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NAN LI

Difference Cartan-Nevanlinna theory and meromorphic solutions of functional equations

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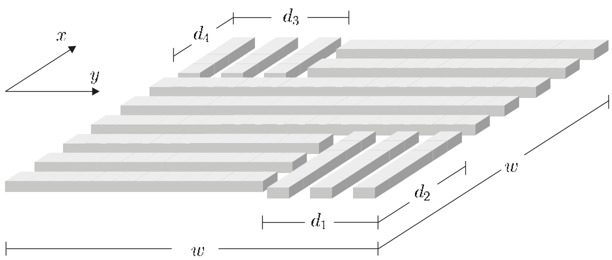

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ABSTRACT

The survey of this thesis begins with some background on classical Nevanlinna and Cartan theories, and their difference analogues. Secondly, we introduce notation and corresponding properties of a good linear operator, which satisfies certain regularity conditions in terms of value distribution theory. This concept will be used as a tool for the study of general classes of functional equations. In addition, this survey also presents some recent results on the properties of solutions to certain types of functional equations. Finally, short summaries of the papers being part of this survey are included.

2010 Mathematics Subject Classification: 39B32; 30D35; 39A10.
Keywords: Nevanlinna theory; Cartan theory; Functional Equation.

Preface

First of all, I am deeply indebted to my supervisor at the University of Eastern Finland, Professor Risto Korhonen, for his kind guidance, valuable advice and thoughtful assistance since I began to further my study of complex analysis.

I sincerely thank my supervisor at Shandong University, Professor Lianzhong Yang, for introducing me to Nevanlinna theory, and continuously guiding me in all the stages of my postgraduate studies.

I give my warm thanks to the faculty and the staff of the Joensuu Physics and Mathematics department for providing me with a friendly environment, and for all the help they offered me. I also wish to thank the University of Eastern Finland for financial support for these two years.

I am grateful to Professor Tingbin Cao and Professor Indrajit Lahiri for their careful reviews and helpful comments. I would also like to thank Professor Katsuya Ishizaki for acting as my opponent, to thank Professor Roy Goldblatt for language checking for this survey.

Finally, I wish to thank my family and all my friends, for their support and encouragement.

Joensuu June, 2015 *Nan Li*

LIST OF PUBLICATIONS

I Risto Korhonen, Nan Li and Kazuya Tohge, *Difference analogue of Cartan's second main theorem for slowly moving periodic targets*, to appear in Ann. Acad. Sci. Fenn. Math.

II Nan Li, *On the existence of solutions of a Fermat-type difference equation*, Ann. Acad. Sci. Fenn. Math. 40 (2015), 907-921.

III Nan Li, Risto Korhonen and Lianzhong Yang, *Good linear operators and meromorphic solutions of functional equations*, Journal of Complex Analysis. 2015. Article ID 960204, 8 pages.

IV Nan Li and Lianzhong Yang, *Growth of solutions to second-order complex differential equations*, Electron. J. Differential Equations. **51** (2014), 1–12.

V Nan Li, Risto Korhonen and Lianzhong Yang, *Nevanlinna uniqueness of linear difference polynomials*, to appear in Rocky Mountain Journal of Mathematics.

AUTHOR'S CONTRIBUTION

The ideas of Papers I–III come from research work done at the University of Eastern Finland, and all authors have an equal contribution.

Papers IV, V originate from research work done at Shandong Univesity, and all authors have an equal contribution.

Contents

1 Introduction

In the 1920's, Nevanlinna theory was devised by the Finnish mathematician Rolf Nevanlinna, as one of the few great mathematical events of the twentieth century. It is composed of two main theorems, which are called Nevanlinna's first and second main theorems. Nevanlinna theory has since been solidly developed in itself and widely applied to other fields of complex analysis such as the uniqueness of meromorphic functions, complex differential and difference equations, and several complex variables.

Since Nevanlinna theory is concerned with the distribution of values of a holomorphic map in the complex projective line, in 1933, Cartan [4] extended the theory to higher dimensional cases. This generalization is a strong result in the value distribution of holomorphic curves in the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$, as well as an efficient tool for certain problems in the complex plane $\mathbb{C}$, or the study of Gauss maps of minimal surfaces in $\mathbb{R}^3$, hyperbolic complex spaces or Diophantine approximation. In this thesis, we extend the difference analogue of Cartan's second main theorem obtained by Halburd, Korhonen, and Tohge [26] for the case of slowly moving periodic hyperplanes, and introduce two different natural ways to find a difference analogue of the truncated second main theorem. As applications, we obtain a new Picard type theorem and difference analogues of the deficiency relation for holomorphic curves.

In order to study the properties of meromorphic solutions of functional spaces, a notion and corresponding properties of a good linear operator, which satisfies certain regularity conditions in terms of value distribution theory, were introduced as a tool since the methodologies used in the study of meromorphic solutions of differential, difference, and q-difference equations are largely similar.

As an application, we apply our methods to study the growth of meromorphic solutions to the functional equation $M(z, f) + P(z, f) = h(z)$, where $M(z, f)$ is a linear polynomial in f and a good linear operator $L(f)$, $P(z, f)$ a polynomial in f with degree $\deg P \geq 2$, both P and M have small meromorphic coefficients, and $h(z)$ is a meromorphic function.

Let C be the ring of meromorphic functions M, rational functions R, entire functions E or polynomials P, respectively. Recently, many scholars have been devoted to investigating the analogue of Fermat's last theorem for function fields, and Gundersen-Hayman [20] collected the best lower estimates that are known for $F_C(n)$, where $F_C(n)$ is the smallest positive integer k such that the equation

$$f_1^n + f_2^n + \cdots + f_k^n = 1$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C. In this thesis, we investigate a difference analogue of this problem for the rings of M, R, E, P with certain conditions, and obtain lower bounds for G_C, where G_C is the smallest positive integer k such that the equation

$$f_1(z) \cdots f_1(z + (n-1)c) + \cdots + f_k(z) \cdots f_k(z + (n-1)c) = 1$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C.

In this thesis, we also study the solutions to functional equations of another type. We investigate the existence of non-trivial subnormal solutions for second-order linear differential equations. We show that under certain conditions some differential equations do not have subnormal solutions, and the hyper-order of every solution is equal to one.

As for the uniqueness problem, we investigate shared value problems related to an entire function $f(z)$ of the hyper-order less than one and its linear difference operator $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$, where

$a_i, c_i \in \mathbb{C}$. We give sufficient conditions in terms of weighted value sharing and truncated deficiencies, which imply that $L(f) \equiv f$. From the above identity, we can see that f is a solution to a linear difference equation with constant coefficients. Therefore, this problem can also be classified as a problem of solutions to functional equations.

Like the case of Cartan's second main theorem, the difference analogue of Cartan's second main theorem can be viewed as another useful tool for the study of solutions to functional equations of certain types. In this thesis, we develop this tool, and extend the difference analogue of Cartan's second main theorem to the case of slowly moving periodic hyperplanes. Furthermore, we introduce two different natural ways to find a difference analogue of the truncated second main theorem.

The rest of this survey is structured as follows: After recalling the basic notations and classical results of Nevanlinna and Cartan theory, as well as their difference analogues in section 2, notation and corresponding properties of a good linear operator are introduced as a tool for investigating the solutions to functional equations in section 3. In section 4, we give some recent results on the properties of solutions to functional equations of three different types. Finally, the essential contents of the Papers I–V are summarized in section 5.

2 Preliminaries

In this survey, a meromorphic function means meromorphic in the whole complex plane $\mathbb{C}$. As a convenience to the reader, we recall the basic notations, definitions and results of the classical Nevanlinna and Cartan theories and their difference analogues in this section; for details, refer to [27,35,36,62].

2.1 CLASSICAL NEVANLINNA THEORY

Let f be a meromorphic function in the complex plane. For each $r > 0$, the *counting function* $N(r, f)$, which measures the average frequency of the poles of f in the disk $|z| < r$, is now being defined as

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ denotes the number of poles of f in the disk $|z| \leq t$, each pole counted according to its multiplicity.

The *proximity function* $m(r, f)$, which measures the average magnitude of f on the circle $|z| = r$, is defined as

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x := \max\{\log x, 0\}$.

The *Nevanlinna characteristic function* of $f(z) \in \mathbb{C}$ is then defined by

$$T(r, f) = m(r, f) + N(r, f).$$

For $a \in \mathbb{C}$, the definitions for $m\left(r, \frac{1}{f-a}\right)$, $N\left(r, \frac{1}{f-a}\right)$ and $T\left(r, \frac{1}{f-a}\right)$ are immediate.

Based on these notations, the definitions of order, hyper order, exponent of convergence of zeros and poles are given to measure the complexity of meromorphic function $f(z)$ as follows:

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}, \ \sigma_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r, f)}{\log r},$$

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r} \text{ and } \lambda\left(\frac{1}{f}\right) = \limsup_{r\to\infty} \frac{\log^+ N(r, f)}{\log r}.$$

Before introducing Nevanlinna theory, we list the following elementary inequalities according to the properties of the positive logarithmic function and the Nevanlinna counting function; for details, see [35]:

$$m(r, \alpha f + \beta g) \leq m(r, f) + m(r, g) + O(1),$$

$$m(r, fg) \leq m(r, f) + m(r, g),$$

$$N(r, \alpha f + \beta g) \leq N(r, f) + N(r, g),$$

$$N(r, fg) \leq N(r, f) + N(r, g),$$

$$T(r, \alpha f + \beta g) \leq T(r, f) + T(r, g) + O(1),$$

$$T(r, fg) \leq T(r, f) + T(r, g).$$

By applying the Poisson–Jensen formula [27, Theorem 1.1], we can easily obtain the first main theorem, see, e.g., [27, Theorem 1.2], [36, Theorem 2.1.10], which gives a relationship between characteristic functions of $1/(f-a)$ and f.

Theorem 2.1.1 (The First Main Theorem) *Let f be a non-constant meromorphic function. Then for any complex number $a \in \mathbb{C}$,*

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

Throughout this section, $S(r, f)$ denotes a quantity satisfying that $S(r, f)/T(r, f)$ approaches zero as $r \to \infty$ outside a possible exceptional set of finite linear measure.

The second main theorem of the Nevanlinna theory depends essentially on the lemma for the logarithmic derivative [62, Theorem 1.7]:

Theorem 2.1.2 (The logarithmic derivative lemma) *Suppose f is meromorphic and non-constant in the complex plane. Then $m(r, f'/f) = S(r, f)$, and whenever f is of finite order, $m(r, f'/f) = O(\log r)$ without an exceptional set.*

Obviously, supposing that $k \in \mathbb{N}$, then $m(r, f^{(k)}/f) = S(r, f)$. Moreover, whenever f is of finite order, $m(r, f^{(k)}/f) = O(\log r)$ without an exceptional set, and according to an inequality due to Milloux (see [62])

$$T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f).$$

By combining the first main theorem with the logarithmic derivative lemma, we can obtain the following second main theorem [35, Theorem A.9], which is the deepest and the most important result of the value distribution theory. It generalizes the classical Picard theorem.

Theorem 2.1.3 (The Second Main Theorem) *Let f be a non-constant meromorphic function, $q \geq 2$, and $\alpha_j (j = 1, 2, \ldots, q)$ distinct complex numbers. Then*

$$m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - \alpha_j}\right) \leq 2T(r, f) - N_{ram}(r, f) + S(r, f),$$

where $N_{ram}(r, f) := 2N(r, f) + N(r, 1/f') - N(r, f')$ is the integrated counting function for multiple points of f, so that each such multiple point (or a multiple pole) of multiplicity p is here to be counted $p - 1$ times.

It has the following two simple variants [35, Theorems A.10–11], which are used more often.

Theorem 2.1.4 (The Second Main Theorem V1) *Let f be a non-constant meromorphic function, $q \geq 2$, and $\alpha_j (j = 1, 2, \ldots, q)$ distinct complex numbers. Then*

$$(q-2)T(r, f) - \sum_{j=1}^{q} N\left(r, \frac{1}{f - \alpha_j}\right) + N_{ram}(r, f) = S(r, f).$$

Theorem 2.1.5 (The Second Main Theorem V2) *Let f be a non-constant meromorphic function, $q \geq 2$, and $\alpha_j (j = 1, 2, \ldots, q)$ distinct complex numbers. Then*

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - \alpha_j}\right) + S(r, f),$$

where $\overline{N}(r,f)$*, resp.* $\overline{N}(1/(f-\alpha_j))$*, stands for the counting function of distinct poles, resp. of distinct* α_j-*points of* f*.*

Actually, the second main theorem is also valid for the case of small functions; for proof refer to [27, Theorem 2.5], [59, Corollary 1].

For the characteristic function of a rational function of a meromorphic function with small coefficents, we have the following Valiron–Mohon'ko theorem [62].

Theorem 2.1.6 (Valiron–Mohon'ko theorem) *Let* f *be a meromorphic function, and consider an irreducible rational function in* f*,*

$$R(z,f)=\frac{\sum_{i=0}^{p}a_i(z)f^i}{\sum_{j=0}^{q}b_j(z)f^j}$$

with meromorphic coefficients $a_i(z)$*,* $b_j(z)$ *small with respect to* f*. Then*

$$T(r,R(z,f))=dT(r,f)+S(r,f),\quad d:=\max\{p,q\}.$$

2.2 DIFFERENCE NEVANLINNA THEORY

For a meromorphic function $f(z)$ and $c\in\mathbb{C}\backslash\{0\}$, a shift of $f(z)$ is defined as $f(z+c)$, while the forward differences are defined as

$$\triangle_c f(z)=f(z+c)-f(z),\quad \triangle_c^n f(z)=\triangle_c(\triangle_c^{n-1}f),\, n=2,3,\ldots$$

Throughout this section, we denote by $\widetilde{S}(r,f)$ a quantity such that $\widetilde{S}(r,f)/T(r,f)$ approaches zero as $r\to\infty$ outside a possible exceptional set of finite logarithmic measure.

Difference analogues of the logarithmic derivative lemma, which are of great importance in the study of difference analogues of classical Nevanlinna theory, were established by Halburd–Korhonen [22,23], and Chiang–Feng [9,10] independently.

Theorem 2.2.1 *[22, Theorem 2.1] Let $f(z)$ be a non-constant meromorphic function of finite order, and $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{\delta}}\right) = \widetilde{S}(r,f).$$

Theorem 2.2.2 *[9, Corollary 2.6] Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and f be a meromorphic function of finite order ρ. Then, for each $\varepsilon > 0$,*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

Recently, Theorem 2.2.1 has been extended to meromorphic functions of hyper-order strictly less than one by Halburd, Korhonen and Tohge [26].

Theorem 2.2.3 *[26, Theorem 2.1] Let $f(z)$ be a non-constant meromorphic function, and $c \in \mathbb{C}$. If $\sigma_2(f) = \sigma_2 < 1$ and $\varepsilon > 0$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\sigma_2-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

As for the relationship of Nevanlinna characteristics of meromorphic function with its shift, two forms exist. The first one is obtained by Chiang-Feng as follows:

Theorem 2.2.4 *[9, Theorem 2.1] Let $f(z)$ be a meromorphic function with the order $\sigma = \sigma(f) < +\infty$, and $c \in \mathbb{C}\backslash\{0\}$. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r,f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

The second form, which is a special case of [34, Lemma 2.2], can be stated as follows:

Theorem 2.2.5 *Let $f(z)$ be a meromorphic function with the hyper-order less than one, and $c \in \mathbb{C}\backslash\{0\}$. Then we have*

$$T(r, f(z+c)) = T(r,f) + \widetilde{S}(r,f).$$

In order to prove the above form, we need the following theorem:

Theorem 2.2.6 *[26] Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function and $s \in (0, \infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \sigma_2 < 1$$

and $\delta \in (0, 1 - \sigma_2)$ then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

By combining Theorem 2.2.6 with the following inequality from reference [15, P.66],

$$(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z)),$$

we can obtain the conclusion of Theorem 2.2.5.

Next we introduce the difference variants of the second main theorem.

Theorem 2.2.7 *[35, Theorem A.15] Let $c \in \mathbb{C}$, and f be a meromorphic function such that $\sigma_2(f) < 1$ and $f(z) - f(z+c) \not\equiv 0$. Moreover, let $q \geq 2$, and $a_1(z), \ldots, a_q(z)$ be small periodic meromorphic functions with period c. Then*

$$m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq 2T(r, f) - N_{pair}(r, f) + \widetilde{S}(r, f),$$

where

$$N_{pair}(r, f) := 2N(r, f) - N(r, \triangle_c f) + N(r, 1/\triangle_c f).$$

Before giving a more general version, we first introduce the following definitions; refer to [23] for details.

Let f be a meromorphic function, and $c \in \mathbb{C}$. We denote by $n_c(r, a)$ the number of points z_0 in $|z| \leq r$ where $f(z_0) = a$ and

$f(z_0 + c) = a$, counted according to the number of equal terms in the beginning of the Taylor series expansions of $f(z)$ and $f(z+c)$ in a neighborhood of z_0. We call such points c-separated a-pairs of f in the disk $\{z : |z| \leq r\}$.

The integrated counting function is now defined in the usual way:

$$N_c\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n_c(t,a) - n_c(0,a)}{t} dt + n_c(0,a) \log r.$$

Similarly, we have $N_c(r, f)$.

A difference analogue of $\overline{N}(r, f)$ can then be obtained as

$$\widetilde{N}_c\left(r, \frac{1}{f-a}\right) := N\left(r, \frac{1}{f-a}\right) - N_c\left(r, \frac{1}{f-a}\right),$$

which counts the number of those a-points (or poles) of f which are not in c-separated pairs. With this notation, we obtain

Theorem 2.2.8 *[35, Theorem A.16] Let $c \in \mathbb{C}$, and f be a meromorphic function such that $\rho_2(f) < 1$, and $f(z) - f(z+c) \not\equiv 0$. Let $q \geq 2$, and $a_1(z), \ldots, a_q(z)$ be distinct small periodic functions with period c. Then*

$$(q-1)T(r,f) \leq \widetilde{N}_c(r,f) + \sum_{k=1}^{q} \widetilde{N}_c\left(r, \frac{1}{f-a_k}\right) + \widetilde{S}(r,f).$$

Next, we give the difference analogues of the Clunie Lemma and Mohon'ko-Mohon'ko Theorem [35, Theorems A.17, A.19].

Theorem 2.2.9 (Difference Clunie Lemma) *Let f be a transcendental meromorphic solution of finite order of*

$$H(z,f)P(z,f) = Q(z,f),$$

where H, P, Q are difference polynomials (with small coefficients) such that the total degree n of $H(z,f)$ in f and its shifts is $\geq \deg Q(z,f)$. Assuming that $H(z,f)$ contains just one term of maximal total degree in f and its shift, then

$$m(r, P(z,f)) = \widetilde{S}(r,f).$$

Theorem 2.2.10 (Difference Mohon'ko-Mohon'ko Theorem) *Let $f(z)$ be a nonconstant meromorphic solution of $P(z, f) = 0$, , where $\sigma_2(f) < 1$ and $P(z, f)$ is difference polynomial in $f(z)$ and its shift. Assuming that $a(z)$ satisfies $T(r, a) = \widetilde{S}(r, f)$, and that $P(z, f)$ does not vanish identically, then*

$$m\left(r, \frac{1}{f-a}\right) = \widetilde{S}(r, f).$$

2.3 CARTAN'S VERSION OF NEVANLINNA THEORY

Several self-contained monographs about Cartan theory on holomorphic curves of the complex plane in projective spaces as well as its related fields exist, for example [14, 33, 37, 47, 50]. Here we only give a short review on this topic.

Since the second fundamental theorem in Nevanlinna theory is concerned with the value distribution of a holomorphic map in the complex projective line, Cartan extended the theory to higher dimensional cases. Like Nevanlinna, Cartan introduced the characteristic function measuring the growth of those holomorphic curves.

Let $g : \mathbb{C} \to \mathbb{P}^{p-1}$ be a holomorphic curve, with homogeneous coordinate $g = [g_1 : \cdots : g_p]$. Its order of growth is defined by

$$\sigma(g) = \limsup_{r\to\infty} \frac{\log^+ T_g(r)}{\log r},$$

where $\log^+ x = \max\{0, \log x\}$ for all $x \geq 0$, and

$$T_g(r) := \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} - u(0),\ u(z) = \sup_{k\in\{1,\ldots,p\}} \log|g_k(z)|$$

is the Cartan characteristic function of g. The hyper-order is defined by

$$\sigma_2(g) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T_g(r)}{\log r}.$$

Before introducing the Cartan Theorem, we first recall some properties of the Cartan characteristic function; refer to [20, 26, 37, 50], for details.

If $g = [g_1 : \cdots : g_p]$ with $p \geq 2$ is a reduced representation of a non-constant holomorphic curve g, then $T_g(r) \to \infty$ as $r \to \infty$, and if at least one quotient g_j/g_m is a transcendental function, then $T_g(r)/\log r \to \infty$ as $r \to \infty$. Moreover, if $f_1, \ldots, f_q$ are q linear combinations of the functions $g_1, \ldots, g_p$ over $\mathbb{C}$, where $q > p$, such that any p of the q functions $f_1, \ldots, f_q$ are linearly independent, then

$$T\left(r, \frac{f_\mu}{f_\nu}\right) \leq T_g(r) + O(1), \tag{2.1}$$

where $r \to \infty$, and μ and ν are distinct integers in the set $\{1, \ldots, q\}$.

The following theorem gives the relationship between the Cartan characteristic function and the Nevanlinna characteristic function:

Theorem 2.3.1 *[4] Let h_1 and h_2 be two linearly independent entire functions that have no common zeros, and set $f = h_1/h_2$. For positive r, set*

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta}) d\theta - v(0),$$

where

$$v(z) = \sup\{\log |h_1(z)|, \log |h_2(z)|\}.$$

Then

$$T(r) = T(r, f) + O(1) \textit{ as } r \to \infty.$$

The order of a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ is independent of the reduced representation of f. For detail proofs, refer to [26,35].

Gundersen-Hayman [20] generalized Cartan's theorem [3,4], and considered a system of p entire functions on $\mathbb{C}$ instead of a holomorphic curve of $\mathbb{C}$ to $\mathbb{P}^{p-1}(\mathbb{C})$. Their theorem is a strong result in the value distribution of holomorpic curves in the $p-1$-dimensional complex projective space $\mathbb{P}^{p-1}(\mathbb{C})$ [14,33,37,50], as well as an efficient tool for certain problems in the complex plane $\mathbb{C}$ [20,28], or for the study of hyperbolic complex spaces [33], etc.

Theorem 2.3.2 (Cartan-Gundersen-Hayman) *[4,28,37] Let g_1, g_2, $\ldots$, g_p be linearly independent entire functions, where $p \geq 2$. Suppose that* *for each complex number z, we have* $\max\{|g_1(z)|, |g_2(z)|, \ldots, |g_p(z)|\} > 0$. *For positive r, set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta - u(0), \quad \textit{where} \quad u(z) = \sup_{j\in\{1,\ldots,p\}} \log|g_j(z)|.$$

Let $f_1, f_2, \ldots, f_q$ *be q linear combinations of the p functions* $g_1, g_2, \ldots, g_p$, *where* $q > p$, *such that any p of the q functions* $f_1, f_2, \ldots, f_q$ *are linearly independent. Let H be the meromorphic functions defined by*

$$H = \frac{f_1 f_2 \cdots f_q}{W(g_1, g_2, \cdots, g_p)},$$

where $W(g_1, g_2, \cdots, g_p)$ *is the Wronskian of* $g_1, g_2, \cdots, g_p$. *Then*

$$(q-p)T(r) \leq N(r, 0, H) - N(r, H) + S(r),\ r > 0,$$

where $S(r)$ *is a quantity satisfying*

$$S(r) = O(\log T(r)) + O(\log r) \textit{ as } r \to \infty \textit{ n.e.}$$

We have

$$N(r, 0, H) \leq \sum_{j=1}^{q} N_{p-1}(r, 0, f_j),$$

and this gives

$$(q-p)T(r) \leq \sum_{j=1}^{q} N_{p-1}(r, 0, f_j) - N(r, H) + S(r),\ r > 0.$$

If at least one of the quotients g_j/g_m *is a transcendental function, then*

$$S(r) = o(T(r)) \textit{ as } r \to \infty \textit{ n.e.},$$

whereas if all the quotients g_j/g_m *are rational functions, then*

$$S(r) \leq -\frac{1}{2}p(p-1)\log r + O(1) \textit{ as } r \to \infty.$$

Furthermore, if all the quotients g_j/g_m *are rational functions, then there exist polynomials* $h_1, h_2, \ldots, h_p$, *and an entire function* ϕ, *such that*

$$g_j = h_j e^{\phi},\ j = 1, 2, \ldots, p.$$

Here $N(r,0,H)$ and $N(r,H)$ denote the ordinary Nevanlinna counting functions of zeros and poles of the meromorphic function H, and $N_{p-1}(r,0,f_j)$ is the truncated function in which a zero of f_j of multiplicity m is counted exactly $\min\{m, p-1\}$ times.

The abbreviation n.e. for nearly everywhere means "everywhere in $\mathbb{R}_{\geq 0}$, except possibly for a set of finite linear measure".

The second fundamental theorem is a particular example of Theorem 2.3.2; for detailed reasoning refer to [35, p. 205]. Borel's theorem can be deduced as a corollary of Cartan's theorem above; here we give its simpler statement.

Theorem 2.3.3 *[37, Corollary 6.2] Let $g_1, \ldots, g_n$ be entire functions without zeros (units in the ring of entire functions). Suppose that*

$$g_1 + \cdots + g_n = 1.$$

Then $g_1, \ldots, g_n$ are linearly dependent if $n \geq 2$.

Of course, this statement is a generalization of Picard's little theorem stating that all holomorphic mappings $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \backslash \{a, b, c\}$ are constants.

Fujimoto [12] and Green [16] interpret Borel's theorem in the context of holomorphic curves of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, as another natural generalization of Picard's theorem.

Before introducing Theorem 2.3.4, we give the following definitions for convenience of the reader.

Here a *hyperplane H* is the set of all points $x \in \mathbb{P}^n$, $x = [x_0 : \cdots : x_n]$, such that

$$\alpha_0 x_0 + \cdots + \alpha_n x_n = 0,$$

where $\alpha_j \in \mathbb{C}$ for $j = 0, \ldots, n$. The hyperplanes H_k, $k = 0, \ldots, m$ defined by $\alpha_{0,k} x_0 + \cdots + \alpha_{n,k} x_n = 0$ are said to be in *general position* if $m \geq n$ and any $n+1$ of the vectors $\alpha_k = (\alpha_{0,k}, \ldots, \alpha_{n,k}) \in \mathbb{C}^{n+1}$ are linearly independent.

Theorem 2.3.4 *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve. Assume that the image of f lies in the complement of the $n + p$ hyperplanes in* *general position, that is, f omits those hyperplanes where $p \in \{1, \ldots, n+1\}$. Then the image of f is contained in a linear subspace at most of dimension $[n/p]$.*

Further extensions of Picard's theorem for holomorphic curves lacking hyperplanes can be found in [13, 14, 17, 18, 33, 37, 50] etc.

2.4 DIFFERENCE CARTAN THEORY

In a similar way as Cartan's value distribution theory extends Nevanlinna theory, a natural generalization of the difference variant of Nevanlinna theory for holomorphic curves in the complex projective space exits. We will introduce this difference variant of Cartan's theory in this section.

Let $g(z)$ be a meromorphic function and $c \in \mathbb{C}$. For a fixed c, we will use the short notation

$$g(z+c) \equiv \overline{g},\ g(z+2c) \equiv \overline{\overline{g}},\ \ldots,\ g(z+nc) \equiv \overline{g}^{[n]}$$

to suppress the explicit z-dependence of $g(z)$. Then the Casorati determinant of $g_1, \ldots, g_n$ ($n \in \mathbb{N}$) is defined by

$$C(g_1, \ldots, g_n) = \begin{vmatrix} g_1 & g_2 & \cdots & g_n \\ \overline{g}_1 & \overline{g}_2 & \cdots & \overline{g}_n \\ \vdots & \vdots & \ddots & \vdots \\ \overline{g}_1^{[n-1]} & \overline{g}_2^{[n-1]} & \cdots & \overline{g}_n^{[n-1]} \end{vmatrix}. \tag{2.2}$$

As Cartan generalized the second main theorem by expressing the ramification term in terms of the Wronski determinant of a set of linearly independent entire functions, Halburd, Korhonen and Tohge [26] gave the difference analogue of Cartan's result by replacing the ramification term in terms of the Casorati determinant of entire functions, which are linearly independent over the field $\mathcal{P}_c^1$ of c-periodic meromorphic functions of the hyper-order less than 1.

Theorem 2.4.1 *[26, Theorem 2.1] Let $n \geq 1$, and $g_0, \ldots, g_n$ be entire functions, linearly independent over $\mathcal{P}_c^1$, such that*

$$\max\{|g_0(z)|, \ldots, |g_n(z)|\} > 0$$

for each $z \in \mathbb{C}$, and

$$\sigma_2(g) < 1, \quad g := [g_0 : \cdots : g_n].$$

Let $\varepsilon > 0$. If $f_0, \ldots, f_q$ are $q+1$ linear combinations of the $n+1$ functions $g_0, \ldots, g_n$ over $\mathbb{C}$, where $q > n$, such that any $n+1$ of the $q+1$ functions $f_0, \ldots, f_q$ are linearly independent over $\mathcal{P}^1_c$, and

$$L := \frac{f_0 f_1 \cdots f_q}{C(g_0, g_1, \ldots, g_n)},$$

then

$$(q-n)T_g(r) \leq N\left(r, \frac{1}{L}\right) - N(r, L) + o\left(\frac{T_g(r)}{r^{1-\rho_2-\varepsilon}}\right) + O(1),$$

where r approaches infinity outside of an exceptional set E of finite logarithmic measure (i.e. $\int_{E\cap[1,\infty)} dt/t < \infty$).

A special case of Theorem 2.4.1 has been also given, independently, by Wong-Law-Wong [57] for a class of finite-order holomorphic curves with finite-order coordinate functions. Their result is also an extension of the results by Halburd-Korhonen in [22] to holomorphic curves into $\mathbb{P}^n(\mathbb{C})$.

Before giving the difference analogue of Green and Fujimoto's Theorem 2.3.4, we first introduce the definition of *forward invariant*.

A preimage of a hyperplane $H \subset \mathbb{P}^n$ under f is said to be *forward invariant* with respect to the translation $\tau(z) = z + c$ if

$$\tau(f^{-1}(\{H\})) \subset f^{-1}(\{H\}), \tag{2.3}$$

where $f^{-1}(\{H\}))$ and $\tau(f^{-1}(\{H\}))$ are multisets in which each point is repeated according to its multiplicity.

For example, let $\varphi(z)$ be an entire function given by the pullback divisor of the hyperplane H. Suppose that

$$\varphi(z) = \frac{\varphi^{(i)}(z_0)}{i!}(z-z_0)^i + O((z-z_0)^{i+1}),\ \varphi^{(i)}(z_0) \neq 0,$$

and

$$\varphi(z+c) = \frac{\varphi^{(j)}(z_0)}{j!}(z-z_0)^j + O((z-z_0)^{j+1}),\ \varphi^{(j)}(z_0) \neq 0,$$

for all z in a neighborhood of z_0. If $j \geq i > 0$, the point z_0 is a forward invariant element in a preimage of H with respect to $\tau(z)$, while if $i > j$, then z_0 is not a forward invariant element.

Finitely many exceptional values are allowed in inclusion (2.3) if the holomorphic curve f is transcendental.

Theorem 2.4.2 *[26, Theorem 1.1] Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve such that $\rho_2(f) < 1$, $c \in \mathbb{C}$ and $p \in \{1, \ldots, n+1\}$. If the $n+p$ hyperplanes in general position have forward invariant preimages under f with respect to the translation $\tau(z) = z + c$, then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq [n/p]$.*

As an immediate consequence of Theorem 2.4.2, we also have

Corollary 2.4.1 *[26, Corollary 1.3] Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve such that $\rho_2(f) < 1$, and $c \in \mathbb{C}$. If the $2n+1$ hyperplanes $H_j (j = 0, 1, \ldots, 2n)$ in $\mathbb{P}^n(\mathbb{C})$ are located in general position and satisfy the condition*

$$\tau(f^{-1}(\{H_j\})) \subset f^{-1}(\{H_j\})$$

about their preimages under f with respect to the translation $\tau_c(z) = z + c$, then f is periodic with period c.

This result indicates us that if the preimages $f^{-1}H_j$ under f are all empty, then f reduces to a constant. Therefore, Corollary 2.4.1 is also regarded as a generalization of Green's Picard-type theorem in [16] for holomorphic curves of the hyper-order less than one.

As Nevanlinna's second main theorem follows from Cartan's result, the analogue of Nevanlinna's second main theorem for the difference operator $\triangle_c$ for constant targets follows from Theorem 2.4.1; for proof refer to [35, p. 216-217].

3 Good Linear Operators

Nevanlinna theory provides us with many tools applicable to the study of value distribution of meromorphic solutions to differential equations. Analogues of some of these tools have been recently developed for difference and q-difference equations. In many cases, the methodologies used in the study of meromorphic solutions to differential, difference and q-difference equations are largely similar. Thus, to collect some of these tools in a common toolbox is urgently needed for the study of general classes of functional equations. In this chapter, we deal with this problem by introducing notation and corresponding properties of a good linear operator, which satisfies certain regularity conditions in terms of value distribution theory.

3.1 INTRODUCTION

The lemma on the logarithmic derivative is an important technical tool in the study of value distribution of meromorphic solutions of differential equations. It plays an important role in the proof of the Clunie lemma [11] and in a theorem due to A. Z Mohon'ko and V. Z Mohon'ko [43], both of which are applicable to large classes of differential equations. Similarly, the difference analogues of the lemma on the logarithmic derivatives due to Halburd and Korhonen [22, 23], and Chiang and Feng [9, 10] are of great importance in studying large classes of difference equations, often by using methods similar to the case of differential equations. A q-difference analogue of the lemma on the logarithmic derivatives [2], as well as an analogous result on the proximity function of polynomial compositions of meromorphic functions [34] are applicable to corresponding classes of q-difference equations and functional equations much in the same way. Therefore, to present all these results under one general framework is naturally needed. For value distribution of meromorphic functions, this was done by Halburd and Korhonen in [25], where a second main theorem was given for general linear operators, operating on a subfield of meromorphic functions for which a suitable analogue of the logarithmic derivative lemma exists. The purpose of this chapter is to develop this method further so that it is applicable to a general class of functional equations. This will be done in section 3.2 by introducing the notion and corresponding properties of a good linear operator, which encompasses such operators as $L(f) = f'$, $L_q(f) = f(qz)$ and $E_c(f) = f(z+c)$.

3.2 GOOD LINEAR OPERATORS

The lemma on the logarithmic derivative and its difference analogues all produce different types of exceptional sets. Therefore, in order to include this phenomenon in our set–up, we first need to give the following notion. We say $\mathbb{P}$ is an *exceptional set property* if for any two sets $E_1 \subset (0,\infty)$ and $E_2 \subset (0,\infty)$ having the property $\mathbb{P}$ it follows that $E_1 \cup E_2$ also has property $\mathbb{P}$. For instance, "finite linear measure", "finite logarithmic measure" and "zero logarithmic density" are all exceptional set properties from the corresponding definitions.

With the above notion, we redefine $S(r,f)$, a quantity satisfying that $S(r,f)/T(r,f)$ approaches zero as $r \to \infty$ outside of an exceptional set with the exceptional set property $\mathbb{P}$ throughout the rest of this survey for unity and simplicity. A meromorphic function a is small with respect to f if $T(r,a) = S(r,f)$.

Denote by $\mathcal{M}$ the field of meromorphic functions in the complex plane, and let $\mathcal{N} \subset \mathcal{M}$. We say that a linear operator $L : \mathcal{N} \to \mathcal{N}$ is a *good linear operator for $\mathcal{N}$ with an exceptional set property* $\mathbb{P}$ if the following two properties hold:

(1) For any $f \in \mathcal{N}$,

$$m\left(r, \frac{L(f)}{f}\right) = S(r,f).$$

(2) The counting functions $N(r, f)$ and $N(r, L(f))$ are asymptotically equivalent, i.e., there is a constant $K \geq 1$ such that

$$\frac{1}{K}N(r, f) + S(r, f) \leq N(r, L(f)) \leq KN(r, f) + S(r, f). \quad (3.1)$$

Next, we will give two examples of a good linear operator. Let $\mathcal{N} = \mathcal{M}$ and $L(f) = f'$, then property (1) is satisfied by the lemma on the logarithmic derivatives, i.e. Theorem 2.1.2 with $\mathbb{P}$ being "finite linear measure". Property (2) holds with $K = 2$ from the definition of counting functions, even without an error term and an exceptional set. Another example is given by taking $\mathcal{N}$ to be the set of all meromorphic functions of the hyper-order strictly less than one, and $L(f(z)) = f(z + 1)$. Then property (1) is satisfied by the difference analogue of the lemma on the logarithmic derivative, i.e. Theorem 2.2.3 with $\mathbb{P}$ being "finite logarithmic measure". In this case property (2) holds with $K = 1$ by using Theorem 2.2.6.

The following result shows that a composition of two good operators is also a good operator.

Theorem 3.2.1 *[38] If L_1 and L_2 are good linear operators for $\mathcal{N}$ with an exceptional set property $\mathbb{P}$, then $L_1 \circ L_2$ is a good linear operator for $\mathcal{N}$ with the same exceptional set property $\mathbb{P}$.*

Proof. Since the linearity follows immediately by the linearity of L_1 and L_2, we only need to check that properties (1) and (2) hold for $L_1 \circ L_2$.

First, for any $f \in \mathcal{N}$, we have

$$m\left(r, \frac{L_1(L_2(f))}{f}\right) \leq m\left(r, \frac{L_1(L_2(f))}{L_2(f)}\right) + m\left(r, \frac{L_2(f)}{f}\right).$$

Therefore, since $f \in \mathcal{N}$ and $L_2(f) \in \mathcal{N}$, and by the assumption that L_1 and L_2 are good operators, we have

$$m\left(r, \frac{L_1(L_2(f))}{f}\right) = S(r, L_2(f))) + S(r, f). \quad (3.2)$$

But since

$$
\begin{aligned}
T(r, L_2(f)) &= m(r, L_2(f)) + N(r, L_2(f)) \\
&\leq m(r, f) + m\left(r, \frac{L_2(f)}{f}\right) + KN(r, f) + S(r, f) \\
&\leq KT(r, f) + S(r, f),
\end{aligned}
$$

equation (3.2) becomes

$$
m\left(r, \frac{L_1(L_2(f))}{f}\right) = S(r, f).
$$

Thus property (1) holds for the operator $L_1 \circ L_2$.

To show that property (2) also holds, we observe that since $L_2(f) \in \mathcal{N}$ and $f \in \mathcal{N}$,

$$
\begin{aligned}
\frac{1}{K_1} N(r, L_2(f)) + S(r, L_2(f)) &\leq N(r, L_1(L_2(f))) \\
&\leq K_1 N(r, L_2(f)) + S(r, L_2(f))
\end{aligned}
$$

and

$$
\frac{1}{K_2} N(r, f) + S(r, f) \leq N(r, L_2(f)) \leq K_2 N(r, f) + S(r, f),
$$

it follows by (3.2) that

$$
\frac{1}{K_1 K_2} N(r, f) + S(r, f) \leq N(r, L_1(L_2(f))) \leq K_1 K_2 N(r, f) + S(r, f).
$$

Thus property (2) is valid for $L_1 \circ L_2$, and hence it is a good linear operator for $\mathcal{N}$ with an exceptional set property $\mathbb{P}$. □

Note, however, that the sum of two good linear operators is not necessarily a good operator, since the lower bound in (3.1) may fail to be valid.

The next theorem shows that a composition of single–term differential and difference operators of an arbitrary order is a good linear operator for sufficiently slowly growing meromorphic functions.

Theorem 3.2.2 *[38] Let $c \in \mathbb{C}$ and $k \in \mathbb{N} \cup \{0\}$, and $\mathcal{N}_1$ be the field of meromorphic functions of the hyper-order strictly less than one. The operator*

$$L(f(z)) = f^{(k)}(z+c)$$

is a good linear operator in $\mathcal{N}_1$ with $\mathbb{P} =$ "finite logarithmic measure".

Proof. By Theorem 3.2.1 it is sufficient to show that the operators $L_1(f) = f^{(k)}$ and $L_2(f(z)) = f(z+c)$ are good linear operators in $\mathcal{N}_1$ with the exceptional set property $\mathbb{P}$. The operator L_1 is in fact good in all of $\mathcal{M}$ with a weaker exceptional set property. Namely, property (1) is satisfied by the lemma on the logarithmic derivative, and property (2) holds since

$$\frac{1}{k+1}N(r,f) \leq N(r,L_1(f)) \leq (k+1)N(r,f) \tag{3.3}$$

for all meromorphic functions $f \in \mathcal{M}$ and for all $r \geq 1$. By combining (3.3) with the lemma on the logarithmic derivative, or by using the Milloux inequality [62], it follows that

$$T(r,L_1(f)) \leq 2T(r,f) + S(r,f).$$

Therefore, if $f \in \mathcal{N}_1$, $L_1(f) \in \mathcal{N}_1$ and thus $L_1 : \mathcal{N}_1 \to \mathcal{N}_1$ is also a good linear operator in $\mathcal{N}_1$ with the exceptional set property $\mathbb{P}$.

If $f \in \mathcal{N}_1$, it follows by Theorem 2.2.3 that

$$m\left(r, \frac{L_2(f)}{f}\right) = m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f). \tag{3.4}$$

Therefore property (1) is satisfied for L_2 in $\mathcal{N}_1$ with the exceptional set property "finite logarithmic measure". Moreover,

$$N(r-|c|, f(z)) \leq N(r, L_2(f)) \leq N(r+|c|, f(z))$$

for all $r \geq |c|$, and so by Theorem 2.2.6, we have

$$N(r,f(z)) + S(r,f) \leq N(r,L_2(f)) \leq N(r,f(z)) + S(r,f). \tag{3.5}$$

Hence property (2) holds for L_2 in $\mathcal{N}_1$ with the exceptional set property "finite logarithmic measure". Finally, by combining (3.4) and (3.5), it follows that

$$T(r, L_2(f)) \leq T(r, f) + o(T(r, f))$$

as $r \to \infty$ outside of a set of finite logarithmic measure. Hence $L_2(\mathcal{N}_1) \subset \mathcal{N}_1$, and thus $L_2 : \mathcal{N}_1 \to \mathcal{N}_1$ is a good linear operator in $\mathcal{N}_1$ with the exceptional set property $\mathbb{P}$. This completes the proof of Theorem 3.2.2. □

Here, for the exceptional set, we can deduce that if the set is of finite linear measure, then it is of finite logarithmic measure. Thus, this composition makes sense.

4 Solutions of functional equations

In this chapter, we recall some results about the uniqueness, growth, and existence of meromorphic solutions in three different classes of functional equations.

4.1 SOLUTIONS TO FUNCTIONAL EQUATIONS OF THE TYPE $L(F) + P(Z, F) = H(Z)$

In 2001, C.C.Yang [60] studied transcendental entire solutions of finite order of

$$L(f) - p(z)f^n = h(z),$$

where $L(f)$ denotes a linear differential polynomial in f with polynomial coefficients, $p(z)$ is a non-vanishing polynomial, $h(z)$ is entire and $n \geq 3$. In particular, he showed that f must be unique, unless $L(f) \equiv 0$.

Later on, Heittokangas, Korhonen, and Laine [29] generalized $p(z), h(z)$ and the coefficients of $L(f)$ from entire functions to a meromorphic case, and obtained the following result.

Theorem 4.1.1 *[29] Consider a differential equation*

$$p(z)f^n - L(z, f) = h, \tag{4.1}$$

where $p(z)$ is a small function of f of degree n, $L(z, f)$ a linear differential polynomial in f, and h a meromorphic function. If $n \geq 4$, then equation (4.1) *may admit at most n distinct entire solutions.*

Specific to $L(f) - p(z)f^3 = h(z)$, Heittokangas *et. al* [29] also considered the existence and uniqueness of meromorphic solutions with only few poles and obtained the following result.

Theorem 4.1.2 *[29] Let f be a transcendental meromorphic function. If f satisfies the nonlinear differential equation*

$$a_1(z)f' + a_0(z)f - p(z)f^3 = h(z), \tag{4.2}$$

then one of the following situations holds:

(a) Equation (4.2) has f as its unique transcendental meromorphic solution such that $N(r, f) = S(r, f)$.

(b) Equation (4.2) has exactly three transcendental meromorphic solutions f_j, $j = 1, 2, 3$ such that $N(r, f_j) = S(r, f_j)$ for $j = 1, 2, 3$. Moreover $a_1(z)f_j' + a_0(z)f_j \equiv 0$, and $h(z) = -p(z)f_j^3$ for all $j = 1, 2, 3$.

In particular, letting $h(z) = c \sin bz$, they obtained

Theorem 4.1.3 *[29] Let p be a non-vanishing polynomial, and b, c be nonzero complex numbers. If p is nonconstant, then the differential equation*

$$f^3 + p(z)f'' = c \sin bz \tag{4.3}$$

admits no transcendental entire solutions, while if p is constant, then equation (4.3) *admits three distinct transcendental entire solutions, provided $(pb^2/27)^3 = \frac{1}{4}c^2$.*

For the growth of the solutions to a slightly more general form,

$$L(f) + P(z, f) = h(z), \tag{4.4}$$

where $L(f) = a_0(z)f + a_1(z)f' + \cdots + a_k(z)f^{(k)}$ is a linear differential polynomial in f with meromorphic coefficients, and $P(z, f) = b_2 f^2 + \cdots + b_n(z)f^n$ is a polynomial in f with meromorphic coefficients, and $h(z)$ a polynomial in f with meromorphic coefficients, they obtained the following result.

Theorem 4.1.4 *[29] Given $L(f)$, $P(z, f)$, $h(z)$ as above, and $P(z, f) \not\equiv 0$, denote by $\mathcal{F}$ the family of meromorphic solutions of (4.4) such that* *whenever $f \in \mathcal{F}$, all coefficients of (4.4) are small meromorphic functions of f, and $N(r, f) = S(r, f)$. If $f, g \in \mathcal{F}$, then*

$$T(r, g) = O(T(r, f)) + S(r, f).$$

Moreover, if $\alpha > 1$, then for some $r_\alpha > 0$,

$$T(r, g) = O(T(\alpha r, f))$$

for all $r \geq r_\alpha$.

Recently, a difference variant of Nevanlinna theory was established by Halburd and Korhonen [22, 23], Chiang and Feng [9, 10] independently. Based on this background, differential-difference analogues of Theorem 4.1.1 to Theorem 4.1.4 were obtained by Yang-Laine in [61]. Before giving the corresponding results, we first introduce the following definition.

A difference polynomial, resp. *a differential-difference polynomial, in* f is a finite sum of difference products of f and its shifts, resp. of products of f, derivatives of f and of their shifts, with all the coefficients of these monomials being small functions of f.

Theorem 4.1.5 *[61] A nonlinear difference equation*

$$f^3(z) + q(z) f(z+1) = c \sin bz, \tag{4.5}$$

where $q(z)$ is a nonconstant polynomial and $b, c \in \mathbb{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a nonzero constant, then equation (4.5) *possesses three distinct entire solutions of finite order, provided $b = 3\pi n$ and $q^3 = (-1)^{n+1} \frac{27}{4} c^2$ for a nonzero integer n.*

An example in a special case is given as follows [61]: the equation

$$f^3(z) + \frac{3}{4} f(z+1) = -\frac{1}{4} \sin 3\pi z,$$

has three entire solutions of finite order, i.e.

$$f_1(z) = \sin \pi z = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z}),$$

$$f_2(z) = \frac{1}{2i}(\varepsilon e^{i\pi z} - \varepsilon^2 e^{-i\pi z}) = -\frac{1}{2}\sin \pi z + \frac{\sqrt{3}}{2}\cos \pi z,$$

$$f_3(z) = \frac{1}{2i}(\varepsilon^2 e^{i\pi z} - \varepsilon e^{-i\pi z}) = -\frac{1}{2}\sin \pi z - \frac{\sqrt{3}}{2}\cos \pi z,$$

where $\varepsilon := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a cubic root of unity.

Theorem 4.1.6 *[61] Let $M(z, f)$ be a linear differential–difference polynomial of f, not vanishing identically, h a meromorphic function of finite order, and $n \geq 4$ an integer. Then the differential-difference equation*

$$f^n + M(z, f) = h \tag{4.6}$$

possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $M(z, f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.

4.2 SOLUTIONS TO FERMAT-TYPE EQUATIONS

Let $\mathbf{C}$ be a ring, and $n\ (\geq 2)$ be an integer. Let $F_{\mathbf{C}}(n)$ denote the smallest positive integer k such that we have a nontrivial representation

$$x_1^n + x_2^n + \ldots + x_k^n = 1,$$

where $x_j \in \mathbf{C}$ for $j = 1, 2, \ldots, k$.

For the case $\mathbf{C} = \mathbb{Z}$, on the one hand, according to the famous Fermat's last theorem, which was proved by Wiles [55] and Taylor and Wiles [51], no nonzero rational numbers x, y, nor an integer n exist, where $n \geq 3$, such that

$$x^n + y^n = 1.$$

On the other hand, Ramanujan noted that $9^3 + (10)^3 + (-12)^3 = 1$. Thus combining the above two facts, we have $F_{\mathbb{Z}}(3) = 3$.

Next, we recall the functional case. Let M, R, E and P denote the rings of meromorphic functions, rational functions, entire functions and polynomials, respectively. Therefore, if $\mathbf{C}$ is equal to M, R, E

or P, and n is an integer satisfying $n \geq 2$, then $F_C(n)$ denotes the smallest positive integer k such that the equation

$$f_1^n + f_2^n + \ldots + f_k^n = 1 \tag{4.7}$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C. Obviously, k depends on n.

Many scholars have investigated this and related problems; for details see [13, 18, 28, 40, 41, 46, 52, 63, 65]. Gundersen and Hayman [20] collected the best lower estimates known for every n as follows:

a $F_P(n) > 1/2 + \sqrt{n + 1/4}$,

b $F_R(n) > \sqrt{n+1}$,

c $F_E(n) \geq 1/2 + \sqrt{n + 1/4}$,

d $F_M(n) \geq \sqrt{n+1}$.

There is another way to express the above inequalities, which can be stated as follows. For C equal to P, R, E, M, there are no k nonconstant functions $f_1, \ldots, f_k$ in C satisfying (4.7) when $n \geq k^2 - k$, $n \geq k^2 - 1$, $n \geq k^2 - k + 1$, or $n \geq k^2$, respectively. In addition, Cartan's theorem was used in [28] to prove all the above four inequalities.

Next, we will recall the upper bound of $F_C(n)$. Let b and n be integers satisfying $1 \leq b \leq n$, for $1 \leq \nu \leq b$, we set $\omega_\nu = \exp\{2\pi i \nu / b\}$. From the identity (see [44])

$$\sum_{\nu=1}^{b} (1 + \omega_\nu z^n)^n = A_0 + A_1 z^{nb} + A_2 z^{2nb} + \ldots + A_{[n/b]} z^{nb[n/b]}, \tag{4.8}$$

where each A_ν is a positive integer, and $[x]$ denotes the greatest integer that is less than or equal to x for a real number x, there exist $k = b + [n/b]$ nonconstant polynomials $f_1, \ldots, f_k$ satisfying equation (4.7). Since the minimum over all b of $k = b + [n/b]$ is $[\sqrt{4n+1}]$ (see [46]), it follows that $F_P(n) \leq \sqrt{4n+1}$, $n \geq 2$.

Next we consider the cases where **C** is equal to E, R or M. From the identity (see [46])

$$\sum_{\nu=1}^{b} \frac{\omega_\nu (1+\omega_\nu z^n)^n}{z^{(b-1)n}} = B_0 + B_1 z^{bn} + B_2 z^{2bn} + \dots + B_{[(n+1)/b]-1} z^{([(n+1)/b]-1)bn},$$

where each B_ν is a positive integer, we find that there exist $k = b + [(n+1)/b] - 1$ nonconstant functions $f_1, \dots, f_k$ in C satisfying (4.7). Since the minimum over all b of $k = b + [(n+1)/b] - 1$ is $[\sqrt{4n+5}] - 1$ (see [28]), we find that $F_C(n) \leq \sqrt{4n+5} - 1$, $n \geq 2$.

Examples concerning the above results are given as follows.

When $n = 2$, on one hand (see [28]), from

$$\left(\frac{1+z}{\sqrt{2}}\right)^2 + \left(\frac{1-z}{\sqrt{2}}\right)^2 + (iz)^2 = 1,$$

we have $F_P(2) \leq 3$. On the other hand (see [32]), if f and g are nonconstant entire solutions of $f^2 + g^2 = 1$, then we obtain entire solutions $f = \cos\omega, g = \sin\omega$; and rational solutions $f = 2z(z^2+1)^{-1}, g = (z^2-1)(z^2+1)^{-1}$. Therefore, we have $F_P(2) = 3$, $F_E(2) = F_M(2) = 2$, and $F_R(2) = 2$.

When $n = 3$, let $b = 2$ in (4.8), and we obtain

$$\frac{1}{2}(1+z^3)^3 + \frac{1}{2}(1-z^3)^3 - 3(z^2)^3 = 1. \tag{4.9}$$

On the other hand, f and g are nonconstant meromorphic solutions of $f^3 + g^3 = 1$ if and only if f and g are certain nonconstant elliptic functions composed of an entire function; see [1]. Combining this result with (4.9) gives $F_M(3) = 2$ and $F_P(3) = F_R(3) = F_E(3) = 3$.

When $n = 4$, let $b = 3$ in (4.9), and we get

$$\frac{1}{18}\left(\frac{1+z^4}{z^2}\right)^4 + \frac{e^{2\pi i/3}}{18}\left(\frac{1+e^{2\pi i/3}z^4}{z^2}\right)^4 + \frac{e^{4\pi i/3}}{18}\left(\frac{1+e^{4\pi i/3}z^4}{z^2}\right)^4 = 1. \tag{4.10}$$

By combining the results [a,b,c,d] collected by Gundersen and Hayman with (4.10), we find that $F_M(4) = F_R(4) = F_E(4) = 3$. For more examples and references concerning equation (4.7), see [19].

4.3 SOLUTIONS TO SECOND-ORDER COMPLEX DIFFERENTIAL EQUATIONS

Consider the second–order homogeneous linear periodic differential equation

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{4.11}$$

where $P(z)$ and $Q(z)$ are polynomials in z and not constants. It is well known that every solution f of (4.11) is entire.

For a meromorphic funtion f, define

$$\sigma_e(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{r},$$

as the e-type order of f. If $f \not\equiv 0$ is a solution of (4.11) satisfying $\sigma_e(f) = 0$, then we say that f is a nontrivial subnormal solution of (4.11).

Wittich [56] investigated subnormal solutions of (4.11), and obtained the form of all subnormal solutions in the following theorem.

Theorem 4.3.1 *[56] If $f \not\equiv 0$ is a subnormal solution of (4.11), then f must have the form*

$$f(z) = e^{cz}(h_0 + h_1 e^z + \cdots + h_m e^{mz}),$$

where $m \geq 0$ is an integer and $c, h_0, \ldots, h_m$ are constants with $h_0 \neq 0$ and $h_m \neq 0$.

Gundersen and Steinbart [21] refined Theorem 4.3.1 and obtained the following theorem.

Theorem 4.3.2 *[21] Under the assumptions of Theorem 4.3.1, the following statements hold.*

(i) If $\deg P > \deg Q$ and $Q \not\equiv 0$, then, any subnormal solution $f \not\equiv 0$ of (4.11) must have the form

$$f(z) = \sum_{k=0}^{m} h_k e^{-kz},$$

where $m \geq 1$ is an integer and $h_0, h_1, \ldots, h_m$ are constants with $h_0 \neq 0$ and $h_m \neq 0$.

(ii) *If* $\deg P \geq 1$ *and* $Q = 0$*, then any subnormal solution of equation* (4.11) *must be a constant.*

(iii) *If* $\deg P < \deg Q$*, then the subnormal solution of equation* (4.11) *is* $f = 0$*.*

Chen and Shon [6] investigate the problem concerning which condition will guarantee that (4.11) does not have a non-trivial subnormal solution, and obtained the following result.

Theorem 4.3.3 *[6] Let*

$$P(e^z) = c_n e^{nz} + \cdots + c_1 e^z + c_0,$$

$$Q(e^z) = d_s e^{sz} + \cdots + d_1 e^z + d_0,$$

where $c_n d_s \neq 0$*,* c_j*,* d_k $(j = 0, 1, \ldots, n; k = 0, 1, \ldots, s)$ *are constants. Suppose that* P *and* Q *satisfy any one of the following three additional hypotheses:*

(i) $s > n$*;*

(ii) $n > s$ *and* $c_0 = d_0 = 0$*;*

(iii) $n > s$ *and equation* $x^2 - c_0 x + d_0 = 0$ *has no positive integer solution.*

Then (4.11) *has no non-trivial subnormal solution, and every non-trivial solution* f *satisfies* $\sigma_2(f) = 1$*.*

For $\deg P = \deg Q$, Xiao [58] studied the case, and proved the following result.

Theorem 4.3.4 *[58] Let*

$$P(e^z) = a_n e^{nz} + \cdots + a_1 e^z + a_0,$$

$$Q(e^z) = b_n e^{nz} + \cdots + b_1 e^z + b_0,$$

where a_i*,* b_i $(i = 0, \ldots, n)$ *are constants,* $a_n b_n \neq 0$*,* $\deg(Q - \frac{b_n}{a_n} P) \geq 1$*. Suppose that any one of the following two hypotheses holds:*

(i) $a_0 a_n = 2b_n$ *and* $a_0^2 = 4b_0$;

(ii) $x^2 - c_0 x + d_0 = 0$ *has no positive integer solution, where* $c_0 = a_0 - 2\frac{b_n}{a_n}$, $d_0 = b_0 - \frac{a_0 b_n}{a_n} + \frac{b_n^2}{a_n^2}$.

Then (4.11) *has no non-trivial subnormal solution, and every non-trivial solution* f *satisfies* $\sigma_2(f) = 1$.

Xiao also considered another problem, namely, which condition will guarantee that the general equation

$$f'' + P(e^{\alpha z})f' + Q(e^{\beta z})f = 0, \ (\alpha \neq \beta), \tag{4.12}$$

where $P(z), Q(z)$ are polynomials in z, α, β are complex constants, does not have a non-trivial subnormal solution, and obtained the following results.

Theorem 4.3.5 *[58] Let*

$$P(e^{\alpha z}) = a_n e^{n\alpha z} + \cdots + a_1 e^{\alpha z} + a_0,$$

$$Q(e^{\beta z}) = b_m e^{m\beta z} + \cdots + b_1 e^{\beta z} + b_0,$$

where $n\,(\geq 1)$, $m\,(\geq 1)$ *are integers,* a_i, b_j $(i = 0, \ldots, n; j = 0, \ldots, m)$, α, β *are constants,* $a_n b_m \neq 0$, $\alpha\beta \neq 0$. *Suppose any one of the following two hypotheses holds:*

(i) $\arg \alpha \neq \arg \beta$;

(ii) $n\alpha = cm\beta\,(0 < c < 1)$.

Then equation (4.12) *has no non-trivial subnormal solution and every non-trivial solution* f *satisfies* $\sigma_2(f) = 1$.

Theorem 4.3.6 *[58] Let*

$$P^*(e^{\alpha z}) = a_n e^{n\alpha z} + \cdots + a_1 e^{\alpha z},$$

$$Q^*(e^{\beta z}) = b_m e^{m\beta z} + \cdots + b_1 e^{\beta z},$$

where $n\,(\geq 1)$, $m\,(\geq 1)$ *are integers,* a_i, b_j $(i = 1, \ldots, n; j = 1, \ldots, m)$, α, β *are constants,* $a_n b_m \neq 0$, $\alpha\beta \neq 0$. *Suppose that* $n\alpha = cm\beta\,(c > 1)$. *Then equation* (4.12) *has no non-trivial subnormal solution and every non-trivial solution* f *satisfies* $\sigma_2(f) = 1$.

In 2007, Chen and Shon [5] studied the existence of subnormal solutions of a more general form

$$f'' + (P_1(e^z) + P_2(e^{-z}))f' + (Q_1(e^z) + Q_2(e^{-z}))f = 0, \tag{4.13}$$

and obtained the following results.

Theorem 4.3.7 *[5] Let $P_j(z)$, $Q_j(z)$ $(j = 1, 2)$ be the polynomials in z. If*

$$\deg Q_1 > \deg P_1 \ \textit{or} \ \deg Q_2 > \deg P_2$$

then (4.13) *has no nontrivial subnormal solution, and every solution of* (4.13) *satisfies $\sigma_2(f) = 1$.*

Theorem 4.3.8 *[5] Let $P_j(z)$, $Q_j(z)$ $(j = 1, 2)$ be the polynomials in z. If*

$$\deg Q_1 < \deg P_1 \ \textit{and} \ \deg Q_2 < \deg P_2$$

and $Q_1 + Q_2 \not\equiv 0$, then (4.13) *has no nontrivial subnormal solution, and every solution of* (4.13) *satisfies $\sigma_2(f) = 1$.*

5 Summary of Papers I-V

In the following summaries, some notations used in the original papers have been introduced in previous chapters.

5.1 SUMMARY OF PAPER I

Viewed as another useful tool (just like the case of Cartan's second main theorem) in the study of solutions to functional equations of certain types, the difference analogue of Cartan's second main theorem is extended in paper I to the case of slowly moving periodic hyperplanes . In addition, two different natural ways are also introduced to find a difference analogue of the truncated second main theorem. As applications, we obtain a new Picard type theorem and difference analogues of the deficiency relation for holomorphic curves.

5.1.1 Introduction

In order to state the Cartan second main theorem for differences, we define the n-dimensional complex projective space $\mathbb{P}^n$ as the quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where

$$(a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots, b_n)$$

if and only if

$$(a_0, a_1, \ldots, a_n) = \lambda(b_0, b_1, \ldots, b_n)$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Comparing the differential operator $Df = f'$ and difference operator $\Delta f = f(z+1) - f(z)$, a natural difference analogue of constant targets for f' is the periodic targets case for Δf. For instance, linear differential equations with constant coefficients can be exactly solved modulo arbitrary constants, while for linear difference equations the same statement is true but with arbitrary periodic functions. Moreover, take the solution space of $L(f) = 0$ as an example, where L is a linear operator mapping a subclass $\mathcal{N}$ of the meromorphic functions in $\mathbb{C}$ into itself. Allowing $L(f) = Df$ gives constants as targets, while the choice $L(f) = \Delta f$ yields periodic functions. Furthermore, as in Theorem 2.4.1, the condition "entire functions $g_1, g_2, \ldots, g_p$ linearly independent over $\mathbb{C}$" is changed naturally to "linearly independent over $\mathcal{P}_c^1$". In the light of this, a natural difference analogue of Cartan's second main theorem would therefore be suitable for slowly moving periodic target hyperplanes, rather than constants, as is the case in Theorem 2.4.1. In paper I, we remedy this situation by introducing the following theorem.

Theorem 5.1.1 *Let $n \geq 1$, and $g = [g_0 : \ldots : g_n]$ be a holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with $\sigma_2(g) = \sigma_2 < 1$, where $g_0, \ldots, g_n$ are linearly independent over $\mathcal{P}_c^1$. If*

$$f_j = \sum_{i=0}^{n} a_{ij} g_i \quad j = 0, \ldots, q, \, q > n,$$

where a_{ij} are c-periodic entire functions satisfying $T(r, a_{ij}) = o(T_g(r))$, such that any $n+1$ of the $q+1$ functions $f_0, \ldots, f_q$ are linearly independent over $\mathcal{P}_c^1$, and

$$L = \frac{f_0 f_1 \cdots f_q}{C(g_0, g_1, \ldots, g_n)},$$

then

$$(q-n)T_g(r) \leq N\left(r, \frac{1}{L}\right) - N(r, L) + o(T_g(r)),$$

where r approaches infinity outside of an exceptional set E of finite logarithmic measure.

Theorem 5.1.1 implies the difference analogue of the second main theorem obtained in [22, Theorem 2.5] in the general case of slowly moving periodic targets, while Theorem 2.4.1 implies the special case of constant targets.

5.1.2 Picard's theorem

As an application of the difference analogue of Cartan's theorem, Halburd, Korhonen and Tohge obtained a difference analogue of Picard's theorem for holomorphic curves in [26].

Theorem 5.1.2 *[26] Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve such that $\sigma_2(f) < 1$, $c \in \mathbb{C}$ and $p \in \{1, \ldots, n+1\}$. If the $n+p$ hyperplanes in general position have forward invariant preimages under f with respect to the translation $\tau(z) = z + c$, then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq [n/p]$.*

As mentioned in the introduction, a natural difference analogue of Picard's theorem would have periodic moving targets. In order to state our generalization in that direction, we first need to define what we exactly mean by a moving periodic hyperplane.

First, we fix the numbers n and $q \ (\geq n)$, and observe q moving hyperplanes $H_j(z)$ associated with $\mathbf{a}_j = (a_{j0}(z), \ldots, a_{jn}(z))$. Let us write $Q := \{0, \ldots, q\}$ and $N := \{0, \ldots, n\}$ for convenience. By $\mathcal{K}$ we denote a field containing all the $a_{jk}(z)$ ($j \in Q, k \in N$) and also $\mathbb{C}$, where $a_{jk}(z)$ are c-periodic entire functions.

Let $H(z)$ be an arbitrary moving hyperplane over the field $\mathcal{K}$ in $\mathbb{P}^n$, that is, a hyperplane given by

$$H(z) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : a_0(z)x_0 + \cdots + a_n(z)x_n = 0\},$$

where $a_0, \ldots, a_n$ are c-periodic entire functions. Thus $H(z)$ is associated with a holomorphic mapping

$$\mathbf{a}(z) = (a_0(z), \ldots, a_n(z)) : \mathbb{C} \to \mathbb{C}^{n+1}.$$

Letting $x = [x_0 : \cdots : x_n]$, we denote

$$L_H(x, \mathbf{a}(z)) = \langle x, \mathbf{a}(z) \rangle = a_0(z)x_0 + \cdots + a_n(z)x_n.$$

For $x = g = [g_0 : \cdots : g_n]$, we then obtain

$$L_H(g, \mathbf{a}(z)) = \langle g(z), \mathbf{a}(z) \rangle = a_0(z)g_0(z) + \cdots + a_n(z)g_n(z),$$

and state that the curve g and the moving hyperplane H is *free* if $L_H(g, \mathbf{a}(z)) \not\equiv 0$.

Moving hyperplanes

$$H_j(z) = \left\{ [x_0 : \cdots : x_n] : \sum_{i=0}^{n} a_{ji}(z) x_i = 0 \right\}$$

in $\mathbb{P}^n$ over $\mathcal{K}$, and holomorphic mappings $\mathbf{a}_j(z) = (a_{j0}(z), \ldots, a_{jn}(z))$ of $\mathbb{C}$ into $\mathbb{C}^{n+1}$ associated with $H_j(z)$, $j = 0, \ldots, q$, are given. We state that $H_0(z), \ldots, H_q(z)$ are in *general position over* $\mathcal{K}$, if $q \geq n$ and any $n+1$ of the vectors $\mathbf{a}_j(z)$, $j = 0, \ldots, q$, are linearly independent over $\mathcal{K}$.

In order to measure the growth of holomorphic mappings associated with moving hyperplanes, we need a modified version of the Cartan characteristic function, and the corresponding notion of hyper-order.

Let $\mathbf{a}(z) = (a_0(z), \ldots, a_n(z)) : \mathbb{C} \to \mathbb{C}^{n+1}$ be a holomorphic mapping. Then

$$T^*_{\mathbf{a}}(r) = \int_0^{2\pi} \sup_{j \in \{0,\ldots,n\}} \log |a_j(re^{i\theta})| \frac{d\theta}{2\pi}$$

is the characteristic function of $\mathbf{a}$, and

$$\sigma_2^*(\mathbf{a}) = \limsup_{r \to \infty} \frac{\log^+ \log^+ T^*_{\mathbf{a}}(r)}{\log r}$$

is the hyper-order of $\mathbf{a}$.

We can now state our generalization of Theorem 5.1.2.

Theorem 5.1.3 *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve such that $\sigma_2(f) < 1$, $c \in \mathbb{C}$, and $p \in \{1, \ldots, n+1\}$. If the $n+p$ moving c-periodic hyperplanes H_j in general position with associated holomorphic mappings $\boldsymbol{a}_j(z) = (a_{j0}(z), \ldots, a_{jn}(z))$ have forward invariant preimages under f with respect to the translation $\tau(z) = z + c$, and*

$$\boldsymbol{a}_{i_1 \cdots i_{n+2}} = (a_{i_1 0}, \ldots, a_{i_1 n}, \ldots, a_{i_{n+2} 0}, \ldots, a_{i_{n+2} n}) \tag{5.1}$$

satisfying $\sigma_2^*(\boldsymbol{a}_{i_1 \cdots i_{n+2}}) < 1$ for all $i_1 \cdots i_{n+2}$, then the image of f is contained in a projective linear subspace over $\mathcal{P}_c^1$ of dimension $\leq [n/p]$.

In order to state the relevant growth condition for the coordinate functions a_{ji} of $\mathbf{a}_0, \ldots, \mathbf{a}_{n+p}$ in a condensed form, we have introduced holomorphic mapping (5.1). Alternatively, this assumption could be replaced with a simpler but stronger condition that each of the coordinate functions a_{ji} satisfies $\sigma_2(a_{ji}) < 1$. Note that in either case we do not need every element of $\mathbf{a}_j$ to be of growth $o(T_g(r))$; what is needed here is just that the hyper-order of holomorphic mapping (5.1) is strictly less than 1.

The following examples are used to demonstrate the sharpness of Theorem 5.1.3.

Since $g(z) := \pi/\Gamma(1-z) = (sin\pi z)\Gamma(z)$ is an entire function with only single zeros on the set of positive integers, it follows that $g^{-1}(\{0\}) = \mathbb{Z}_{>0}$ is forward invariant under the shift $\tau(z) = z+1$. On the other hand, the entire function $h(z) := (\sin\pi z)/\Gamma(z)$ has single zeros on $\mathbb{Z}_{>0}$ and double zeros on the set of non-positive integers $\mathbb{Z}_{\leq 0}$. The set of the zeros of $h(z)$ is still forward invariant with respect to $\tau(z)$ in our definition. Note also that the gamma function $\Gamma(z)$ is a meromorphic function of order 1 and the maximal type in the plane; in fact, $T(r,\Gamma) = (1+o(1))\frac{r}{\pi}\log r$ ([8, Proposition 7.3.6]), while $T(r, \sin\pi z) = 2r + O(1) = o(T(r,\Gamma))$ (see also [8, p. 27]). Further, $\sin\pi z \in \mathcal{P}_1^1$ but $\Gamma \notin \mathcal{P}_1^1$.

Let us consider the holomorphic curve

$$f := \left[\frac{1}{\Gamma(z)} : \frac{1}{\Gamma(z)} : \frac{1}{\Gamma(z+1/2)}\right] = \left[1 : 1 : \frac{\Gamma(z)}{\Gamma(z+1/2)}\right] : \mathbb{C} \to \mathbb{P}^2,$$

which has its image in a subset of $\mathbb{P}^2$ of dimention 1. Take the four moving hyperplanes H_j over $\mathcal{P}_f$ with $c=1$, each of which is given respectively by the vectors

$$(\sin\pi z, 0, 0), \quad (0, \sin\pi(z+1), 0), \quad (0, 0, \sin\pi(z+1/2)),$$

and

$$(\sin\pi z, \sin\pi(z+1), \sin\pi(z+1/2))$$

in $(\mathcal{P}_c^1)^3$ in general position. It is easy to see that each of these hyperplanes has a forward invariant preimage under f. For example,

$f^{-1}(\{H_1\})$ coincides with the zeros of the above entire function $h(z)$. This shows that Theorem 5.1.3 is sharp in the case where $n = p = 2$.

Similarly, when $n = 3$ and $p = 2, 3$, the bound $[n/p] = 1$ is attained by the six hyperplanes given by the following vectors in $(\mathcal{P}^1_1)^4$ in general position with the primitive fourth root of unity ω:

$$(\sin \pi z)(1, 0, 0, 0), \quad (\sin \pi z)(0, 1, 0, 0), \quad (\sin \pi z)(1, \omega, \omega^2, \omega^3),$$

$$(\cos \pi z)(0, 0, 1, 0), \quad (\cos \pi z)(0, 0, 0, 1), \quad (\cos \pi z)(1, 1, 1, 1)$$

and the curve $f : \mathbb{C} \to \mathbb{P}^3$ is given by

$$\begin{aligned} f : &= \left[\frac{1}{\Gamma(z)} : -\frac{1}{\Gamma(z)} : \frac{\omega}{\Gamma(z+\frac{1}{2})} : -\frac{1}{\Gamma(z+\frac{1}{2})}\right] \\ &= \left[1 : -1 : \omega\frac{\Gamma(z)}{\Gamma(z+\frac{1}{2})} : -\frac{\Gamma(z)}{\Gamma(z+\frac{1}{2})}\right]. \end{aligned}$$

This f is linearly degenerate in the sense that

$$f(\mathbb{C}) = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3 | z_1 + z_2 = 0, z_3 + \omega z_4 = 0\} \simeq \mathbb{P}^1.$$

A counter-example is also given to show the best-possibility of the restriction of the hyper-order less than one.

Consider the holomorphic curve $f(z) := [1 : \exp e^{2\pi i z}] : \mathbb{C} \to \mathbb{P}^1$, and three two-dimensional constant vectors $(1, 0)$, $(0, -1)$, $(1, -1)$ associating to three hyperplanes of $\mathbb{P}^1$ in general position. It is easy to see that the roots of the linear equation

$$\langle (1, \exp e^{2\pi i z}), (1, -1) \rangle = 1 - \exp e^{2\pi i z} = 0$$

are forward invariant with respect to $\tau(z) = z + 1$, since they are of the form

$$z = \frac{1}{2\pi i} \log(2m\pi) \pm \frac{1}{4} + k$$

for $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$. (For $\tau(z) = \frac{1}{2\pi i} \log(2m\pi) \pm \frac{1}{4} + (k + 1)$.) On the other hand, $[n/p] = [1/2] = 0$, but $f(z)$ is a non-constant meromorphic function in $\mathbb{C}$.

5.1.3 Difference analogues of the truncated second main theorem

Next, we introduce two alternative difference analogues of the truncated second main theorem, and provide corresponding difference deficiency relations. We begin with a definition of the difference counterpart of the concept of truncation.

Let $n \in \mathbb{N}$, $c \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{P}$. An a-point z_0 of a meromorphic function $h(z)$ is said to be *n-successive* and *c-separated*, if the n entire functions $h(z+\nu c)(\nu = 1, \dots, n)$ take the value a at $z = z_0$ with multiplicity there not less than that of $h(z)$. All the other a-points of $h(z)$ are called *n-aperiodic* of *pace c*. By $\widetilde{N}_g^{[n,c]}(r, L_H)$ we denote the counting function of *n-aperiodic* zeros of the function $L_H(g, \mathbf{a}) = \langle g(z), \mathbf{a}(z) \rangle$ of pace c.

Therefore, when all the zeros of $L_H(g, \mathbf{a})$, taking their multiplicities into account, are located periodically with period c, we obtain $\widetilde{N}_g^{[n,c]}(r, L_H) \equiv 0$. So in the case when the hyperplane H is forward invariant by g with respect to the translation $\tau_c(z) = z + c$, i.e. $\tau_c(g^{-1}(\{H\})) \subset g^{-1}(\{H\})$ holds by definition.

In addition, we denote

$$N_g(r, L_H) = N\left(r, \frac{1}{L_H(g, \mathbf{a})}\right) = N\left(r, \frac{1}{\langle g(z), \mathbf{a}(z) \rangle}\right)$$

and

$$N_C(r, 0) = N\left(r, \frac{1}{C(g_0, \dots, g_n)}\right).$$

With these definitions and notations in hand, we obtain the following auxiliary result:

Theorem 5.1.4 *Let g be a holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, $n \in \mathbb{N}$ and $q \in \mathbb{N}$ be such that $q \geq n$, and let*

$$\boldsymbol{a}_j(z) = (a_{j0}, \dots, a_{jn}), \quad j \in \{0, \dots, q\},$$

where $a_{jk}(z)$ are c-periodic entire functions satisfying $T(r, a_{jk}) = o(T_g(r))$ for all $j, k \in \{0, \dots, q\}$. If the moving hyperplanes

$$H_j(z) = \left\{[x_0 : \cdots : x_n] : L_{H_j}(x, \boldsymbol{a}_j(z)) = 0\right\}, \; j \in \{0, \dots, q\},$$

are located in general position, then

$$\sum_{j=0}^{q} N_g(r, L_{H_j}) - N_C(r, 0) \leq \sum_{j=0}^{q} \widetilde{N}_g^{[n,c]}(r, L_{H_j}) + o(T_g(r)).$$

By combining Theorem 5.1.4 with Theorem 5.1.1, we get the following difference analogue of the truncated second main theorem.

Theorem 5.1.5 (Difference Cartan Second Main Theorem) *Let $n \geq 1$, and $g = [g_0 : \ldots : g_n]$ be a holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with $\sigma_2(g) = \sigma_2 < 1$, where $g_0, \ldots, g_n$ are linearly independent over $\mathcal{P}_c^1$. Let*

$$\boldsymbol{a}_j(z) = (a_{j0}, \ldots, a_{jn}), \quad j \in \{0, \ldots, q\},$$

where $a_{jk}(z)$ are c-periodic entire functions satisfying $T(r, a_{jk}) = o(T_g(r))$ for all $j, k \in \{0, \ldots, q\}$. If the moving hyperplanes

$$H_j(z) = \left\{ [x_0 : \cdots : x_n] : L_{H_j}(x, \boldsymbol{a}_j(z)) = 0 \right\}, \ j \in \{0, \ldots, q\},$$

are located in general position, then

$$(q - n)T_g(r) \leq \sum_{j=0}^{q} \widetilde{N}_g^{[n,c]}(r, L_{H_j}) + o(T_g(r))$$

for all r outside of a set E with finite logarithmic measure.

From Theorem 5.1.5, we can obtain a difference analogue of the truncated deficiency relation for holomorphic curves.

Corollary 5.1.1 *Under the assumptions of Theorem 5.1.5, we obtain*

$$\sum_{j=0}^{q} \delta_g^{[n,c]}(0, L_{H_j}) \leq n + 1,$$

where

$$\delta_g^{[n,c]}(0, L_{H_j}) = 1 - \limsup_{r \to \infty} \frac{\widetilde{N}_g^{[n,c]}(r, L_{H_j})}{T_g(r)}.$$

Instead of n-successive points, we can also consider points with different separation properties. For instance, we say that a is a *derivative-like* paired value of f with the separation c if the following property holds for all except at most finitely many a-points of f: whenever $f(z) = a$ with the multiplicity m, then also $f(z+c) = a$ with the multiplicity $\max\{m-1, 0\}$.

As for the definition of the usual truncated counting function, refer, for instance, to Chapter 5.4 or [20] for details.

With this definition we may state the second difference analogue of the truncated second main theorem.

Theorem 5.1.6 *Assume that the hypotheses of Theorem 5.1.1 hold, and* 0 *is a derivative-like paired value of* f_i *with the separation* c *for all* $i \in \{0, \ldots, q\}$*. Then we obtain*

$$N(r, 0, L) \leq \sum_{j=0}^{q} N_n(r, 0, f_j) + O(1),$$

and this gives

$$(q-n)T_g(r) \leq \sum_{j=0}^{q} N_n(r, 0, f_j) - N(r, L) + o(T_g(r)),$$

where r *approaches infinity outside of an exceptional set of finite logarithmic measure.*

Theorem 5.1.6 immediately implies the following deficiency relation for derivative-like paired values of holomorphic curves.

Corollary 5.1.2 *Under the assumption of Theorem 5.1.6, we obtain*

$$\sum_{j=0}^{q} \delta_g^{[n]}(0, f_j) \leq n+1,$$

where

$$\delta_g^{[n]}(0, f_j) = 1 - \limsup_{r\to\infty} \frac{N_n\left(r, \frac{1}{f_j}\right)}{T_g(r)}.$$

For the growth of a holomorphic curve to be relatively fast, Theorem 5.1.1 can also be used to obtain a sufficient condition in terms of value distribution. With regard to the definition of exceptional paired value, refer to Chapter 5.2 or [22] for details.

Corollary 5.1.3 *Let $n \geq 1$, and $g = [g_0 : \dots : g_n]$ be a holomorphic curve of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, where $g_0, \dots, g_n$ are linearly independent over $\mathcal{P}_c^1$. If*

$$f_j = \sum_{i=0}^{n} a_{ij} g_i \quad j = 0, \dots, q, \, q > n,$$

where a_{ij} are c-periodic entire functions satisfying $T(r, a_{ij}) = o(T_g(r))$, such that any $n+1$ of the $q+1$ functions $f_0, \dots, f_q$ are linearly independent over $\mathcal{P}_c^1$, and 0 is an exceptional paired value of f_i for all $i \in \{0, \dots, q\}$, then we have $\sigma_2(g) \geq 1$.

5.2 SUMMARY OF PAPER II

The purpose of paper II is to formulate and study a difference analogue of Fermat's last theorem for function fields M, R, E, P.

In the light of the fact that a natural difference analogue of the Taylor series expansion is the factorial series [42, p. 272], we intend to consider the difference monomial $x(x-1)\cdots(x-n+1)$ as a discrete analogue of x^n. Therefore, in paper II we study the difference equation

$$f_1 \overline{f}_1 \cdots \overline{f}_1^{[n-1]} + f_2 \overline{f}_2 \cdots \overline{f}_2^{[n-1]} + \cdots + f_k \overline{f}_k \cdots \overline{f}_k^{[n-1]} = 1, \qquad (5.2)$$

and denote by G_C the smallest positive integer k such that the above equation (5.2) has a solution consisting of k nonconstant functions $f_1, \dots, f_k$ in C.

In order to state our results, we need the following definition and notations. Refer to [22, 62] for more details. Let f and g be meromorphic functions and a be a complex number. Let z_n $(n = 1, 2, \dots)$ be zeros of $f - a$. If z_n $(n = 1, 2, \dots)$ are also zeros of $g - a$ (ignoring multiplicity), we denote

$$f = a \Rightarrow g = a \quad \text{or} \quad g = a \Leftarrow f = a.$$

Let $\nu(n)$ be the multiplicity of the zero z_n. If z_n $(n = 1, 2, \ldots)$ are also $\nu(n)$ $(n = 1, 2, \ldots)$ multiple zeros of $g - a$ at least, we write

$$f = a \to g = a \quad \text{or} \quad g = a \leftarrow f = a.$$

If $f = a \rightleftharpoons g = a$, it is said that f and g share a CM; If $f = a \Leftrightarrow g = a$, it is said that f and g share a IM; if $f = a \to \overline{f} = a$ except for at most finitely many a-points of f, it is said that a is *an exceptional paired value of f with the separation c* (as defined in [22]).

Let $\widetilde{M}$ be the collection of all nonconstant meromorphic functions of the hyper-order less than one such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{M}$ satisfies the following properties

(i) f_i and $1/f_j$ $(i, j = 1, \ldots, k, i \neq j)$ have no common zeros;

(ii) $f_i = \infty \rightleftharpoons \overline{f}_i = \infty$ for all $i = 1, \ldots, k$;

(iii) 0 is an exceptional paired value of f_i for all $i = 1, \ldots, k$.

In the case of meromorphic functions, compared to the lower bound of F_M, we obtain a corresponding result for $G_{\widetilde{M}}$.

Theorem 5.2.1 *Let n (≥ 2) be an integer. Then*

$$G_{\widetilde{M}}(n) \geq \sqrt{n+1}.$$

Let $\widetilde{E}$ be the collection of all nonconstant entire functions of hyper-order less than one such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{E}$ satisfies the property that $f_i = 0 \Rightarrow \overline{f}_i = 0$ for all $i = 1, \ldots, k$.

In particular, for the case of entire functions, analogously to the lower bound of F_E, we give a better lower estimate for $G_{\widetilde{E}}$.

Theorem 5.2.2 *Let n (≥ 2) be an integer. Then*

$$G_{\widetilde{E}}(n) \geq 1/2 + \sqrt{n + 1/4}.$$

The following example is given to show the condition that the hyper-order of less than one cannot be deleted.

Take $f(z) = \exp\{e^z\}$, $c = i\pi$ and $n = 2$. Since 0 and ∞ are Picard exceptional values of $f(z)$, they are also automatically exceptional paired values of $f(z)$. Moreover, the function also satisfies the conditions $f = 0 \Rightarrow \overline{f} = 0$ and $f = \infty \Rightarrow \overline{f} = \infty$. The hyper-order of $f(z)$ is 1, and

$$f(z) \cdot f(z+c) = \exp\{e^z\} \cdot \exp\{e^{z+i\pi}\} = \exp\{e^z\} \cdot \exp\{-e^z\} = 1.$$

But $k = 1$ is strictly less than $1/2 + \sqrt{2+1/4} = \frac{3+1}{2} = 2$ and $\sqrt{2+1}\,(>1)$.

The next example shows the sharpness of the lower bound of G_C, where C is equal to $\widetilde{M}$ and $\widetilde{E}$.

Let $c = 2\pi$, $f_1 = \sin z$ and $f_2 = \cos z$. Then $\overline{f}_1 = \sin(z+2\pi) = \sin z$ and $\overline{f}_2 = \cos(z+2\pi) = \cos z$. Clearly $f_i\,(i = 1,2)$ satisfy $f_i = 0 \Rightarrow \overline{f}_i = 0$ and

$$f_1\overline{f}_1 + f_2\overline{f}_2 = \sin^2 z + \cos^2 z = 1.$$

Furthermore, 0 is an exceptional paired value of f_i for $i = 1,2$. Thus we have $G_{\widetilde{M}}(2) \leq 2$ and $G_{\widetilde{E}}(2) \leq 2$. On the other hand, by Theorems 5.2.1 and 5.2.2, we obtain $G_C(n) > 1$ for $C = \widetilde{M}, \widetilde{E}$. Therefore, $G_{\widetilde{M}}(2) = G_{\widetilde{E}}(2) = 2$.

Let $\widetilde{R}$ be the collection of all nonconstant rational functions such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{R}$ satisfies the property that zeros and poles are of a multiplicity positive integer multiple of n.

In the case of rational functions, compared to the lower bound for F_R, we get a corresponding estimate for $G_{\widetilde{R}}$.

Theorem 5.2.3 *Let $n\,(\geq 2)$ be an integer. Then*

$$G_{\widetilde{R}}(n) > \sqrt{n+1}.$$

Let $\widetilde{P}$ be the collection of all nonconstant polynomial functions such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{P}$ satisfies the property that zeros are of a multiplicity no less than n.

Moreover, in the case of polynomials, we give a better lower estimate for $G_{\widetilde{P}}$, as an analogue to the entire case.

Theorem 5.2.4 *Let $n\,(\geq 2)$ be an integer. Then*

$$G_{\widetilde{P}}(n) > 1/2 + \sqrt{n+1/4}.$$

5.3 SUMMARY OF PAPER III

In Paper III, we apply the concept and corresponding properties of a good linear operator to study meromorphic solutions of

$$M(z,f)+P(z,f)=h(z), \tag{5.3}$$

where $M(z,f)$ denotes a linear polynomial in f and $L(f)$ with L being a good linear operator, $P(z,f)$ is a polynomial in f and $h(z)$ a meromorphic function.

Equation (5.3) is an extension of a differential equation studied by J. Heittokangas *et. al* [29] in 2002. They considered the growth of meromorphic solutions of equation (4.4) and obtained Theorem 4.1.4.

Specific to $L(f)-p(z)f^3=h(z)$, J. Heittokangas *et. al* [29] also considered the existence and uniqueness of meromorphic solutions with only few poles and obtained Theorem 4.1.2.

Difference-differential counterparts of Theorem 4.1.4 and Theorem 4.1.2 were obtained by Laine and Yang in [61]. They investigated equations (4.5) and (4.6), and obtained Theorem 4.1.5 and Theorem 4.1.6. Further results on difference and differential-difference related to (4.6) can be found, e.g., in [49,53,54].

In the following theorem we apply the concept and corresponding properties of a good linear operator introduced in chapter 3 to obtain a natural extension of Theorem 4.1.4 and of its difference analogue to a general class of functional equations.

Theorem 5.3.1 *Let $\mathcal{N}\subset\mathcal{M}$ such that for any $f\in\mathcal{N}$,*

$$N(r,f)=o(T(r,f))$$

as $r\to\infty$ outside of a set E with an exceptional property $\mathbb{P}$, and let $\{L_k : k\in J\}$ be a finite collection of good linear operators for $\mathcal{N}$ with an exceptional set property $\mathbb{P}$. If $f_1\in\mathcal{N}$ and $f_2\in\mathcal{N}$ are any two meromorphic solutions of the equation

$$M(z,f)+P(z,f)=h(z), \tag{5.4}$$

where $P(z, f) = b_2(z)f^2 + \cdots b_n(z)f^n$ *is a polynomial in* f *with small meromorphic coefficients,* $h \in \mathcal{M}$ *and* $M(z, f)$ *is a linear polynomial in* f *and* $L_k(f)$, $k \in J$, *with small meromorphic coefficients, then*

$$T(r, f_2) = O(T(r, f_1)) + o(T(r, f_1)),$$

where $r \to \infty$ *outside of an exceptional set* E *with the property* $\mathbb{P}$.

The following corollary of Theorem 5.3.1 is obtained by choosing $\mathcal{N}$ as the family of meromorphic functions of the hyper-order strictly less than one with relatively few poles, and by taking $L_1, \ldots L_{l+1}$ such that $L_k(f) = f(z + c_k)$, $k = 1, \ldots, l$ and $L_{l+1}(f) = f'$.

Corollary 5.3.1 *Let* $M(z, f)$ *be a linear differential-difference polynomial in* f. *If* f_1 *and* f_2 *are any two meromorphic solutions of the equation* (5.4) *of the hyper-order strictly less than one such that* $N(r, f_1) = S(r, f_1)$ *and* $N(r, f_2) = S(r, f_2)$, *then*

$$T(r, f_2) = O(T(r, f_1)) + S(r, f_1).$$

Moreover, if $\alpha > 1$, *then for some* $r_\alpha > 0$,

$$T(r, f_2) = O(T(\alpha r, f_1))$$

for all $r \geq r_\alpha$. *In addition, every meromorphic solution such that the hyper-order* $\sigma_2 < 1$ *and* $N(r, f) = S(r, f)$ *satisfies* $\rho(f) = \rho(h)$.

In particular, let $P(z, f) = -p(z)f^3(z)$, where $p(z)$ is a small meromorphic function, then we obtain the following result on the existence of meromorphic solutions.

Theorem 5.3.2 *Let* f *be an transcendental meromorphic function of the hyper-order* $\sigma_2 < 1$, $M(z, f)$ *a linear differential-difference polynomial of* f *with small meromorphic coefficients, not vanishing identically, and* h *a meromorphic function. Set* $\lambda_f = \max\{\lambda(f), \lambda(\frac{1}{f})\}$. *If* f *satisfies the nonlinear differential-difference equation*

$$M(z, f) - p(z)f(z)^3 = h(z), \tag{5.5}$$

where $p(z)$ $(\not\equiv 0)$ *is a small function of* f, *then one of the following situations holds:*

(a) Equation (5.5) has f as its unique transcendental meromorphic solution such that $\lambda_f < \sigma_f$.

(b) Equation (5.5) has exactly three transcendental meromorphic solutions f_j, $j = 1, 2, 3$ such that $\lambda_{f_j} < \sigma_{f_j}$ for $j = 1, 2, 3$. Moreover $M(z, f_j) \equiv 0$, and $h(z) = -p(z) f_j^3$ for all $j = 1, 2, 3$.

5.4 SUMMARY OF PAPER IV

The goal of paper IV is to study the growth of solutions to second-order linear differential equations of a certain type. We show that under certain conditions some differential equations do not have subnormal solutions, and the hyper-order of every solution equals one.

Given the background knowledge in section 4.3, it is natural to ask what will happen when $\deg P_1 = \deg Q_1$ and $\deg P_2 = \deg Q_2$ for equation (4.13).

We consider the above question and obtain the following theorem.

Theorem 5.4.1 *Let*

$$\begin{aligned}
P_1(z) &= a_n z^n + \cdots + a_1 z + a_0, \\
Q_1(z) &= b_n z^n + \cdots + b_1 z + b_0, \\
P_2(z) &= c_m z^m + \cdots + c_1 z + c_0, \\
Q_2(z) &= d_m z^m + \cdots + d_1 z + d_0,
\end{aligned}$$

where a_i, b_i $(i = 0, \ldots, n)$, c_j, d_j $(j = 0, \ldots, m)$ are constants, $a_n b_n c_m d_m \neq 0$. Suppose that $a_n d_m = c_m b_n$ and any one of the following three hypotheses holds:

(i) there exists i satisfying $\left(-\frac{b_n}{a_n}\right) a_i + b_i \neq 0$, $0 < i < n$;

(ii) there exists j satisfying $\left(-\frac{b_n}{a_n}\right) c_j + d_j \neq 0$, $0 < j < m$;

(iii)

$$\left(-\frac{b_n}{a_n}\right)^2 + \left(-\frac{b_n}{a_n}\right)(a_0 + c_0) + b_0 + d_0 \neq 0.$$

Then (4.13) *has no non-trivial subnormal solution, and every non-trivial solution f satisfies $\sigma_2(f) = 1$.*

The following example is given to show the sharpness of restrictions (i)–(iii) in Theorem 5.4.1.

The equation

$$f'' + (e^{2z} + e^{-z} + 1)f' + (2e^{2z} + 2e^{-z} - 2)f = 0$$

has a subnormal solution $f_0 = e^{-2z}$. Here $n = 2$, $m = 1$, $a_2 = 1$, $b_2 = 2$, $a_1 = b_1 = 0$, $c_1 = 1$, $d_1 = 2$, $a_0 + c_0 = 1$, $b_0 + d_0 = -2$, $(-\frac{b_2}{a_2}) \cdot a_1 + b_1 = 0$, and $(-\frac{b_2}{a_2})^2 + (-\frac{b_2}{a_2})(a_0 + c_0) + b_0 + d_0 = 0$.

Another problem we consider in paper IV is which condition will guarantee that the more general form

$$f'' + (P_1(e^{\alpha z}) + P_2(e^{-\alpha z}))f' + (Q_1(e^{\beta z}) + Q_2(e^{-\beta z}))f = 0, \quad (5.6)$$

where $P(z), Q(z)$ are polynomials in z, α, β are complex constants, does not have a non-trivial subnormal solution. We prove the following two theorems.

Theorem 5.4.2 *Let*

$$\begin{aligned} P_1(z) &= a_{1m_1}z^{m_1} + \cdots + a_{11}z + a_{10}, \\ P_2(z) &= a_{2m_2}z^{m_2} + \cdots + a_{21}z + a_{20}, \\ Q_1(z) &= b_{1n_1}z^{n_1} + \cdots + b_{11}z + b_{10}, \\ Q_2(z) &= b_{2n_2}z^{n_2} + \cdots + b_{21}z + b_{20}, \end{aligned}$$

where $m_k \geq 1$, $n_k \geq 1$ $(k = 1, 2)$ are integers, a_{1i_1} $(i_1 = 0, 1, \ldots, m_1)$, a_{2i_2} $(i_2 = 0, 1, \ldots, m_2)$, b_{1j_1} $(j_1 = 0, 1, \ldots, n_1)$, b_{2j_2} $(j_2 = 0, 1, \ldots, n_2)$, α and β are complex constants, $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2} \neq 0$, $\alpha\beta \neq 0$. Suppose $m_1\alpha = c_1n_1\beta$ $(0 < c_1 < 1)$ or $m_2\alpha = c_2n_2\beta$ $(0 < c_2 < 1)$. Then (5.6) *has no non-trivial subnormal solution and every non-trivial solution f satisfies $\sigma_2(f) = 1$.*

Theorem 5.4.3 *Let $P_1(z)$, $P_2(z)$, $Q_1(z)$, $Q_2(z)$ be defined as in Theorem 5.4.2. Suppose $m_1\alpha = c_1n_1\beta$ $(c_1 > 1)$ and $m_2\alpha = c_2n_2\beta$ $(c_2 > 1)$. Then* (5.6) *has no non-trivial subnormal solution and every non-trivial solution f satisfies $\sigma_2(f) = 1$.*

The following example shows that the restrictions that $m_1\alpha = c_1 n_1 \beta$ $(c_1 > 1)$ and $m_2\alpha = c_2 n_2 \beta\,(c_2 > 1)$ cannot be omitted.

Note that the solution $f_0 = e^{-z} + 1$ satisfying the equation

$$f'' - [e^{3z} + e^{2z} + e^{-z}]f' - [e^{2z} + e^{-z}]f = 0$$

is subnormal. Here $\alpha = \frac{1}{2}$, $\beta = 1/3$, $m_1 = 6$, $m_2 = 2$, $n_1 = 6$, $n_2 = 3$, $m_1\alpha = \frac{3}{2}n_1\beta$ and $m_2\alpha = n_2\beta$.

5.5 SUMMARY OF PAPER V

As a result of recent interest in existence, value distribution and growth of meromorphic solutions of difference equations (see, e.g., [9,24]), a difference variant of Nevanlinna theory has emerged. With the development of new tools in value distribution theory suited to study solutions of difference equations, research into the general value distribution properties of meromorphic functions can be studied from a new perspective. Shared value problems of meromorphic functions and their shifts (see, e.g., [7, 30, 31, 39, 48]) are a new active direction of study.

The purpose of paper V is to investigate shared value problems related to an entire function $f(z)$ of the hyper-order less than one and its linear difference operator $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$, where $a_i, c_i \in \mathbb{C}$. We give sufficient conditions in terms of weighted value sharing and truncated deficiencies, which imply that

$$L(f) \equiv f. \tag{5.7}$$

Equation (5.7) also implies that f is a solution to a linear difference equation with constant coefficients. Therefore, the exact form of f can be, at least in principle, determined by using the characteristic equation for linear difference equations.

For simplicity, the family of all small meromorphic functions with respect to f, i.e. of the growth $S(r, f)$ which was defined in chapter 3.2, is denoted by $S(f)$. Moreover, $\hat{S}(f) = S(f) \cup \{\infty\}$.

Heittokangas et al. proved that if a finite-order meromorphic function $f(z)$ and $f(z + \eta)$ share three distinct periodic functions

$a_j \in \widehat{S}(f)$ $(j = 1;2;3)$ with period η CM, then f is a periodic function with period η (see [30, Theorem 2.1(a)]). They also showed that the 3 CM assumption can be replaced by 2 CM + 1 IM, and the same conclusion holds (see [31, Theorem 2]). Chen and Yi [7] considered the case where $f(z)$ and $\Delta f(z)$ share three distinct values a, b, ∞ CM as follows:

Theorem 5.5.1 *[7] Let $f(z)$ be a transcendental meromorphic function such that its order of growth $\sigma(f)$ is not an integer or infinite, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share three distinct finite values a, b, ∞ CM, then $f(z+\eta) \equiv 2f(z)$.*

In the case of only one CM value, but with the function f being entire and additionally having a finite Borel exceptional value, Chen and Yi obtained the following theorem.

Theorem 5.5.2 *[7] Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a, and $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share the value a CM, then $a = 0$ and*

$$\frac{f(z+\eta) - f(z)}{f(z)} = A,$$

where A is a nonzero constant.

Comparing the aforementioned results of Heittokangas et al. with Theorems 5.5.1 and 5.5.2, a natural question swift arises to end. Can the CM condition in these theorems be weakened to IM? Another question is whether we can extend these results in a natural way to general linear operators, rather than just the difference $\Delta f(z)$ or the shift operator. In paper IV, we studied these problems from the point of view of weighted value sharing. In order to explain what exactly do we mean by this we need to first introduce some additional notation.

Let l be a non-negative integer or infinite. Denote by $E_l(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted

m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f and g share (a, l). It is easy to see that if f and g share (a, l), then f and g share (a, p) for $0 \leq p \leq l$. We also note that f and g share the value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞), respectively.

Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{p)}\left(r, \frac{1}{f-a}\right)$ to denote the counting function of the zeros of $f - a$ whose multiplicities are not greater than p, $N_{(p+1}\left(r, \frac{1}{f-a}\right)$ to denote the counting function of the zeros of $f - a$ whose multiplicities are not less than $p+1$, and we use $\overline{N}_{p)}\,(r, \frac{1}{f-a})$ and $\overline{N}_{(p+1}(r, \frac{1}{f-a})$ to denote their corresponding reduced counting functions (ignoring multiplicities), respectively. We use $\overline{E}_{p)}(a, f)$ ($\overline{E}_{(p+1}(a, f)$) to denote the set of zeros of $f - a$ with multiplicities $\leq p$ ($\geq p+1$) (ignoring multiplicity), respectively. We also use $N_p\left(r, \frac{1}{f-a}\right)$ to denote the counting function of the zeros of $f - a$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then by defining the truncated deficiency as

$$\delta_p(a, f) = 1 - \limsup_{r \to +\infty} \frac{N_p(r, \frac{1}{f-a})}{T(r, f)},$$

it follows that $\delta_p(a, f) \geq \delta(a, f)$, where $\delta(a, f)$ is the usual Nevanlinna deficiency of f.

Our results give sufficient conditions in terms of weighted value sharing and truncated deficiencies for a transcendental entire function of relatively slow growth to be mapped to itself by a linear difference operator.

Theorem 5.5.3 *Let $f(z)$ be a transcendental entire function with the hyper-order less than* 1, *and a_j, $c_j \in \mathbb{C}$ be constants such that $L(f) := \sum_{j=1}^{k} a_j f(z + c_j) \not\equiv 0$, and $c_i \neq c_j$ when $i \neq j$. Assume that $f(z) - 1$ and $L(f) - 1$ share the value* $(0, l)$*. Then*

$$L(f) \equiv f$$

if one of the following assumptions holds:

(1) $l \geq 2$ *and*

$$\delta_2(0, f) + \delta(0, f) + \delta(1, f) > 1; \tag{5.8}$$

(2) $l = 1$ *and*

$$\frac{1}{2}\delta_2(0, f) + \frac{3}{4}\delta(0, f) + \frac{1}{2}\delta(1, f) > \frac{3}{4}; \tag{5.9}$$

(3) $l = 0$ *(i.e.* $f - 1$ *and* $L(f) - 1$ *share the value* 0 *IM) and*

$$\delta_2(0, f) + 3\delta(0, f) + \Theta(0, f) + \delta(1, f) > 4. \tag{5.10}$$

In Theorem 5.5.2 it was assumed that $a = 0$ is a Borel exceptional value of an entire function f. Therefore, since such an f is always of regular growth, the condition of Borel exceptionality of 0 implies that conditions (5.8), (5.9) and (5.10) are automatically satisfied.

Theorem 5.5.4 *Let* f *and* $L(f)\ (\not\equiv 0)$ *be defined as in Theorem 5.5.3. Assume that* $f - 1$ *and* $L(f) - 1$ *share the value* $(0, l)$ *and* $\overline{E}_{(i}(0, f) \subseteq \overline{E}_{(i}(0, L(f)))$ $(i \geq 3)$. *Then*

$$L(f) \equiv f$$

if one of the following assumptions holds:

(1) $l \geq 2$ *and*

$$2\delta_2(0, f) + \delta(1, f) > 1;$$

(2) $l = 1$ *and*

$$\frac{5}{4}\delta_2(0, f) + \frac{1}{2}\delta(1, f) > \frac{3}{4};$$

(3) $l = 0$ *(i.e.* $f - 1$ *and* $L(f) - 1$ *share the value* 0 *IM) and*

$$2\delta_2(0, f) + \frac{1}{2}\Theta(0, f) + \frac{1}{2}\delta(1, f) > 2.$$

If, instead of assuming that $i \geq 3$ as in Theorem 5.5.4, we consider the more general case $i \geq 2$, we have to impose slightly stronger conditions to obtain the same assertion.

Theorem 5.5.5 *Let f and $L(f)(\not\equiv 0)$ be defined as in Theorem 5.5.3. Assume that $f-1$ and $L(f)-1$ share the value $(0,l)$ and $\overline{E}_{(i}(0,f) \subseteq \overline{E}_{(i}(0,L(f))$ $(i \geq 2)$. Then*

$$L(f) \equiv f$$

if one of the following assumptions holds:

(1) $l \geq 2$ *and*

$$2\delta_2(0,f) + \delta(1,f) > 1;$$

(2) $l = 1$ *and*

$$\delta_2(0,f) + \frac{1}{4}\Theta(0,f) + \frac{1}{2}\delta(1,f) > \frac{3}{4};$$

(3) $l = 0$ *(i.e. $f-1$ and $L(f)-1$ share the value 0 IM) and*

$$\delta_2(0,f) + \frac{3}{2}\Theta(0,f) + \frac{1}{2}\delta(1,f) > 2.$$

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Nan Li

Difference Cartan-Nevanlinna theory and meromorphic solutions of functional equations

This thesis considers the properties of meromorphic solutions of some functional equations in the complex plane. In addition, a generalization of the difference Cartan second main theorem is also introduced, which can be used as a tool for the study of functional equations.

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