EXACT SOLUTION OF SOME NONLINEAR DIFFERENTIAL EQUATIONS BY HIROTA METHOD

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ABSTRACT

The Hirota Bilinear Method is applied to construct exact analytical one solitary wave solutions of some class of nonlinear differential equations. First one is the system of multidimensional nonlinear wave equation with the reaction part in form of the third order polynomial determined by three distinct constant vectors. Second one is the mixed diffusion wave equation in one dimension. The bilinear representation is derived by extracting one of the vector roots (unstable in general). This allows us reduce the cubic nonlinearity to a quadratic one. In our approach, the velocity of solitary wave is fixed by truncating the Hirota perturbation expansion and it is found in terms of all three roots. Furthermore, Hirota Bilinear Method is also proposed to solve Brusselator reaction model. The simulations of solutions are illustrated for different polynomial roots and parameters as well.

ÖZET

Lineer olmayan bazi diferansiyel denklemlerin tam analitik tek soliteryen dalga çözümleri Hirota ikili lineer metot uygulanarak elde edildi. İlki reaksiyon kismi üçüncü dereceden polinom formunda, üç farkli sabit vektör ile belirlenen çok boyutlu lineer olmayan dalga denklemi. İkincisi bir boyutlu karma difüzyon dalga denklemi. İkili lineer gösterim vektör köklerinden birinin (genellikle kararsiz olan) alinmasiyla üretilir. Bu bize kübik lineer olmayan yapiyi kuadratik forma indirgemeyi mümkün kilar. Yaklaşimimizda, soliteryen dalgalarin hizi, üç kökün tümüne göre Hirota pertürbe açilimlarinin kesimiyle belirlendi. Ayrica Hirota ikili metot Brusselator reaksiyon modelini çözmek için de kullanildi. Çözümlerin simülasyonlari farkli polinom kökler ve parametreler için örneklendirildi.

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CHAPTER 1

INTRODUCTION

The study of physical phenomena by means of mathematical models is an essential element in both theoretical and applied sciences. Often such models lead to nonlinear systems, and, in a surprisingly large number of cases, to certain prototypical equations. Solitons are an intrinsically nonlinear phenomena, and their history is intimately connected to the development of the theory of nonlinear wave equations (Ablowitz and Segur 1981).

A soliton is a permanent localized hump and kept its localized form under interactions with other solitons in nonlinear wave. In the following part, we submit certain distinguished features of the solitons:

Integrability: Before the discovery of solitons, mathematicians and people who used math were under the impression that we could not solve nonlinear partial differential equations exactly. However, solitons led us to recognize that through a combination of such diverse subjects as quantum physics and algebraic geometry, we can actually solve some nonlinear equations exactly, which gives us a tremendous "window" into what is possible in nonlinearity.

Nonlinear Superposition: In linear equations there is a simple way to make new solutions from known ones, just multiply them by scalars and add! This is known as superposition. Before the discovery of solitons, there was no analogue of this construction for nonlinear equations, but the way that a 2-soliton solution can be viewed as a combination (though not a simple linear combination) of two 1-soliton solutions leads to a recognition that (at least for soliton equations) there is a nonlinear superposition principle as well. (This is now understood algebraically in terms of the symmetries known as Darboux (Matveev and Salle 1991) and Backlund transformations (Schief and Rogers 2002).

Particle-like behavior: The particle like behavior of solitons (that they are localized and preserved under collisions) leads to a large number of applications for solitons. On the one hand, we hope to be able to use solitons to better understand real elementary

particles. There are macroscopic phenomena, such as internal waves on the ocean, spontaneous transparency, and the behavior of light in fiber optic cable just to name a few, which are know understood in terms of solitons (Griffiths 1987).

Algebraic-Geometry: Perhaps the most surprising thing about solitons is the way they interface the subjects of differential equations and algebraic geometry. This has been used to the benefit of each subject. For example, there are large classes of solutions to soliton equations that we know only because they are provided by classical results in algebraic geometry (Belokolos et al. 1987).

Solitons have practical applications and they are also interesting subjects of theoretical study. Over the past thirty-five years, soliton research has been conducted in fields as diverse as particle physics, molecular biology, quantum mechanics, geology, meteorology, oceanography, astrophysics and cosmology. But the area of soliton research which is technologically most significant is currently the study of solitons in optical fibers, where the sought-after goal is to use soliton pulses as the information carrying "bits" in optical fibers. Optical communications is a wonderful example of how the interplay between mathematics and more applied research has generated significant technological advances: Historically, soliton research and optical communications started as two separate fields. In the last twenty years, however, technological advances in conventional transmission systems have often inspired parallel developments in soliton systems and vice versa, up to the point where the two transmission techniques have become so similar that they can be considered just as two different realizations of the same type of system the nonlinear optical transmission line (Mu et al. 2000).

The first observation of a solitary wave occurred in August 1834, when Scottish engineer Scott Russell described a large "wave of translation," propagating for miles in a narrow barge canal (Russell 1872). Russell's discovery generated a lively controversy as to whether the inviscid equations of water waves would possess such solutions.

Several people contributed to the effort of trying to understand the phenomenon, including Airy, (Airy 1845), Boussinesq, (Boussinesq 1872) and Korteweg and de Vries KdV , who in 1895 derived and studied a nonlinear wave equation- which now bears their name as a model for the evolution of long waves in a shallow one-dimensional water channel (Korteweg and Vries 1895).

The controversy was then resolved when it was shown that the KdV equation ad-

mits solitary wave solutions. It would be almost seventy years before these findings would be taken any further. In the meantime, other problems appeared which would eventually be connected to Russell's discovery. In 1847 Stokes had obtained a series solution to the water wave equations, describing a periodic nonlinear wave train in deep water (Stokes 1847). The series was formally proven to be convergent by Levi Civita in 1926 (Civita 1926). However, Benjamin and Feir showed in 1967 that the Stokes wave and in fact any uniform wave train in sufficiently deep water is modulationally unstable (Benjamin et al. 1967).

Meanwhile, in 1965 Fermi, Pasta and Ulam numerically integrated the ordinary differential equations which describe a set of coupled anharmonic oscillators-now called the Fermi-Pasta-Ulam FPU problem in order to understand the mechanisms which lead to thermal equilibrium. The equipartition of energy suggested that the nonlinearity would quickly cause the energy contained in the initial condition to be redistributed among all the modes, thus driving the system to equilibrium. However, to their great surprise, they found that only a very small number of modes were actually participating in the dynamics, and that the energy flowed back and forth among them in a recurrent pattern (Fermi et al. 1974).

The explanation of all these effects came with a break- through in our understanding of nonlinear phenomena. In 1965 Zabusky and Kruskal, while studying the FPU problem, re-derived the KdV equation as a continuum approximation, and they discovered through numerical simulations that the solitary wave solutions to the KdV equation had a property which was previously unknown: namely, the fact that these solutions interacted "elastically" with each other (Zabusky and Kruskal 1965) . Because of this particle-like property, they termed these solutions solitons.

A solutions of solitonic nonlinear differential equations are obtained by using the number of techniques. One of the most popular classical technique is the inverse scattering method. Gardner, Greene, Kruskal and Miura developed into a new method of mathematical physics by solving the initial value problem for the KdV equation using the ideas of inverse scattering method (Gardner et al. 1967). One year later, Lax generalized these ideas, (Lax 1968) and in 1971 Zakharov and Shabat showed that the method worked for another physically significant nonlinear evolution equation, the nonlinear Schrödinger, NLS, equation (Zakharov and Shabat 1972) which is the underlying mechanism for the

Benjamin-Feir instability in water waves. Then in 1974 Ablowitz, Kaup, Newell and Segur showed how those techniques could be used to solve a wide class of nonlinear evolution equations. They also showed that these methods are the analog of the Fourier transform for nonlinear problems (Ablowitz et al. 1974). They called the procedure the inverse scattering transform IST . This method is very powerful, but at the same time it is most complicated and needs information about analytic behavior of scattering data. In addition it is applicable only to integrable systems.

The second most popular classical method is the Lie method and its generalizations, as nonclassical method of group-invariant solutions,direct method for finding symmetry reductions, etc. (Clarkson and Mansfield 1993), based on using the Lie symmetry of a given PDE in order to construct exact solution. The Lie method is not efficient for equations which have poor Lie symmetry and become cumbersome (Cherniha 1998). The other method, based on travelling wave ansatz (Khater et al. 2002), reduces nonlinear PDE to a nonlinear ODE. But in ODE system, the travelling wave speed is an unknown parameter that must be fixed by the analysis and choosing a special trial trajectory (Scott 1999). It appears that if resulting ODE is of the Painleve type, then it can be solved explicitly, leading to exact solution of the original equation. This way Ablowitz and Zeppetella (Ablowitz and Zeppetella 1979) obtained an exact travelling wave solution of Fisher's equation by finding the special wave speed for which the resulting ODE is of the Painleve type. However, the method seems does not allow extensions to more than one travelling waves, higher space dimensions and multi-component order parameter. In the case of a multi-species (phases, cells, reactants, populations, and so on) system, where reaction term has several steady states, one should expect similar travelling wave solution that join steady states. The variety of spatial wave phenomena in multi-species reaction-diffusion system and in space for analysis of travelling wavefront solutions increases, such multi-species solutions are more difficult to determine analytically.

From another site, during the last 30 years the direct method proposed by Hirota becomes a powerful tool for constructing multisoliton solutions to integrable nonlinear evolution equations (Hirota 1973). This, relatively simple and algebraic rather than analytic method, a loss one to avoid many analytic difficulties of more sophisticated the inverse scattering method. The general idea of the method is first to transform the nonlinear equation under consideration into a bilinear equation or system of equations, and

then use the formal power series expansion to solve it. For integrable systems the series admit exact truncation for an arbitrary number of solitons. While for periodic solutions it includes an infinite number of terms. The truncation of Hirota's perturbation series for non-integrable case, similarly to the Painleve reduction (Ablowitz and Zeppetella 1979), fixes the velocity of soliton.

The properties of Hirota Method are summarized in the next section.

1.1 Some Properties of Hirota Bilinear Operators

In this section we give some properties of the Hirota derivative operators D_t, D_x (Hirota 1980), which are convenient to introduce as a various products of them are defined by following equation:

$$D_t^n D_x^m (f \cdot g) = (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m f(x,t) g(x',t')|_{x=x', t=t'} \qquad (1.1)$$

There are some first few derivatives of above equation are given explicitly as follows:

$$D_x(f \cdot g) = f'g - g'f$$

$$D_x^2(f \cdot g) = f''g - 2f'g' + g''f$$

$$D_x^3(f \cdot g) = f'''g - 3f''g' + 3g''f' - g'''f$$

$$D_x^4(f \cdot g) = f^{IV}g - 4f'''g' + 6f''g'' - 4f'g''' + fg^{IV}$$

$$\dots$$

The following properties are easily seen from the definition

1. $D_x^m(f \cdot 1) = (\frac{\partial}{\partial x})^m f$

2. $D_x^m(f \cdot g) = (-1)^m D_x^m(g \cdot f)$

3. $D_x^m(f \cdot f) = 0$ for odd m

4. $D_x^m(f \cdot g) = D_x^{m-1}(f_x \cdot g - f \cdot g_x)$

5. $D_x^m(f \cdot f) = 2D_x^{m-1}(f_x \cdot f)$ for even m

6. $D_x D_t(f \cdot f) = 2D_x(f_t \cdot f) = 2D_t(f_x \cdot f)$

7. $D_x^m exp(p_1 x) \cdot exp(p_2 x) = (p_1 - p_2)^m exp[(p_1 + p_2)x]$

8. $exp(\epsilon D_x)(f(x) \cdot g(x)) = f(x + \epsilon)g(x - \epsilon)$

9. $exp(\epsilon D_x)(fg \cdot hl) = [exp(\epsilon D_x)(f \cdot h)][exp(\epsilon D_x)(g \cdot l)]$

10. $D_x(fg \cdot h) = (\frac{\partial f}{\partial x})gh + f(D_x(g \cdot h)$

11. $D_x^2(fg \cdot h) = (\frac{\partial^2 f}{\partial x^2})gh + 2(\frac{\partial f}{\partial x})D_x(g \cdot h) + fD_x^2(g \cdot h)$

12. $D_x^m exp(px)f \cdot exp(px)g = exp(2px)D_x^m f \cdot g$

The following equalities are useful for transforming nonlinear differential equations into the bilinear forms.

13. $exp(\epsilon \frac{\partial}{\partial x})[f/g] = [exp(\epsilon D_x)f \cdot g]/[\cosh(\epsilon D_x)g \cdot g]$

14. $\frac{\partial}{\partial x}(\frac{f}{g}) = \frac{D_x(f \cdot g)}{g^2}$

15. $\frac{\partial^2}{\partial x^2}\left(\frac{f}{g}\right) = \frac{D_x^2(f \cdot g)}{g^2} - \left(\frac{f}{g}\right)\frac{D_x^2(f \cdot g)}{g^2}$

16. $\frac{\partial^3}{\partial x^3}\left(\frac{f}{g}\right) = \frac{D_x^3(f \cdot g)}{g^2} - 3\left[\frac{D_x^2(f \cdot g)}{g^2}\frac{D_x^2(f \cdot f)}{g^2}\right]$

17. $2\cosh(\epsilon\frac{\partial}{\partial x})\log f = \log[\cosh(\epsilon D_x)f \cdot f]$

18. $\frac{\partial^2}{\partial x^2}(\log f) = \frac{D_x^2(f \cdot f)}{2f^2}$

19. $\frac{\partial^4}{\partial x^4}(\log f) = \frac{D_x^4(f \cdot f)}{2f^2} - 6\left[\frac{D_x^2(f \cdot f)}{2f^2}\right]^2$

1.2   Outline

The idea to use Hirota method for solution of nonlinear evolution equations including vector shock solitons equations with three distinct roots belongs to Pashaev and Tanoğlu (Pashaev and Tanoğlu 2005a) and (Tanoğlu and Pashaev 2005b). Here following main strategy of their paper we extend analysis to some more general equations.

In Chapter 2, we demonstrate effectiveness of the Hirota method to construct one shock soliton solution for the n-component multidimensional nonlinear wave equation. By using Hirota ansatz first, the equation is transformed into the bilinear system of n+1 differential equations, then we solve them exactly using Hirota's perturbation approach.

In Chapter 3, we apply the Hirota bilinear method to find the exact one soliton solution of the nonlinear mixed-diffusion wave equation. We show that if the background root of our ansatz is the mean value of the other two roots, the velocity parameter is static. Then solutions' simulations which confirm theoretical findings are illustrated by varying the equation's parameters.

In Chapter 4, we study the Brussaletor reaction model. By the Hirota method we construct exact one soliton solution for this model. Furthermore, certain simulations related to the model are presented by varying the model's parameters.

Finally, we discuss the effectiveness of the Hirota bilinear method for finding one soliton solutions of the different type of nonlinear differential equations.

CHAPTER 2

VECTOR SHOCK SOLITON SOLUTION

In this chapter we demonstrate the application of the Hirota method for constructing shock soliton solution of n-component and three space dimensions of following partial differential equation. We consider the dependent variable U which consists of n component vectors $\mathbf{U}(x,y,z) = (U^1(x,y,z), U^2(x,y,z), ..., U^n(x,y,z))$.

$$\alpha \frac{\partial^2 \mathbf{U}}{\partial t^2} + \beta \frac{\partial \mathbf{U}}{\partial t} + \gamma \frac{\partial \mathbf{U}}{\partial x} + \tau \frac{\partial \mathbf{U}}{\partial y} + \lambda \frac{\partial \mathbf{U}}{\partial z} = D\nabla^2 \mathbf{U} - (\mathbf{U} - \mathbf{a}_1, \mathbf{U} - \mathbf{a}_2)(\mathbf{U} - \mathbf{a}_3) \quad (2.1)$$

where $\mathbf{a_j} = (a_j{}^1, a_j{}^2, ..., a_j{}^n) \in R^n$, $(j = 1, 2, 3)$, are three distinct constant vectors, and $(\mathbf{U} - \mathbf{a_1}, \mathbf{U} - \mathbf{a_2}) \equiv \sum_{i=1}^{n}(U^i - a_1^i)(U^i - a_2^i)$ means the Euclidean scalar product of vectors $\mathbf{U} - \mathbf{a_1}$ and $\mathbf{U} - \mathbf{a_2}$ and $\nabla^2 = \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator. When $n = 1$ in the one space dimension the model reduces to the well known nonlinear equations appearing in a different fields sometimes with different names: for $\alpha = \gamma = \tau = \lambda = 0$ and $\beta = D = 1$ equation (2.1) denotes that Fitzhugh-Nagumo equation with ($a_1 = 0, a_2 = 1, a_3 = a$) arising in population genetics and models the transmission of nerve impulse, autocatalytic chemical reaction model introduced by Schlögl, generalized Fisher equation, Newell-Whitehead equation or or Kolmogorov-Petrovsky-Piscounov equation with ($a_1 = 0, a_2 = 1, a_3 = -1$), Huxley equation with ($a_1 = 0, a_2 = 0, a_3 = 1$). For $\alpha = \gamma = \tau = \lambda = 0$ the equation has been solved by Pashaev and Tanoğlu (Pashaev and Tanoğlu 2005a).

In our work, the velocity of soliton is fixed by truncation the Hirota perturbation expansion and it is found in terms of all three roots. Shock soliton for extensions of the model, by including the first and second order time derivatives with space variables are derived.

2.1 Solution of the Equation

To reduce equation (2.1) with cubic nonlinearity and distinct roots $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, to the bilinear form, we have to modify the standard Hirota anzast by exracting one of the

vector roots (Pashaev and Tanoğlu, 2005a). The solution of the problem is assumed to have a form

$$\mathbf{U} = \mathbf{a}_3 + \frac{\mathbf{g}}{f}, \tag{2.2}$$

where $\mathbf{g}(\mathbf{x}, t)$ is a n-component vector function and $f(\mathbf{x}, t)$ is a real function. In order to obtain bilinear system of equations, all derivatives with respect to the dependent variables in equation (2.1) are expressed as

$$\mathbf{U}_{tt} = \frac{D_t^2(\mathbf{g} \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_t(f \cdot f)}{f^2}, \tag{2.3a}$$

$$\mathbf{U}_t = \frac{D_t(\mathbf{g} \cdot f)}{f^2}, \tag{2.3b}$$

$$\mathbf{U}_x = \frac{D_x(\mathbf{g} \cdot f)}{f^2} \tag{2.3c}$$

$$\mathbf{U}_y = \frac{D_y(\mathbf{g} \cdot f)}{f^2}, \tag{2.3d}$$

$$\mathbf{U}_z = \frac{D_z(\mathbf{g} \cdot f)}{f^2}, \tag{2.3e}$$

$$\nabla^2\mathbf{U} = \frac{D_x^2(\mathbf{g} \cdot f)}{f^2} + \frac{D_y^2(\mathbf{g} \cdot f)}{f^2} + \frac{D_z^2(\mathbf{g} \cdot f)}{f^2}$$

$$- \frac{\mathbf{g}}{f}\frac{D_x^2(f \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_y^2(f \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_z^2(f \cdot f)}{f^2},$$

where the Hirota derivative according to x_i is defined as

$$D_{x_i}^n(a \cdot b) = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i'}}\right)^n (a(\mathbf{x})b(\mathbf{x'}))|_{\mathbf{x}=\mathbf{x'}}. \tag{2.4}$$

After substituting equations (2.3) into equation (2.1), the following expression is obtained :

$$\alpha[\frac{D_t^2(\mathbf{g} \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_t(f \cdot f)}{f^2}] + \beta[\frac{D_t(\mathbf{g} \cdot f)}{f^2}] + \gamma[\frac{D_x(\mathbf{g} \cdot f)}{f^2}] + \tau[\frac{D_y(\mathbf{g} \cdot f)}{f^2}] + \lambda[\frac{D_z(\mathbf{g} \cdot f)}{f^2}]$$

$$-D[\frac{D_x^2(\mathbf{g} \cdot f)}{f^2} + \frac{D_y^2(\mathbf{g} \cdot f)}{f^2} + \frac{D_z^2(\mathbf{g} \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_x^2(f \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_y^2(f \cdot f)}{f^2} - \frac{\mathbf{g}}{f}\frac{D_z^2(f \cdot f)}{f^2}]$$

$$+\frac{\mathbf{g}}{f}((\frac{\mathbf{g}}{f} - (\mathbf{a}_1 - \mathbf{a}_3))(\frac{\mathbf{g}}{f} - (\mathbf{a}_2 - \mathbf{a}_3))) = 0$$

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Since in equation (2.3) instead of one n-dimensional vector function $\mathbf{U}$ we introduced n-dimensional vector function $\mathbf{g}$ and one more scalar function f, we have freedom to decouple the above system of n-equations as the bilinear system of n+1 equations

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g} \cdot f) = 0, \tag{2.5a}$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(f \cdot f) = -(\mathbf{g}, \mathbf{g}) + ((\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2), \mathbf{g})f \tag{2.5b}$$

where $\tilde{\mathbf{a}}_1 \equiv \mathbf{a}_1 - \mathbf{a}_3$ and $\tilde{\mathbf{a}}_2 \equiv \mathbf{a}_2 - \mathbf{a}_3$. To solve this system in the Hirota method, the functions f and vector function $\mathbf{g}$ suppose to have form of the formal perturbation series in a parameter ϵ

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \ldots, \tag{2.6a}$$

$$\mathbf{g} = \mathbf{g}_0 + \epsilon \mathbf{g}_1 + \epsilon^2 \mathbf{g}_2 + \ldots \tag{2.6b}$$

Substituting (2.6) into the system (2.5) and equating coefficients of the same powers of ϵ converts (2.5) into a sequence of the zeroth, first, second order and higher order, bilinear equations are found.

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_0 \cdot 1) = 0, \tag{2.7a}$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(1 \cdot 1) = -(\mathbf{g}_0, \mathbf{g}_0) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0), \tag{2.7b}$$

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_0 \cdot f_1 + \mathbf{g}_1 \cdot 1) = 0, \tag{2.8a}$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(2 \cdot f_1) = -2(\mathbf{g}_0, \mathbf{g}_1) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0 f_1 + \mathbf{g}_1), \tag{2.8b}$$

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_0 \cdot f_2 + \mathbf{g}_1 \cdot f_1 + \mathbf{g}_2 \cdot 1) = 0,$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(2 \cdot f_2 + f_1 \cdot f_1) = -2(\mathbf{g}_0, \mathbf{g}_2) - (\mathbf{g}_1, \mathbf{g}_1) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0 f_2 + \mathbf{g}_1 f_1 + \mathbf{g}_2)$$

:.

2.1.1 The Zeroth Order Calculation

We assume that all components of $\mathbf{g}_0$ are constants, then the first equation in (2.7) is satisfied automatically. From the second equation we get

$$(\mathbf{g}_0, \mathbf{g}_0) - (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0) + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2) = (\mathbf{g}_0 - \tilde{\mathbf{a}}_1, \mathbf{g}_0 - \tilde{\mathbf{a}}_2) = 0. \tag{2.9}$$

Solutions of the equation (2.9) can be found as:

$$\mathbf{g}_0 = \tilde{\mathbf{a}}_1, \quad \mathbf{g}_0 = \tilde{\mathbf{a}}_2 \tag{2.10a}$$

$$(\mathbf{g}_0 - \tilde{\mathbf{a}}_1) \perp (\mathbf{g}_0 - \tilde{\mathbf{a}}_2). \tag{2.11a}$$

2.1.2 The First Order Calculation

In this subsection we find the first order solutions, namely, $\mathbf{g}_1$, $\mathbf{f}_1$. For simplicity we assume that $\mathbf{g}_0 = \tilde{\mathbf{a}}_1$. As a next step we are going to find the first order solutions, $\mathbf{g}_1$ and f_1. Then equations (2.8) may be rewritten as a linear system

$$\tilde{\mathbf{a}}_1(\alpha\partial_t^2 - \beta\partial_t - \gamma\partial_x - \tau\partial_y - \lambda\partial_z - D\Delta)f_1 + (\alpha\partial_t^2 + \beta\partial_t + \gamma\partial_x + \tau\partial_y + \lambda\partial_z - D\Delta)\mathbf{g}_1 = 0$$

$$2D\Delta f_1 - 2\alpha\partial_t^2 f_1 + 2(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)f_1 + 2(\tilde{\mathbf{a}}_1, \mathbf{g}_1) - (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \tilde{\mathbf{a}}_1 f_1 + \mathbf{g}_1) = 0.$$

where Δ denotes three dimensional Laplace operator. The simplest nontrivial solution of this system has the form

$$\mathbf{g}_1 = \mathbf{a}\, e^{\eta_1}, \qquad f_1 = b\, e^{\eta_1} \tag{2.12}$$

where $\mathbf{a} = (a^1, a^2, ..., a^n)$ and b are n+1 constants, and $\eta_1 = \mathbf{kx} + \omega t + \delta_0$. Here $\mathbf{kx} = k_x x + k_y y + k_z z$ means the three dimensional scalar product. Unknown constants, the wave vector $\mathbf{k}$ and frequency ω are fixed by a dispersion relation, while n constants $(a^1, a^2, ..., a^n)$ are fixed by equations (2.12).

After substituting equations (2.12) into the system of second order sequence for variables $(a^1, a^2, ..., a^n)$ and b we have $(n + 1) \times (n + 1)$ homogeneous linear algebraic

system:

$$A\xi = \begin{bmatrix} F(\omega, \mathbf{k}) & 0 & 0 & . & \tilde{a}_1{}^1 \tilde{F}(\omega, \mathbf{k}) \\ 0 & F(\omega, \mathbf{k}) & 0 & . & \tilde{a}_1{}^2 \tilde{F}(\omega, \mathbf{k}) \\ . & . & . & . & . \\ 0 & 0 & . & F(\omega, \mathbf{k}) & \tilde{a}_1{}^n \tilde{F}(\omega, \mathbf{k}) \\ (\tilde{a}_1{}^1 - \tilde{a}_2{}^1) & . & . & (\tilde{a}_1{}^n - \tilde{a}_2{}^n) & G(\omega, \mathbf{k}) \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

where $F(\omega, \mathbf{k})$, $\tilde{F}(\omega, \mathbf{k})$, $G(\omega, \mathbf{k})$ and $\vec{\chi}$ are expressed as follows;

$$F(\omega, \mathbf{k}) \;=\; \alpha \omega^2 + \beta \omega + (\mathbf{k} \cdot \vec{\chi}) - D\mathbf{k}^2, \tag{2.13a}$$

$$\tilde{F}(\omega, \mathbf{k}) \;=\; \alpha \omega^2 - \beta \omega - (\mathbf{k} \cdot \vec{\chi}) - D\mathbf{k}^2, \tag{2.13b}$$

$$G(\omega, \mathbf{k}) \;=\; 2(D\mathbf{k}^2 - \alpha \omega^2) + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1), \tag{2.13c}$$

$$\vec{\chi} \;=\; (\gamma, \tau, \lambda). \tag{2.13d}$$

Nontrivial solution of this system of equations exists only if $Det(A) = 0$. This determinant can be evaluated by expansion along the last row so that we have expression

$$Det(A) = F(\omega, \mathbf{k})^{n-1}[F(\omega, \mathbf{k})G(\omega, \mathbf{k}) - \tilde{F}(\omega, \mathbf{k}) \sum_{i=1}^{n} \tilde{\mathbf{a}}_1^i (\tilde{\mathbf{a}}_1 - \tilde{\mathbf{a}}_2)^i]. \tag{2.14}$$

This gives us the following dispersion relations:

$$\alpha \omega^2 + \beta \omega + (\mathbf{k} \cdot \vec{\chi}) - D\mathbf{k}^2 = 0 \tag{2.15}$$

and

$$\alpha \omega^2 + \beta \omega + (\mathbf{k} \cdot \vec{\chi}) - D\mathbf{k}^2 - (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1) = 0. \tag{2.16}$$

The dispersion relation (2.15) gives us a constant solution, thus we use dispersion (2.16) for the further calculations. Solving the first n equations in the above $(n+1) \times (n+1)$ linear algebraic system of equations we have :

$$\mathbf{a} = b\nu_1 \, \tilde{\mathbf{a}}_1 \tag{2.17}$$

where

$$\nu_1 = \frac{2(D\mathbf{k}^2 - \alpha \omega^2) - (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)}{(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)} \tag{2.18}$$

or combining this with equation (2.16), we find the other representation for ν_1

$$\nu_1 = \frac{2(\beta\omega + (\mathbf{k} \cdot \overrightarrow{\chi})) - (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)}{(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)}. \qquad (2.19)$$

Thus, we have

$$\mathbf{g}_1 = b\nu_1\,\tilde{\mathbf{a}}_1 e^{\eta_1}, \quad f_1 = be^{\eta_1}. \qquad (2.20)$$

2.1.3   The Second Order Calculation

In this subsection, we show that we do not need to make further calculations for the order of ϵ, greater than 2, by assuming $\mathbf{g}_2 = 0$ and $f_2 = 0$, since for those of choices, the terms in the expansion (2.6) will disappear after $\mathrm{O}(\epsilon)$.

Substituting $\mathbf{g}_1$, f_1 to the first equation of second order bilinear equations and using property of bilinear Hirota operators we get

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_1 \cdot f_1) = 0$$

$\tilde{\mathbf{a}}_1(\alpha\partial_t^2 - \beta\partial_t - \gamma\partial_x - \tau\partial_y - \lambda\partial_z - D\Delta)f_2 + (\alpha\partial_t^2 + \beta\partial_t + \gamma\partial_x + \tau\partial_y + \lambda\partial_z - D\Delta)\mathbf{g}_2 = 0.$ Then, from the last equation of second order bilinear equations we find additional constraint on $\mathbf{g}_1$, f_1:

$$(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)f_1^2 = -(\mathbf{g}_1, \mathbf{g}_1) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_1)f_1. \qquad (2.21)$$

Next substituting solution (2.20) into equation (2.21) results in the following relation:

$$(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2) = -(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)\nu_1^2 + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \tilde{\mathbf{a}}_1)\nu_1 \qquad (2.22)$$

which can be written in the form:

$$(\nu_1 - 1)(\tilde{\mathbf{a}}_1, (\nu_1\tilde{\mathbf{a}}_1 - \tilde{\mathbf{a}}_2)) = 0. \qquad (2.23)$$

Assuming $\nu_1 = 1$, leads to the trivial result $\mathbf{k} = 0$. Thus if $\nu_1 \neq 1$ from equation (2.23) the following nontrivial solution can be found :

$$\nu_1 = \frac{(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)}{(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)}. \qquad (2.24)$$

2.1.4 The Explicit One Soliton Solution

Combining the equations (2.18) and (2.19) with equation (2.24) respectively, we find restrictions on allowed values of the length for the wave vector $\mathbf{k}$

$$A_1\mathbf{k}^2 + A_2(\mathbf{k} \cdot \overrightarrow{\chi})^2 + A_3(\mathbf{k} \cdot \overrightarrow{\chi}) + A_4 = 0 \tag{2.25}$$

where

$$A_1 = 8D\beta^2(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)^2$$

$$A_2 = -8\alpha(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)^2$$

$$A_3 = 8\alpha(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)[(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2) + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)](\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)$$

$$A_4 = -4\beta^2(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)[(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1) - (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)]$$

The solution of the equation (2.16) with respect to ω is given by the equation:

$$\omega_{(1,2)} = \frac{-\beta \mp \sqrt{4D\alpha k^2 - 4\alpha(\mathbf{k} \cdot \overrightarrow{\chi}) + \beta^2 + 4\alpha(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2 - \tilde{\mathbf{a}}_1)}}{2\alpha} \tag{2.26}$$

Next, after substituting solution equation (2.25) into equation (2.26), the frequency number can be obtained in an explicit form.

Finally, the velocity vector is given by the following formula:

$$\mathbf{v} = -\omega \frac{\mathbf{k}}{|\mathbf{k}|^2}. \tag{2.27}$$

It is easy to show that each bilinear equation, which has order greater than 2, has simple solution as, $\mathbf{g}_i = \mathbf{0}$ and $f_i = 0$, for $i > 2$. Therefore, we have only finite number of terms in the expansion (2.6). After substituting f and $\mathbf{g}$ in the equation (2.3), we find the following exact solution to our problem

$$\mathbf{U} = \mathbf{a}_3 + (\mathbf{a}_1 - \mathbf{a}_3)\frac{1 + \nu_1 e^{\eta_1}}{1 + e^{\eta_1}} \tag{2.28}$$

where $\nu_1 = (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)/(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_1)$. We note that the constant b appearing only in front of exponential terms can be absorbed by the arbitrary constant λ in equation (2.12) and

leads just to shift of the soliton's origin.In terms of original vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ it acquires the final form

$$\mathbf{U} = \mathbf{a}_3 + \frac{\mathbf{a}_1 - \mathbf{a}_3}{|\mathbf{a}_1 - \mathbf{a}_3|} \frac{((\mathbf{a}_1 - \mathbf{a}_3), (\mathbf{a}_1 - \mathbf{a}_3 + (\mathbf{a}_2 - \mathbf{a}_3)e^{\eta_1}))}{1 + e^{\eta_1}} \qquad (2.29)$$

2.2 Discussion

This solution is the shock-solitary wave with asymptotics $\mathbf{U} \to \mathbf{a}_1$, for $\eta_1 \to -\infty$ and $\mathbf{U} \to \mathbf{a}_3 + (\mathbf{a}_1 - \mathbf{a}_3)\nu_1$, for $\eta_1 \to +\infty$, at a fixed time. As easy to see, these asymptotic solutions are the stationary $\mathbf{U}_t = 0$ homogenous ($\mathbf{U}_x = \mathbf{U}_y = \mathbf{U}_z = 0$) solutions of equation (2.3). The parameter ν_1 in this solution has meaning of ratio $\nu_1 = (|\tilde{\mathbf{a}}_2|/|\tilde{\mathbf{a}}_1|) \cos \alpha$, where $0 \le \alpha \le \pi$ is the angle between vectors $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2$. Since equation (2.12) has infinite number of solutions, our shock soliton interpolates between the "vacuum" solution determined by vector $\mathbf{a}_1$ and the solution with vector $\mathbf{a}_3 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)\tilde{\mathbf{a}}_1/|\tilde{\mathbf{a}}_1|^2$ which is valued in the continuum set.

If in equation (2.9) for $\mathbf{g}_0$ we choose another root $\mathbf{g}_0 = \tilde{\mathbf{a}}_2$, then we have another shock soliton solution of equation (2.3)

$$\mathbf{U} = \mathbf{a}_3 + (\mathbf{a}_2 - \mathbf{a}_3)\frac{1 + \nu_2 e^{\eta_1}}{1 + e^{\eta_1}} \qquad (2.30)$$

where $\nu_2 = (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)/(\tilde{\mathbf{a}}_2, \tilde{\mathbf{a}}_2)$. Solution (2.29) has asymptotic $\mathbf{U} \to \mathbf{a}_2$, for $\eta_1 \to -\infty$ and $\mathbf{U} \to \mathbf{a}_3 + (\mathbf{a}_2 - \mathbf{a}_3)\nu_2$, for $\eta_1 \to +\infty$, at a fixed time. The parameter ν_2 has meaning of the ratio

$\nu_2 = (|\tilde{\mathbf{a}}_1|/|\tilde{\mathbf{a}}_2|) \cos \alpha$, where α as above for the ν_1, is the angle between vectors $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2$.

CHAPTER 3

NONLINEAR MIXED DIFFUSION WAVE EQUATION

In this chapter, the Hirota Bilinear Method is applied to find exact analytical solution of one dimensional nonlinear mixed diffusion-wave equation with nonlinear reaction part in the form of the third order polynomial. It provides a simple approach to construct one soliton solution for non-integrable model. Furthermore, relationships between three simple roots and the wave speed of the soliton and some simulations are given for the different arbitrary of equations' parameters.

We consider following scalar nonlinear wave equation

$$\alpha U_{tt} + \beta U U_t + \gamma U U_x + \tau U_t + \lambda U_x = D U_{xx} - (U - a_1)(U - a_2)(U - a_3) \tag{3.1}$$

where $a_1, a_2, a_3 > 0$ are distinct real numbers and $\alpha, \beta, \gamma, \tau, \lambda$ and D are constant parameters. U is the function of space variable x and time variable t.

3.1 Solution of the Equation

The solution of the equation (3.1) is assumed to have a form

$$U = a_i + \frac{g}{f} \quad \text{where} \quad i = 1, 2, 3. \tag{3.2}$$

In this equation $g(x, t)$ and $f(x, t)$ are real functions. Without loss of generality we can assume that $a_i = a_3$.

In order to obtain bilinear representation of the equation (3.1) all derivatives are expressed in terms of the bilinear operators as follows:

$$U_t = \frac{D_t(g \cdot f)}{f^2}, \tag{3.3a}$$

$$U U_t = \frac{g}{f} \frac{D_t(g \cdot f)}{f^2}, \tag{3.3b}$$

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$$U_x = \frac{D_x(g \cdot f)}{f^2}, \tag{3.3c}$$

$$UU_x = \frac{g}{f}\frac{D_x(g \cdot f)}{f^2}, \tag{3.3d}$$

$$U_{tt} = \frac{D_t^2(g \cdot f)}{f^2} - \frac{g}{f}\frac{D_t(f \cdot f)}{f^2}, \tag{3.3e}$$

$$U_{xx} = \frac{D_x^2(g \cdot f)}{f^2} - \frac{g}{f}\frac{D_x^2(f \cdot f)}{f^2} \tag{3.3f}$$

where the Hirota derivative according to x_i is defined as

$$D_{x_i}^n(a \cdot b) = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i'}}\right)^n (a(x)b(x'))|_{x=x'}. \tag{3.4}$$

After substituting equations (3.3) into equation (3.1), the following expression is obtained

$$\alpha[\frac{D_t^2(g \cdot f)}{f^2} - \frac{g}{f}\frac{D_t(f \cdot f)}{f^2}] + \beta[(a_3 + \frac{g}{f})\frac{D_t(g \cdot f)}{f^2}] + \gamma[(a_3 + \frac{g}{f})\frac{D_x(g \cdot f)}{f^2}]$$

$$+\tau[\frac{D_t(g \cdot f)}{f^2}] + \lambda[\frac{D_x(g \cdot f)}{f^2}] - D[\frac{D_x^2(g \cdot f)}{f^2} - \frac{g}{f}\frac{D_x^2(f \cdot f)}{f^2}]$$

$$+\frac{g}{f}(\frac{g}{f} - (a_1 - a_3))(\frac{g}{f} - (a_2 - a_3)) = 0.$$

Then deduce the following system of bilinear form after collecting the terms which do not consist g/f with $1/f^2$ and the rest of the terms with g/f parentheses in above equation and equating them zero separately.

$$(\alpha D_t^2 - DD_x^2 + \beta a_3 D_t + \gamma a_3 D_x + \tau D_t + \lambda D_x)(g \cdot f) = 0 \tag{3.5a}$$

$$(DD_x^2 - \alpha D_t^2 + \tilde{a}_1 \tilde{a}_2)(f \cdot f) = -g^2 + (\tilde{a}_1 + \tilde{a}_2 - \beta D_t - \gamma D_x)(g \cdot f) \tag{3.5b}$$

where $\tilde{a}_1 \equiv a_1 - a_3$ and $\tilde{a}_2 \equiv a_2 - a_3$. To solve this system in the Hirota method, the functions f and g suppose to have of the formal perturbation series in parameter ϵ.

$$f(x,t) = \sum_{i=0}^{\infty} \epsilon^i f_i(x,t), \qquad g(x,t) = \sum_{i=0}^{\infty} \epsilon^i g_i(x,t) \tag{3.6}$$

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without loss of generality assume that $f_0 = 1$.

Substituting equations (3.6) into the system (3.5) and equating coefficients of the same powers of ϵ converts equations (3.5) into a sequence of the zeroth, first, second order and so one, bilinear equations given as respectively,

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_0 \cdot 1) = 0,$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(1 \cdot 1) = -(\mathbf{g}_0, \mathbf{g}_0) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0),$$

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_0 \cdot f_1 + \mathbf{g}_1 \cdot 1) = 0,$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(2 \cdot f_1) = -2(\mathbf{g}_0, \mathbf{g}_1) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0 f_1 + \mathbf{g}_1),$$

$$(\alpha D_t^2 + \beta D_t + \gamma D_x + \tau D_y + \lambda D_z - D\Delta)(\mathbf{g}_0 \cdot f_2 + \mathbf{g}_1 \cdot f_1 + \mathbf{g}_2 \cdot 1) = 0,$$

$$(D\Delta - \alpha D_t^2 + (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2))(2 \cdot f_2 + f_1 \cdot f_1) = -2(\mathbf{g}_0, \mathbf{g}_2) - (\mathbf{g}_1, \mathbf{g}_1) + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_2, \mathbf{g}_0 f_2 + \mathbf{g}_1 f_1 + \mathbf{g}_2),$$

3.1.1 The Zeroth Order Calculation

We assume that the value of g_0 is constant, then the first equation of zeroth order equations is satisfied automatically. From the second equation we get

$$g_0^2 - (\tilde{a}_1 + \tilde{a}_2)g_0 + \tilde{a}_1\tilde{a}_2 \;\;=\;\; 0 \qquad (3.7)$$

$$(g_0 - \tilde{a}_1)(g_0 - \tilde{a}_2) \;\;=\;\; 0 \qquad (3.8)$$

which is the quadratic equation of g_0. Solution of the second equation can be foud as $g_0 = \tilde{a}_1$ or $g_0 = \tilde{a}_2$. As a next step we are going to find the first order solutions g_1 and f_1.

3.1.2 The First Order Calculation

In this subsection we find the first order real solutions, g_1 and f_1. For simplicty we assume that $g_0 = \tilde{a}_1$ without loss of generality. Then first order sequences may be rewritten by using the definition of the Hirota bilinear operator and reduced to a linear

system

$$\alpha\tilde{a}_1 f_{1tt} - D\tilde{a}_1 f_{1xx} - \beta a_3\tilde{a}_1 f_{1t} - \gamma a_3\tilde{a}_1 f_{1x} - \tau\tilde{a}_1 f_{1t} - \lambda\tilde{a}_1 f_{1x}$$

$$(3.9)$$

$$+\alpha g_{1tt} - D g_{1xx} + \beta a_3 g_{1t} + \gamma a_3 g_{1x} + \tau g_{1t} + \lambda g_{1x} = 0$$

$$2D f_{1xx} - 2\alpha f_{1tt} - \tilde{a}_1\gamma f_{1x} - \tilde{a}_1\beta f_{1t} + \beta g_{1t} + \gamma g_{1x}$$

$$(3.10)$$

$$+2\tilde{a}_1\tilde{a}_2 f_1 + 2\tilde{a}_1 g_1 - (\tilde{a}_1 + \tilde{a}_2)(\tilde{a}_1 f_1 + g_1) = 0.$$

We are interested in finding one soliton solutions, which correspond to the exponential solution of equations (3.9) and (3.10). Then the simplest nontrivial solution of this system has the form

$$g_1 = c_1\, e^{\eta_1}, \qquad f_1 = c_2\, e^{\eta_1} \tag{3.11}$$

where c_1 and c_2 are constants, and $\eta_1 = kx + wt + \delta$. Coefficients k and w are found by a dispersion relation. Then, substituting equations (3.11) into the system equations (3.9) and (3.10) the homogeneous linear algebraic system is constructed for unknown variables c_1 and c_2.

$$\tilde{a}_1[\alpha\omega^2 - Dk^2 - \omega(\beta a_3 + \tau) - k(\gamma a_3 + \lambda]c_1$$

$$(3.12)$$

$$+[\alpha\omega^2 - D\tilde{a}_1 k^2 + \omega(\beta a_3 + \tau) + k(\gamma a_3 + \lambda]c_2 = 0$$

$$[2(Dk^2 - \alpha\omega^2) - \tilde{a}_1(\beta\omega + \gamma k + \tilde{a}_1 - \tilde{a}_2)]c_1 + [\beta\omega + \gamma k + \tilde{a}_1 - \tilde{a}_2]c_2 = 0.$$

Then we compose the matrix with respect to c_1 and c_2 as follows:

$$A\xi = \begin{bmatrix} \psi_1(\omega, k) & \psi_2(\omega, k) \\[2mm] \psi_3(\omega, k) & \psi_4(\omega, k) \end{bmatrix} \begin{bmatrix} c_1 \\[2mm] c_2 \end{bmatrix} = \begin{bmatrix} 0 \\[2mm] 0 \end{bmatrix}$$

where

$$\psi_1(\omega, k) = \tilde{a}_1[\alpha\omega^2 - Dk^2 - \omega(\beta a_3 + \tau) - k(\gamma a_3 + \lambda] \tag{3.13}$$

$$\psi_2(\omega, k) = [\alpha\omega^2 - D\tilde{a}_1 k^2 + \omega(\beta a_3 + \tau) + k(\gamma a_3 + \lambda] \tag{3.14}$$

$$\psi_3(\omega, k) = [2(Dk^2 - \alpha\omega^2) - \tilde{a}_1(\beta\omega + \gamma k + \tilde{a}_1 - \tilde{a}_2)] \tag{3.15}$$

$$\psi_4(\omega, k) = [\beta\omega + \gamma k + \tilde{a}_1 - \tilde{a}_2]. \tag{3.16}$$

Nontrivial solution of this system of equations exists only if $Det(A) = 0$. This gives us the following dispersion relation:

$$\alpha\omega^2 + a_1(\beta\omega + \gamma k) + \lambda k + \omega\tau - Dk^2 + (a_1 - a_3)(a_1 - a_2) = 0. \tag{3.17}$$

3.1.3   The Second Order Calculation

After substituting f_2, g_2 into the first equation of second order equations we get

$$(\alpha D_t^2 - DD_x^2 + \beta a_3 D_t + \gamma a_3 D_x + \tau D_t + \lambda D_x)(\tilde{a}_1 \cdot f_2 + c_1 e^{\eta_1} \cdot c_2 e^{\eta_1} + g_2 \cdot 1) = 0. \tag{3.18}$$

Then using property of bilinear Hirota operators we have

$$c_1 c_2(\alpha D_t^2 - DD_x^2 + \beta a_3 D_t + \gamma a_3 D_x + \tau D_t + \lambda D_x)(e^{\eta_1} \cdot e^{\eta_1}) = 0 \tag{3.19}$$

where we choose f_2, g_2 equal to zero. After substituting $f_2 = 0$ and $g_2 = 0$ into the second equation of second order equations, we can rewritten it as follows

$$(DD_x^2 - \alpha D_t^2 + \tilde{a}_1\tilde{a}_2)(f_1 \cdot f_1) = -g_1^2 + (a_1 + a_2)g_1 f_1 \tag{3.20}$$

then

$$\tilde{a}_1\tilde{a}_2 f_1^2 = -g_1^2 + (\tilde{a}_1 + \tilde{a}_2)(g_1 f_1). \tag{3.21}$$

Next substitute the equation (3.11) into the equation (3.21) to find the arbitrary constant c_1 and c_2. These substitutions result in

$$\tilde{a}_1\tilde{a}_2 c_2^2 = -c_1^2 + (\tilde{a}_1 + \tilde{a}_2)(c_1 c_2). \tag{3.22}$$

The roots of quadratic equation can be found as

$$\frac{c_2}{c_1} = \frac{1}{\tilde{a}_1} \quad \text{or} \quad \frac{c_2}{c_1} = \frac{1}{\tilde{a}_2}. \tag{3.23}$$

For the ratio constants we obtain:

$$\frac{c_2}{c_1} = \frac{-\beta\omega - \gamma k - (a_1 - a_2)}{2\lambda k + 2\omega\tau + (a_1 + a_3)(\beta\omega + \gamma k) + (a_1 - a_3)(a_1 - a_2)}. \tag{3.24}$$

By equating equation (3.24) to $1/(a_2 - a_3)$ we get the relation between ω and k as follows:

$$\omega = \frac{\tilde{a_2}^2 - \tilde{a_1}^2 - \gamma k(a_1 + a_2) - 2\lambda k}{\beta(a_1 + a_2) + 2\tau}. \tag{3.25}$$

Combining (3.25) with dispersion formula (3.17), we find quadratic equation fixing the wave number k:

$$Ak^2 + Bk + C = 0 \tag{3.26}$$

where

$$A = \frac{(a_1 + a_2)\alpha\gamma[4\lambda + (a_1 + a_2)\gamma] + 4\alpha\lambda^2 - [(a_1 + a_2)\beta + 2\tau]^2 D}{[(a_1 + a_2)\beta + 2\tau]^2}$$

$$B = \frac{(a_1\gamma + \lambda)[(a_1 + a_2)\beta + 2\tau]^2}{[(a_1 + a_2)\beta + 2\tau]^2}$$

$$+ \frac{[-2\alpha(\tilde{a_2}^2 - \tilde{a_1}^2) + ((a_1 + a_2)\beta + 2\tau)][(a_1 + a_2)\gamma + 2\lambda]}{[(a_1 + a_2)\beta + 2\tau]^2}$$

$$C = \frac{(a_1 - a_2)(a_1 - a_3)[(a_1 + a_2)\beta + 2\tau]^2}{[(a_1 + a_2)\beta + 2\tau]^2}$$

$$+ \frac{(\tilde{a_2}^2 - \tilde{a_1}^2)[(\tilde{a_2}^2 - \tilde{a_1}^2)\alpha + ([(a_1 + a_2)\beta + 2\tau](a_1\beta + \tau))]}{[(a_1 + a_2)\beta + 2\tau]^2}.$$

Substituting solution of (3.26) into (3.25), we find expression for ω.

The velocity ν is given by the equation

$$\nu = -\frac{\omega}{k}. \tag{3.27}$$

The velocity equation will be used in following section to determine the speed of the solitons.

Finally after putting $\epsilon = 1$ the expansion for f and g has the form

$$f = 1 + \frac{1}{\tilde{a_2}}e^\eta \tag{3.28}$$

$$g = \tilde{a_1} + e^\eta. \tag{3.29}$$

From the simple calculation, it can be seen that the terms in the expansion will be vanished after $O(\epsilon)$.

The solution of our problem becomes:

If we assume the solution $U = a_1 + g/f$ we obtain

$$U = a_1 + \frac{\tilde{\tilde{a}}_2 + e^\eta}{1 + \frac{1}{\tilde{a}_3}e^\eta} \tag{3.30}$$

$$U = a_1 + \frac{(a_2 - a_1) + e^\eta}{(a_3 - a_1) + e^\eta}(a_3 - a_1) \tag{3.31}$$

If we assume the solution $U = a_2 + g/f$ we have

$$U = a_2 + \frac{\tilde{\tilde{a}}_1 + e^\eta}{1 + \frac{1}{\tilde{a}_3}e^\eta} \tag{3.32}$$

$$U = a_2 + \frac{(a_1 - a_2) + e^\eta}{(a_3 - a_2) + e^\eta}(a_3 - a_2). \tag{3.33}$$

If we assume the solution $U = a_3 + g/f$ we get

$$U = a_3 + \frac{\tilde{a}_1 + e^\eta}{1 + \frac{1}{\tilde{a}_2}e^\eta} \tag{3.34}$$

$$U = a_3 + \frac{(a_1 - a_3) + e^\eta}{(a_2 - a_3) + e^\eta}(a_2 - a_3). \tag{3.35}$$

where

$$\tilde{a}_1 = a_1 - a_3 \quad , \quad \tilde{a}_2 = a_2 - a_3 \quad , \quad \tilde{a}_3 = a_3 - a_2 \tag{3.36}$$

$$\tilde{\tilde{a}}_1 = a_1 - a_2 \quad , \quad \tilde{\tilde{a}}_2 = a_2 - a_1 \quad , \quad \tilde{\tilde{a}}_3 = a_3 - a_1. \tag{3.37}$$

3.2   Simulations

In this section we simulate some solitary wave solutions of the equation (3.1) for varying parameters. The purpose of these simulations is to understand influence of parameters on the solution of equations under consideration.

3.2.1   Fisher's Equation with Cubic Nonlinearity

If the parameters in equation (3.1) choose such as $\alpha = \beta = \gamma = \lambda = 0$ and $\tau = D = 1$ the equation (3.1) transform the following equation which is known as Fisher's

equation with three zeros.

$$U_t = U_{xx} - (U - a_1)(U - a_2)(U - a_3) \tag{3.38}$$

The wave scalar is found as follows:

$$|k| = \frac{1}{\sqrt{2}}|\tilde{a}_1 - \tilde{a}_2|. \tag{3.39}$$

Next the frequency becomes

$$\omega = \frac{\tilde{a}_2{}^2 - \tilde{a}_1{}^2}{2}. \tag{3.40}$$

Finally from the equation (3.27) the velocity is expressed as

$$|\nu| = \frac{1}{\sqrt{2}}(\tilde{a}_1 + \tilde{a}_2). \tag{3.41}$$

The solution of the equation (3.28) with respect to a_3 +g/f is given

$$U = a_3 + \frac{(a_1 - a_3) + e^\eta}{(a_2 - a_3) + e^\eta}(a_2 - a_3). \tag{3.42}$$

In Figure 3.1 we exhibit the solution of the equation (3.38) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = 0.7071$, $\omega = 0.1$ and $\nu = 0.1414$.

As can be seen in this figure the characteristic lines of the equation have different slopes.

One solitary wave solution for the equation (3.38) is exhibited in Figure 3.2 by constant parameters $a_1 = 0, a_2 = 1, a_3 = 0.5$, $k = 0.7071$, $\omega = 0$ and $\nu = 0$.

The Figure 3.2 shows that the characteristic lines of the equation have same slopes.

3.2.2 Burgers Equation with Cubic Nonlinearity

For $\alpha = \beta = \lambda = 0$ and $\tau = \gamma = D = 1$ the equation (3.1) transform the equation (3.36) which is known as Burger's Equation with cubic nonlinearity. Then the roots considered as $a_1 = 0$, $a_2 = 1$, $a_3 = 0.4$ and the equation appeared as follow

$$U_t + UU_x = U_{xx} - U(U - 1)(U - 0.4). \tag{3.43}$$

The wave scalar is found as follows

$$k_1 = a_1 - a_2 \quad or \quad k_2 = \frac{a_2 - a_1}{2}. \tag{3.44}$$

Next the frequency becomes

$$\omega_1 = (a_2 - a_1)(a_1 + a_2 - a_3) \tag{3.45}$$

$$\omega_2 = \frac{(a_2 - a_1)[-0.5(a_1 + a_2) + (a_1 + a_2 - 2a_3)]}{2}. \tag{3.46}$$

Finally from the equation (3.27) the velocity is expressed as

$$\nu_1 = (a_1 + a_2 - a_3) \tag{3.47}$$

$$\nu_2 = [-0.5(a_1 + a_2) - (a_1 + a_2 - 2a_3)]. \tag{3.48}$$

In Figure 3.3 we simulate the solution of the equation (3.43) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = -1$, $\omega = 0.6$ and $\nu = 0.6$. In figure (3.4) we exhibit the solution of the equation (3.43) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = 0.5$, $\omega = -0.15$ and $\nu = 0.3$.

In Figure 3.3 when we add the transport term to the fisher equation (3.38) the speed of the velocity gets bigger and solution gets sharper.

3.2.3 Nonlinear Evolution Equation with Transport Term

We consider the nonlinear wave equation with transport term for $\beta = \gamma = \tau = D = 1$ and $\alpha = \lambda = 0$. Then the roots considered as $a_1 = 0$, $a_2 = 1$, $a_3 = 0.4$. After substituting the parameters into the equation (3.1) the following equation is found.

$$U_t + UU_x + UU_t = U_{xx} - U(U - 1)(U - 0.4) \tag{3.49}$$

The frequency, and the wave scalar expressed as respectively,

$$\omega_{(1,2)} = \frac{(a_1 + a_2)[-a_1(2 + a_1) + a_2(2 + a_2)]}{2[(a_1 + a_2)(4 + a_1 + a_2) + 4](2 + a_1 + a_2)}$$

$$\mp \frac{(a_1 + a_2)[(a_1 - a_2)(2 + a_1 + a_2)\sqrt{4(a_1 + a_2)(1 + a_3) + 8a_3 + 9}]}{2[(a_1 + a_2)(4 + a_1 + a_2) + 4](2 + a_1 + a_2)}$$

$$+ \frac{(a_2 - a_1)(a_1 + a_2 - 2a_3)}{(2 + a_1 + a_2)}$$

$$k_{(1,2)} = \frac{(a_2 - a_1)[1 \pm \sqrt{9 + 8a_3 + 4(a_1 + a_2)(1 + a_3)}]}{-2(2 + a_1 + a_2)}$$

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In figure 3.5 we simulate the solution of the equation (3.49) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = -0.8698$, $\omega = 0.3566$ and $\nu = 0.4099$. In Figure 3.6 we illustrate the solution of the equation (3.49) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = 0.5365$, $\omega = -0.1121$ and $\nu = 0.2090$.

In Figure 3.5 when we add the transport term in time direction to the burger equation (3.43), the solution gets sharper.

3.2.4 Nonlinear Evolution Equation with Convection and Transport Terms

We consider the nonlinear wave equation with convection and transport term for $\beta = \gamma = \tau = \lambda = D = 1$ and $\alpha = 0$. Then the roots considered as $a_1 = 0$, $a_2 = 1$, $a_3 = 0.4$. After substituting the parameters into the equation (3.1) the following equation is obtained:

$$U_t + UU_x + UU_t + U_x = U_{xx} - U(U-1)(U-0.4) \tag{3.50}$$

The wave scalar, the frequency and the velocity expressed as respectively,

$$k_{1,2} = \mp(a_1 - a_2)\sqrt{\frac{a_3 + 1}{2 + a_1 + a_2}} \tag{3.51}$$

$$\omega_{1,2} = (a_2 - a_1)\frac{a_1 + a_2 - 2a_3 \mp \sqrt{\frac{a_3+1}{2+a_1+a_2}}(2 + a_1 + a_2)}{2 + a_1 + a_2} \tag{3.52}$$

$$\nu_{1,2} = \mp\frac{(a_1 + a_2 - 2a_3)}{\sqrt{\frac{a_3+1}{2+a_1+a_2}}} \pm (2 + a_1 + a_2). \tag{3.53}$$

In Figure 3.7 we illustrate the solution of the equation (3.50) for $k = -0.6831$, $\omega = 0.7497$ and $\nu = 1.0975$. The polynomial roots $a_1 = 0, a_2 = 1, a_3 = 0.4$. In Figure 3.8 we show that the solution of the equation (3.50) for $k = 0.6831$, $\omega = -0.6164$ and $\nu = 0.9024$. The polynomial roots $a_1 = 0, a_2 = 1, a_3 = 0.4$.

In Figure 3.7 when we add the convection term in space direction to the equation (3.49) the solution is more sharper than the solution of the equation (3.49).

3.2.5   Nonlinear Mixed Diffusion Wave Equation

We consider the nonlinear mixed diffusion wave equation for $\alpha = 0.99, \beta = \gamma = \tau = \lambda = D = 1$. Then the roots considered as $a_1 = 0$, $a_2 = 1$, $a_3 = 0.4$. After substituting the parameters into the equation (3.1) the following equation is obtained:

$$U_{tt} + UU_t + UU_x + U_t + U_x = U_{xx} - U(U-1)(U-0.4) \tag{3.54}$$

The wave scalar, the frequency and the velocity expressions are submitted respectively,

$$k_{(1,2)} = \frac{C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2}}{C_3}$$

$$\omega_{(1,2)} = \frac{C_3(a_2^2 - a_1^2) - (a_1 + a_2)(C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2})}{C_3(a_1 + a_2) + 2}$$

$$- \frac{2(C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2})}{C_3(a_1 + a_2) + 2}$$

$$v_{(1,2)} = -\frac{C_3(a_2^2 - a_1^2) - (a_1 + a_2)(C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2})}{((a_1 + a_2) + 2)(C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2})}$$

$$- \frac{2(C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2})}{C_3((a_1 + a_2) + 2)(C_1 - (a_1 - a_2)(2 + a_1 + a_2)\sqrt{C_2})}$$

where

$$C_1 = -1.98[a_1^3 - a_2(a_2 + 2)(a_2 - 2a_3) + a_1^2(2 + a_2 - 2a_3) - a_1(\pm a_2^2 + 4a_3)]$$

$$C_2 = 0.08(1 + a_3) + 3.96(a_1^2 + a_2^2) - 14.76(a_1 + a_2) + 7.92(2a_3 + a_1 a_2)$$

$$C_3 = -0.02[(a_1 + a_2)^2 + 4(1 + a_1 + a_3)].$$

In Figure 3.9 we simulate the solution of the equation (3.54) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = -16.1219$, $\omega = 16.1886$ and $\nu = 1.0041$. In Figure 3.10 we exhibit the solution of the equation (3.54) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = 2.9219$, $\omega = -2.8552$ and $\nu = 0.9771$.

In Figure 3.9 when we add the second order time derivative to the equation (3.50) the solution gets the sharpest situation.

3.2.6   Dissipative Nonlinear Wave Equation

We consider the dissipative nonlinear wave equation for $\beta = \gamma = \lambda = 0$ and $\alpha = \tau = D = 1$ and the polynomial roots $a_1 = 0, a_2 = 1, a_3 = 0.4$.

After substituting the parameters and the roots into the equation (3.1) the following equation is found.

$$U_{tt} + U_t = U_{xx} - U(U - 1)(U - 0.4) \tag{3.55}$$

After applying the same procedure as before, we can find the wave scalar, the frequency and the velocity respectively, we have

$$k = \pm \sqrt{\frac{\tilde{a_2}^2 - \tilde{a_1}^2}{2}[1 + \frac{\tilde{a_2}^2 - \tilde{a_1}^2}{2}] + (a_1 - a_2)(a_1 - a_3)} \tag{3.56}$$

$$\omega = \frac{\tilde{a_2}^2 - \tilde{a_1}^2}{2} \tag{3.57}$$

$$|\nu| = \frac{\tilde{a_2}^2 - \tilde{a_1}^2}{2\sqrt{\frac{\tilde{a_2}^2 - \tilde{a_1}^2}{2}[1 + \frac{\tilde{a_2}^2 - \tilde{a_1}^2}{2}] + (a_1 - a_2)(a_1 - a_3)}}. \tag{3.58}$$

We submit the solution of the equation (3.41) for different cases in Figure 3.11 and Figure 3.12.

In Figure 3.11 we simulate the solution of the equation (3.55) for $k = -0.7141$, $\omega = 0.1$ and $\nu = 0.1400$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$. In Figure 3.12 we exhibit the solution of the equation (3.55) for $a_1 = 0, a_2 = 1, a_3 = 0.4$, $k = -0.7141$, $\omega = 0.1$ and $\nu = 0.1400$.
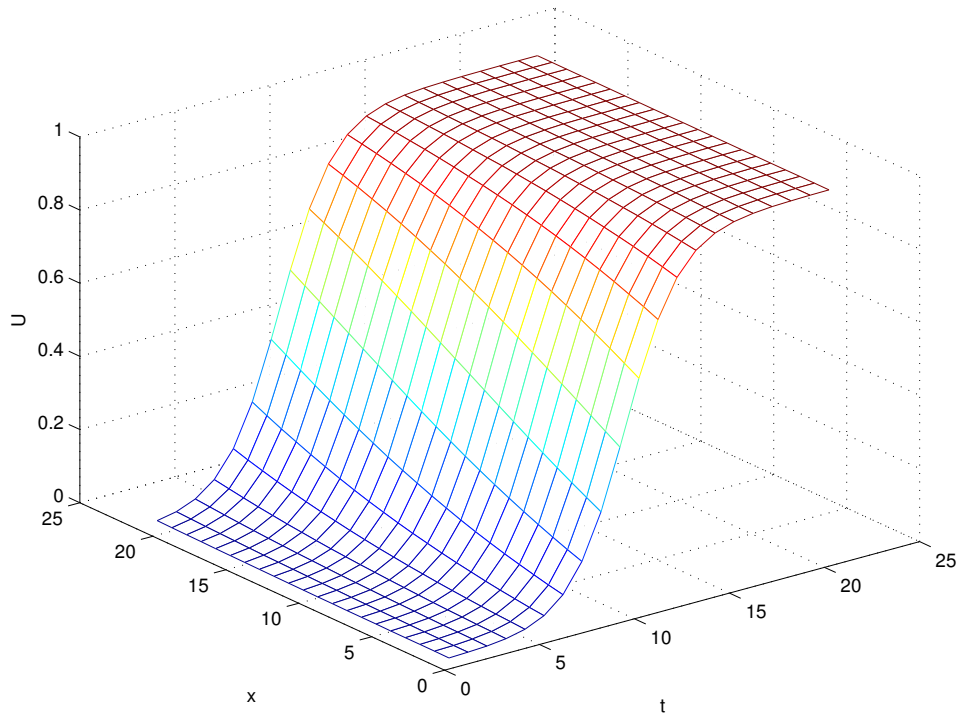
Figure 3.1. Meshgrid simulation of evolution of the travelling wave with $k = 0.7071$, $\omega = 0.1$ and $\nu = 0.1414$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

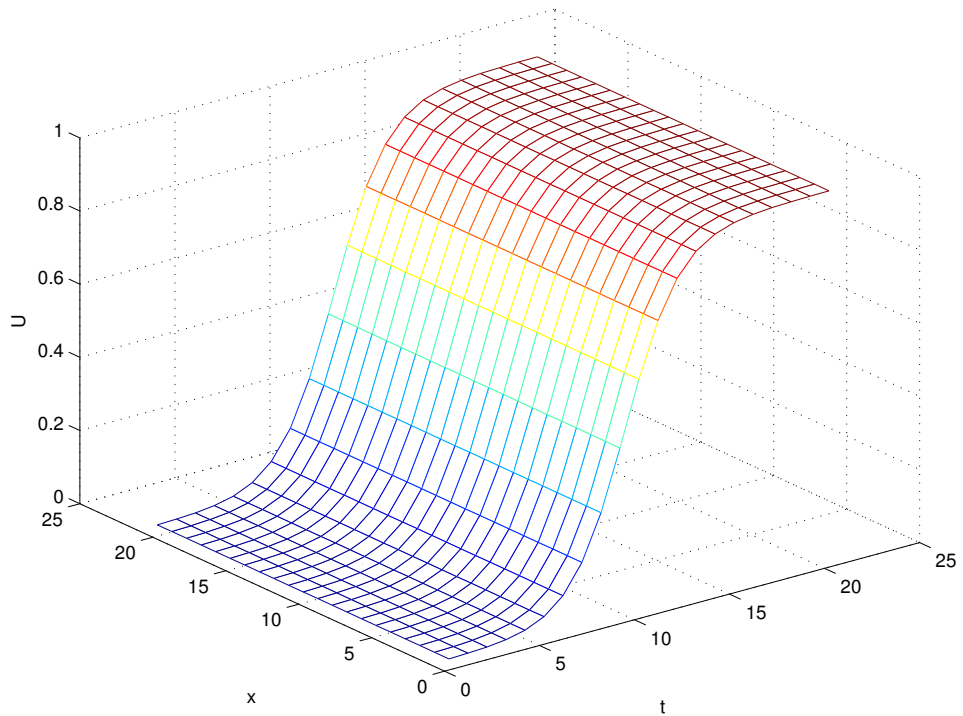Figure 3.2. Meshgrid simulation of evolution of the travelling wave with $k = 0.7071$, $\omega = 0$ and $\nu = 0$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.5$
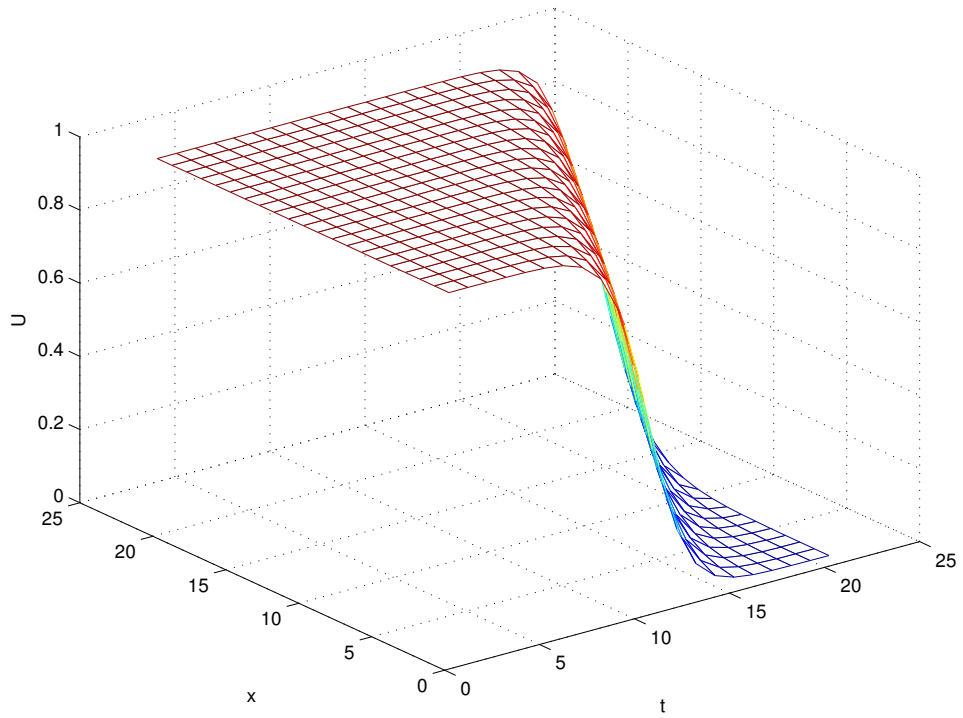
Figure 3.3. Meshgrid simulation of evolution of the travelling wave with $k = -1$, $\omega = 0.6$ and $\nu = 0.6$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$
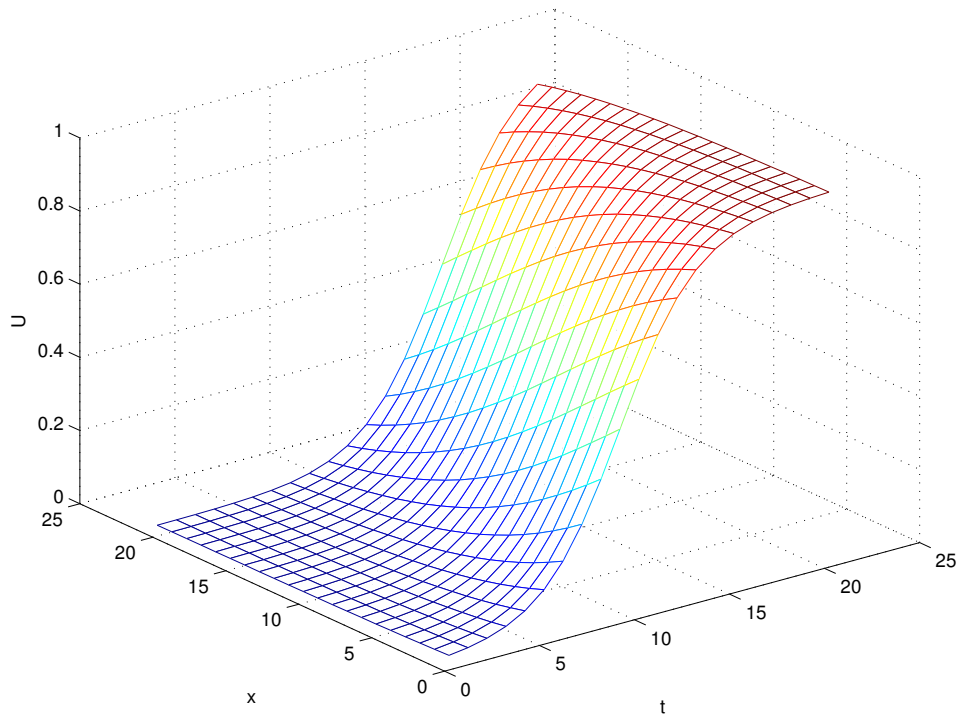

Figure 3.4. Meshgrid simulation of evolution of the travelling wave with $k = 0.5$, $\omega = -0.15$ and $\nu = 0.3$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$
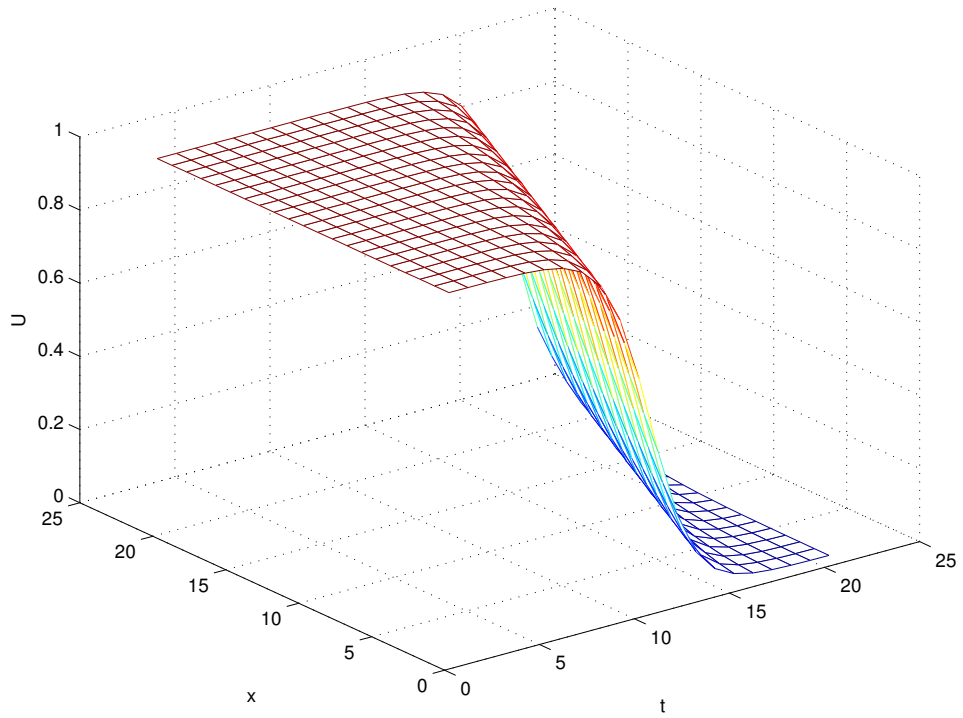
Figure 3.5. Meshgrid simulation of evolution of the travelling wave with $k = -0.8698$, $\omega = 0.3566$ and $\nu = 0.4099$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

Figure 3.6. Meshgrid simulation of evolution of the travelling wave with $k = 0.5365$, $\omega = -0.1121$ and $\nu = 0.2090$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

Figure 3.7. Meshgrid simulation of evolution of the travelling wave with $k = -0.6831$, $\omega = 0.7497$ and $\nu = 1.0975$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

Figure 3.8. Meshgrid simulation of evolution of the travelling wave with $k = 0.6831$, $\omega = -0.6164$ and $\nu = 0.9024$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

Figure 3.9. Meshgrid simulation of evolution of the travelling wave with $k = -16.1219$, $\omega = 16.1886$ and $\nu = 1.0041$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

Figure 3.10. Meshgrid simulation of evolution of the travelling wave with $k = 2.9219$, $\omega = -2.8552$ and $\nu = 0.9771$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$
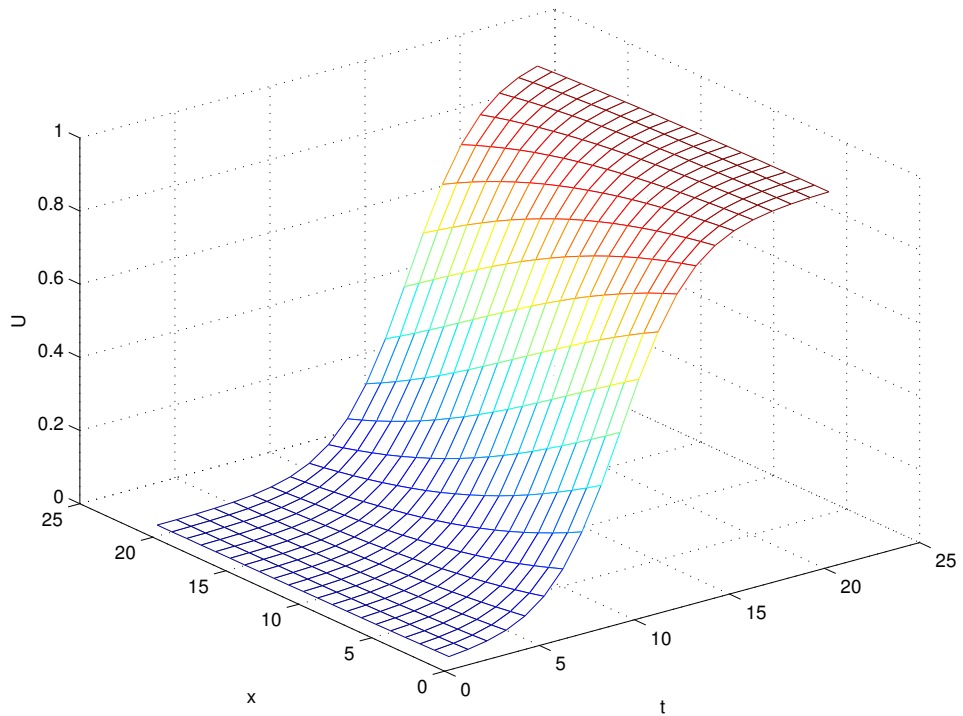
Figure 3.11. Meshgrid simulation of evolution of the travelling wave with $k = -0.7141$, $\omega = 0.1$ and $\nu = 0.1400$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$
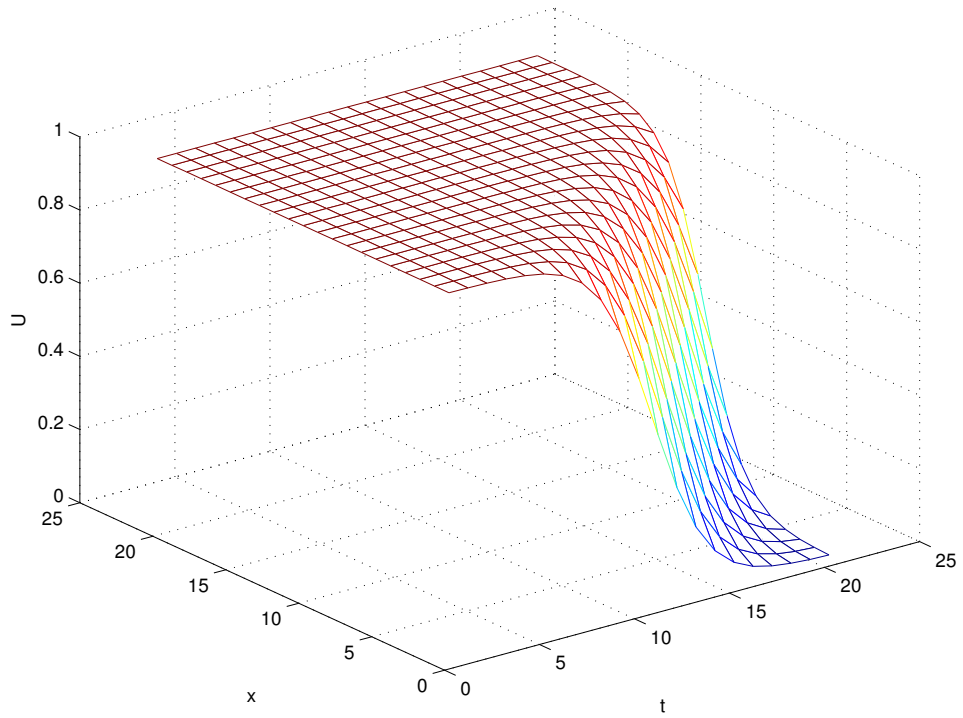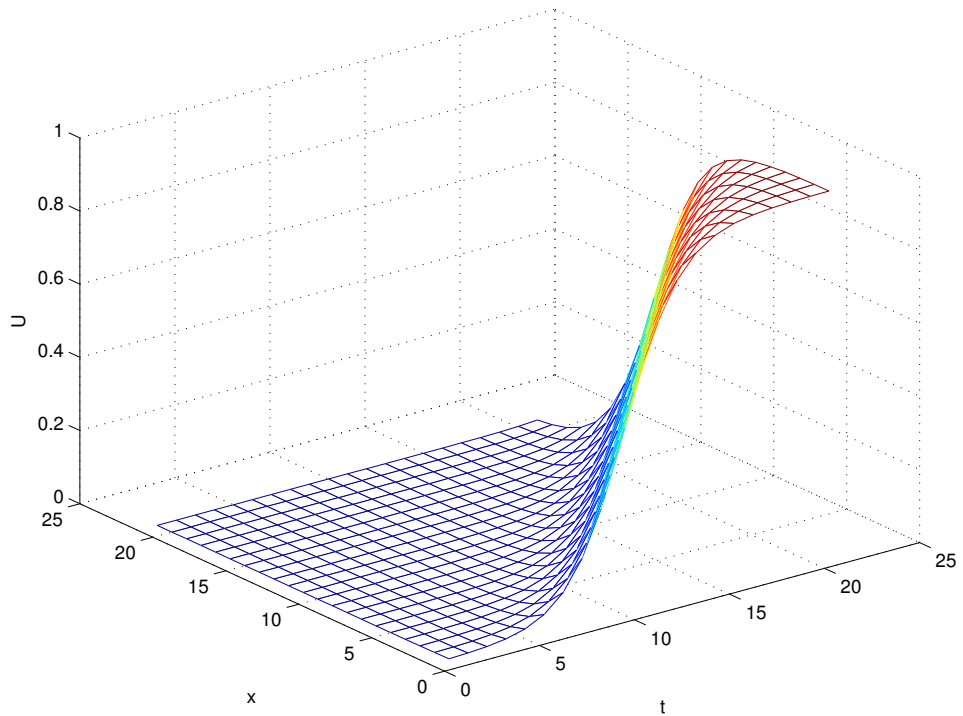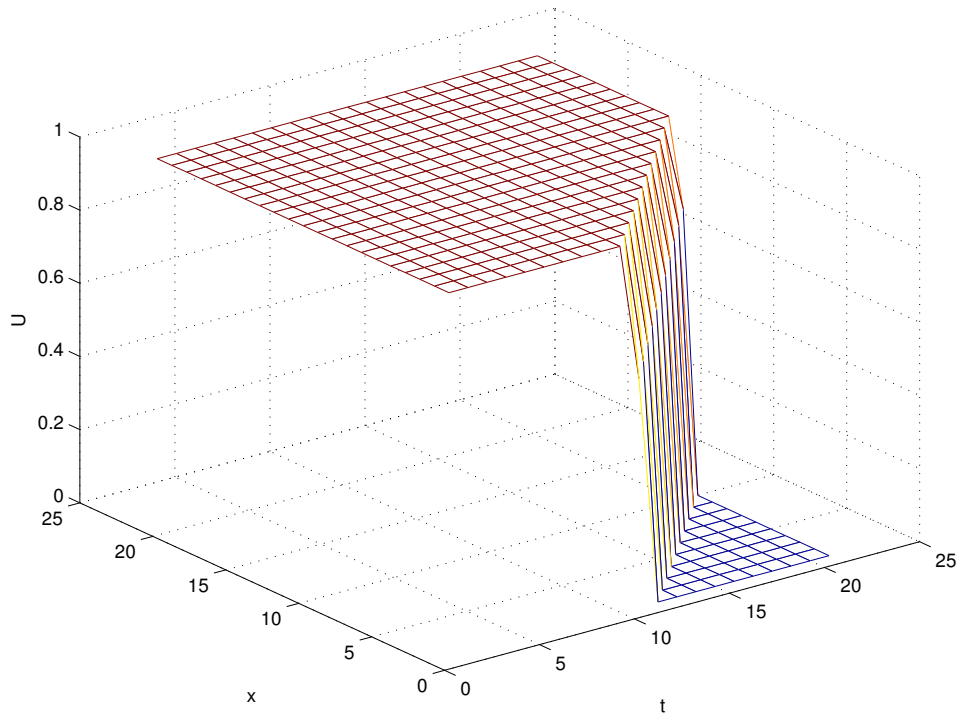

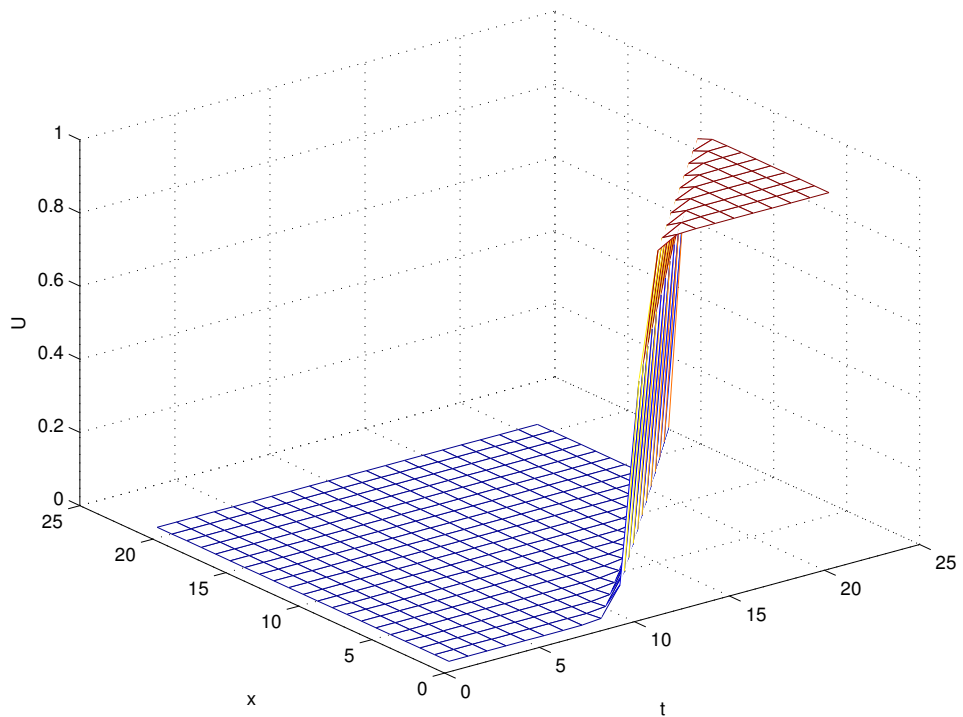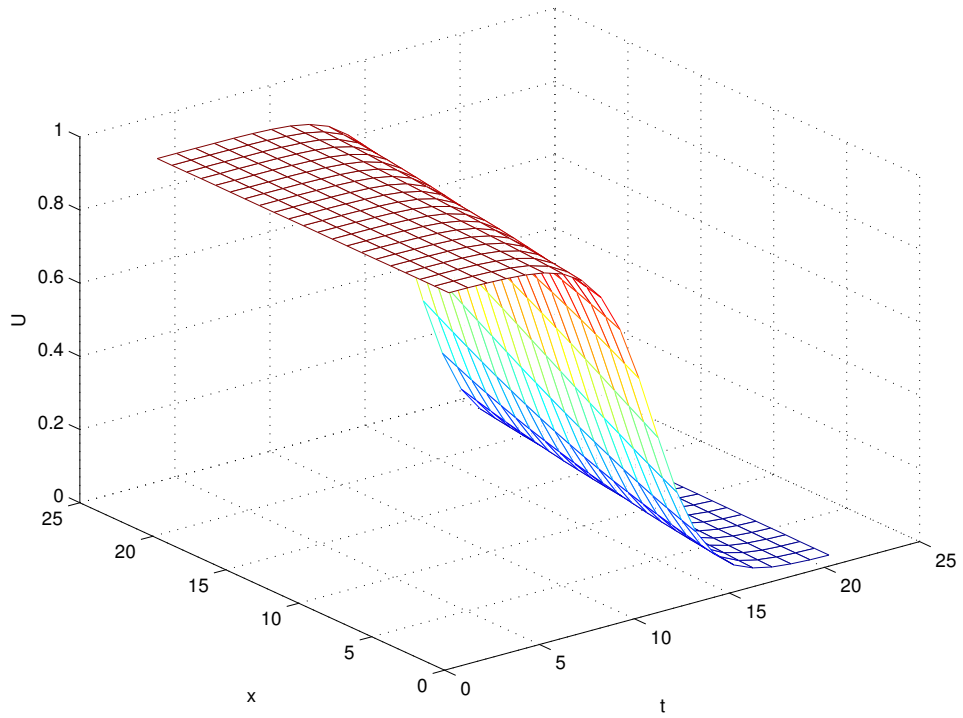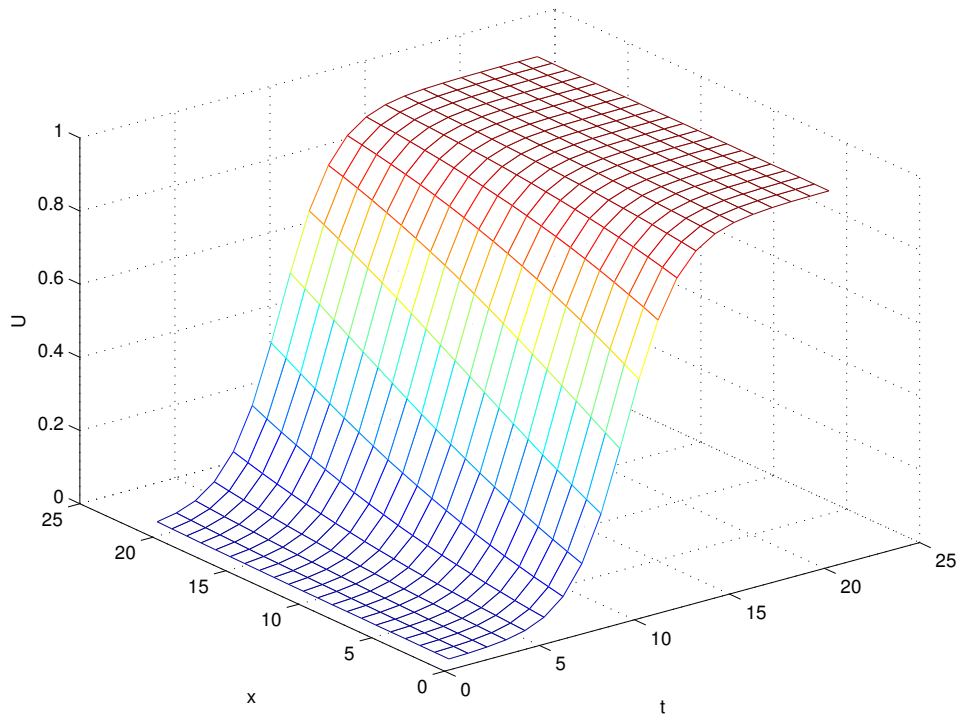Figure 3.12. Meshgrid simulation of evolution of the travelling wave with $k = -0.7141$, $\omega = 0.1$ and $\nu = -0.1400$. The parameters $a_1 = 0, a_2 = 1, a_3 = 0.4$

CHAPTER 4

BRUSSELATOR REACTION MODEL

In this chapter, we consider the Hirota bilinear method, to construct one solitary wave solution of Brusselator reaction model given in system (4.1).

$$U_t \;\; = \;\; KU_{xx} + U^2 V - \beta U \tag{4.1a}$$

$$V_t \;\; = \;\; KV_{xx} - U^2 V + \beta U \tag{4.1b}$$

where $K > 0, \beta > 0$. This model was proposed by Prigogine and Lefever and reviewed by Lefever et al (Prigogine and Lefever 1968). by adding spatial diffusion and altering slightly the reaction mechanism. By using the sine-cosine method (Cherniha 1998), the several types of solutions of these equations are obtained.

In our work, we are proposing Hirota Bilinear Method to construct solitary wave solution for the system (4.1).

4.1 Solution of the Model

To solve the system of non-linear equations (4.1), first we add them, then following heat equation is obtained.

$$(U + V)_t = K(U + V)_{xx}. \tag{4.2}$$

A simplest solution to equation (4.2) can be taken as

$$U(x,t) + V(x,t) = \tau \tag{4.3}$$

where τ is a real constant. Substituting this in the first equations (4.1) we get an equation in the variable u only as follows:

$$U_t = KU_{xx} - U^3 + \tau U^2 - \beta U. \tag{4.4}$$

The solution of the problem is assumed to have a form

$$U = \frac{g}{f}, \tag{4.5}$$

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where $g(x,t)$ and $f(x,t)$ are real functions.

In order to obtain bilinear system of equations, all derivatives with respect to the dependent variables in equation (4.4) are expressed as

$$U_t = \frac{D_t(g \cdot f)}{f^2} \tag{4.6a}$$

$$U_{xx} = \frac{D_x^2(g \cdot f)}{f^2} - \frac{g}{f}\frac{D_x^2(f \cdot f)}{f^2} \tag{4.6b}$$

where the Hirota derivative according to x_i is defined as

$$D_{x_i}^n(a \cdot b) = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i'}}\right)^n (a(x)b(x'))|_x = x'. \tag{4.7}$$

After substituting equationsc(4.6) into equation (4.4), the following expression is obtained :

$$\frac{D_t(g \cdot f)}{f^2} - K[\frac{D_x^2(g \cdot f)}{f^2} - \frac{g}{f}\frac{D_x^2(f \cdot f)}{f^2}] - (\frac{g}{f})^3 + \tau(\frac{g}{f})^2 - \beta(\frac{g}{f}) = 0. \tag{4.8}$$

Then reduces equation (4.8) to bilinear form as follows

$$(D_t - KD_x^2)(g \cdot f) = 0 \tag{4.9a}$$

$$(KD_x^2)(f \cdot f) = -g^2 + \tau gf - \beta f^2. \tag{4.9b}$$

To solve this system in the Hirota method, the functions f and g have form of the formal perturbation series in a parameter ϵ

$$f = \sum_{i=0}^{\infty} \epsilon^i f_i, \qquad g = \sum_{i=0}^{\infty} \epsilon^i g_i \tag{4.10}$$

without loss of generality assume that $f_0 = 1$.

Substituting equations (4.10) into the system (4.9) and equating coefficients of the same powers of ϵ converts equations (4.9) into a sequence of the zeroth, first, second order and so one, bilinear equations given by equations (4.11), (4.12) and (4.13), respectively,

$$(D_t - KD_x^2)(g_0 \cdot 1) = 0 \tag{4.11a}$$

$$(KD_x^2)(1 \cdot 1) = -g_0^2 + \tau g_0 - \beta, \tag{4.11b}$$

$$(D_t - KD_x^2)(g_0 \cdot f_1 + g_1 \cdot 1) = 0 \tag{4.12a}$$

$$(KD_x^2)(2 \cdot f_1) = -2(g_0 \cdot g_1) + \tau(g_0 \cdot f_1 + g_1 \cdot 1) - 2\beta \cdot f_1, \tag{4.12b}$$

$$(D_t - KD_x^2)(g_0 \cdot f_2 + g_1 \cdot f_1 + g_2 \cdot 1) = 0 \tag{4.13a}$$

$$(KD_x^2)(2 \cdot f_2 + f_1 \cdot f_1) = -2(g_0, g_2 + g_1 \cdot g_1 + g_2 \cdot g_0)$$

$$+\tau(g_0 \cdot f_2 + g_1 \cdot f_1 + g_2 \cdot 1) \tag{4.13b}$$

$$-\beta(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1).$$

4.1.1   The Zeroth Order Calculation

We assume that the value of g_0 is constant, then the first equation in equation (4.11) is satisfied automatically. From the second equation we get

$$g_0^2 - \tau g_0 + \beta = 0. \tag{4.14}$$

The solutions of the equation (4.14) can be found as:

$$\Delta_1 = \tau^2 - 4\beta \tag{4.15}$$

where $\Delta_1 > 0$

$$g_0^{(1,2)} = \frac{\tau \pm \sqrt{\Delta_1}}{2}. \tag{4.16}$$

4.1.2   The First Order Calculation

In this subsection we are going to find the first order real solutions, g_1 and f_1. Then equations (4.12) may be rewritten as a linear system

$$g_0(-\partial_t - K\partial_{xx})f_1 + (\partial_t - K\partial_{xx})g_1 = 0 \tag{4.17a}$$

$$2K\partial_{xx}f_1 + 2g_0g_1 - \tau(g_0f_1 + g_1.1) + 2\beta f_1 = 0. \tag{4.17b}$$

We are interested in finding one soliton solutions, which correspond to the exponential solution of equations (4.17) . Then the simplest nontrivial solution of this system has the form

$$g_1 = a\, e^{\eta_1}, \qquad f_1 = b\, e^{\eta_1} \tag{4.18}$$

where a and b are constants, and η_1=kx+wt+γ. Coefficients k and w are found by a dispersion relation. Then, substituting equations (4.17) into the system equations (4.18) the homogeneous linear algebric system is constructed for unknown variables a and b.

$$A\xi = \begin{bmatrix} (w - Kk^2) & (-g_0 w - Kg_0 k^2) \\ \\ (2g_0 - \tau) & (2Kk^2 - \tau g_0 + 2\beta) \end{bmatrix} \begin{bmatrix} a \\ \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

Nontrivial solution of this system of equations exists only if $Det(A) = 0$. This gives us the following dispersion relation:

$$\omega = -\frac{g_0^2 Kk^2 - K^2 k^4 - Kk^2 \beta}{g_0^2 + Kk^2 + \beta - g_0 \tau}. \tag{4.19}$$

Substituting g_1, f_1 to the first system of equations (4.13) and using property of bilinear Hirota operators,we have

$$(D_t - D_x^2)(g_1 \cdot f_1) = (D_t - D_x^2)(e^{\eta_1} \cdot e^{\eta_1}) = 0. \tag{4.20}$$

4.1.3 The Second Order Calculation

In this part, we find the following set of equations for f_2, g_2.
After substituting f_2, g_2 into the first equation(4.13) we get

$$g_0(-\partial_t - K\partial_{xx})f_2 + (\partial_t - K\partial_{xx})g_2 = 0. \tag{4.21}$$

Similar to the first system (4.17) the simplest solution for these equations is the trivial one $g_2 = 0$ and $f_2 = 0$. Then, from the last equation (4.13) we find additional constraint on g_1, f_1:

$$g_1^2 - \tau g_1 f_1 + \beta f_1^2 = 0. \tag{4.22}$$

We substitute equations (4.18) into equation (4.22), to find the arbitrary constant b we get:

$$\beta b^2 - a\tau b + a^2 = 0 \tag{4.23}$$

$$\Delta_2 = \tau^2 a^2 - 4\beta a^2 \tag{4.24}$$

where $\Delta_2 > 0$

$$b^{(1,2)} = \frac{a\tau \pm \sqrt{\Delta_2}}{2\beta}. \tag{4.25}$$

Combining this equation with equation (4.16), we find the relation between b and g_0 as follows:

$$b_1 = \frac{\mid a \mid g_0^{(1)}}{\beta} \quad , \quad b_2 = \frac{\mid a \mid g_0^{(2)}}{\beta}. \tag{4.26}$$

Next, we can find k using equations (4.26)with second row of matrix:

$$\frac{(2Kk^2 - \tau g_0 + 2\beta)}{-(2g_0 - \tau)} = g_0. \tag{4.27}$$

Then, the wave vector is given by the following formula:

$$k = \pm\sqrt{\frac{g_0(2\tau - g_0) - 2\beta}{2K}}. \tag{4.28}$$

Next, after substituting equation (4.5) into the equation (4.19), the following explicit form for the frequency is found as follows

$$\omega = -\frac{(2\tau - 3g_0)(2\beta - 2\tau g_0 + g_0^2)}{2g_0}. \tag{4.29}$$

Finally, the velocity vector is given by the following formula

$$\upsilon = -\frac{\omega}{\mid \vec{k} \mid}. \tag{4.30}$$

The wave vector which is given in equation (4.28) and the frequency given by equation (4.29). For the speed of solitary wave, we have the expression as follows:

$$\upsilon = \frac{(2\tau - 3g_0)(2\beta - 2\tau g_0 + g_0^2)}{2g_0\sqrt{\frac{g_0(2\tau-g_0)-2\beta}{2K}}}. \tag{4.31}$$

After substituting f and g in the equation (4.5), we find the following exact solutions

$$U_1 = \frac{g_0^{(2)} + e^{\eta_1}}{1 + \frac{g_0^{(1)}}{\beta}e^{\eta_1}} \qquad U_2 = \frac{g_0^{(1)} + e^{\eta_1}}{1 + \frac{g_0^{(2)}}{\beta}e^{\eta_1}}. \tag{4.32}$$

At the end, we compose v with respect to u solutions as follows

$$V_1 = \tau - \frac{g_0^{(2)} + e^{\eta_1}}{1 + \frac{g_0^{(1)}}{\beta}e^{\eta_1}} \qquad V_2 = \tau - \frac{g_0^{(1)} + e^{\eta_1}}{1 + \frac{g_0^{(2)}}{\beta}e^{\eta_1}}. \tag{4.33}$$

4.2   Simulations

We simulate the solution of the system given in (4.1) concerning varies choice of equation's parameters.

Figures 4.1-4.3 exhibit the time evolution solution of the system (4.1) for fixed $\tau = 5, \beta = 2$ and K =0.01, 0.1, 0.5 respectively.

As can be seen in these figures, as K increases the curves representing solutions U and V get far away from each other. In other words, the curvature of the curves gets bigger.

Figures 4.4-4.6 illustrate the time evolution solution of the system (4.1) for fixed K=0.1, $\beta = 0.1$ and $\tau =$2, 3, 4 respectively.

In these figures, as K increases the representing solution curves U and V intersect each other.

Figure 4.1. Meshgrid simulation of evolution of the solutions U and V at the same window. The parameters $\tau = 5, \beta = 2, K = 0.01$

Figure 4.2. Meshgrid simulation of evolution of the solutions U and V at the same window. The parameters $\tau = 5, \beta = 2, K = 0.1$

Figure 4.3. Meshgrid simulation of evolution of the solutions U and V at the same window. The parameters $\tau = 5, \beta = 2, K = 0.5$

Figure 4.4. Meshgrid simulation of evolution of the solutions U and V at the same window. The parameters $\tau = 2, \beta = 0.1, K = 0.1$

Figure 4.5. Meshgrid simulation of evolution of the solutions U and V at the same window. The parameters $\tau = 3, \beta = 0.1, K = 0.1$
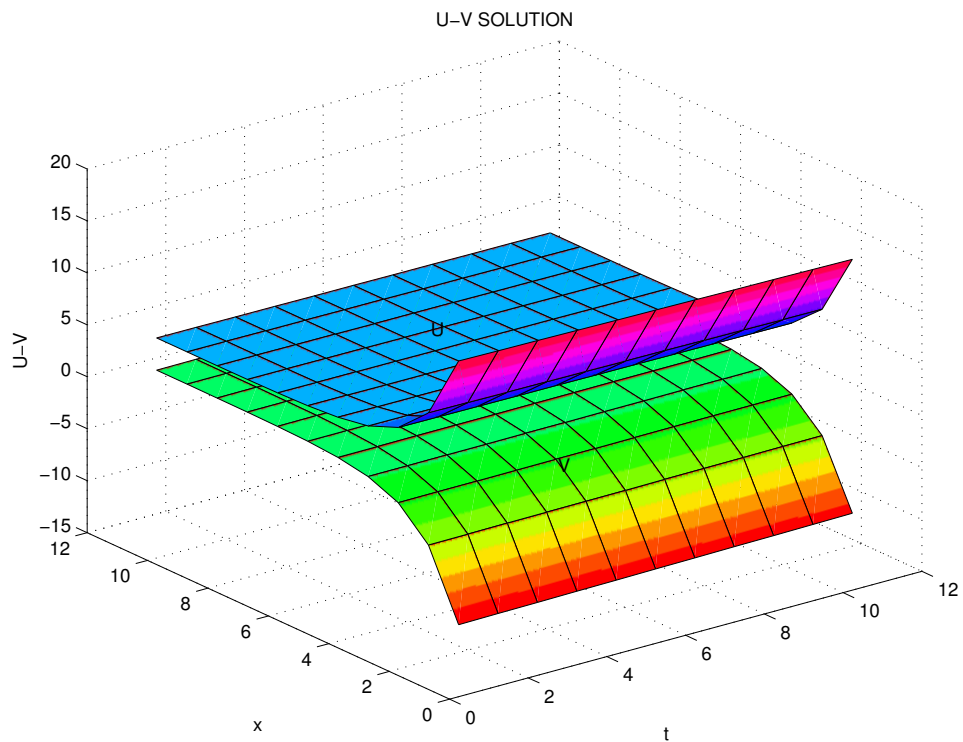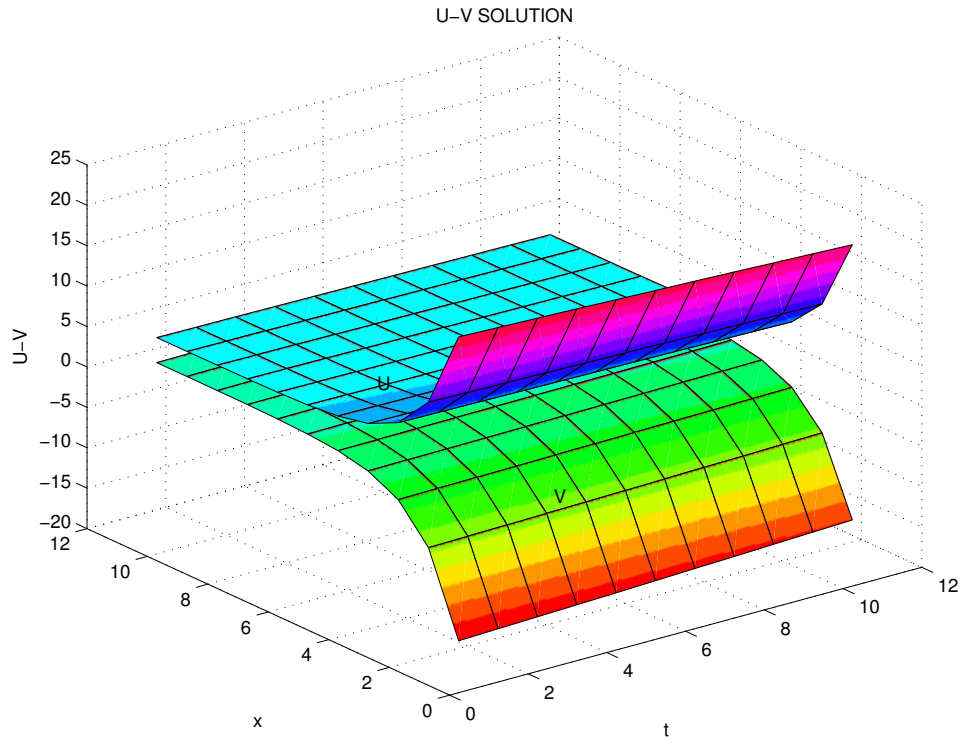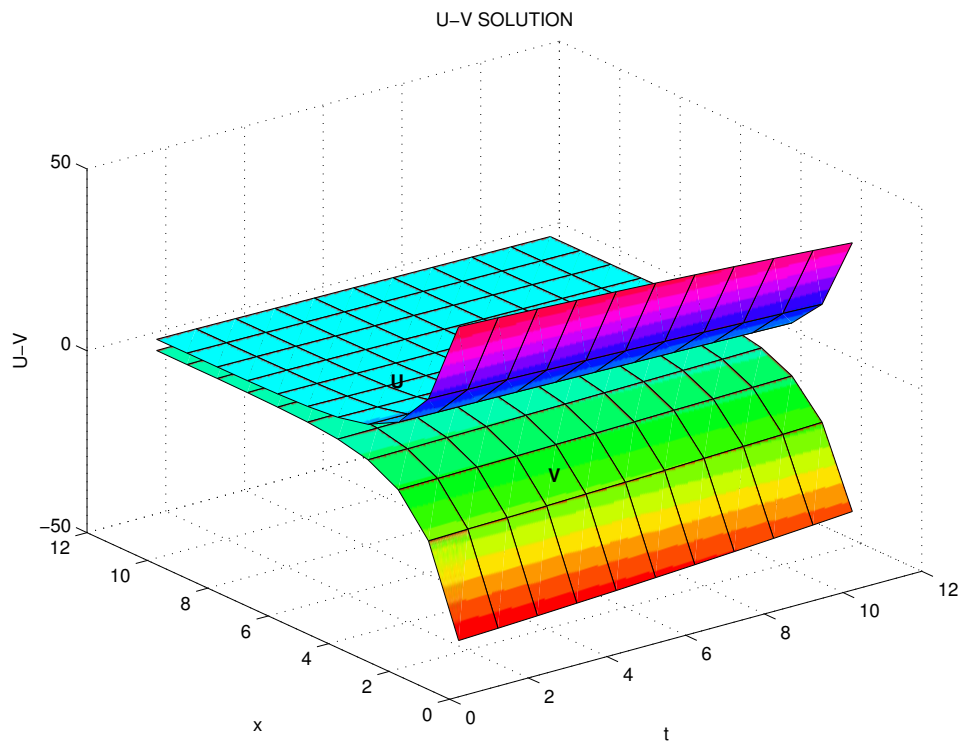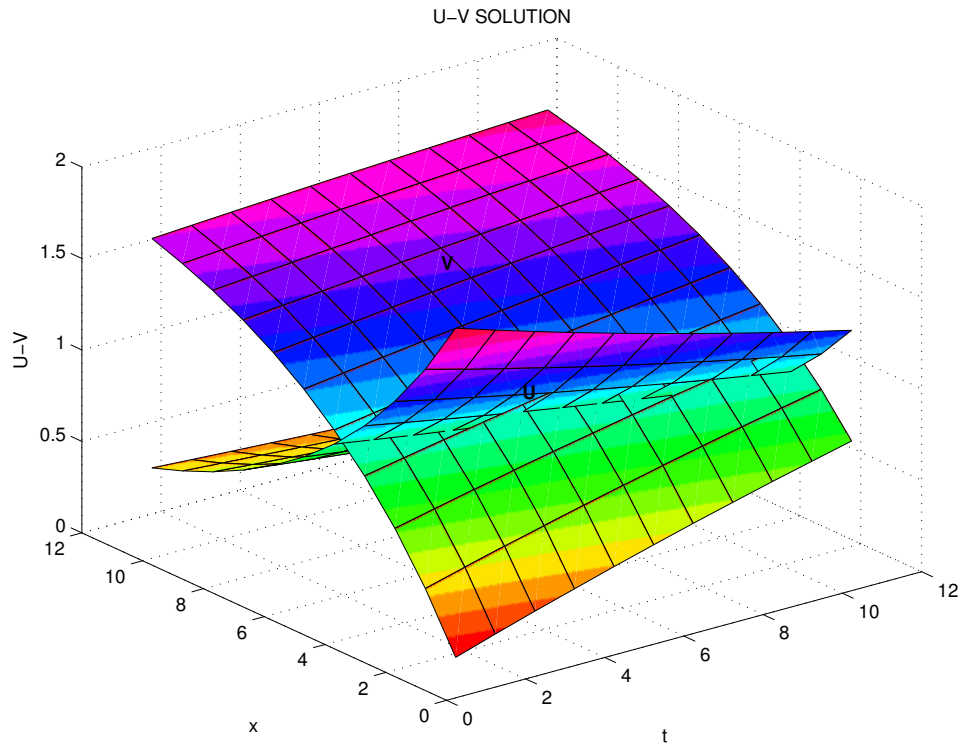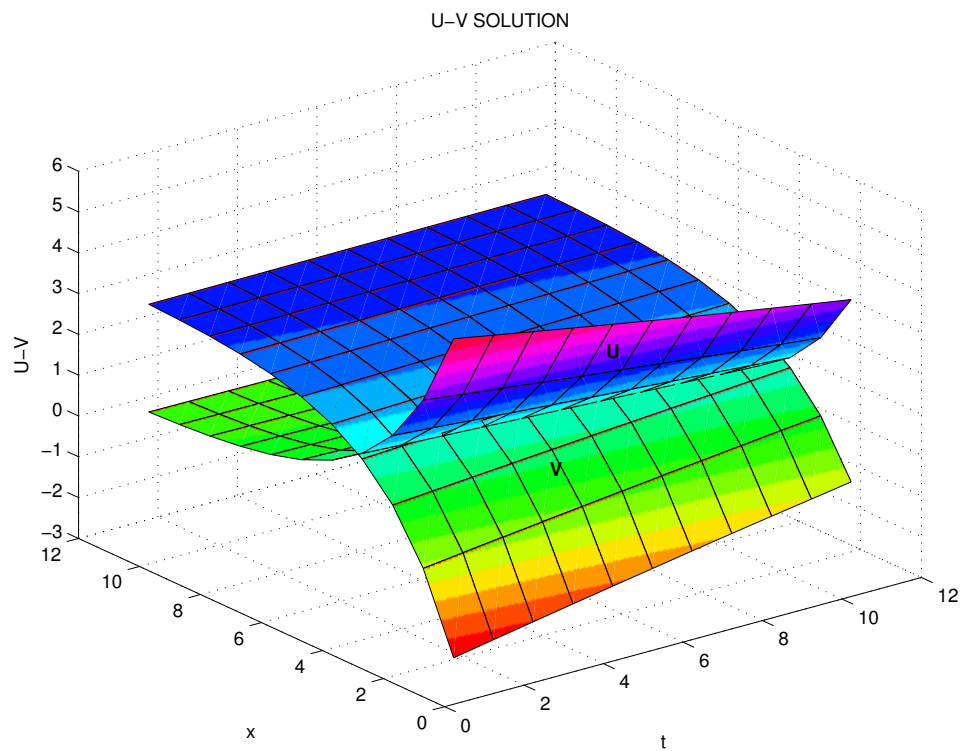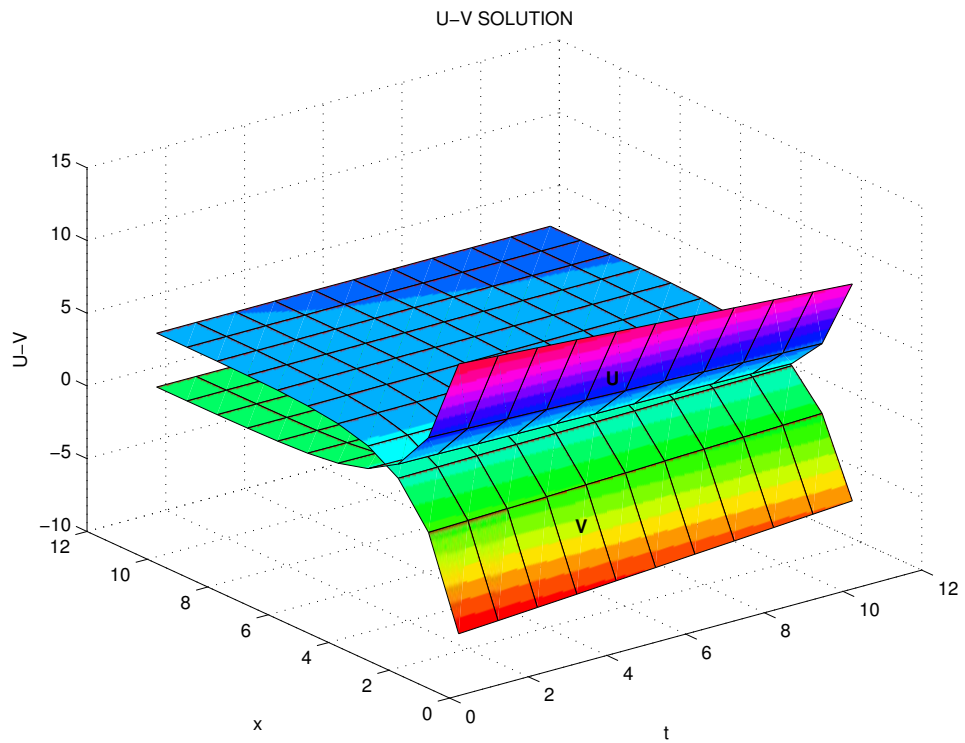
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Figure 4.6. Meshgrid simulation of evolution of the solutions U and V at the same window. The parameters $\tau = 4, \beta = 0.1, K = 0.1$
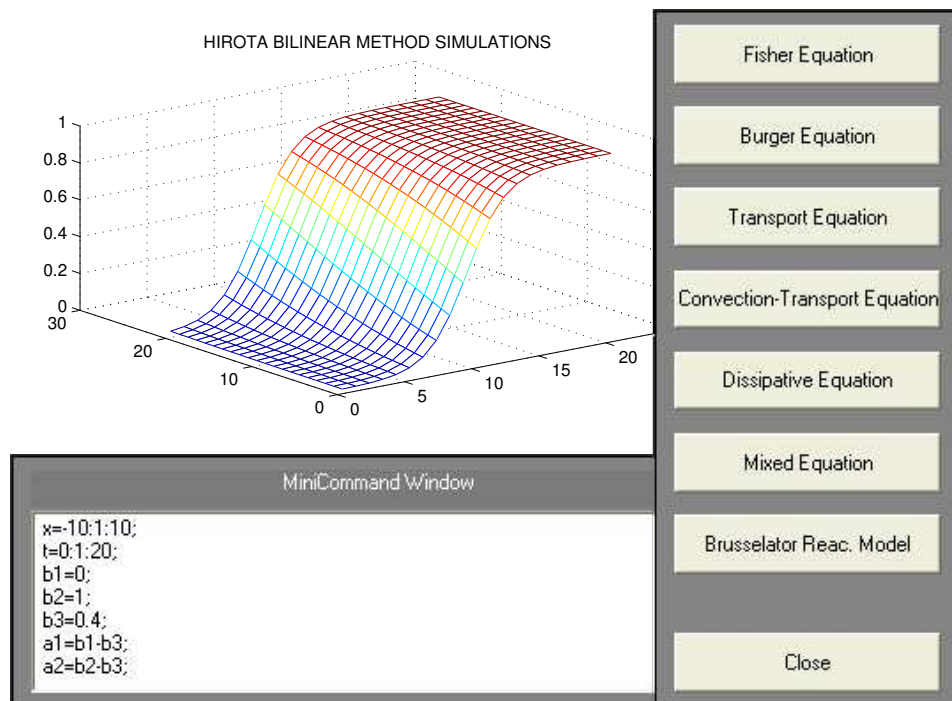

Figure 4.7. All simulations related to some class of nonlinear differential equations and Brusselator reaction model at the same window.

CHAPTER 5

SUMMARY

In this thesis, we applied the Hirota bilinear method to construct exact analytical one solitary wave solutions of some class of nonlinear differential equations. First we found an exact shock one soliton solution of the system dissipative convection-reaction-diffusion equation for n-component vector order parameter and with reaction part as the third order polynomial, determined by three distinct constant vectors, in more then one space dimensions. The bilinear representation is derived by extracting one of the vector roots (unstable in general), which allowed us to reduce the cubic nonlinearity to a quadratic one. The vector shock soliton solution, implementing transition between other two roots, as a fixed points of the potential from continuum set of the values, is constructed in a simple way. Truncation of the perturbation series in our second order calculations, restricts value of wave number, frequency and velocity of the travelling wave.

We illustrated effectiveness of the Hirota bilinear method by constructing one soliton solution of the nonlinear mixed diffusion wave equation with cubic nonlinearity in one space dimension. The related equation has been converted into bilinear system by Hirota transformation then the formal power series expansion is used to find wave scalar, frequency and velocity. In our approach, the relationship between tree zeros of the nonlinear part and the speed of the propagation of travelling waves depending on parameters are given. Simulations of the particular cases of the equation were also exhibited.

Finally, the Hirota bilinear method was proposed to solve Brusselator Reaction Model. The bilinear system was derived by using Hirota operators and truncation of the perturbation series similarly to the way Ablowitz and Zeppetella (Ablowitz and Zeppetella 1979) obtained on exact travelling wave solution of the model by finding the special wave speed. Simulations for special cases of the model are given.

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APPENDIX A

COMPUTER SOFTWARE

In Appendix A, the Matlab codes are given for the solution of the related some class of nonlinear differential equations.

```
function deniz(action);

if nargin<1,
action='initialize';

end;

if strcmp(action,'initialize'),

figNumber=figure( ...
'Name','The simulation of Hirota solution', ...
'NumberTitle','off', ...
'Visible','off');
% Give the figure a nice black background
colordef(figNumber,'black')
axes( ...
'Units','normalized', ...
'Position',[0.10 0.45 0.60 0.45]);

%  clf;
  %==================================
% Set up the MiniCommand Window
top=0.35;
left=0.05;
```

```
right=0.75;

bottom=0.05;

labelHt=0.05;

spacing=0.005;

promptStr=str2mat(' ',' % Press the buttons at the right to see', ...
    ' %The simulation of Hirota solution ');
% First, the MiniCommand Window frame
frmBorder=0.02;
frmPos=[left-frmBorder bottom-frmBorder ...
    (right-left)+2*frmBorder (top-bottom)+2*frmBorder];
uicontrol( ...
    'Style','frame', ...
    'Units','normalized', ...
    'Position',frmPos, ...
    'BackgroundColor',[0.50 0.50 0.50]);
% Then the text label
labelPos=[left top-labelHt (right-left) labelHt];
uicontrol( ...
    'Style','text', ...
    'Units','normalized', ...
    'Position',labelPos, ...
    'BackgroundColor',[0.50 0.50 0.50], ...
    'ForegroundColor',[1 1 1], ...
    'String','MiniCommand Window');
% Then the editable text field
mcwPos=[left bottom (right-left) top-bottom-labelHt-spacing];
mcwHndl=uicontrol( ...
    'Style','edit', ...
    'HorizontalAlignment','left', ...
    'Units','normalized', ...
    'Max',10, ...
    'BackgroundColor',[1 1 1], ...
```

```matlab
    'Position',mcwPos, ...
    'Callback','deniz(''eval'')', ...
    'String',promptStr);
% Save this handle for future use
set(gcf,'UserData',mcwHndl);


%===================================
% Information for all buttons
labelColor=[0.8 0.8 0.8];
top=0.95;
left=0.70;
btnWid=0.27;
btnHt=0.08;
% Spacing between the button and the next command's label
spacing=0.03;


%===================================
% The CONSOLE frame
frmBorder=0.02;
yPos=0.05-frmBorder;
frmPos=[left-frmBorder yPos btnWid+2*frmBorder 0.9+2*frmBorder];
uicontrol( ...
    'Style','frame', ...
    'Units','normalized', ...
    'Position',frmPos, ...
    'BackgroundColor',[0.50 0.50 0.50]);


%===================================
% The Brusselator button
btnNumber=7;
yPos=top-(btnNumber-1)*(btnHt+spacing);
labelStr='Brusselator';
```

```
callbackStr='deniz(''grafbutton'')';
cmdStr=str2mat( ...
    'x=0:0.1:1;t=0:0.1:1;',...
    'K=0.01;',...
    'beta=2;',...
    'to=5;',...
    'delta=(to*to-4*beta);',...
    'r1=(to+sqrt(delta))/(2);',...
    'r2=(to-sqrt(delta))/(2);',...
    'k=sqrt((r1*(2*to-r1)-2*beta)/2*K);',...
    'w=-((2*to-3*r1)*(2*beta-2*to*r1+r1*r1))/(2*r1);',...
    '[xx,tt] = meshgrid(x,t);',...
    'g=(r2+exp(k*xx+w*tt));',...
    'f=(1+(r1/beta)*exp(k*xx+w*tt));',...
    'uu=g*(f.^(-1));',...
    'vv=to-uu;',...
    'figure(1);',...
    ' surfl(uu);',...
    ' hold on;',...
    ' surfl(vv);',...
    'figure(2);',...
    ' surfl(uu);',...
    'figure(3);',...
    ' surfl(vv);'...
        );
    % Generic button information
    %
    btnPos=[left yPos-btnHt btnWid btnHt];
    uicontrol( ...
    'Style','pushbutton', ...
    'Units','normalized', ...
    'Position',btnPos, ...
```

```
        'String',labelStr,...
        'Callback',callbackStr, ...
        'UserData',cmdStr);
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % The Fisher Equation button
    btnNumber=1;
yPos=top-(btnNumber-1)*(btnHt+spacing);
labelStr='Fisher Equation';
callbackStr='deniz(''grafbutton'')';
cmdStr=str2mat( ...
    'x=-10:1:10;',...
    't=0:1:20;',...
    'b1=0;',...
    'b2=1;',...
    'b3=0.4;',...
    'a1=b1-b3;',...
    'a2=b2-b3;',...
    '[xx,tt] = meshgrid(x,t);',...
    'yy=a1+exp(0.1*tt+(1/sqrt(2))*xx);',...
    'zz=a2+exp(0.1*tt+(1/sqrt(2))*xx);',...
    'uu1=b3+a2*(yy./zz);',...
    ' [xx,tt] = meshgrid(x,t);',...
    'yy=a1+exp(0*tt+(1/sqrt(2))*xx);',...
    'zz=a2+exp(0*tt+(1/sqrt(2))*xx);',...
    'uu2=b3+a2*(yy./zz);',...
    'figure(1);',...
    'mesh(uu1);',...
    'figure(2);',...
    'mesh(uu1);',...
    'figure(3);',...
    'mesh(uu2);'...
        );
```

```matlab
% Generic button information
btnPos=[left yPos-btnHt btnWid btnHt];
uicontrol( ...
    'Style','pushbutton', ...
    'Units','normalized', ...
    'Position',btnPos, ...
    'String',labelStr, ...
    'Callback',callbackStr, ...
    'UserData',cmdStr);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The Burger Equation button
    btnNumber=2;
    yPos=top-(btnNumber-1)*(btnHt+spacing);
    labelStr='Burger Equation';
    callbackStr='deniz(''grafbutton'')';
    cmdStr=str2mat( ...
        'x=-10:1:10;',...
        't=0:1:20;',...
        'b1=0;',...
        'b2=1;',...
        'b3=0.4;',...
        'a1=b1-b3;',...
        'a2=b2-b3;',...
        '[xx,tt] = meshgrid(x,t);',...
        'yy=a1+exp(0.6*tt-1*xx);',...
        'zz=a2+exp(0.6*tt-1*xx);',...
        'uu1=b3+a2*(yy./zz);',...
        '[xx,tt] = meshgrid(x,t);',...
        'yy=a1+exp(-0.15*tt+0.5*xx);',...
```

```
'zz=a2+exp(-0.15*tt+0.5*xx);',...
'uu2=b3+a2*(yy./zz);',...
'figure(1);',...
'mesh(uu1);',...
'figure(2);',...
'mesh(uu1);',...
'figure(3);',...
'mesh(uu2);'...
 );




% Generic button information
btnPos=[left yPos-btnHt btnWid btnHt];
uicontrol( ...
    'Style','pushbutton', ...
    'Units','normalized', ...
    'Position',btnPos, ...
    'String',labelStr, ...
    'Callback',callbackStr, ...
    'UserData',cmdStr);


%===================================
% The Transport Equation button
btnNumber=3;
yPos=top-(btnNumber-1)*(btnHt+spacing);
labelStr='Transport Equation';
callbackStr='deniz(''grafbutton'')';
cmdStr=str2mat( ...
    'x=-10:1:10;',...
    't=0:1:20;',...
    'b1=0;',...
```

```
        'b2=1;',...
        'b3=0.4;',...
        'a1=b1-b3;',...
        'a2=b2-b3;',...
        '[xx,tt] = meshgrid(x,t);',...
        'yy=a1+exp(0.356611*tt - 0.869834*xx);',...
        'zz=a2+exp(0.356611*tt - 0.869834*xx);',...
        'uu1=b3+a2*(yy./zz);',...
        '[xx,tt] = meshgrid(x,t);',...
        'yy=a1+exp(-0.112167*tt + 0.536501*xx);',...
        'zz=a2+exp(-0.112167*tt + 0.536501*xx);',...
        'uu2=b3+a2*(yy./zz);',...
        'figure(1);',...
        'mesh(uu1);',...
        'figure(2);',...
        'mesh(uu1);',...
        'figure(3);',...
        'mesh(uu2);'...
         );


% Generic button information
btnPos=[left yPos-btnHt btnWid btnHt];
uicontrol( ...
    'Style','pushbutton', ...
    'Units','normalized', ...
    'Position',btnPos, ...
    'String',labelStr, ...
    'Callback',callbackStr, ...
    'UserData',cmdStr);


%==================================
% The Convection-Transport Equation button
```

```
btnNumber=4;

yPos=top-(btnNumber-1)*(btnHt+spacing);

labelStr='Convection-Transport Equation';

callbackStr='deniz(''grafbutton'')';

cmdStr=str2mat( ...

    'x=-10:1:10;',...

    't=0:1:20;',...

    'b1=0;',...

    'b2=1;',...

    'b3=0.4;',...

    'a1=b1-b3;',...

    'a2=b2-b3;',...

    '[xx,tt] = meshgrid(x,t);',...

    'yy=a1+exp(0.749797*tt - 0.68313*xx);',...

    'zz=a2+exp(0.749797*tt - 0.68313*xx);',...

    'uu1=b3+a2*(yy./zz);',...

    '[xx,tt] = meshgrid(x,t);',...

    'yy=a1+exp(-0.616463*tt + 0.68313*xx);',...

    'zz=a2+exp(-0.616463*tt + 0.68313*xx);',...

    'uu2=b3+a2*(yy./zz);',...

    'figure(1);',...

    'mesh(uu1);',...

    'figure(2);',...

    'mesh(uu1);',...

    'figure(3);',...

    'mesh(uu2);'...

      );




% Generic button information

btnPos=[left yPos-btnHt btnWid btnHt];
```

```
uicontrol( ...
    'Style','pushbutton', ...
    'Units','normalized', ...
    'Position',btnPos, ...
    'String',labelStr, ...
    'Callback',callbackStr, ...
    'UserData',cmdStr);


%=================================

% The Dissipative Equation button
btnNumber=5;
yPos=top-(btnNumber-1)*(btnHt+spacing);
labelStr='Dissipative Equation';
labelList=' cyan| red| green| blue| magenta| yellow';
cmdStr=str2mat( ...
    'x=-10:1:10;',...
    't=0:1:20;',...
    'b1=0;',...
    'b2=1;',...
    'b3=0.4;',...
    'a1=b1-b3;',...
    'a2=b2-b3;',...
    '[xx,tt] = meshgrid(x,t);',...
    'yy=a1+exp(0.1*tt - 0.714143*xx);',...
    'zz=a2+exp(0.1*tt - 0.714143*xx);',...
    'uu1=b3+a2*(yy./zz);',...
    '[xx,tt] = meshgrid(x,t);',...
    'yy=a1+exp(0.1*tt + 0.714143*xx);',...
    'zz=a2+exp(0.1*tt + 0.714143*xx);',...
    'uu2=b3+a2*(yy./zz);',...
    'figure(1);',...
```

```
        'mesh(uu1);',...
        'figure(2);',...
        'mesh(uu1);',...
        'figure(2);',...
        'mesh(uu1);',...
        'figure(3);',...
        'mesh(uu2);'...
            );




% Generic button information
btnPos=[left yPos-btnHt btnWid btnHt];
uicontrol( ...
        'Style','pushbutton', ...
        'Units','normalized', ...
        'Position',btnPos, ...
        'String',labelStr, ...
        'Callback',callbackStr, ...
        'UserData',cmdStr);


%====================================
% The Mixed Equation button
btnNumber=6;
yPos=top-(btnNumber-1)*(btnHt+spacing);
labelStr='Mixed Equation';
callbackStr='deniz(''grafbutton'')';
cmdStr=str2mat( ...
        'x=-10:1:10;',...
        't=0:1:20;',...
        'b1=0;',...
        'b2=1;',...
```

```
    'b3=0.4;',...

    'a1=b1-b3;',...

    'a2=b2-b3;',...

    '[xx,tt] = meshgrid(x,t);',...

    'yy=a1+exp(1.10401*tt - 1.03735*xx);',...

    'zz=a2+exp(1.10401*tt - 1.03735*xx);',...

    'uu1=b3+a2*(yy./zz);',...

    '[xx,tt] = meshgrid(x,t);',...

    'yy=a1+exp(-0.837348*tt + 0.904015*xx);',...

    'zz=a2+exp(-0.837348*tt + 0.904015*xx);',...

    'uu2=b3+a2*(yy./zz);',...

    'figure(1);',...

    'mesh(uu1);',...

    'figure(2);',...

    'mesh(uu1);',...

    'figure(2);',...

    'mesh(uu1);',...

    'figure(3);',...

    'mesh(uu2);'...

        );




% Generic button information

btnPos=[left yPos-btnHt btnWid btnHt];

uicontrol( ...

    'Style','pushbutton', ...

    'Units','normalized', ...

    'Position',btnPos, ...

    'String',labelStr, ...

    'Callback',callbackStr, ...

    'UserData',cmdStr);
```

```
%==================================

% The CLOSE button
uicontrol( ...
    'Style','push', ...
    'Units','normalized', ...
    'Position',[left bottom btnWid btnHt], ...
    'String','Close', ...
    'Callback','close(gcf)');

% Now uncover the figure
set(figNumber,'Visible','on');
 title([' Samples']);

elseif strcmp(action,'grafbutton'),
cmdStr=get(gco,'UserData');
mcwHndl=get(gcf,'UserData');
set(mcwHndl,'String',cmdStr);
evalmcw(mcwHndl);


elseif strcmp(action,'eval'),
cla reset;
mcwHndl=get(gcf,'UserData');
evalmcw(mcwHndl);

elseif strcmp(action,'info'),
helpwin(mfilename);

end;    % if strcmp(action, ...
```