterizes the Carleson measures in question.

If $n > 1$, then there are two different characterizations of Carleson measures for $B_2^\sigma(\mathbb{B}_n)$. The first was obtained by Arcozzi, Rochberg and Sawyer, [1], and is given in terms of integration operators on trees (dyadic structures on the ball $\mathbb{B}_n$). The other characterization, closer to the approach we will follow in these notes, was obtain by E. Tchoundja, [10] and is in terms of T(1) conditions of a certain operator.

These results give the characterization of Carleson measures for all n, when $\sigma \in \left[0, \frac{1}{2}\right] \cup \left[\frac{n}{2}, \infty\right)$, but leave a giant island where the result is unknown.

Question 3.1. Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$. Namely, give a geometric characterization of the measures μ for which:

$$\int_{\mathbb{B}_n} |f(z)|^2\, d\mu(z) \leq \|\mu\|^2_{CM(B_2^\sigma(\mathbb{B}_n))} \|f\|_{B_2^\sigma(\mathbb{B}_n)}\,.$$

The goal in the remaining parts of these notes is to explain the main ideas that go into obtaining the characterization of these measures. In the interest of making the notes largely self-contained, the results from related papers are included.

3.2. Operator Theoretic Characterization of Carleson Measures

In this section we rephrase the Carleson measure problem as one about the boundedness of a certain integral operator. Let $\mathcal{H}$ be a Hilbert space of functions on a domain Ω with reproducing kernel function K_x, i.e.,

$$f(x) = \langle f, K_x \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}.$$

We wish to recast the Carleson measure question in terms of the boundedness of a certain integral operator. While this recasting might seem to simply be "moving symbols around", it turns out that doing so allows for certain tools in harmonic analysis to play a role.

Recall that a measure μ is $\mathcal{H}$-Carleson exactly if the inclusion map ι from $\mathcal{H}$ to $L^2(\Omega;\mu)$ is bounded, or

$$\int_{\Omega} |f(z)|^2 \, d\mu(z) \leq \|\mu\|_{CM(\mathcal{H})}^2 \, \|f\|_{\mathcal{H}}^2 .$$

We can give a characterization of Carleson measures for the space $\mathcal{H}$ in terms of information about the boundedness of a certain linear operator related to the reproducing kernel K_x.

Lemma 3.2. ([1, Arcozzi, Rochberg, Sawyer]) *A measure μ is a $\mathcal{H}$-Carleson measure if and only if the linear map*

$$f(z) \mapsto T(f)(z) = \int_{\Omega} \mathrm{Re}\big(K_x(z)\big) f(x) d\mu(x)$$

is bounded on $L^2(\Omega;\mu)$.

Proof. The inclusion map ι is bounded from $\mathcal{H}$ to $L^2(\Omega;\mu)$ if and only if the adjoint map ι^* is bounded from $L^2(\Omega;\mu)$ to $\mathcal{H}$, namely,

$$\|\iota^* f\|_{\mathcal{H}}^2 = \langle \iota^* f, \iota^* f \rangle_{\mathcal{H}} \leq C \|f\|_{L^2(\Omega;\mu)}^2, \quad \forall f \in L^2(\Omega;\mu).$$

For an $x \in \Omega$ we have

$$\begin{aligned} \iota^* f(x) &= \langle \iota^* f, K_x \rangle_{\mathcal{H}} = \langle f, \iota K_x \rangle_{L^2(\Omega:\mu)} \\ &= \int_{\Omega} f(w) \overline{K_x(w)} d\mu(w) = \int_{\Omega} f(w) K_w(x) d\mu(w). \end{aligned}$$

Using this computation, we obtain that

$$\begin{aligned}\|\iota^* f\|_{\mathcal{H}}^2 &= \langle \iota^* f, \iota^* f\rangle_{\mathcal{H}}\\ &= \left\langle \int_{\Omega} K_w(\cdot) f(w) d\mu(w), \int_{\Omega} K_{w'}(\cdot) f(w') d\mu(w') \right\rangle_{\mathcal{H}}.\end{aligned}$$

Those computations then give

$$\begin{aligned}\|\iota^* f\|_{\mathcal{H}}^2 &= \int_{\Omega}\int_{\Omega} \langle K_w, K_{w'}\rangle_{\mathcal{H}} f(w) d\mu(w) \overline{f(w')} d\mu(w')\\ &= \int_{\Omega}\int_{\Omega} K_w(w') f(w) d\mu(w) \overline{f(w')} d\mu(w').\end{aligned}$$

The continuity of ι^* for general f is equivalent to having it for real f. Without loss of generality, we can suppose that f is real. In that case we can continue the computation with

$$\|\iota^* f\|_{\mathcal{H}}^2 = \int_{\Omega}\int_{\Omega} \operatorname{Re} K_w(w') f(w) f(w') d\mu(w) d\mu(w') = \langle Tf, f\rangle_{L^2(\Omega;\mu)}.$$

But, the last quantity satisfies the required estimates exactly when the operator T is bounded. □

3.2.1. Calderón-Zygmund Kernels and Calderón-Zygmund Estimates

We apply the results of Lemma 3.2 to the space $B_2^\sigma(\mathbb{B}_n)$. We know that these are reproducing kernel Hilbert spaces with kernels given by

$$K_\lambda^\sigma(z) = \frac{1}{(1-\overline{\lambda} z)^{2\sigma}}.$$

The discussion above implies that we study the operator

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \operatorname{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w) d\mu(w)$$

and determine necessary and sufficient conditions for when this operator is bounded on $L^2(\mathbb{B}_n; d\mu)$.

Motivated by ideas in harmonic analysis, we want to show that the kernel of the operator above satisfies Calderón-Zygmund estimates with respect to a certain metric. Once we have this, then there is a proof strategy that can be exploited to obtain the boundedness.

3.2.2. The Appropriate Metric

Define

$$\Delta(z,w) \equiv \begin{cases} \left||z|-|w|\right| + \left|1-\frac{z\overline{w}}{|z||w|}\right|, & z,w \in \mathbb{B}_n \setminus \{0\}, \\ |z|+|w|, & \text{otherwise.} \end{cases}$$

We have the following lemma.

Lemma 3.3. *The following properties of $\Delta(z,w)$ hold:*

(1) $\Delta(z,w)$ *is a pseudo-distance on* $\mathbb{B}_n$*;*

(2) $\Delta(z,w)$ *is invariant under rotation.*

Proof. The second point follows by inspection. For the first point, one uses that $d(z,w) = |1-\overline{z}w|^{1/2}$ is a pseudo-metric on the unit ball and that the sum of metrics is a metric. □

We also have the following additional properties of the metric. The proof of this lemma is purely computational and we outline it now. Though the interested reader can find it in [2].

Lemma 3.4. *For every* $z \in \mathbb{B}_n$ *and* r_0 *with* $0 < r_0 < 1$*, setting* $z_0 = (r_0, 0, \ldots, 0)$ *we have*

(1) $|1 - z_1 r_0| \geq \frac{1}{3}\Delta(z,z_0)$*;*

(2) $|z_1 - r_0| \leq \Delta(z,z_0)$*;*

(3) $\sum_{k=2}^{n} |z_k|^2 \leq 2\Delta(z,z_0)$*;*

(4) $|1 - z z_0| \leq 1 - r_0^2 + \Delta(z,z_0)$*.*

Proof. For (1), observe

$$\begin{aligned}\left|1-\frac{z_1}{|z|}\right| &\leq |1-z_1 r_0| + \left|z_1 r_0 - \frac{z_1}{|z|}\right| \leq |1-z_1 r_0| + 1 - r_0\,|z| \\ &\leq 2\,|1-z_1 r_0|\,.\end{aligned}$$

It is easy to see that $||z|-r_0| \leq |1-z_1 r_0|$ (treat the cases $|z| > r_0$ and $|z| \leq r_0$ separately and estimate in an obvious manner). Combining these estimates easily gives that $\Delta(z,z_0) \leq 3\,|1-z_1 r_0|$.

For (2), note that

$$\begin{aligned}|z_1 - r_0| \leq \big|z_1 - |z|\big| + \big||z| - r_0\big| &\leq |z|\left|1 - \frac{\overline{z_1}}{|z|}\right| + \big||z| - r_0\big| \\ &\leq \Delta(z, z_0).\end{aligned}$$

For (3), we have that

$$\begin{aligned}\sum_{k=2}^{n} |z_k|^2 = |z|^2 - |z_1|^2 \leq 2\,\big||z| - |z_1|\big| \leq 2\,\big|z_1 - |z|\big| &\leq 2\left|1 - \frac{z_1}{|z|}\right| \\ &\leq 2\Delta(z, z_0).\end{aligned}$$

Finally, for (4) we have

$$\begin{aligned}|1 - z z_0| = |1 - z_1 r_0| \leq 1 - r_0^2 + \big|r_0^2 - z_1 r_0\big| &\leq 1 - r_0^2 + |r_0 - z_1| \\ &\leq 1 - r_0^2 + \Delta(z, z_0)\end{aligned}$$

with the last inequality following from (2). □

Then Δ is a pseudo-metric and makes the ball into a space of homogeneous type. Recall that a space of homogenous type is a topological set X, a pseudo-distance ρ and a positive Borel measure μ on X such that for any $x \in X$, the balls of radius $r > 0$ form a basis of open neighborhoods of x. And, there is a constant A such that for all $x \in X$ and $r > 0$, $0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty$ (the measure satisfies a doubling property). This means that much of the "standard" harmonic analysis can be carried out on the ball $\mathbb{B}_n$ using Δ as a replacement for the regular Euclidean distance. In particular, one can prove versions of standard covering lemmas, and one can prove versions of well-known theorems in harmonic analysis (e.g., the boundedness of Calderón-Zygmund operators).

We now come to the key property of the kernel of the operator in question: the kernel is a Calderón-Zygmund kernel relative to the metric $\Delta(z, w)$.

Lemma 3.5. *The kernel* $\mathrm{Re}\left(1/(1 - \overline{w}z)^{2\sigma}\right)$ *satisfies the following properties:*

(1) *There exists a constant C such that*

$$\left|\mathrm{Re}\left(\frac{1}{(1 - \overline{w}z)^{2\sigma}}\right)\right| \leq \frac{C}{\Delta(z, w)^{2\sigma}};$$

(2) *There exist constants* C_1 *and* C_2 *such that for all* $z, w, \xi \in \mathbb{B}_n$ *if* $\Delta(z,\xi) > C_1\Delta(w,\xi)$ *then*

$$\left|\mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) - \mathrm{Re}\left(\frac{1}{(1-\overline{\xi}z)^{2\sigma}}\right)\right| \leq C_2\frac{\Delta(w,\xi)^{\frac{1}{2}}}{\Delta(z,\xi)^{2\sigma+\frac{1}{2}}}.$$

(3) $\mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) \leq \frac{1}{\max\left\{(1-|z|^2)^{2\sigma}, (1-|w|^2)^{2\sigma}\right\}}.$

Proof. For the proof of the first claim, observe that if we let $\rho(w) = (|w|, 0, \ldots, 0)$ denote the rotation of the point $w \in \mathbb{B}_n$ and use the rotation invariance we have:

$$\Delta(z,w) = \Delta\big(\rho(z), \rho(w)\big) \leq 3\,|1-\rho(z)\rho(w)| = 3\,|1-\overline{z}w|\,.$$

Taking $C = 3^{2\sigma}$ proves the claim. Here we have used the key properties of the metric Δ.

The second claim is more involved, though we essentially use the Fundamental Theorem of Calculus, compute a derivative and estimate. Because of rotation invariance, we may assume that $\xi = (r_0, 0, \ldots, 0)$. Then observe that

$$\begin{aligned}\mathrm{Re}&\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) - \mathrm{Re}\left(\frac{1}{(1-\overline{\xi}z)^{2\sigma}}\right)\\ &= \int_0^1 \mathrm{Re}\left(\frac{2\sigma z(\overline{w}-\overline{\xi})}{(1-z\overline{w}-tz(\overline{z}-\overline{\xi}))^{2\sigma+1}}\right)dt.\end{aligned}$$

Taking absolute values we find,

$$\begin{aligned}&\left|\mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) - \mathrm{Re}\left(\frac{1}{(1-\overline{\xi}z)^{2\sigma}}\right)\right|\\ &\qquad\leq \int_0^1 \left|\left(\frac{2\sigma z(\overline{w}-\overline{\xi})}{(1-z\overline{w}-tz(\overline{z}-\overline{\xi}))^{2\sigma+1}}\right)\right|dt.\end{aligned}$$

A simple computation gives

$$\left|z(\overline{w}-\overline{\xi})\right| \leq |w_1 - r_0| + \left(\sum_{k=2}^{n}|z_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=2}^{n}|w_k|^2\right)^{\frac{1}{2}}.$$

By the basic properties of the metric Δ in Lemma 3.4 and for $C_1 > 0$

sufficiently large

$$\begin{aligned}
\left|z(\overline{w}-\overline{\xi})\right| &\leq 2\Delta(w,\xi)^{\frac{1}{2}}\left(\Delta(w,\xi)^{\frac{1}{2}}+\Delta(z,\xi)^{\frac{1}{2}}\right)\\
&\leq \frac{4}{\sqrt{C_1}}\Delta(w,\xi)^{\frac{1}{2}}\Delta(z,\xi)^{\frac{1}{2}}\\
&\leq \frac{C}{\sqrt{C_1}}\Delta(w,\xi)^{\frac{1}{2}}\left|1-\overline{z}\xi\right|^{\frac{1}{2}} \leq \frac{1}{2}\left|1-z\overline{\xi}\right|.
\end{aligned}$$

Setting $\eta=(1-t)w+t\xi$ we have that

$$\begin{aligned}
\left|1-z\overline{\eta}\right| &= \left|1-z\overline{w}-tz(\overline{\xi}-\overline{w})\right|\\
\left|z(\overline{\eta}-\overline{\xi})\right| &= \left|(1-z\overline{\xi})-(1-z\overline{\eta})\right| = (1-t)\left|z(\overline{w}-\overline{\xi})\right|.
\end{aligned}$$

Again, for C_1 large enough we have that $|1-z\overline{\eta}|\geq\frac{1}{2}\left|1-z\overline{\xi}\right|$. Combining these estimates one can easily deduce the claimed estimate on the difference of the kernel. The reader can see that the same proof in fact applies to the kernel

$$\frac{1}{(1-\overline{w}z)^{2\sigma}}$$

(the real part need not be taken). The third claim is a trivial estimate. □

The key point behind this lemma is that it says that the operator

$$T_{\mu,2\sigma}(f)(z)=\int_{\mathbb{B}_n}\mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right)f(w)d\mu(w)$$

is a Calderón-Zygmund operator on the set $\mathbb{B}_n$ relative to the metric $\Delta(z,w)$, and one should seek out necessary and sufficient conditions for when a Calderón-Zygmund operator is bounded. It turns out that there exists a beautiful result in harmonic analysis that suggests how one should proceed. For motivation, we now state the result in the Euclidean setting. Suppose that we have a kernel $K(x,y)$ that is locally integrable and satisfies conditions of the form:

$$\begin{aligned}
|K(x,y)| &\lesssim \frac{1}{|x-y|^n}\\
\left|K(x,y)-K(x',y)\right| &\lesssim \frac{|x-x'|^\delta}{|x-y|^{n+\delta}}
\end{aligned}$$

provided that $|x-y| \gtrsim |x'-y|$. We then define

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy.$$

Making sense of this integral does require some care, but we largely ignore that issue in these notes. For rigorous definitions and properties of Calderón-Zygmund kernels we refer the reader to the monograph [12]. We are interested in necessary and sufficient conditions so that the Calderón-Zygmund operator T is bounded on $L^2(\mathbb{R}^n)$. This is captured by the following.

Theorem 3.6. (David and Journé, [4]) *If T is a Calderón-Zygmund operator then $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if $T(1), T^*(1) \in$ BMO$(\mathbb{R}^n)$ and T is weak bounded. Equivalently, $T\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if for all cubes Q*

$$\int_Q |T1_Q(x)|^2 \, dx \le C_1 \, |Q|$$
$$\int_Q |T^*1_Q(x)|^2 \, dx \le C_2 \, |Q|$$

Moreover, $\left\| T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \right\| \approx C_1 + C_2$.

The point behind this theorem is that it reduces the boundedness to a certain 'testing condition' only on the operator. Based on this theorem and what appears above, we would suppose that it is possible to simply apply the proof strategy of the David-Journé to the problem at hand. By and large this is the right idea, but requires some modifications.

3.3. A Real Variable Version of the Problem

As sometimes happens in complex analysis, it is beneficial to change the problem to a 'real variable' question and attempt to resolve the real-variable problem using results in harmonic analysis. We now reformulate what is going on for this problem in the language of non-homogeneous harmonic analysis.

We are interested in Calderón-Zygmund operators that do not satisfy the standard estimates. In particular, these kernels will live in $\mathbb{R}^d$ but will only satisfy estimates as if they live in $\mathbb{R}^m$ for $m \le d$.

More precisely, we are interested in Calderón-Zygmund operators whose kernels satisfy the following estimates

$$|k(x,y)| \leq \frac{C_{CZ}}{|x-y|^m},$$

and

$$|k(y,x) - k(y,x')| + |k(x,y) - k(x',y)| \leq C_{CZ} \frac{|x-x'|^\tau}{|x-y|^{m+\tau}}$$

provided that $|x - x'| \leq \frac{1}{2}|x-y|$, with some (fixed) $0 < \tau \leq 1$ and $0 < C_{CZ} < \infty$. Once the kernel has been defined, then we say that an $L^2(\mathbb{R}^d;\mu)$ bounded operator is a Calderón–Zygmund operator with kernel k if,

$$T_\mu f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \forall\, x \notin \operatorname{supp} f\,.$$

If k is a nice function, then the integral above can be defined for all x and it gives a Calderón–Zygmund operators with kernel k. It is for this reason, that when given a 'bad' Calderón–Zygmund kernel people consider the sequence of "cut-off" kernels and the *uniform boundedness* of this sequence. In the application at hand, all our Calderón-Zygmund operators can be considered a priori bounded (for example in future arguments, we can always think that μ is compactly supported inside the (complex) unit ball), and we will be interested in *the effective bound*, in terms of the parameters C_{CZ}, τ and in terms of a certain $T1$ condition we explain below. Frequently, this is the better view point to adopt instead of the "cut-off" approach.

For kernels that satisfy these types of estimates and for virtually arbitrary underlying measures, a deep theory has been developed by Nazarov, Treil and Volberg in [5–8, 13]. The essential core of this theory showed that in this situation, one can develop the majority of Calderón–Zygmund theory and study the boundedness of the associated singular integral operators via a "T1 Condition".

Having in mind the application to Carleson measures in the complex unit ball, we wish to extend the theory in [5–8, 13] to the case of singular integral operators that arise naturally as "Bergman–type" operators. These will be operators that will satisfy the Calderón–Zygmund estimates from above, but we (again having in mind the above mentioned application, see further) additionally allow them to have the following property

$$|k(x,y)| \leq \frac{1}{\max\left(d(x)^m, d(y)^m\right)},$$

where $d(x) \equiv \operatorname{dist}(x, \mathbb{R}^d \setminus H)$ and H is an open set in $\mathbb{R}^d$. The examples that the reader should keep in mind are the Calderón-Zygmund kernels built from the function $k(x,y) = (1 - x \cdot y)^{-m}$ where $H = \mathbb{B}_d$, the unit ball in $\mathbb{R}^d$. These are the standard "Bergman–type" kernels that arise naturally when looking at the Carleson measure problem for Besov-Sobolev spaces. When we have a kernel k that satisfies the Calderón-Zygmund estimates and the additional property of measuring "distance to some open set" as above, we will let

$$T_{\mu,m}(f)(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y).$$

In applications, we will be viewing the Calderón–Zygmund kernels that arise as living on certain fixed sets. Accordingly, we will say that k is a Calderón–Zygmund kernel on a closed $X \subset \mathbb{R}^d$ if $k(x,y)$ is defined only on $X \times X$ and the previous properties of k are satisfied whenever $x, x', y \in X$. In this context one can then prove the following theorem, whose proof we turn to later in the notes.

Theorem 3.7. ([11, Volberg and Wick]) *Let $k(x,y)$ be a Calderón-Zygmund kernel of order $m \leq d$ on $X \subset \mathbb{R}^d$, with Calderón-Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and suppose that all balls such that $\mu(B(x,r)) > r^m$ lie in an open set H. Let also*

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) \equiv \operatorname{dist}(x, \mathbb{R}^d \setminus H)$. Finally, suppose that for all cubes Q a "$T1$ Condition" holds for the operator $T_{\mu,m}$ with kernel k and for the operator $T^*_{\mu,m}$ with kernel $k(y,x)$:

$$\|T_{\mu,m}\mathbf{1}_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q),$$
$$\|T^*_{\mu,m}\mathbf{1}_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q).$$

Then $\|T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq C(A,m,d,\tau)$.

While this theorem is stated in a Euclidean setting, it is in fact possible to recast the theorem in a space of homogeneous type (such as the set $\mathbb{B}_n$ with metric $\Delta(z,w)$), or even in the Euclidean setting with a different metric than the standard one. This flexibility will allow us to obtain a resolution to the Carleson measure question.

The proof of Theorem 3.7 follows prior work in non-homogeneous harmonic analysis. The incredibly rough sketch of the proof is as follows (details appear later). We proceed by duality to study the behavior of the operator $T_{\mu,m}$ and so for $f,g\in L^2(\mathbb{R}^d,\mu)$ we need to control:

$$\langle T_{\mu,m}f,g\rangle_{L^2(\mathbb{R}^d,\mu)}.$$

One decomposes $\mathbb{R}^d$ into (random) dyadic lattices $\mathcal{D}_1$ and $\mathcal{D}_2$. One defines martingale difference operators to express the functions f and g as:

$$f=\Lambda f+\sum_{Q\in\mathcal{D}_1}\Delta_Q f,\quad g=\Lambda g+\sum_{R\in\mathcal{D}_2}\Delta_R g$$

where Λf denotes the average of f over the cube Q_0. An easy application of the testing conditions lets one further reduce to the case that:

$$f=\sum_{Q\in\mathcal{D}_1}\Delta_Q f,\quad g=\sum_{R\in\mathcal{D}_2}\Delta_R g,$$

namely that the functions have mean value zero. Using two different lattices provides access to a certain averaging technique of Nazarov, Treil and Volberg that allows one to further restrict the sums above to "good" dyadic cubes. One than expands via these good cubes:

$$\begin{aligned}\langle T_{\mu,m}f,g\rangle_{L^2(\mathbb{R}^d,\mu)} &= \sum_{Q,R}\langle T_{\mu,m}\Delta_Q f,\Delta_R g\rangle_{L^2(\mathbb{R}^d,\mu)}\\ &= \sum_{\text{"Diagonal"}}+\sum_{\text{"Upper Triangle"}}\\ &\quad +\sum_{\text{"Lower Triangle"}}\langle T_{\mu,m}\Delta_Q f,\Delta_R g\rangle_{L^2(\mathbb{R}^d,\mu)}.\end{aligned}$$

The upper/lower triangular terms are controlled by the Carleson Embedding Theorem (a slight variant of what we met in section 1!) and the testing conditions. The diagonal terms are controlled by the testing conditions and simpler estimates. For full details behind this proof strategy we refer the reader to [5–8, 11, 13].

3.3.1. Connection to the Characterization of Carleson Measures

We can now formulate the characterization of the Carleson measures for all Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$.

Theorem 3.8. (Characterization of Carleson Measures for Besov–Sobolev Spaces $B_2^\sigma(\mathbb{B}_n)$) *Suppose that* $0 < \sigma$. *Let* μ *be a positive Borel measure in* $\mathbb{B}_n$. *Then the following conditions are equivalent:*

(a) μ *is a* $B_2^\sigma(\mathbb{B}_n)$*-Carleson measure;*

(b) $T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ *is bounded;*

(c) *There is a constant* C *such that*

  (i) $\|T_{\mu,2\sigma}1_Q\|^2_{L^2(\mathbb{B}_n;\mu)} \le C\,\mu(Q)$ *for all* Δ*-cubes* Q*;*

  (ii) $\mu(B_\Delta(x,r)) \le C\,r^{2\sigma}$ *for all balls* $B_\Delta(x,r)$ *that intersect* $\mathbb{C}^n \setminus \mathbb{B}_n$.

Some parts of this theorem require some explanation. Above, the sets B_Δ are balls measured with respect to the metric Δ. The discussion above clearly shows that the operator $T_{\mu,2\sigma}$ is a Bergman-type Calderón–Zygmund operator with respect to this metric Δ for which we can apply (an extended version) of Theorem 3.7. And, the set Q is a "cube" defined with respect to the metric Δ. The reason these are referred to as "cubes" are because in the proof of Theorem 3.7 the cubes considered arise from a naturally constructed dyadic lattice. In the standard Euclidean case, Theorem 3.7, we simply will take the standard dyadic lattice. However, for our characterization on the ball we will need to transfer certain parallelepiped regions to a spherical neighborhood of the sphere. This situation arises since, in a neighborhood of the sphere, the metric Δ will look like a variant of the standard Euclidean metric but with different powers appearing. It would be geometrically "nicer" if these shapes were actual (non-Euclidean) balls. However, this difficulty can be overcome since one actually can replace these non-Euclidean cubes by non-Euclidean balls. This change between cubes and balls is *not* a total triviality because the operator $T_{\mu,m}$ does not have a positive kernel. In the Euclidean setting, we refer the reader to [5] for this passage from cubes to balls, and remark that these changes can be adapted to handle the case of a non-Euclidean metric verbatim.

Sketch of Proof of Theorem 3.8. We have seen that (a) and (b) are equivalent by Lemma 3.2. It is clear that (b) implies (c)(i) since for any subset E of $\mathbb{B}_n$, letting 1_E denote the indicator of the set we have that

$$\begin{aligned}\|T_{\mu,2\sigma}1_E\|^2_{L^2(\mathbb{B}_n;\mu)} &\le \|T_{\mu,2\sigma}\|^2_{L^2(\mathbb{B}_n;\mu)\to L^2(\mathbb{B}_n;\mu)}\,\|1_E\|^2_{L^2(\mathbb{B}_n;\mu)}\\ &= \|T_{\mu,2\sigma}\|^2_{L^2(\mathbb{B}_n;\mu)\to L^2(\mathbb{B}_n;\mu)}\,\mu(E).\end{aligned}$$

Applying this to the cubes in question gives the claim. While (b) implies (c)(ii) since (b) is equivalent to (a) and since (c)(ii) is equivalent to the obvious necessary geometric testing condition obtained by checking the embedding on the reproducing kernels.

All the work has to go into proving that (c) implies (b). However, this follows from (a variant of) Theorem 3.7. We know that the operator $T_{\mu,2\sigma}$ is a Calderón-Zygmund operator relative to the metric Δ, and so the testing conditions in (c)(i) are simply the testing condition in Theorem 3.7. The reason we only have one set of testing conditions in our theorem is because the kernel is self-adjoint. Setting $H = \mathbb{B}_n$ we also have for our kernel that

$$\left|\mathrm{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right)\right| \lesssim \frac{1}{\max(d(z)^{2\sigma}, d(w)^{2\sigma})}\,,$$

where $d(z) \equiv \mathrm{dist}_\Delta(z, \mathbb{C}^n \setminus \mathbb{B}_n)$. Finally, the well-known necessary condition for the space $B_2^\sigma(\mathbb{B}_n)$ to be imbedded in $L^2(\mathbb{B}_n;\mu)$ is

$$\mu(B_\Delta(\zeta,r)) \le C_1\, r^{2\sigma}, \quad \forall\, \zeta \in \partial\mathbb{B}_n\,. \tag{3.1}$$

This is easily seen by testing the embedding condition on the reproducing kernel for the space of functions. However, (3.1) can be rewritten in a form akin to the conditions on the measure in Theorem 3.7. Namely, of course (3.1) is equivalent to (with another constant)

$$\mu(B_\Delta(\zeta,r)) \le C_2\, r^{2\sigma}, \quad \forall\, B_\Delta(\zeta,r) :\; B_\Delta(\zeta,r) \cap \mathbb{C}^d \setminus \mathbb{B}_n \ne \emptyset\,. \tag{3.2}$$

In turn, (3.2) can be rephrased as saying

> every metric ball such that $\mu(B_\Delta(\zeta,r)) > C_2\, r^{2\sigma}$
> is contained in the unit ball $\mathbb{B}_n$.

We have now essentially mapped the conditions appearing in (c) to the hypotheses of Theorem 3.7. Appropriate modifications of the proof of Theorem 3.7 then conclude the discussion. □

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