ANALYSIS OF FINITE DIFFERENCE METHODS FOR CONVECTION-DIFFUSION PROBLEM

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ABSTRACT

We consider finite difference methods for one dimensional convection-diffusion problem. An error analysis shows that the solution of the upwind scheme is not uniformly convergent in the discrete maximum norm due to its behavior in the layer. Then, we introduce and analyze a numerical method, Il'in-Allen-Southwell scheme, that is first-order uniformly convergent in the discrete maximum norm throughout the domain. Finally, we present numerical results that confirm theoretical findings.

ÖZ

Konveksiyon-difüzyon probleminin bir boyutlu çözümleri için sonlu farklar metodu ele alınmaktadır. Geri fark denkleminin çözümünün sürekli olmayan maksimum normda, tabakadaki davranışından dolayı düzgün yakınsak olmadığı hata analizi ile gösterilmektedir. Ardından alan boyunca, sürekli olmayan maksimum normda, birinci dereceden düzgün yakınsaklık gösteren Il'in-Allen-Southwell metodu tanıtılmakta ve analizi yapılmaktadır. Son olarak, teorik bulgular nümerik testlerle desteklenmektedir.

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Chapter 1

INTRODUCTION

In this work, we study the numerical solution techniques using the finite difference method for the convection-diffusion problem. The governing equations of the problem are given by

$$\begin{aligned} Lu &= -\epsilon u'' + a(x)u' = f(x) \quad , \quad 0 < x < 1 \qquad (1.1) \\ u(0) &= \alpha \\ u(1) &= \beta \quad , \quad a(x) > a_0 > 0 \end{aligned}$$

where ϵ is small parameter $0 < \epsilon \leq 1$ is used to measure the relative amount of diffusion to convection. $a(x)$ and $f(x)$ are smooth functions.

The convection-diffusion problem (1.1) comes from a reduction of a partial differential equation to an ordinary differential equation due to cylindrical or spherical symmetry. It arises in diverse areas such as the moisture transport in dessicated soil, the potential function of fluid injection through one side of a long vertical channel, the potential for a semiconductor device modelling, and the steady flow of a viscous, incompressible axisymmetric fluid between two rotating coaxial disks. It also comes in the problem of meridional angle change of the deformed middle surface and stress function in the theory of shells of revolution. Although the equation (1.1) may not be applied directly to real applications, it is important to find its solution, because it is an important stage in investigation of many practical applications. There is a lot of work in literature dealing with the numerical solution of singularly perturbed problems, showing the interest in this type of problems [1, 8, 10].

The major difficulty in the numerical solution of (1.1) is to find a numerical approximation scheme, which is uniformly accurate in ϵ, and a solution cost, which does not grow with decreasing ϵ. The standard finite difference scheme of upwind and centered type on a uniform mesh does not belong to this class. Because, the pointwise error is not necessarily reduced by successive uniform refinement of the mesh in contrast to solving unperturbed problems. Furthermore, although the standard centered-difference scheme is order of $O(h^2)$, it is numerically unstable and gives oscillatory solution unless the meshsize is fine. In order to remove these oscillatory solutions, it is necessary to use sufficiently small stepsize h compared to ϵ. But it is not practical to use finer mesh than ϵ in real application when ϵ is very small. On the other hand, Kellogg and Tsan [5] have analyzed the behavior of error of the standard upwind scheme for solving a general linear, singular perturbation problem on an even mesh. They showed that the method is not ϵ-uniform. In other words, the upwind scheme is not uniformly convergent in the discrete maximum norm due to its behavior in the layer. Therefore, we introduce and analyze a numerical method, the Il'in-Allen-Southwell scheme, which is uniformly convergent in the discrete maximum norm.

In Chapter 2, we introduce convection-diffusion problem and describe finite difference operators. We also display analytical behavior of one-dimensional convection-diffusion model problem.

In Chapter 3 and Chapter 4, centered-difference and backward(upwind) difference schemes are introduced and analyzed. The convergence and error estimates for the upwind scheme is presented and proved. Furthermore, some numerical results which confirm theoretical findings are demonstrated.

In Chapter 5, we consider a numerical method, a uniformly convergent Il'in-Allen-Southwell method, with better accuracy throughout the domain for full range of ϵ. We again present some numerical results in Section 5.2.

Chapter 2

OVERVIEW OF CONVECTION-DIFFUSION PROBLEM

In this chapter, we describe the convection-diffusion problem and then introduce a convection-diffusion equation in one-dimension on the interval $[0, 1]$. Finally, a short history of the finite difference methods are given and difference operators are introduced.

2.1 The Problem Statement

In this section, we consider the convection-diffusion equation. Imagine a river flowing strongly and smoothly. Some ink pours into the water at a certain point. Two physical processes operate here: 1) Convection alone would carry the ink along a one-dimensional curve on the surface. If the flow is fast, this is the dominant mechanism. 2) The ink diffuses slowly through the water. It makes curve spread out gradually.

When convection and diffusion are both present in a differential equation, we have a convection-diffusion problem.

Convection-diffusion problems have many practical applications in fluid flows, water quality problems, convective heat transfer problems and simulation of semiconductor devices. Also this equation arise, from the linearization of the Navier-Stokes equation and the drift-diffusion equation of semiconductor device modelling. Therefore it is especially important to devise effective numerical methods for their approximate solution.

2.2 The Analytical Behavior of Convection-Diffusion Problem

We now consider the following convection-diffusion problem of type (1.1):

$$\begin{aligned} -\epsilon u_{xx} + a u_x &= 1 \quad \text{on} \quad [0,1] \quad \text{where } a = 1. \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned} \tag{2.1}$$

which we can solve exactly:

$$u(x) = x - \frac{\exp\left(-\frac{1-x}{\epsilon}\right) - \exp\left(-\frac{1}{\epsilon}\right)}{1 - \exp\left(-\frac{1}{\epsilon}\right)}. \tag{2.2}$$

If ϵ is big enough, the solution will be smooth and then standard numerical methods will yield good results. However, as ϵ tends to zero, there is a boundary layer around $x = 1$.

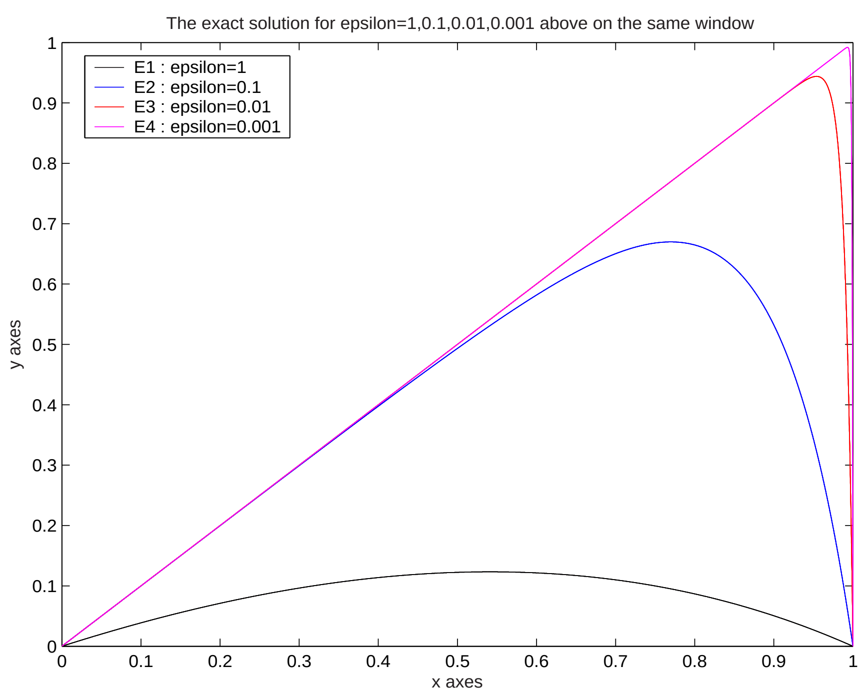

Figure 2.1: Exact solution of the problem (2.1) for several values of ϵ.

2.3 Finite Difference Approximations

In this section, we begin with a short history of the finite difference method and then extend our discussion to describe several finite difference approximations of interest.

We start the short history with the 1930s and further development of the finite difference method. Although some ideas may be traced back further, we begin the fundamental paper by Courant, Friedrichs and Lewy (1928) on the solutions of the problems of mathematical physics by means of finite differences. A finite difference approximation was first defined for the wave equation and the CFL stability condition was shown to be necessary for convergence. Error bounds for difference approximations of elliptic problems were first derived by Gerschgorin (1930) whose work was based on a discrete analogue of the maximum principle for Laplace's equation. This approach was pursued through the 1960s and various approximations of elliptic equations and associated boundary conditions were analyzed.

The finite difference theory for general initial value problems and parabolic problems then had an intense period of development during 1950s and 1960s, when the concept of stability was explored in the Lax equivalence theorem and the Kreiss matrix lemmas. Independently of the engineering applications, a number of papers appeared in the mathematical literature in the mid-1960s which were concerned with the construction and analysis of finite difference schemes by the Rayleigh-Ritz procedure with piecewise linear approximating functions.

The history of numerical methods for convection-diffusion problems begins about 30 years ago, in 1969. In this year, two significant Russian papers [11, 6] analyzed new numerical methods for convection-diffusion ODEs. In [6], Bakhvalov considered an upwind difference scheme on a layer-adapted graded mesh. Such meshes are based on a logarithmic scale. They are very fine inside the boundary layer and coarse outside. The fineness of the mesh means that the added artificial diffusion is very small inside the layer, and consequently the layer is not smeared excessively. In 1990 the Russian mathematician Grisha Shishkin showed that instead one could use a simpler piecewise uniform mesh. This idea has been propagated throughout the 1990s by a group of Irish mathematicians: Miller, O'Riordan, Hegarty and Farrell. During the next 20 years, researchers from many countries developed Ilin-type schemes for many singularly perturbed ODEs and some PDEs. The original Il'in paper used a complicated technique called the "double-mesh principle" to analyze the difference scheme. This became obsolete overnight when in 1978 Kellogg and Tsan published a revolutionary and famous paper [5] that was gratefully seized on by other researchers in the area. Their paper showed how to design barrier or comparison functions to convert truncation errors to computed errors, and also gave for the first time sharp a prior estimates for the solution of the convection-diffusion ODE. Late 20th-century mathematicians who have worked on numerical methods for convection-diffusion problems include Goering, Tobiska, Roos, Lube, Felgenhauer, John, Matthies, Risch, Schieweck.

Now let us introduce finite difference methods that we will employ, in the sequel, on an equidistant grid with meshsize h. We set $x_i = ih$ for $i = 0, ..., n+1$ with $x_0 = 0$ and $x_{n+1} = 1$. $x_{i+1} = x_i + h$ and $x_{i-1} = x_i - h$, $h = \frac{x_{n+1}-x_0}{n}$, $(x_{n+1} - x_0)$ is the length of the interval.

We refer to scheme (2.3) as the forward difference scheme because the forward difference approximation is used for the derivative.

$$(D^+u)(x) = \frac{u(x+h) - u(x)}{h} \approx \frac{u_{i+1} - u_i}{h} \tag{2.3}$$

Similarly (2.4) is referred to as the backward difference scheme.

$$(D^-u)(x) = \frac{u(x) - u(x-h)}{h} \approx \frac{u_i - u_{i-1}}{h} \tag{2.4}$$

Each of these formulas gives a *first order accurate approximation* to $u'(x)$, meaning that the size of the error is roughly proportional to h itself.

A finite difference method comprises a discretization of the differential equation using the grid points x_i, where the unknowns u_i (for $i = 0, ..., n+1$) are approximations to $u(x_i)$. It is also customary to approximate $u'(x)$ by the *centered-difference*

$$(D^o u)(x) = \frac{u(x+h) - u(x-h)}{2h} \approx \frac{u_{i+1} - u_{i-1}}{2h} \tag{2.5}$$

In fact $(D^o u)(x)$ gives a *second order accurate* approximation, the error is proportional to h^2 and, hence, is much smaller than the error in a first order approximation when h is small.

Composing the *forward* and *backward* differences, we get the following central approximations for $u''(x)$:

$$
\begin{aligned}
(D^+D^-u)(x) &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \qquad (2.6)\\
&\approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.
\end{aligned}
$$

Chapter 3

CENTERED-DIFFERENCE METHOD FOR CONVECTION-DIFFUSION PROBLEM

In this chapter, we present and analyze centered-difference approximation for convection-diffusion problem. Then we simulate some numerical results and explain the computer program.

3.1 Implementation of Centered-Difference Method for Convection-Diffusion Problem

We study with centered-difference method for the equation (2.1). Let us approximate diffusion term by second order central difference operator and convective term by centered-difference operator

$$-\epsilon(D^+D^-u)(x) + a(D^o u)(x) = 1 \text{ , on } [0,1] \text{ ,}$$

where a is fixed. Combining terms with the same indices, we get

$$a_1 u_{i+1} - 2b_1 u_i + c_1 u_{i-1} = 1 \tag{3.1}$$

$$\text{where } a_1 = \frac{-\epsilon}{h^2} + \frac{a}{2h} \text{ , } b_1 = \frac{-\epsilon}{h^2} \text{ and } c_1 = \frac{-\epsilon}{h^2} - \frac{a}{2h}. \tag{3.2}$$

We will briefly present some numerical results for the convection-diffusion problem. In Figures (3.1)-(3.4), the exact solution and centered-difference approximation are plotted on the same window for different values of ϵ. We divide the interval into 50 subintervals and take the convection coefficient to be 1.

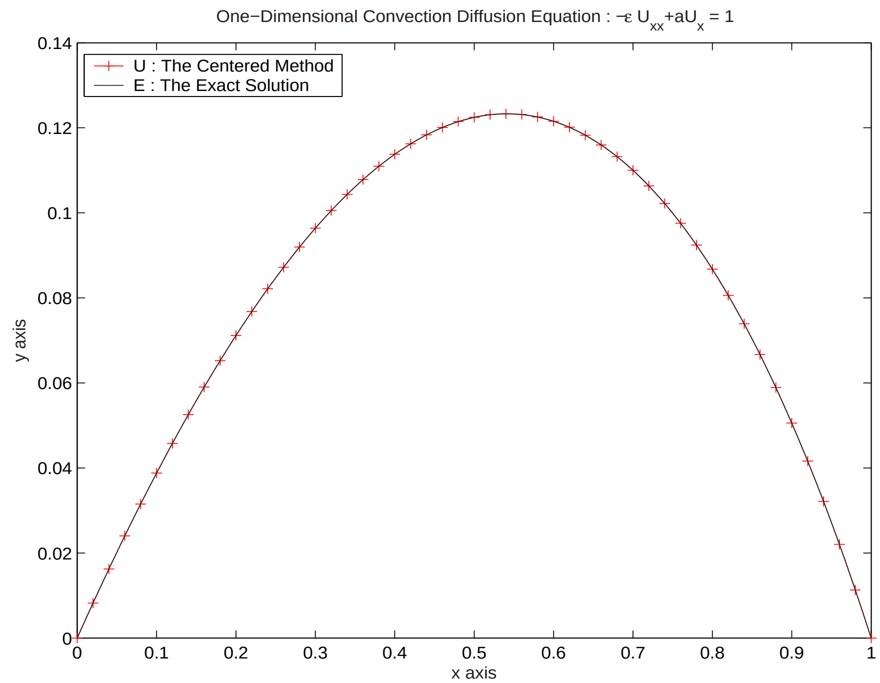

Figure 3.1: The centered-difference approximation with $n = 50$, $a = 1$ and $\epsilon = 1$.

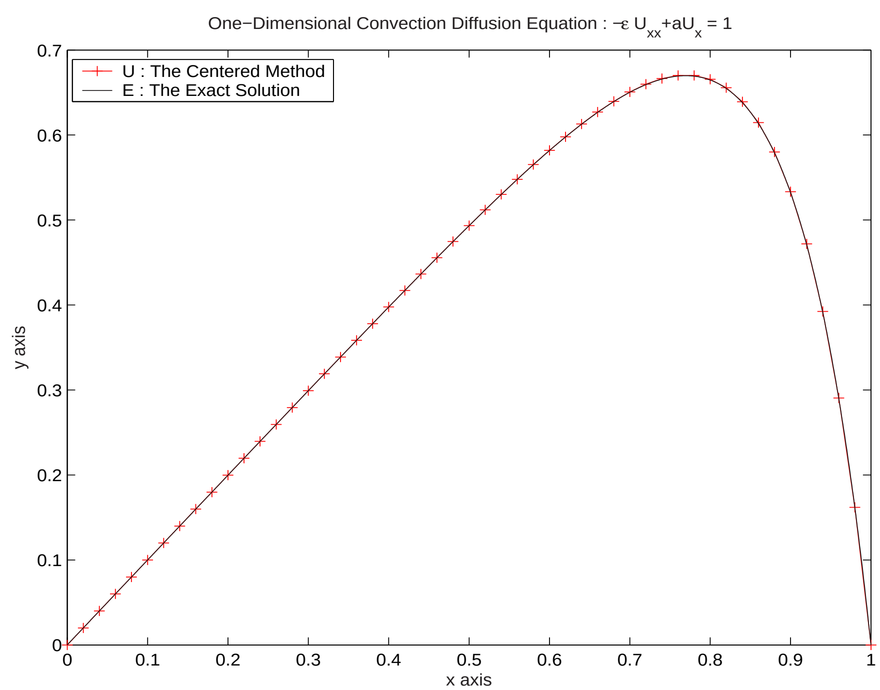

Figure 3.2: The centered-difference approximation with $n = 50$, $a = 1$ and $\epsilon = 0.1$.

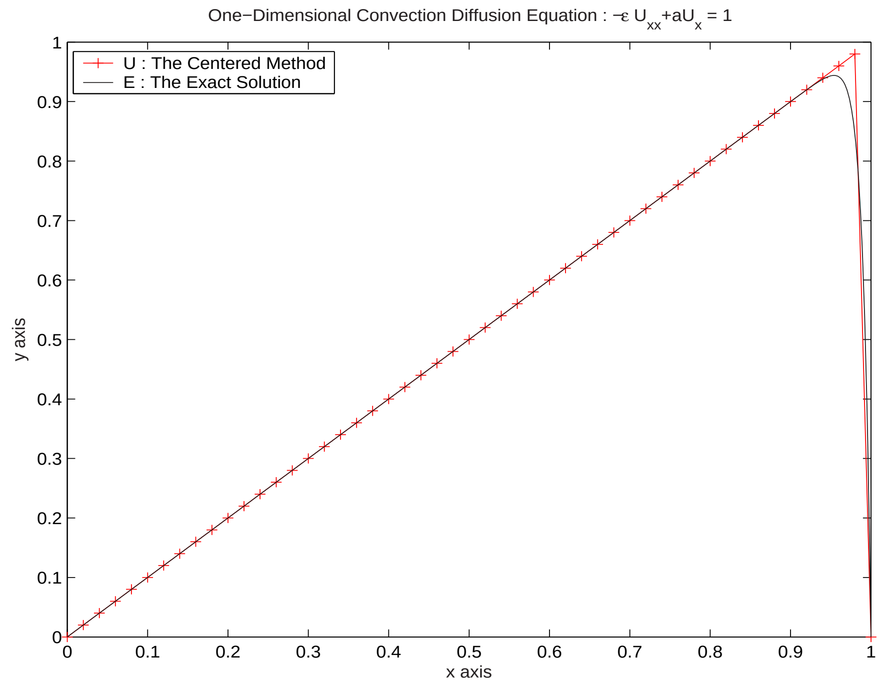

Figure 3.3: The centered-difference approximation with $n = 50$, $a = 1$ and $\epsilon = 0.01$.

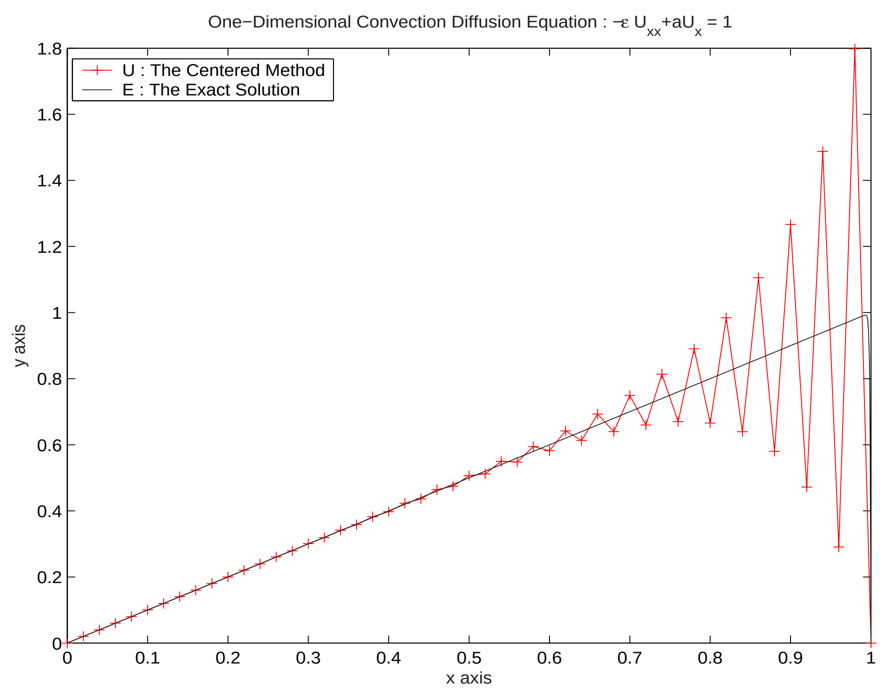

Figure 3.4: The centered-difference approximation with $n = 50$, $a = 1$ and $\epsilon = 0.001$.

3.2 Analysis of Centered-Difference Approximation

We can go back to the equation (3.1) and establish the solution of the difference equation. First, we consider homogeneous case and try $u_i = r^i$

$$a_1 r^{i+1} - 2b_1 r^i + c_1 r^{i-1} = 0$$

$$\Rightarrow \ a_1 r^2 - 2b_1 r + c_1 = 0$$

$$\Rightarrow \ r_{1,2} = \frac{2b_1 \mp \sqrt{4b_1^2 - 4a_1 c_1}}{2a_1} = \frac{b_1 \mp \sqrt{b_1^2 - a_1 c_1}}{a_1}$$

From the equation (3.2), we result in

$$b_1^2 - a_1 c_1 = (\frac{a_1 + c_1}{2})^2 - a_1 c_1 = \frac{a_1^2 + 2a_1 c_1 + c_1^2 - 4a_1 c_1}{4} = (\frac{a_1 - c_1}{2})^2,$$

then we get

$$r_{1,2} = \frac{b_1 \mp (\frac{a_1 - c_1}{2})}{a_1} \qquad \text{or equivalently}$$

$$r_1 = 1 \quad \text{and} \quad r_2 = \frac{c_1}{a_1}.$$

The second root r_2 can be rewritten as follows

$$r_2 = \frac{c_1}{a_1} = \frac{\frac{-2\epsilon - ah}{2h^2}}{\frac{-2\epsilon + ah}{2h^2}} = \frac{-2\epsilon - ah}{-2\epsilon + ah} = \frac{-2\epsilon - 2\epsilon\alpha}{-2\epsilon + 2\epsilon\alpha} = \frac{1+\alpha}{1-\alpha} \ \text{where} \ \ \alpha = \frac{ah}{2\epsilon}.$$

Thus, the solution of the difference equation (3.1) is obtained as

$$u_i = d_1 r_1^i + d_2 r_2^i + \frac{x_i}{a} = d_1 + d_2 (\frac{1+\alpha}{1-\alpha})^i + \frac{x_i}{a}. \tag{3.3}$$

This result shows that if $\alpha < 1$ ($\epsilon > 0.01$), we may expect approximate solution to be consistent with the physical configuration. However, if $\alpha > 1$ ($\epsilon < 0.01$), numerical solution oscillates. This is because, when we take $\alpha > 1$, r_2 will be negative. Therefore, we observe oscillations as it can be seen from Figure (3.4) from point to point.

3.3 Computer Programming

We write matlab codes to solve the convection-diffusion problem with different finite difference methods. First of all, we attempt to compute a grid function consisting of values $u_0, u_1, ..., u_n, u_{n+1}$ where u_i is our approximation to the solution $u(x_i)$. Here $x_i = ih$ and $h = 1/n$ is the mesh width, the distance between grid points. From the boundary conditions we know that $u_0 = 0$ and $u_{n+1} = 0$ and so we have n unknown values $u_1, ..., u_n$ to compute. Then we replace $u''(x)$ in the equation (1.1) by the second order central difference approximation and replace $u'(x)$ in the equation (1.1) by centered-difference approximation. We obtain a set of algebraic equations as follows

$$au_{i+1} - 2bu_i + cu_{i-1} = f(x_i)$$

where $a = \frac{-\epsilon}{h^2} + \frac{a}{2h}$, $b = \frac{-\epsilon}{h^2}$ and $c = \frac{-\epsilon}{h^2} - \frac{a}{2h}$ e.t.c and we take $\epsilon = 1, ..., 10^{-6}$, $a = 1$ and $n = 50$.

We now have a linear system of n equations for the n unknowns, which can be written in the form

$$Au = F$$

where u is the vector of unknowns $u = [u_1, ..., u_n]^T$ and

$$\mathbf{A} = \begin{bmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & b & c \\ & & & & a & b \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix}$$

This tridiagonal linear system is solved for u from any right hand side F. Then, all of the results are plotted, after the system of equations is solved. Other methods use the same solution techniques.

Chapter 4

BACKWARD DIFFERENCE METHOD FOR CONVECTION-DIFFUSION PROBLEM

In this chapter, we present backward difference approximation for convection-diffusion problem. Then, we state the theorem which proves convergence and error bounds in backward difference method. At last, we discuss stability and consistency of schemes using some numerical results.

4.1 Implementation of Backward Difference Method for Convection-Diffusion Problem

In this section, we shall study backward difference method for the equation (2.1). Convective term is approximated by backward difference operator

$$-\epsilon(D^+D^-u)(x) + a(D^-u)(x) = 1 \text{ , on } [0,1]$$

where a is fixed. To write the following equation, we combine terms with the same indices

$$-a_2u_{i+1} + b_2u_i + c_2u_{i-1} = 1 \tag{4.1}$$

$$\text{where } a_2 = \frac{\epsilon}{h^2} \text{ , } b_2 = \frac{2\epsilon}{h^2} + \frac{a}{h} \text{ and } c_2 = \frac{-\epsilon}{h^2} - \frac{a}{h}. \tag{4.2}$$

4.2 Analysis of Backward Difference Approximation

Consider the solution of the difference equation (4.1):

$$-a_2r^{i+1} + b_2r^i + c_2r^{i-1} = 0 \text{ where } u_i = r^i$$

$$\Rightarrow \; -a_2 r^2 + b_2 r + c_2 = 0$$

$$\Rightarrow \; r_{1,2} = \frac{-b_2 \mp \sqrt{b_2^2 + 4a_2c_2}}{-2a_2}$$

From the equation (4.2), we result in

$$b_2^2 + 4a_2c_2 = (a_2 - c_2)^2 + 4a_2c_2 \; = \; a_2^2 + 2a_2c_2 + c_2^2 \; = \; (a_2 + c_2)^2,$$

then we have

$$r_{1,2} = \frac{-b_2 \mp (a_2 + c_2)}{-2a_2}.$$

Hence, the roots of the homogeneous part of the the equation (4.1) are found to be $r_1 = 1$ and $r_2 = -\frac{c_2}{a_2}$. To write the solution of the equation (4.1), we need to rewrite r_2

$$r_2 = -\frac{c_2}{a_2} = \frac{\frac{\epsilon}{h^2} + \frac{a}{h}}{\frac{\epsilon}{h^2}} \; = \; \frac{\epsilon + ah}{\epsilon} \; = \; \frac{\epsilon + 2\epsilon\alpha}{\epsilon} \; = \; (1 + 2\alpha).$$

then the solution of the difference equation (4.1) is found as follows

$$u_i = d_1 r_1^i + d_2 r_2^i + \frac{x_i}{a} = d_1 + d_2(1 + 2\alpha)^i + \frac{x_i}{a}. \tag{4.3}$$

Backward difference method is expected to give more stable result than centered-difference approximation. This is because, if we take $\alpha > 1$ or $\alpha < 1$, r_2 produces always positive results and we do not observe oscillations as it can be seen from Figures (4.1)-(4.4). Furthermore, the backward difference method is first-order convergent outside the boundary layer but it is not convergent in the layer. The following lemmas and Theorem 4.2.7 indicate stability estimates of the upwind scheme.

Lemma 4.2.1. ***(Comparison Principle)*** *Suppose that w and w^* are functions in $C^2(0,1) \cap C[0,1]$ that satisfy*

$$\begin{aligned} Lw(x) \; &\leq \; Lw^*(x), \quad \forall\, x \in (0,1) \quad \text{and} \\ w(0) \; &\leq \; w^*(0), \quad w(1) \leq w^*(1). \quad \text{Then} \\ w(x) \; &\leq \; w^*(x), \qquad \forall\, x \in [0,1] \end{aligned}$$

where w^* is so called a barrier function for w.

A complete discussion of maximum and comparison principles for second-order elliptic problems can be found in [4]. Lemma 4.2.1 also implies a uniqueness result for solutions of the boundary value problem (1.1).

We now present Lemma 4.2.2 that gives information about the derivatives of u. Proof of the Lemma 4.2.2, which contains proofs of an a priori estimate for u' and a stability result, can be found in [1].

Lemma 4.2.2. *Let us assume that $a(x) \geq a_0 > 0$. Then for $i = 1, 2, ...,$ the solution u of (1.1) satisfies*

$$| u^{(i)}(x) | \leq \; C(1 + \epsilon^{-i} \exp{(-a_0 \frac{1-x}{\epsilon})}) \quad \text{for} \quad 0 < x < 1.$$

It is standard to use the theory of M-matrices in classical finite difference analysis. The following defines what the M-matrix is.

Definition 4.2.3. A matrix A is an M-matrix if its entries a_{ij} satisfy $a_{ij} \leq 0$ for $i \neq j$ and its inverse A^{-1} exists with $A^{-1} \geq 0$.

We require some inequalities and additional results to derive error bounds for the upwind scheme. The following Lemmas enable us to derive these error bounds.

Lemma 4.2.4. *Let $z_i = 1 + x_i, \quad 0 \leq i \leq n$. Then $L_h z_i \leq C$, where $C > 0$ does not depend on ϵ.*

Lemma 4.2.5. *(a)* $(\frac{h}{\epsilon})^k r_3^{-(1-x_i)/h} \leq C r_2^{-(1-x_i)/h} \leq C r_1^{-(1-x_i)/h}, \quad 0 \leq i < n$, or $0 \leq i \leq n$ if $k = 0$, *where k is a nonnegative integer and C depends only on k;*
(b) $r_2^{-(1-x_i)/h} \leq r_1^{-(1-x_i)/h} \leq \exp{(-a_0 \frac{1-x_i}{ah+\epsilon})}$*;*
(c) $r_2^{-(1-x_i)/h} \leq r_1^{-(1-x_i)/h} \leq C\, \exp{(-a_0^* \frac{1-x_i}{\epsilon})}$, *where $h \leq \epsilon$, and $a_0^* \in (0, a_0)$ is a constant depending only on a_0. We set $r_1 = 1 + \frac{a_0 h}{\epsilon}$, $r_2 = r_1 + \frac{a_0^2 h^2}{\epsilon^2}$, $r_3 = \exp(\frac{a_0 h}{\epsilon})$.*

The next lemma will be used, with Lemmas 4.2.1 and 4.2.4, to convert bounds for the discretization error.

Lemma 4.2.6. *There is a $C > 0$ depending only $a(x)$ and a such that,*

$$L_h r_1^{-(1-x_i)/h} \geq \frac{C}{\max(h,\epsilon)} r_1^{-(1-x_i)/h}.$$

Now we present and prove the main theorem that enables to observe the error behavior of the upwind scheme for the problem (1.1).

Theorem 4.2.7. *Assume that $a \geq a_0 > 0$. Then there exists a positive constant a_0^* which depends only on a_0, such that the error of the upwind scheme (4.1) at the inner grid points $x_i : i = 1, ..., n$ satisfies*

$$|\, u(x_i) - u_i \,| \leq \begin{cases} Ch\{1 + \epsilon^{-1} \exp{(-a_0^* \frac{1-x_i}{\epsilon})}\} & if \quad h \leq \epsilon, \\ Ch + C \exp{(-a_0^* \frac{1-x_i}{\epsilon})} & if \quad h \geq \epsilon. \end{cases}$$

Proof. First, at the grid point x_i, we obtain

$|\, \tau_i \,| := |\, L_h u(x_i) - f(x_i) \,|$ where τ_i is the consistency error.

Lu and $L_h u$ can be written as

$$\begin{aligned} Lu &= -\epsilon u'' + au' = f(x) \\ L_h u &= -\frac{\epsilon}{h^2}[u(x_{i+1}) - 2u(x_i) + u(x_{i-1})] + \frac{a}{h}[u(x_i) + u(x_{i-1})]. \end{aligned}$$

The order of accuracy of every finite difference approximation depends on the smoothness of u. For instance, Taylor's formula yields,

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u'''(x_i) + \frac{1}{3!}\int_{x_i}^{x_i+h} u^{(4)}(t)(x_i - t)^3 dt$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u'''(x_i) + \frac{1}{3!}\int_{x_i}^{x_i-h} u^{(4)}(t)(x_i - t)^3 dt$$

Now we can write

$$\begin{aligned} L_h u(x_i) \quad &= \quad -\frac{\epsilon}{h^2}\,[h^2 u''(x_i) + \frac{1}{3!}\int_{x_i}^{x_i+h} u^{(4)}(t)(x_i - t)^3 dt + \frac{1}{3!}\int_{x_i}^{x_i-h} u^{(4)}(t)(x_i - t)^3 dt] \\ &\quad + \frac{a}{h}\,[hu'(x_i) - \int_{x_i}^{x_i-h} u''(t)(x_i - t)dt]. \end{aligned}$$

Using integration by parts

$$
\begin{aligned}
L_h u(x_i) &= -\frac{\epsilon}{h^2}[h^2 u''(x_i) + \frac{1}{3!}(x_i - t)^3 u^{(3)}(t)|_{x_i}^{x_i+h} + \frac{3}{3!}\int_{x_i}^{x_i+h} u^{(3)}(t)(x_i - t)^2 dt \\
&\quad + \frac{1}{3!}(x_i - t)^3 u^{(3)}(t)|_{x_i}^{x_i-h} + \frac{3}{3!}\int_{x_i}^{x_i-h} u^{(3)}(t)(x_i - t)^2 dt] \\
&\quad + \frac{a}{h}[hu'(x_i) - \int_{x_i}^{x_i-h} u''(t)(x_i - t)dt] \\
&= -\frac{\epsilon}{h^2}[h^2 u''(x_i) + \frac{1}{3!}(x_i - x_i - h)^3 u^{(3)}(t) + \frac{1}{2}\int_{x_i}^{x_i+h} u^{(3)}(t)(x_i - t)^2 dt \\
&\quad + \frac{1}{3!}(x_i - x_i + h)^3 u^{(3)}(t) + \frac{1}{2}\int_{x_i}^{x_i-h} u^{(3)}(t)(x_i - t)^2 dt] \\
&\quad + \frac{a}{h}[hu'(x_i) - \int_{x_i}^{x_i-h} u''(t)(x_i - t)dt] \\
&= -\frac{\epsilon}{h^2}[h^2 u''(x_i) + \frac{1}{2}\int_{x_i}^{x_i+h} u^{(3)}(t)(x_i - t)^2 dt + \frac{1}{2}\int_{x_i}^{x_i-h} u^{(3)}(t)(x_i - t)^2 dt] \\
&\quad + \frac{a}{h}[hu'(x_i) - \int_{x_i}^{x_i-h} u''(t)(x_i - t)dt] \\
L_h u(x_i) &= -\epsilon u''(x_i) - \frac{\epsilon}{2h^2}[\int_{x_i}^{x_i+h} u^{(3)}(t)(x_i - t)^2 dt + \int_{x_i}^{x_i-h} u^{(3)}(t)(x_i - t)^2 dt] \\
&\quad + au'(x_i) - \frac{a}{h}\int_{x_i}^{x_i-h} u''(t)(x_i - t)dt
\end{aligned}
$$

then we obtain

$$
\begin{aligned}
&\mid \tau_i \mid := \mid L_h u(x_i) - f(x_i) \mid \leq \mid -\epsilon u''(x_i) - \frac{\epsilon}{2h^2}[\int_{x_i}^{x_i+h} u^{(3)}(t)(x_i - t)^2 dt \\
&+ \int_{x_i}^{x_i-h} u^{(3)}(t)(x_i - t)^2 dt] + au'(x_i) - \frac{a}{h}\int_{x_i}^{x_i-h} u''(t)(x_i - t)dt - (-\epsilon u''(x_i) + au'(x_i)) \mid
\end{aligned}
$$

$$
\mid \tau_i \mid \leq C \int_{x_{i-1}}^{x_{i+1}} (\epsilon \mid u^{(3)}(t) \mid + a \mid u''(t) \mid) dt \tag{4.4}
$$

Using Lemma 4.2.2,

$$\begin{aligned} | u^{(3)}(t) | &\leq C(1+\epsilon^{-3}\exp{(-a_0\frac{1-t}{\epsilon})}) \\ | u''(t) | &\leq C(1+\epsilon^{-2}\exp{(-a_0\frac{1-t}{\epsilon})}) \end{aligned}$$

By computing $| \tau_i |$

$$\begin{aligned} | \tau_i | &\leq C\int_{x_{i-1}}^{x_{i+1}}[\epsilon C(1+\epsilon^{-3}\exp{(-a_0\frac{1-t}{\epsilon})}) + C(1+\epsilon^{-2}\exp{(-a_0\frac{1-t}{\epsilon})})]dt \\ &\leq C\int_{x_{i-1}}^{x_{i+1}}[C(\epsilon+1)+2C\epsilon^{-2}\exp{(-a_0\frac{1-t}{\epsilon})}]dt \\ &\leq C[C(\epsilon+1)t|_{x_{i-1}}^{x_{i+1}} + C\epsilon^{-2}\int_{x_{i-1}}^{x_{i+1}}\exp{(-a_0\frac{1-t}{\epsilon})}dt] \\ &\leq Ch + C\epsilon^{-2}\int_{x_{i-1}}^{x_{i+1}}\exp{(-a_0\frac{1-t}{\epsilon})}dt \\ &\leq Ch + C\epsilon^{-2}\frac{\exp{(-a_0\frac{1-t}{\epsilon})}}{a_0/\epsilon}|_{x_{i-1}}^{x_{i+1}} \\ &\leq Ch + \frac{C\epsilon^{-1}}{a_0}[\exp{(-a_0\frac{1-(x_i+h)}{\epsilon})} - \exp{(-a_0\frac{1-(x_i-h)}{\epsilon})}] \\ &\leq Ch + C\epsilon^{-1}[\exp{(-a_0\frac{1-x_i}{\epsilon})}(\exp{(\frac{a_0h}{\epsilon})} - \exp{(\frac{-a_0h}{\epsilon})})] \end{aligned}$$

we get the inequality

$$| \tau_i | \leq Ch + C\epsilon^{-1}\sinh(\frac{a_0h}{\epsilon})\exp{(-a_0\frac{1-x_i}{\epsilon})}.$$

We consider the boundary value problem, in first the case $h \leq \epsilon$. Then $\frac{a_0h}{\epsilon}$ is bounded. Since $\sinh t \leq Ct$ for t bounded, we obtain, using Lemma 4.2.5(a)

$$| \tau_i | \leq Ch\{1+\epsilon^{-2}\exp{(-a_0\frac{1-x_i}{\epsilon})}\}.$$

Since $L_h(u(x_i)-u_i) = \tau_i$, we may use Lemmas 4.2.4 and 4.2.6 to obtain

$$\begin{aligned} | \tau_i | &\leq Ch\{1+\epsilon^{-1}\epsilon^{-1}\exp{(-a_0\frac{1-x_i}{\epsilon})}\} \\ &\leq Ch\{L_h(1+x_i)+\epsilon^{-1}(\epsilon^{-1}\exp{(-a_0\frac{1-x_i}{\epsilon})})\} \end{aligned}$$

$$\leq \; Ch\{L_h(1+x_i) + \epsilon^{-1} L_h \exp(-a_0 \frac{1-x_i}{\epsilon})\}$$

$$L_h \mid u(x_i) - u_i \mid \; \leq \; L_h\{Ch(1+x_i) + \frac{Ch}{\epsilon} \exp(-a_0 \frac{1-x_i}{\epsilon})\}.$$

From Lemmas 4.2.1 and 4.2.5(c), we obtain the desired inequality as follows

$$\mid u(x_i) - u_i \mid \; \leq \; Ch(1+x_i) + \frac{Ch}{\epsilon} \exp(-a_0 \frac{1-x_i}{\epsilon})$$

$$\leq \; Ch\{1 + \epsilon^{-1} \exp(-a_0^* \frac{1-x_i}{\epsilon})\}.$$

Now, in the case $h \geq \epsilon$, we use the splitting

$$u(x) = -u_0(1) \exp(-a(1) \frac{1-x}{\epsilon}) + z(x)$$

Imitating the proof of Lemma 4.2.2 [1] we find that

$$\mid z^{(i)}(x) \mid \leq \; C(1 + \epsilon^{1-i} \exp(-a(1) \frac{1-x}{\epsilon})). \tag{4.5}$$

Set

$$v(x) = -u_0(1) \exp(-a(1) \frac{1-x}{\epsilon})$$

and define v_h and z_h by

$L_h v_h = L v$ and $L_h z_h = L z$,

where v_h and z_h agree with v and z, respectively, on the boundary. Then

$$\mid u(x_i) - u_i \mid = \mid v(x_i) + z(x_i) - (v_i + z_i) \mid \leq \mid v(x_i) - v_i \mid + \mid z(x_i) - z_i \mid$$

For the consistency error due to z, we obtain as before

$$\mid \tau_i(z) \mid \leq \; C \int_{x_{i-1}}^{x_{i+1}} (\epsilon \mid z^{(3)}(t) \mid + a \; \mid z''(t) \mid) dt \tag{4.6}$$

Using the equation (4.5) and calculating the integral in the equation (4.5),

$$\mid z^{(3)}(t) \mid \leq \; C(1 + \epsilon^{-2} \exp(-a(1) \frac{1-t}{\epsilon}))$$

$$\mid z''(t) \mid \leq \; C(1 + \epsilon^{-1} \exp(-a(1) \frac{1-t}{\epsilon}))$$

we have the inequality

$$| \tau_i(z) | \leq Ch + C \sinh(\frac{a_0 h}{\epsilon}) \exp(-a_0 \frac{1-x_i}{\epsilon}).$$

For $h \geq \epsilon$, we now use $\sinh t \leq Ce^t$ and get

$$\begin{aligned} | \tau_i(z) | &\leq Ch + C \exp(\frac{a_0 h}{\epsilon}) \exp(-a_0 \frac{1-x_i}{\epsilon}) \\ &\leq Ch + C \exp(\frac{a_0 h}{\epsilon}(1 - x_i - h)) \\ &\leq Ch + C \exp(\frac{a_0 h}{\epsilon}(1 - x_{i+1})) \end{aligned}$$

It remains to bound the consistency error due to v. From the definition of v,

$$\begin{aligned} v(x) &= -u_0(1) \exp(-a(1)\frac{1-x}{\epsilon}) \\ Lv &= -\epsilon v'' + av' \\ v' &= -\frac{u_0(1)a(1)}{\epsilon} \exp(-a(1)\frac{1-x}{\epsilon}) \\ v'' &= -\frac{u_0(1)a(1)^2}{\epsilon^2} \exp(-a(1)\frac{1-x}{\epsilon}) \end{aligned}$$

$$\begin{aligned} Lv &= -\epsilon(-\frac{u_0(1)a(1)^2}{\epsilon^2} \exp(-a(1)\frac{1-x}{\epsilon})) + a(-\frac{u_0(1)a(1)}{\epsilon} \exp(-a(1)\frac{1-x}{\epsilon}) \\ &= \frac{1}{\epsilon} u_0(1) \exp(-a(1)\frac{1-x}{\epsilon})(a(1)^2 - a(1)\, a) \end{aligned}$$

we have

$$| Lv(x) | \leq C\epsilon^{-1} \; | v(x) | .$$

Thus

$$| (L_h v_h)_i | = | Lv(x_i) | \leq C\epsilon^{-1} \exp(-a_0 \frac{1-x_i}{\epsilon}).$$

Again invoking the discrete comparison principle, we get

$$| v(x_i) - v_i | \leq | v(x_i) | + | v_i | \leq C \exp(-a_0 \frac{1-x_i}{\epsilon}).$$

Using the Lemmas 4.2.1 and 4.2.6 proves the result for the case $h \geq \epsilon$. □

Example 4.2.8. $-\epsilon u'' - u' = 0,\ \ u(0) = 0,\ \ u(1) = 1$, which has a layer at $x = 0$. Exact solution of this problem

$$u(x) = c_1 + c_2 \exp\left(\frac{-x}{\epsilon}\right)$$

and apply the boundary conditions

$$u(0) = c_1 + c_2 \exp(0) = 0$$

$$u(1) = c_1 + c_2 \exp\left(\frac{-1}{\epsilon}\right) = 1$$

$$\Rightarrow\ c_1 = \frac{1}{1 - \exp\left(\frac{-1}{\epsilon}\right)} \text{ and } c_2 = -\frac{1}{1 - \exp\left(\frac{-1}{\epsilon}\right)}.$$

Now, we can write

$$u(x) = \frac{1}{1 - \exp\left(\frac{-1}{\epsilon}\right)} - \frac{1}{1 - \exp\left(\frac{-1}{\epsilon}\right)} \exp\left(\frac{-x}{\epsilon}\right) = \frac{1 - \exp\left(\frac{-x}{\epsilon}\right)}{1 - \exp\left(\frac{-1}{\epsilon}\right)}.$$

Then the backward difference scheme yields

$$-\epsilon\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) - \frac{u_i - u_{i-1}}{h} = 0$$

$$u_i = \frac{1 - r^i}{1 - r^n} \ \text{ with }\ r = \frac{\epsilon}{\epsilon + h}.$$

For $h = \epsilon$, we obtain

$$u_1 = \frac{1 - (1/2)^1}{1 - (1/2)^n} = \frac{(1/2)}{1 - (1/2)^n} \quad \text{but} \quad u(x_1) = \frac{1 - \exp(-1)}{1 - \exp\left(\frac{-1}{\epsilon}\right)}.$$

Theorem 4.2.7 shows that outside the boundary layer we have *first-order convergence*. However, theorem does not prove convergence and can not be sharpened near the layer as it can be seen in this example. Furthermore, Figure (4.5) indicates that the error behavior for this example as h varies with ϵ fixed.

We now present some numerical results for various values of ϵ in Figures (4.1)-(4.4). We again divide the interval into 50 subintervals and take the convection coefficient to be 1.

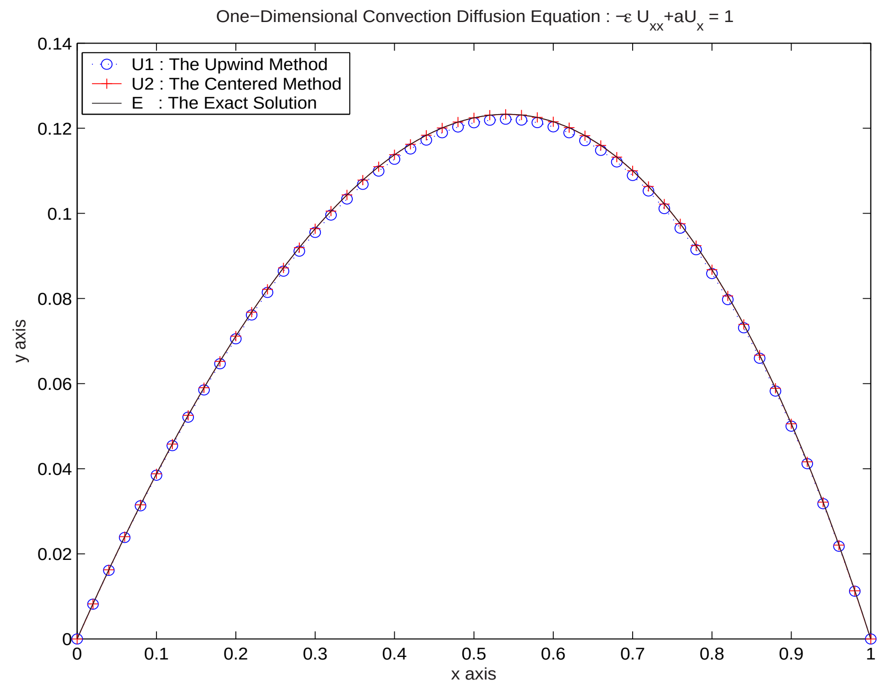

Figure 4.1: The centered-difference and backward difference approximations with $n = 50$, $a = 1$ and $\epsilon = 1$ ($\alpha < 1$).

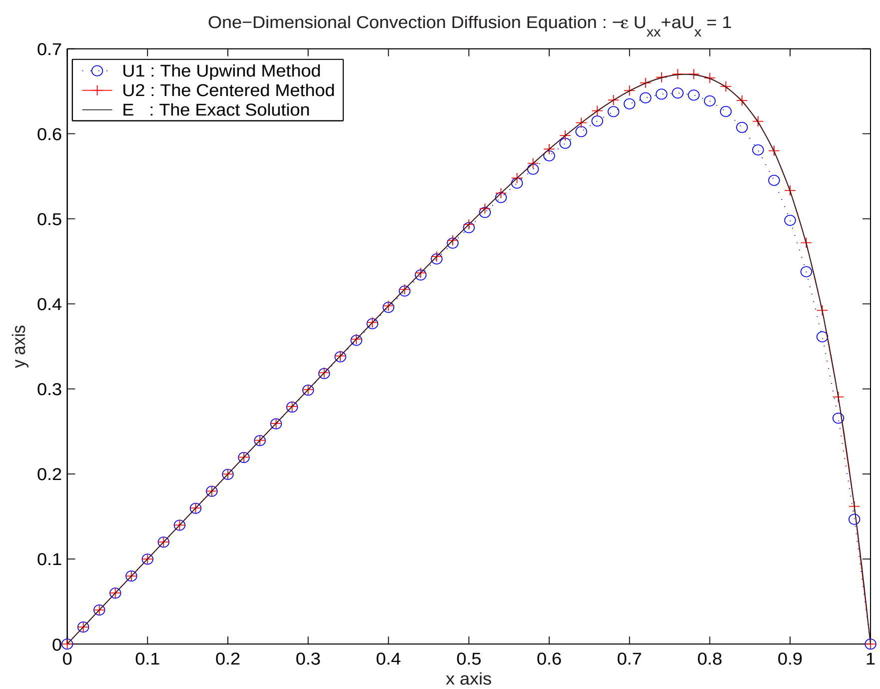

Figure 4.2: The centered-difference and backward difference approximations with $n = 50$, $a = 1$ and $\epsilon = 0.1$ ($\alpha < 1$).

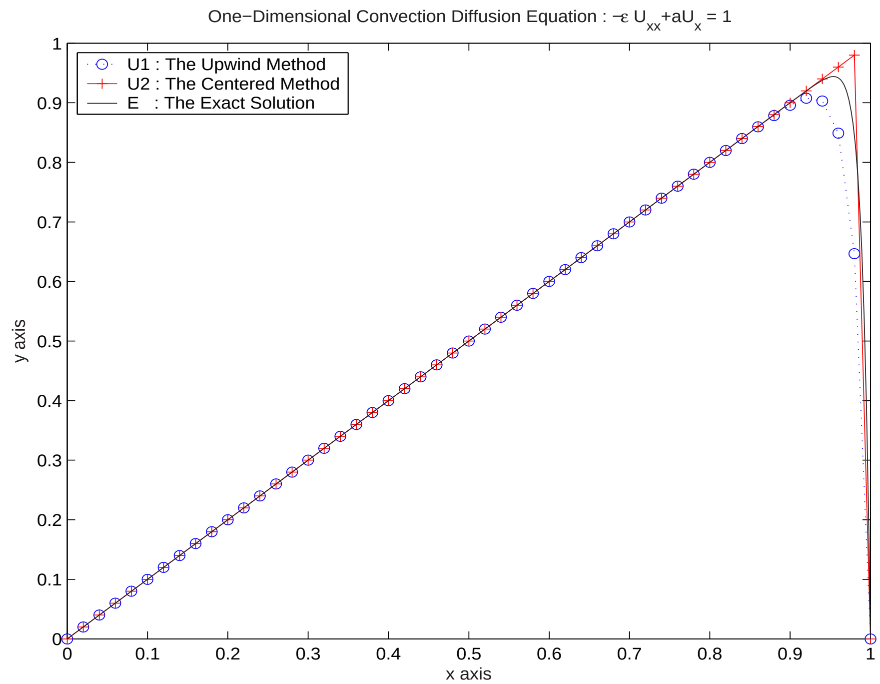

Figure 4.3: The centered-difference and backward difference approximations with $n = 50$, $a = 1$ and $\epsilon = 0.01$ $(\alpha = 1)$.

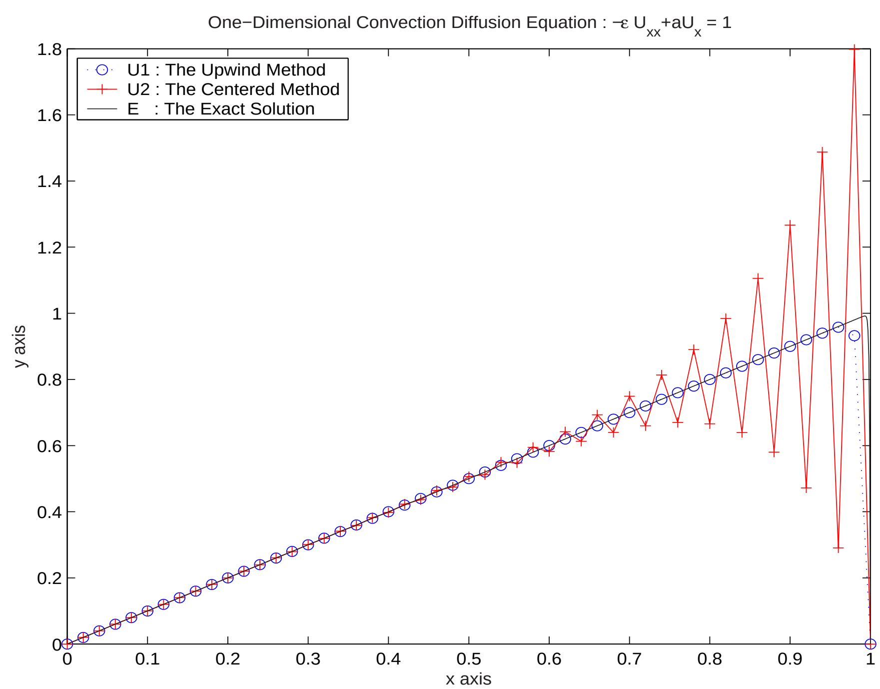

Figure 4.4: The centered-difference and backward difference approximations with $n = 50$, $a = 1$ and $\epsilon = 0.001$ $(\alpha > 1)$.

If ϵ is big enough, backward and centered-difference approximations produce numerical results consistent with the physical configuration of the problem. For $\epsilon \leq 0.001$, centered-difference approximations give oscillatory solutions, see Figure (4.4). However, backward difference approximations are still stable. Although the backward difference method works for all values of ϵ, it is not uniformly convergent: That is, as stepsize decreases, the error may increase for some values of ϵ. We have observed this behavior especially in midrange values of ϵ, see Figure (4.5). Furthermore, in Theorem 4.2.7 we have seen that the upwind scheme is first-order convergent outside the boundary layer but it is not convergent in the layer. Therefore, we now discuss another finite difference method which gives better results than the previous ones.

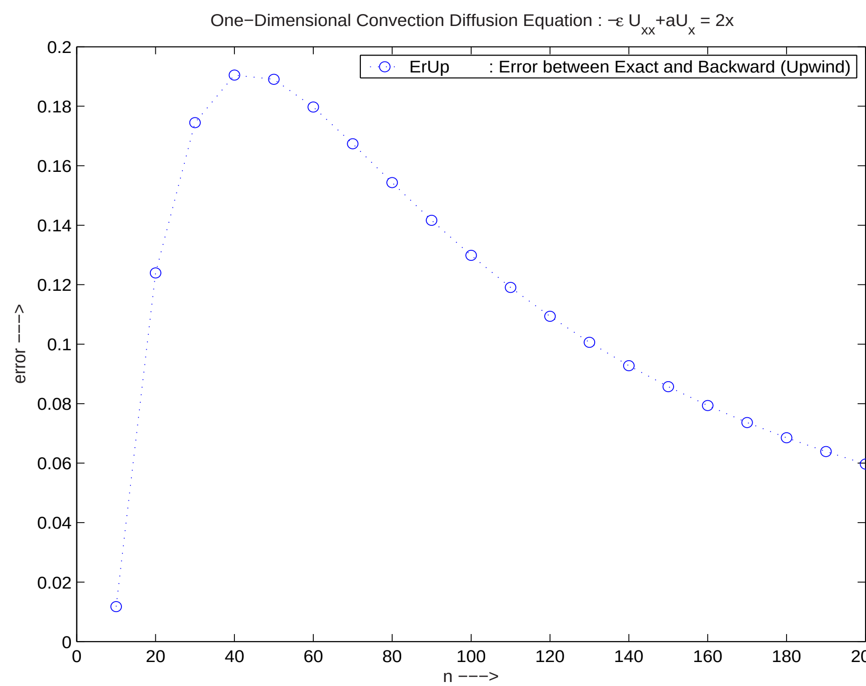

Figure 4.5: The error at the layer for the upwind scheme with $a = 1$ and $\epsilon = 0.01$

Chapter 5

UNIFORMLY CONVERGENT METHOD FOR CONVECTION-DIFFUSION PROBLEM

In this chapter, we consider a uniformly convergent method, called Il'in-Allen-Southwell Scheme. We first show how to construct such a method. Then we analyze the method and present some numerical results.

5.1 Construction of a Uniformly Convergent Method

We describe a way of constructing a uniformly convergent difference scheme. We start with the standard derivation of an exact scheme for the convection-diffusion problem (2.1). Introduce the formal adjoint operator L^* of L

$$Lu = -\epsilon u'' + au' = f\,, \quad u(0) = u(1) = 0\,, \quad a > 0.$$

Let g_i be local Green's function of L^* with respect to the point x_i ; that is

$$L^* g_i = -\epsilon g_i'' - a g_i' = 0 \ \ in \ \ (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \tag{5.1}$$

Let us impose boundary conditions

$$g_i(x_{i-1}) = g_i(x_{i+1}) = 0$$

and impose additional conditions

$$\epsilon(g_i'(x_i^-) - g_i'(x_i^+)) = 1.$$

Now

$$\int_{x_{i-1}}^{x_{i+1}} (Lu) g_i dx = \int_{x_{i-1}}^{x_{i+1}} f g_i dx$$

and multiplying by g_i and then integrating by parts

$$\int_{x_{i-1}}^{x_{i+1}} (-\epsilon u''(x) + au'(x))g_i dx = \int_{x_{i-1}}^{x_{i+1}} f g_i dx$$

$$= \int_{x_{i-1}}^{x_i} (-\epsilon u'' + au')g_i dx + \int_{x_i}^{x_{i+1}} (-\epsilon u'' + au')g_i dx$$

$$= (-\epsilon u' + au)g_i(x)|_{x_{i-1}}^{x_i} + (-\epsilon u' + au)g_i(x)|_{x_i}^{x_{i+1}}$$

$$- \int_{x_{i-1}}^{x_i} (-\epsilon u' + au)g_i' dx - \int_{x_i}^{x_{i+1}} (-\epsilon u' + au)g_i' dx$$

$$= [(-\epsilon u'(x_i^-) + au(x_i))g_i(x_i) - (-\epsilon u'(x_{i-1}) + au(x_{i-1}))g_i(x_{i-1})]$$

$$+ [(-\epsilon u'(x_{i+1}) + au(x_{i+1}))g_i(x_{i+1}) - (-\epsilon u'(x_i^+) + au(x_i))g_i(x_i)]$$

$$- \int_{x_{i-1}}^{x_i} (au)g_i' dx - \int_{x_i}^{x_{i+1}} (au)g_i' dx + \int_{x_{i-1}}^{x_i} (\epsilon u')g_i' dx + \int_{x_i}^{x_{i+1}} (\epsilon u')g_i' dx$$

$$= -\epsilon u'(x_i^-)g_i(x_i) + \epsilon u'(x_i^+)g_i(x_i) + \epsilon u(x)g_i'(x)|_{x_{i-1}}^{x_i} + \epsilon u(x)g_i'(x)|_{x_i}^{x_{i+1}}$$

$$+ \int_{x_{i-1}}^{x_i} (-\epsilon g_i'' - ag_i')u dx + \int_{x_i}^{x_{i+1}} (-\epsilon g_i'' - ag_i')u dx$$

since u' is continuous on (x_{i-1}, x_{i+1}), then we have

$$= [\epsilon u(x_i)g_i'(x_i^-) - \epsilon u(x_{i-1})g_i'(x_{i-1}^+)] + [\epsilon u(x_{i+1})g_i'(x_{i+1}^-) - \epsilon u(x_i)g_i'(x_i^+)]$$

$$= -\epsilon g_i'(x_{i-1})u_{i-1} + u_i + \epsilon g_i'(x_{i+1})u_{i+1}$$

The identity can be written as

$$-\epsilon g_i'(x_{i-1})u_{i-1} + u_i + \epsilon g_i'(x_{i+1})u_{i+1} = f \int_{x_{i-1}}^{x_{i+1}} g_i dx. \tag{5.2}$$

The difference scheme whose i^{th} equation (5.1) is exact. We are able to evaluate each g_i''s exactly.

The solution of the equation (5.1) is given by

$$g_i(x^-) = c_1 + c_2(\frac{-\epsilon}{a})e^{\frac{-ax}{\epsilon}} \;\text{ on }\; (x_{i-1}, x_i) \tag{5.3a}$$

$$g_i(x^+) = c_1^{'} + c_2^{'}(\frac{-\epsilon}{a})e^{\frac{-ax}{\epsilon}} \;\text{ on }\; (x_i, x_{i+1}) \tag{5.3b}$$

We have 4 unknowns c_1 , c_2 , $c_1^{'}$, $c_2^{'}$, therefore we need 4 equations :

$$g_i(x_{i-1}) = 0 \tag{5.4}$$

$$g_i(x_{i+1}) = 0 \tag{5.5}$$

$$-\epsilon(g_i^{'}(x_i^-) - g_i^{'}(x_i^+)) = 1 \tag{5.6}$$

and, from continuity of g_i at $x = x_i$

$$g_i(x_i^-) = g_i(x_i^+) \tag{5.7}$$

Imposing boundary conditions (5.4) and (5.5),

$$g_i(x_{i-1}) = c_1 + c_2(\frac{-\epsilon}{a})e^{\frac{-ax_{i-1}}{\epsilon}} = 0 \tag{5.8}$$

$$g_i(x_{i+1}) = c_1^{'} + c_2^{'}(\frac{-\epsilon}{a})e^{\frac{-ax_{i+1}}{\epsilon}} = 0. \tag{5.9}$$

By taking derivative the equation (5.3)

$$g_i^{'}(x_i^-) = c_2(\frac{-\epsilon}{a})(\frac{-a}{\epsilon})e^{\frac{-ax_i}{\epsilon}}$$

$$g_i^{'}(x_i^+) = c_2^{'}(\frac{-\epsilon}{a})(\frac{-a}{\epsilon})e^{\frac{-ax_i}{\epsilon}}$$

then the equation (5.10) can be written in the following form

$$\epsilon(c_2e^{\frac{-ax_i}{\epsilon}} - c_2^{'}e^{\frac{-ax_i}{\epsilon}}) = 1$$

$$\Rightarrow \;\; c_2 - c_2^{'} = \frac{1}{\epsilon}e^{\frac{ax_i}{\epsilon}}. \tag{5.10}$$

we have written $g_i(x_i^-) = g_i(x_i^+)$ from continuity of g_i at $x = x_i$

$$g_i(x_i^-) - g_i(x_i^+) = 0$$

$$\Rightarrow \ c_1 + c_2(\frac{-\epsilon}{a})e^{\frac{-ax_i}{\epsilon}} - [c_1^{'} + c_2^{'}(\frac{-\epsilon}{a})e^{\frac{-ax_i}{\epsilon}}] = 0$$

and then we have

$$(c_1 - c_1^{'}) + (c_2 - c_2^{'})(\frac{-\epsilon}{a})e^{\frac{-ax_i}{\epsilon}} = 0 \tag{5.11}$$

Let us assume that $\alpha_i = \frac{ax_i}{\epsilon}$, $\rho_i = \frac{ah}{\epsilon}$. We can write

$$e^{\frac{ax_{i+1}}{\epsilon}} = e^{\frac{a(x_i+h)}{\epsilon}} = e^{\frac{ax_i}{\epsilon}+\frac{ah}{\epsilon}} = e^{\alpha_i+\rho_i}$$

$$e^{\frac{ax_{i-1}}{\epsilon}} = e^{\frac{a(x_i-h)}{\epsilon}} = e^{\frac{ax_i}{\epsilon}-\frac{ah}{\epsilon}} = e^{\alpha_i-\rho_i}.$$

Hence, we transform the equations (5.8)-(5.11) into the equations (5.12)-(5.15)

$$c_1 + c_2(\frac{-\epsilon}{a})e^{-\alpha_i+\rho_i} = 0 \tag{5.12}$$

$$c_1^{'} + c_2^{'}(\frac{-\epsilon}{a})e^{-\alpha_i-\rho_i} = 0 \tag{5.13}$$

$$c_2 - c_2^{'} = \frac{1}{\epsilon}e^{\alpha_i} \tag{5.14}$$

$$(c_1 - c_1^{'}) + (c_2 - c_2^{'})(\frac{-\epsilon}{a})e^{-\alpha_i} = 0 \tag{5.15}$$

Plug the equation (5.14) into the equation (5.15), we get

$$(c_1 - c_1^{'}) + \frac{1}{\epsilon}e^{\alpha_i}(\frac{-\epsilon}{a})e^{-\alpha_i} = 0$$

$$(c_1 - c_1^{'}) = \frac{1}{a}. \tag{5.16}$$

Subtracting the equation (5.13) from the equation (5.12), then by using equations

(5.16) and (5.14)

$$(c_1 - c_1^{'}) + (c_2 e^{-\alpha_i+\rho_i} - c_2^{'} e^{-\alpha_i-\rho_i})(\frac{-\epsilon}{a}) = 0$$

$$\frac{1}{a} + (c_2 e^{-\alpha_i+\rho_i} - (c_2 - \frac{1}{\epsilon} e^{\alpha_i}) e^{-\alpha_i-\rho_i})(\frac{-\epsilon}{a}) = 0$$

$$\frac{1}{a} + (c_2 e^{-\alpha_i+\rho_i} - c_2 e^{-\alpha_i-\rho_i} + \frac{1}{\epsilon} e^{\alpha_i} e^{-\alpha_i-\rho_i})(\frac{-\epsilon}{a}) = 0$$

$$e^{-\alpha_i} c_2 (e^{\rho_i} - e^{-\rho_i}) + \frac{1}{\epsilon} e^{-\rho_i} = (\frac{-1}{a})(\frac{-a}{\epsilon})$$

$$e^{-\alpha_i} c_2 (e^{\rho_i} - e^{-\rho_i}) = \frac{1}{\epsilon} - \frac{1}{\epsilon} e^{-\rho_i}. \tag{5.17}$$

Now, we can solve the equation (5.17) for c_2:

$$c_2 = \frac{e^{\alpha_i}}{\epsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})}. \tag{5.18}$$

To find $c_2^{'}$, we plug c_2 into the equation (5.14)

$$c_2^{'} = \frac{e^{\alpha_i}}{\epsilon} (\frac{1 - e^{-\rho_i}}{e^{\rho_i} - e^{-\rho_i}} - 1)$$

$$c_2^{'} = \frac{e^{\alpha_i}}{\epsilon} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \tag{5.19}$$

Plug c_2 into the equation (5.12)

$$c_1 + \frac{e^{\alpha_i}}{\epsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} (\frac{-\epsilon}{a}) e^{-\alpha_i+\rho_i} = 0$$

$$c_1 - \frac{1}{a} \frac{e^{\rho_i} - e^{\rho_i-\rho_i}}{(e^{\rho_i} - e^{-\rho_i})} = 0$$

then we have c_1 as follows

$$c_1 = \frac{1}{a} \frac{e^{\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})}. \tag{5.20}$$

Plug c_1 into the equation (5.16), we obtain $c_1^{'}$ as

$$c_1^{'} = \frac{1}{a} (\frac{e^{\rho_i} - 1 - e^{\rho_i} + e^{-\rho_i}}{e^{\rho_i} - e^{-\rho_i}})$$

$$c_1^{'} = \frac{1}{a} \frac{e^{-\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})}. \tag{5.21}$$

Let us impose the equations (5.18)-(5.21), then we can rewrite the equation (5.3) as follows

$$g_i(x^-) = \frac{1}{a}\frac{e^{\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\epsilon}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}(\frac{-\epsilon}{a})e^{\frac{-ax}{\epsilon}} \tag{5.22a}$$

$$g_i(x^+) = \frac{1}{a}\frac{e^{-\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\epsilon}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}(\frac{-\epsilon}{a})e^{\frac{-ax}{\epsilon}}. \tag{5.22b}$$

Taking derivative the equation (5.22),

$$g_i^{'}(x^-) = \frac{-a}{\epsilon}e^{\frac{-ax}{\epsilon}}(\frac{-1}{a})e^{\alpha_i}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} = \frac{1}{\epsilon}e^{\frac{-ax}{\epsilon}}e^{\frac{ax_i}{\epsilon}}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}$$

$$g_i^{'}(x^+) = \frac{-a}{\epsilon}e^{\frac{-ax}{\epsilon}}(\frac{-1}{a})e^{\alpha_i}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} = \frac{1}{\epsilon}e^{\frac{-ax}{\epsilon}}e^{\frac{ax_i}{\epsilon}}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}$$

we easily obtain equation (5.23) as

$$g_i^{'}(x_{i-1}^-) = \frac{1}{\epsilon}e^{\frac{-ax_{i-1}}{\epsilon}+\frac{ax_i}{\epsilon}}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} = \frac{1}{\epsilon}e^{\frac{ah}{\epsilon}}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}$$

$$g_i^{'}(x_{i-1}^-) = \frac{1}{\epsilon}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})} \tag{5.23a}$$

$$g_i^{'}(x_{i+1}^+) = \frac{1}{\epsilon}e^{\frac{-ax_{i+1}}{\epsilon}+\frac{ax_i}{\epsilon}}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} = \frac{1}{\epsilon}e^{\frac{-ah}{\epsilon}}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}$$

$$g_i^{'}(x_{i+1}^+) = \frac{1}{\epsilon}\frac{(e^{-\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}. \tag{5.23b}$$

Now, we can calculate the integral using by g_i^+ and g_i^-

$$f\int_{x_{i-1}}^{x_{i+1}} g_i dx = f[\int_{x_{i-1}}^{x_i} g_i^- dx + \int_{x_i}^{x_{i+1}} g_i^+ dx] \text{ where } \rho_i = \frac{ah}{\epsilon}, \ \alpha_i = \frac{ax_i}{\epsilon}$$

$$= \int_{x_{i-1}}^{x_i}[\frac{1}{a}\frac{e^{\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\epsilon}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}(\frac{-\epsilon}{a})e^{\frac{-ax}{\epsilon}}]dx$$

$$+\int_{x_i}^{x_{i+1}}[\frac{1}{a}\frac{e^{-\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\epsilon}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}(\frac{-\epsilon}{a})e^{\frac{-ax}{\epsilon}}]dx$$

$$= \frac{1}{a}\frac{e^{\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})}x|_{x_{i-1}}^{x_i} + \frac{-e^{\alpha_i}}{a}\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}\frac{e^{\frac{-ax}{\epsilon}}}{\frac{-a}{\epsilon}}|_{x_{i-1}}^{x_i}$$

$$+\frac{1}{a}\frac{e^{-\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})}x|_{x_i}^{x_{i+1}} + \frac{-e^{\alpha_i}}{a}\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}\frac{e^{\frac{-ax}{\epsilon}}}{\frac{-a}{\epsilon}}|_{x_i}^{x_{i+1}}$$

$$= [\frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}] + [\frac{\epsilon}{a^2}\, e^{\alpha_i} e^{\frac{-ax_i}{\epsilon}}\, \frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}(1-e^{\frac{ah}{\epsilon}})]$$

$$+[\frac{h}{a}\frac{(e^{-\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}] + [\frac{\epsilon}{a^2}\, e^{\alpha_i} e^{\frac{-ax_i}{\epsilon}}\, \frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}(e^{\frac{-ah}{\epsilon}}-1)]$$

$$= \frac{h}{a}\frac{(e^{\rho_i}+e^{-\rho_i}-2)}{(e^{\rho_i}-e^{-\rho_i})} + [\frac{\epsilon}{a^2}\, e^{\alpha_i} e^{\frac{-ax_i}{\epsilon}} (\frac{(1-e^{-\rho_i})(1-e^{\rho_i})+(1-e^{\rho_i})(e^{-\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})})]$$

$$= \frac{h}{a}\frac{(e^{\frac{\rho_i}{2}}-e^{\frac{-\rho_i}{2}})^2}{(e^{\frac{\rho_i}{2}}-e^{\frac{-\rho_i}{2}})(e^{\frac{\rho_i}{2}}+e^{\frac{-\rho_i}{2}})} = \frac{h}{a}[\frac{(e^{\frac{\rho_i}{2}}-e^{\frac{-\rho_i}{2}})}{(e^{\frac{\rho_i}{2}}+e^{\frac{-\rho_i}{2}})}\frac{e^{\frac{\rho_i}{2}}}{e^{\frac{\rho_i}{2}}}] = \frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}+1)}.$$

Finally, it can be written as follows

$$f\int_{x_{i-1}}^{x_{i+1}} g_i dx = f\frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}+1)}.$$

This generates the scheme,

$$-\frac{e^{\rho_i}-1}{e^{\rho_i}-e^{-\rho_i}}u_{i-1} + u_i - \frac{1-e^{-\rho_i}}{e^{\rho_i}-e^{-\rho_i}}u_{i+1} = f\;\frac{h}{a}\;\frac{e^{\rho_i}-1}{e^{\rho_i}+1} \tag{5.24}$$

where $\quad \rho_i = \frac{ah}{\epsilon}$.

5.2 Analysis of a Uniformly Convergent Method: the Il'in-Allen-Southwell Method

The following theorem enables us to understand convergence and stability of the Il'in method in the discrete maximum norm.

Theorem 5.2.1. *The Il'in-Allen-Southwell scheme is first-order uniformly convergent in the discrete maximum norm, i.e.,*

$$\max_i \mid u(x_i) - u_i \mid \leq\; Ch.$$

Proof. This is similar to the proof of Theorem 4.2.7; in particular we use again the splitting $u = v + z$, where v is a boundary layer function and the bound on $\mid z^{(j)} \mid$ has a factor ϵ^{1-j}.

First we estimate $\mid z(x_i) - z_i \mid$. For the corresponding consistency error we obtain

$$\begin{aligned}
\mid \tau_i \mid &\leq C \int_{x_{i-1}}^{x_{i+1}} (\epsilon \mid z^{(3)}(t) \mid + a \mid z''(t) \mid) dt \\
&\leq Ch + C\epsilon^{-1} \int_{x_{i-1}}^{x_{i+1}} \exp{(-a_0 \frac{1-t}{\epsilon})} dt \\
&\leq Ch + C \sinh(\frac{a_0 h}{\epsilon}) \exp{(-a_0 \frac{1-x_i}{\epsilon})}.
\end{aligned}$$

An application of the discrete comparison principle gains us (as in the proof of Theorem 4.2.7) a power of ϵ. We now have

$$\mid z(x_i) - z_i \mid \leq Ch + C\epsilon \sinh(\frac{a_0 h}{\epsilon}) \exp{(-a_0 \frac{1-x_i}{\epsilon})} \quad \text{for } i = 1, ..., n.$$

For $\epsilon \leq h$, we immediately obtain $\mid z(x_i) - z_i \mid \leq Ch$. In the case $h \leq \epsilon$, we use the inequality $1 - e^{-t} \leq ct$ for $t > 0$ and again get the desired estimate.

It is more technical to bound $\mid v(x_i) - v_i \mid$. A direct computation gives

$$Lv = -\frac{a(1)}{\epsilon}(a(1) - a(x))v(x)$$

and at the grid points

$$L_h v = -\frac{2a(x) \sinh q(1) \sinh(q(1) - q(x))}{h \sinh q(x)} v(x) \quad \text{where } q(x) = \frac{a(x)h}{2\epsilon}.$$

These equations reflect the fact that when a(x) is constant, the Il'in-Allen-Southwell scheme is exact. Again using the consistency error and a barrier function, some manipulations yield

$$\mid v(x_i) - v_i \mid \leq C \frac{h^2}{h + \epsilon} \leq Ch$$

(see [5]). This completes the proof of Theorem 5.2.1. □

Now we report some numerical results for the convection-diffusion problem about the backward difference and the uniformly convergent methods. Error plots are shown in Figures (5.1), (5.2) and (5.3) for different values of ϵ. The error between the uniformly convergent method and the exact solution is given by the

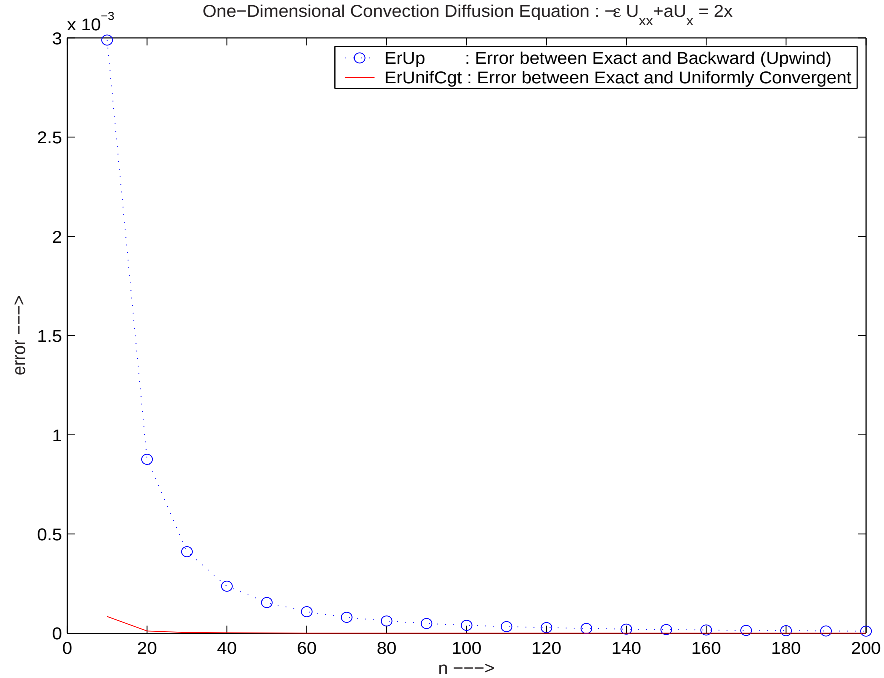

Figure 5.1: The error at the layer for the upwind and the uniformly convergent methods with $a = 1$ and $\epsilon = 1$

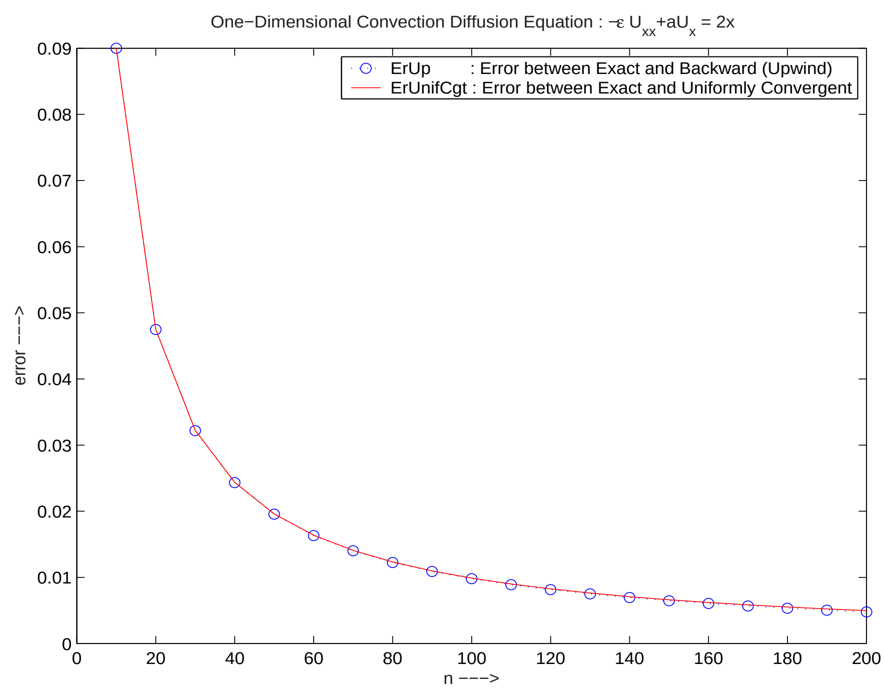

Figure 5.2: The error at the layer for the upwind and the uniformly convergent methods with $a = 1$ and $\epsilon = 0.000001$

solid line and the error between the backward difference method and the exact solution is shown as the curve with the circles.

In Figures (5.1) and (5.2) we see that backward difference method is well-behaved like the uniformly convergent method for smaller and larger values of ϵ. Also, we observe that if stepsize decreases, error decreases and both methods perform well in this values of ϵ.

It is seen that in Figure (5.3), for midrange values of ϵ, even though

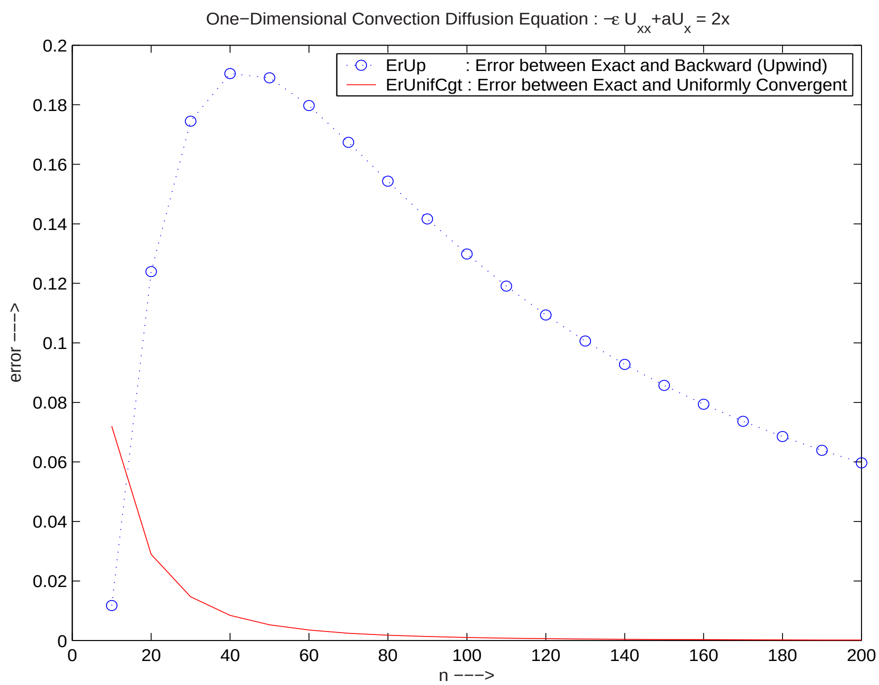

Figure 5.3: The error at the layer for the upwind and the uniformly convergent methods with $a = 1$ and $\epsilon = 0.01$

the uniformly convergent method is well, the behavior of the backward difference method does not get better and in fact becomes worse. Furthermore, for the uniformly convergent method, the error bound decreases when the mesh is refined regardless of the ratio of the parameter h and ϵ.

Figures (5.4)-(5.8) present the exact solution, the backward difference and the uniformly convergent methods are plotted on the same window with $n = 50$, $a = 1$ for different values of ϵ.

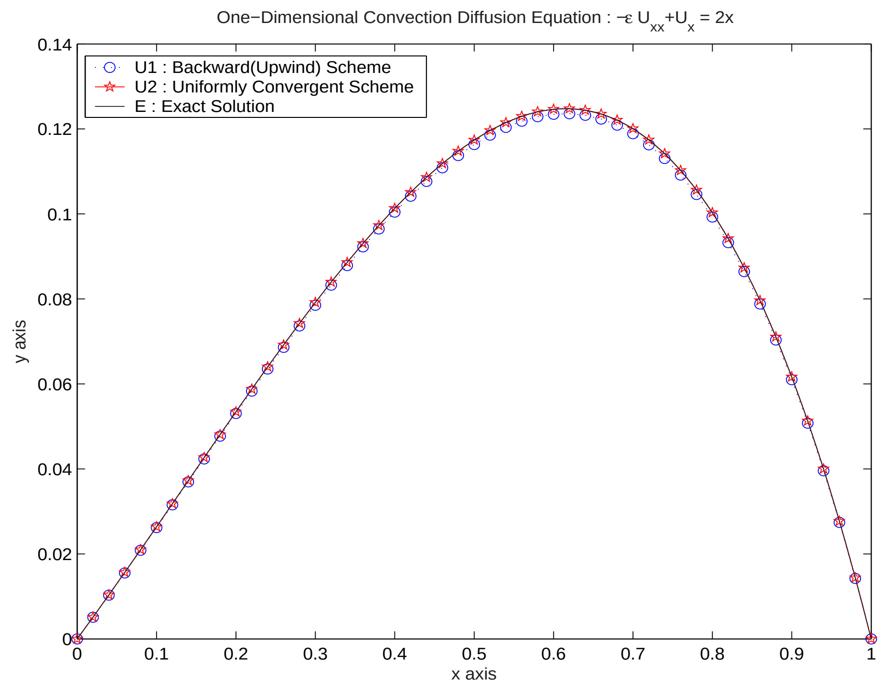

Figure 5.4: The backward difference and the uniformly convergent approximations with $n = 50$, $a = 1$ and $\epsilon = 1$

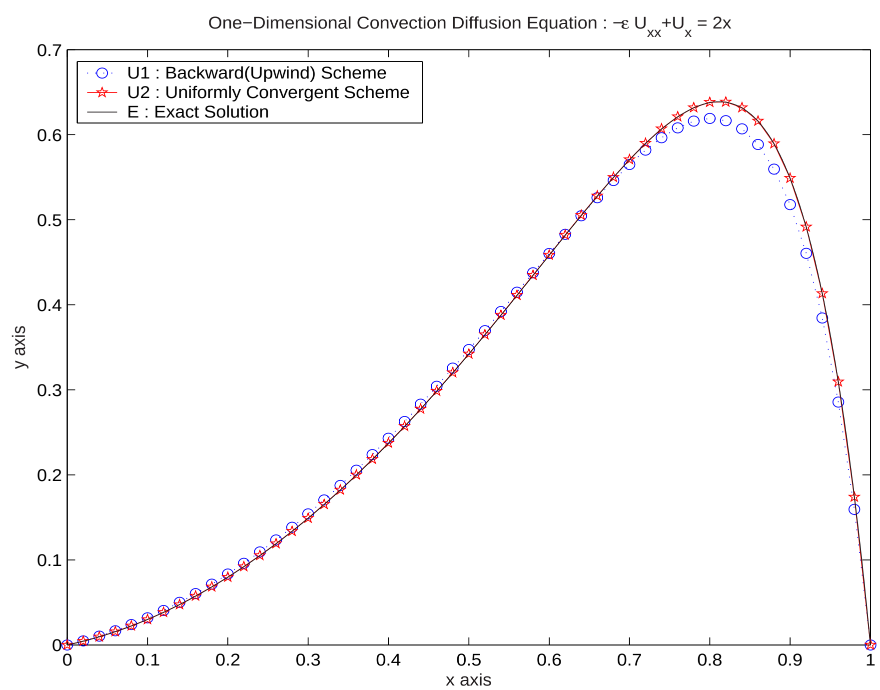

Figure 5.5: The backward difference and the uniformly convergent approximations with $n = 50$, $a = 1$ and $\epsilon = 0.1$

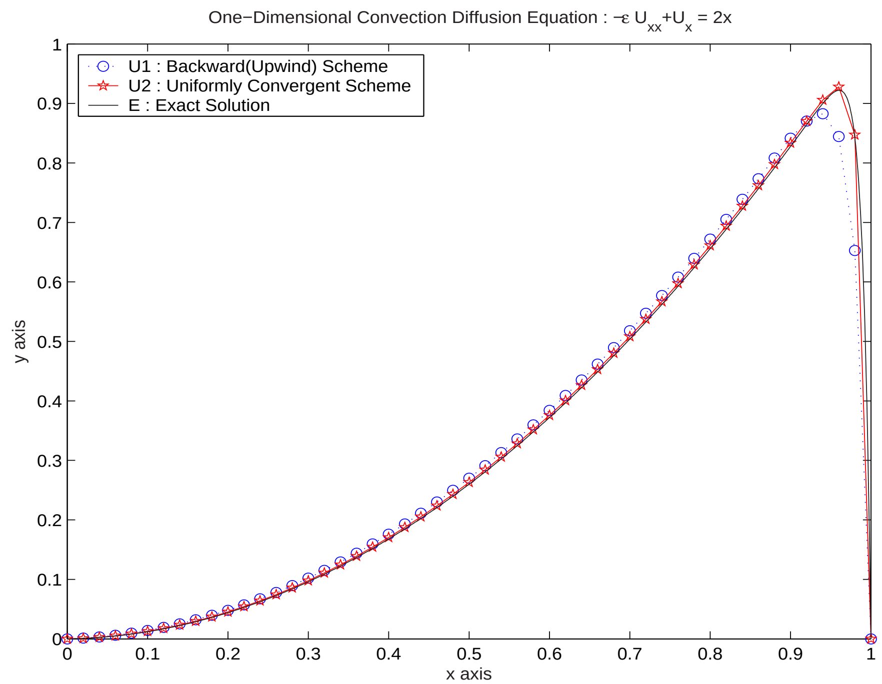

Figure 5.6: The backward difference and the uniformly convergent approximations with $n = 50$, $a = 1$ and $\epsilon = 0.01$

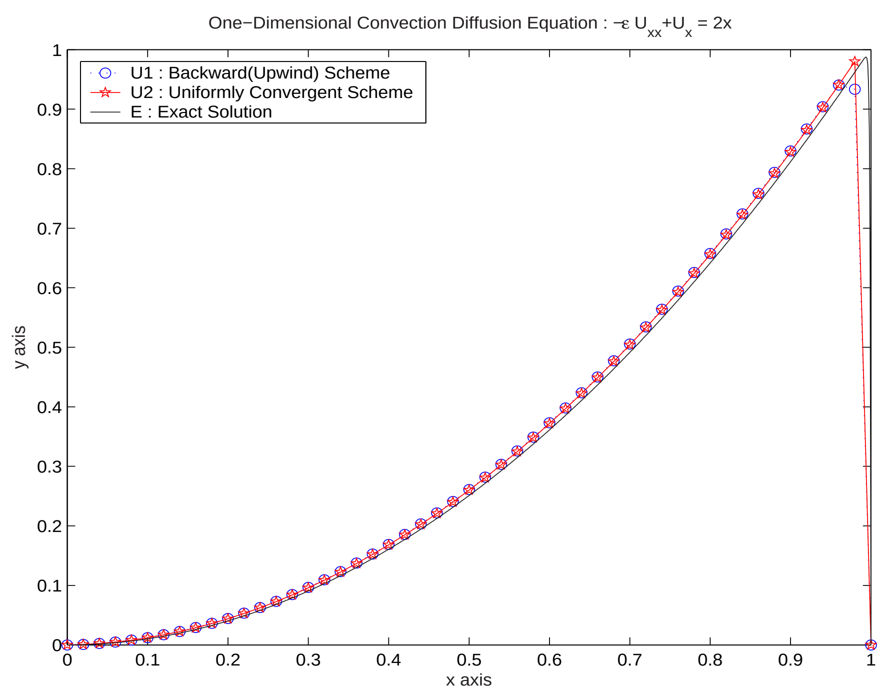

Figure 5.7: The backward difference and the uniformly convergent approximations with $n = 50$, $a = 1$ and $\epsilon = 0.001$

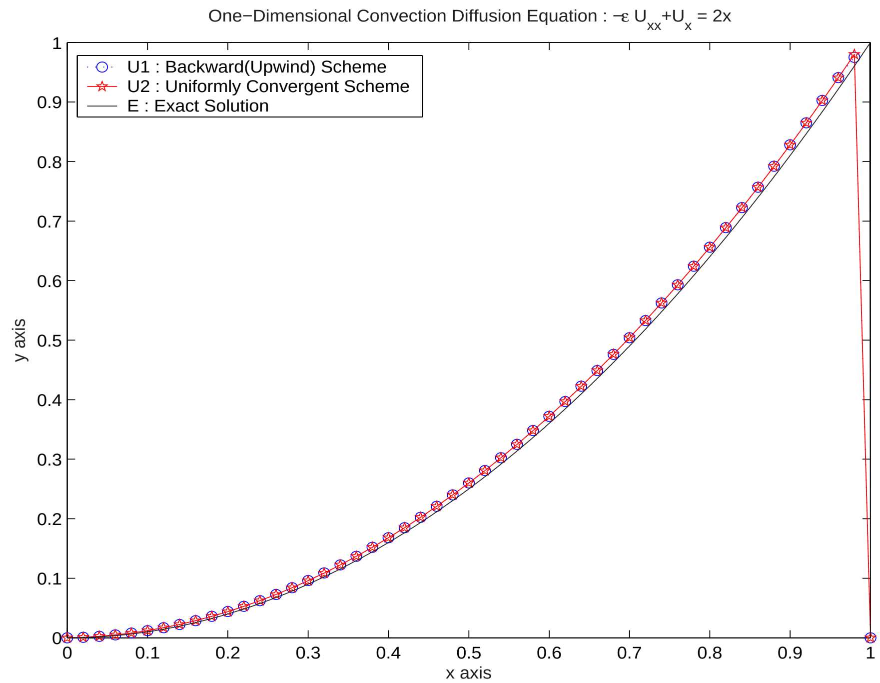

Figure 5.8: The backward difference and the uniformly convergent approximations with $n = 50$, $a = 1$ and $\epsilon = 0.0001$

The uniformly convergent method produces good results and it matches the exact solution for $\epsilon = 1$, and 0.1. However, the solution of the backward difference method is not well-behaved like the solution of the uniformly convergent method, see Figures (5.4) and (5.5). When we decrease the values of ϵ, the uniformly convergent method gives quite good results. However, a similar calculation but using $\epsilon = 0.0001$ shows that the behavior of the backward difference and the uniformly convergent methods are similar in Figure (5.8) at fixed n. Consequently, the uniformly convergent method gives better results than the other methods and the computed and the plotted solutions of this method is well-behaved.

Chapter 6

CONCLUSION

In this thesis, we investigated different finite difference schemes for convection-diffusion problem. We presented analytical behavior of the problem and a short history of the finite difference method and then introduced finite difference operators.

We analyzed centered-difference approximation and we have observed that this method works well for large values of ϵ. However, it fails to approximate for small values of ϵ. Therefore, we have used the backward difference scheme for convection-diffusion equation and then we analyzed it. We saw that the backward difference method produces non-oscillatory results for all values of ϵ. The method is first-order convergent outside the boundary layer, however, it is not convergent in the layer. It is also over diffusive for small values of ϵ. The error, between the exact solution and the backward difference approximation, was simulated and investigated. We have found that the error increases as stepsize h gets smaller for mid-range values of ϵ.

That led us to use a numerical method, a uniformly convergent, called Il'in-Allen-Southwell scheme, with better accuracy throughout the domain for full range of ϵ. We have shown how to construct such a method. The analysis shows that it is first-order uniformly convergent in the discrete maximum norm. Finally, we have written a computer program in MATLAB 6.5 and simulate the method for several cases of interest. We have observed that theoretical findings support the numerical results that we have obtained.

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APPENDIX

Here, the MATLAB codes we wrote to solve the convection diffusion problem is shown.

```
%start program
%Convection Diffusion Equation; -eps(Uxx)+aUx=1 solving for finite
%difference method, using backward and centered  difference schemes.
%Moreover calculate exact solution and plot all of solutions.
%Boundary conditions : U(0)=U(1)=0.
clear all n=input('Enter n : '); a1=input('Enter a : ');
eps=input('Enter epsilon : ');
L=1;        % System size (length) L=b-a
h = L/n;    % Stepsize
%Solution for Backward Difference Scheme :
e=(-eps-a1*h)/(h^2); f=(2*eps+a1*h)/(h^2); g=(-eps/(h^2));
%Solution of linear equation as AU=F; set Matrix A1
for i=1:n-1;
   for j=1:n-1;
      if i==j
         A1(i,j)=f;
      elseif i==j+1
         A1(i,j)=e;
      elseif i==j-1
         A1(i,j)=g;
      else parity=0;
      end
   end
end
%Set Matrix F
for i=1:n-1;
```

```
   for j=1:1;
       F(i,j)=1;
   end
end
%Calculate values of U;
u1=A1\F;
for i=2:n;
       U1(i)=u1(i-1);
end
%Calculate boundary values
U1(1)=0; U1(n+1)=0;
%Solution for Centered Difference Scheme :
x=(-2*eps-a1*h)/(2*(h^2)); y=(2*eps)/(h^2);
z=(-2*eps+a1*h)/(2*(h^2));
%Solution of linear equation as AU=F;
%Set Matrix A2
for i=1:n-1;
   for j=1:n-1;
      if i==j
         A2(i,j)=y;
      elseif i==j+1
         A2(i,j)=x;
      elseif i==j-1
         A2(i,j)=z;
      else parity=0;
      end
   end
end
%Set Matrix F
%Calculate values of U;
u2=A2\F;
for i=2:n;
       U2(i)=u2(i-1);
```

```
end
%Calculate boundary values
U2(1)=0; U2(n+1)=0;
%Exact Solution :
x=[0:.001:1];
E=(x/a1)+(exp((a1*x-a1)/eps)-exp(-a1/eps))/(a1*exp(-a1/eps)-a1);
x1=[0:h:1];
%Set the properties of the plot
plot(x1,U1,'bo:',x1,U2,'r.-',x,E,'k');
%axis([0 1 0 1]);
xlabel('x axis'); ylabel('y axis'); legend('U1 : Backward Scheme
','U2 : Centered Scheme ','E : Exact Solution');
title('One-Dimensional Convection Diffusion Equation :
 -\epsilon U_{xx}+aU_{x} = 1');
%end program

%Start program
%Convection Diffusion Equation; -eps(Uxx)+aUx=2x solving
%for finite difference method, using backward difference schemes.
%Moreover calculate exact solution and uniformly convergent
%solution and plot error. Boundary conditions : U(0)=U(1)=0 , a=1.
clear all eps=input('Enter epsilon : ');
L=1;        % System size (length) L=b-a
count=1; for n=10:10:200;
h = L/n;    % Stepsize
%Solution for Backward(Upwind) Difference Scheme :
a=(-eps-h)/(h^2); b=(2*eps+h)/(h^2); c=(-eps)/(h^2);
%Solution of linear equation as AU=F; set Matrix A1
for i=1:n-1;
   for j=1:n-1;
      if i==j
         A1(i,j)=b;
      elseif i==j+1
```

```
         A1(i,j)=a;
      elseif i==j-1
         A1(i,j)=c;
      else parity=0;
      end
   end
end
%Set Matrix F : x(i)=i*h => f(x(i))=2*(1-x(i))=2*(1-i*h) .
for i=1:n-1;
   for j=1:1;
       F(i,j)=2*i*h;
   end
end
%Calculate values of U;
u1=A1\F;
for i=2:n;
       U1(i)=u1(i-1);
end
%Calculate boundary values
U1(1)=0; U1(n+1)=0;
%Solution for Uniformly Convergent :
x=-(1-exp(-h/eps))/(1-exp(-(2*h)/eps)); y=1;
z=-(exp(-h/eps)-exp(-(2*h)/eps))/(1-exp((-2*h)/eps));
%Solution of linear equation as AU=F;
%Set Matrix A2
for i=1:n-1;
   for j=1:n-1;
      if i==j
         A2(i,j)=y;
      elseif i==j+1
         A2(i,j)=x;
      elseif i==j-1
         A2(i,j)=z;
```

```
      else parity=0;
      end
   end
end
%Set Matrix F :
for i=1:n-1;
   for j=1:1;
       F2(i,j)=2*i*h*h*((1-exp(-h/eps))/(1+exp(-h/eps)));
   end
end
%Calculate values of U;
u2=A2\F2;
for i=2:n;
    U2(i)=u2(i-1);
end
%Calculate boundary values
U2(1)=0; U2(n+1)=0;
%Exact Solution of -ep u'' + u' = 2x;  u(0)=0; u(1)=0;
x=[0:h:1]; E=2*eps*x+(x.^2)+((2*eps+1)*
(exp(-1/eps)-exp((x-1)/eps)))/(1-exp(-1/eps));
%Calculate Difference between exact solution and upwind scheme,
%exact solution and uniformly convergent .
ErUp(count)        = abs(E(n)-U1(n)) ; %exact and upwind
ErUnifCgt(count)   = abs(E(n)-U2(n)) ; %exact and uniformly
count=count+1;
end nn=10:10:200;
%Set the properties of the plot
plot(nn,ErUp,'bo:',nn,ErUnifCgt,'r') xlabel('n'); ylabel('error');
 legend('ErUp        : Error between Exact and Backward (Upwind)',
'ErUnifCgt : Error between Exact and Uniformly Convergent');
 title('One-Dimensional Convection Diffusion Equation:
 -\epsilon U_{xx}+aU_{x} = 2x');
```

```
%x1=[0:h:1];
%Set the properties of the plot
%plot(x1,U1,'ro',x1,U2,'b.:',x,E,'k');
%axis([0 1 -1 2.5]);
%xlabel('x axis');
%ylabel('y axis');
%legend('U1 : Backward(Upwind) Scheme ',
%'U2 : Uniformly Convergent Scheme ','E : Exact Solution');
%title('One-Dimensional Convection Diffusion Equation :
%-\epsilon U_{xx}+U_{x} = 2x');
%end program
```