Q-PERIODICITY, SELF-SIMILARITY AND WEIERSTRASS-MANDELBROT FUNCTION

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ABSTRACT

Q-PERIODICITY, SELF-SIMILARITY AND WEIERSTRASS-MANDELBROT FUNCTION

In the present thesis we study self-similar objects by method's of the q-calculus. This calculus is based on q-rescaled finite differences and introduces the q-numbers, the q-derivative and the q-integral. Main object of consideration is the Weierstrass-Mandelbrot functions, continuous but nowhere differentiable functions. We consider these functions in connection with the q-periodic functions. We show that any q-periodic function is connected with standard periodic functions by the logarithmic scale, so that q-periodicity becomes the standard periodicity. We introduce self-similarity in terms of homogeneous functions and study properties of these functions with some applications. Then we introduce the dimension of self-similar objects as fractals in terms of scaling transformation. We show that q-calculus is proper mathematical tools to study the self-similarity. By using asymptotic formulas and expansions we apply our method to Weierstrass-Mandelbrot function, convergency of this function and relation with chirp decomposition.

ÖZET

Q-PERİODİKLİK, KENDİNE BENZERLİK VE WEIERSTRASS-MANDELBROT FONKSİYONU

Bu tezde q-hesaplama metodlarıyla kendine benzeyen nesneler çalışılmıştır. Bu hesaplama metodu q-yeniden ölçeklendirilen sonlu farklar ve tanımlanan q-sayılar, q-türev ve q-integral temeline dayanmaktadır. Ana nesne olarak her yerde sürekli fakat hiçbir yerde türevi olmayan Weierstrass-Mandelbrot fonksiyonları düşünülmüştür. Bu fonksiyonların q-periyodik fonksiyonlarla bağlantılı olduğu düşünülmüştür. Herhangi bir q-periyodik fonksiyonun, logaritmik ölçek altında standart periyodik fonksiyonlarla bağlantısı gösterilmiş, böylece q-periyodiklik, standart periyodiklik olmuştur. Kendine benzerlik yerine homojen fonksiyonlar tanımlanmış ve bu fonksiyonların özellikleri bazı uygulamalarla birlikte çalışılmıştır. Fraktallar gibi kendine benzeyen nesneler için ölçek dönüşümü altında boyut kavramı tanımlanmıştır. Kendine benzer nesneler üstünde çalışmak için q-hesaplama, özel bir matematiksel metod olarak gösterilmiştir. Bazı asimptotik formüller ve açılımlar kullanılarak Weierstrass-Mandelbrot fonksiyonunun yakınsaklığı ve bu fonksiyonun chirp ayrışması ile ilgisi gösterilmiştir.

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CHAPTER 1

INTRODUCTION

"Geometry has two great treasures; one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name precious jewel. " J. Kepler

From ancient times, proportions in architecture, in human body, in nature, in music,etc. are related with concept of beauty, and become the origin of first mathematical discoveries. Proportions by integer numbers form the counting as an origin of arithmetic, number theory and music. And this proportions are used for the first measurement. Then, the rational numbers and music proportions were explored by Pythagoreans. Counting fractional parts by rational numbers increase the precision of measurement by many times. When different measurement units were invented (depending on instruments), it becomes necessary to connect different measurement scales. These scales are chosen similar geometrical shape. This relativity of scales is related with invariance of an object to different instruments, measuring its size. This way the scale invariance becomes important concept of modern science.

Mathematically the scale invariance is related with dilatation of space and can be formulated as property of the self-similarity. Idea of the homogeneous function fixes in exact form, what means the self-similarity. A function of one variable $f(x)$ is said to be scale invariant if under re-scaling of the argument we get the same function, up to the multiplication constant: $f(qx) = Cf(x)$. In many situations constant C is a function of scale parameter q: $C = F(q)$ so that $f(qx) = F(q)f(x)$. Exact definition of the scale invariant or homogeneous function, fixes this function as some power of q: $F(q) = q^d$. Then we have definition $f(qx) = q^d f(x)$ for some exponent $d \in \mathbb{R}$. This definition can be extended to the homogeneous functions of several variables.

The famous Euler theorem, and homogeneous ordinary differential equations, are two remarkable examples in mathematics, related to these functions. It turns out that these functions describe many interesting objects with self-similar properties. We mention first that the mechanical similarity, when the potential energy is a homogeneous function of coordinates. For example in the Kepler problem. Then possible to make general conclusions about behavior such systems (Landau & Lifshitz, 1960).

Another application field is probability distribution as a homogeneous function, which is subject of study the random walk on hierarchical lattices (Erzan & Gorbon, 1999) and probability densities of homogeneous functions and networks (Blumenfeld, 1988).

In Economics, the homogeneity (or scale invariance) means that on the real market with normal concurrence, does not exist any special time interval. It can be formulated as: Let $X(t)$ describes changes of the price, then the function $\ln X(t)$ has next property: the increment distribution at any time interval, $\ln X(t+d) - \ln X(t)$, does not depend on d (Mandelbrot & Hudson, 2004).

In music, the musical scale is related with musical harmony. In ancient time music was considered as a strictly mathematical discipline, handling with number relationships, ratios and proportions. In the quadrivium (the curriculum of the Pythagorean School) the music was placed on the same level as arithmetic, geometry and astronomy. Music was the science of sound and harmony. People had realized very early that two different notes do not always sound pleasant when play together. Moreover the ancient Greeks discovered that to a note with a given frequency only a those other notes whose frequencies were integer multiples of the first one, could be properly combined. If, for example, a note of the frequency 220 Hz was given the notes of frequencies 440 Hz, 660 Hz, 880 Hz, 1100 Hz and so on sounded best when played together with the first one. Furthermore, examination of different sounds showed that these integer, multiples of the base frequency, always appear in a weak intensity when the basic note is played. If a string whose length defines a frequency 220 Hz is vibrating, the general sound also contains components of the frequencies 440 Hz, 660 Hz, 880 Hz, 1100 Hz and so on. The most important frequency ratio $1:2$ is called octave in the Western system of music notation. The different notes in such a relation are often considered as principally the same, only varying in their pitch but not their character. The Greek saw in octave the 'cyclic identity'. The following ratios build the musical fifth $(2:3)$, fourth $(3:4)$, major third $(4:5)$ and minor third $(5:6)$. These ratios corresponds not only to the sounding frequencies but also relative to string lengths. All this studies of 'harmonic' ratios and proportions were the essence of music during Pythagorean times (Rothwell, 1977).

The human ear has a logarithmic response to sound so that the perceived difference between notes on a scale is the same if their frequencies are spaced as a power law. For keyboard instruments the entire frequency range can be partitioned into a number of discrete notes spaced at equal logarithmic intervals. And in this context appearance of logarithmic scale is natural due to logarithmic spiral structure of our ears.

The logarithm spiral (see Fig.2.2) has the self-similarity property. The self similarity means a similar shape to the original one after scaling. Remarkable geometrical objects realizing idea of self-similarity are fractals, introduced and intensively studied by Mandelbrot (Mandelbrot, 1982). In his paper, Mandelbrot used the coast of Britain to show how an object might have a longer length the smaller the increment of measurement. He described how Britain could be measured with a long ruler to give a rough approximation of the length of the coast. Then make that ruler half as long, and the approximation will be more similar to the true object. One can continue this process many, many times and never level off; one could never know exactly how long the coast of Britain is (if it were a true fractal). In his paper, Mandelbrot used the work of English meteorologist Lewis Richardson, who discovered that the length of a coastline grows the smaller the unit used to measure it. In fractals we have fixed scale for self-similar object so that it is related to the dimension of this object. For fractals this dimension is different from integer valued (topological dimension).

Self-similarity is also used in images and fractal image compression. The advance of the information age the need for mass information storage and fast communication links grows. Storing images in less memory leads to a direct reduction in storage cost and faster data transmissions. These facts justify the efforts, of private companies and universities, on new image compression algorithms. This algorithm is fractal image compression (Hezar, 1997).

Another application of self-similarity is statistically self-similar signals and signal processing with fractals. A random process is statistically self-similar with parameter H if for any real $a > 0$ it obeys the scaling relation $x(t) = a^{-H}x(at)$ (Borgnat & Flandin, 2002). Self-similar solutions: $T(x, t_1)$ at the some moment t_1, is similar to solution $T(x, t_0)$ at some previous moment t_0.

As a non-differentiable objects, prolegomena of fractals started from the end of the nineteenth century, when some mathematicians visualized that it was possible to find a class of functions that were continuous everywhere but nowhere differentiable. Karl Weierstrass was one of the first to propose such functions. Extension of this function by Mandelbrot, called the Weierstrass-Mandelbrot function is an example of fractal in graph of the function. And it appears in the signal processing and wavelets theory. For example, the chirp signal processing; the chirp (Compressed High Intensity Radar Pulse) techniques have been used for a number of years above the water in many commercial and military radar systems. The techniques used to create an electromagnetic chirp pulse have now been modified and adapted for acoustic imaging sonar systems.

It is worse to notice that in XIX century also new type of calculus take the origin the so called q-calculus. The q-Calculus, is based on the finite difference re-scaling. First results in q-Calculus belong to Euler, who discovered Euler's Identities for q-exponential functions and Gauss, who discovered the q-binomial formula. These results lead to an intensive research on q-Calculus in XIX century. Discovery of Heine's formula (Heine, 1846) for a q-Hypergeometric function as a generalization of the hypergeometric series and relation with the Ramanujan product formula; relation between Euler's identities and the Jacobi Triple product identity, are just few of the remarkable results obtained in this field. Euler's infinite product for the classical partition function, Gauss formula for number of sums of two squares, Jacobi's formula for the number sums of four squares are natural outcomes of q-Calculus. The systematic development of q-calculus begins from F.H.Jackson who in 1908 reintroduced the Euler-Jackson q-difference operator (Jackson,1908). Integral as a sum of finite geometric series has been considered by Archimedes, Fermat and Pascal (Andrew & Askey, 1999). Fermat introduced the first q-integral of the particular function $f(x)\,x^{\alpha}$ by introducing the Fermat measure at q-lattice points $x = aq^n$. Then Thomae in 1869 and Jackson in 1910 defined general q-integral on finite interval (Ernst, 2001). Subjects involved in modern q-Calculus include combinatorics, number theory, quantum theory, quantum groups, quantum exactly solvable systems, statistical mechanics. In the last 30 years q-calculus becomes a bridge between mathematics and physics and intensively used by physicist. It turns out that the q-Calculus is best adapted for studying the self-similar systems. A q-periodic functions as a solution of the functional equation $f(qx) = f(x)$ or $D_q f(x) = 0$ plays in the theory of the q-difference equations the role similar to an arbitrary constant in the differential equations. The famous Weierstrass-Mandelbrot function, which is continuous but nowhere differentiable, is related with q-periodic function. In XX century it becomes connected with structure of fractal sets discovered by Mandelbrot (Mandelbrot, 1982). It seams that q-calculus is most suitable mathematical techniques to study the fractals.

In this thesis we are going to study the self-similar objects like fractals in the form of the Weierstrass-Mandelbrot function by the method of the q-calculus. Those functions have proved to be very useful to simulate irregular patterns found in nature and are interesting mathematical objects. The thesis organized as follows;

In Chapter 2 we introduce the basic concepts of q-Calculus as q-number. Specially the q-periodic functions are described in details. Then we introduce the concept of self-similarity and homogeneous functions.Results related with homogeneous functions and some of their applications in geometry and theory of differential equations are presented in Section 2.3.1. In Section 2.3.2 we discuss dimension of the self similar objects. And in Section 2.3.3, relations of these objects with q-Calculus are established. At the end of Chapter 2 the Mellin transform and logarithmic scale are derived. In our study we follow notations from book of Kac and Cheung (Kac & Cheung, 2002).

In Chapter 3 we study some basic asymptotic formulas which are necessary to understand convergency of infinite sum (Weierstrass-Mandelbrot function). We start from definitions and theorems on asymptotic expansions and till Bernoulli polynomials, the Gamma and Beta functions. The last Section 3.1.6 is dealing with the Euler-Maclaurin formula and Stirling's asymptotic formula.

In Chapter 4 we apply all above results to study the Weierstrass-Mandelbrot function and applications in signal processing like the chirp decomposition. In Section 4.1. we discuss the history of continuous but nowhere differentiable function (Weierstrass-Mandelbrot function). And Section 4.1.1 we study the self-similarity property of the Weierstrass-Mandelbrot function. In Section 4.1.2 we show a relation between q-periodic function and the Weierstrass-Mandelbrot function.In Section 4.1.3 we show convergency of Weierstrass-Mandelbrot function and in Section 4.1.4 we get the Mellin expansion of q-periodic functions. We plot the graphs of Weierstrass-Mandelbrot functions in Section 4.1.5. In Section 4.2.1 we consider a relation between the stationarity and the self-similarity. We show that the shift invariant functions are stationary and the scale invariant functions are self similar. And in Section 4.2.2 we define transformation from the shift invariance (Fourier transform) to the scale invariance (Mellin transform), which we call the Lamperti transformation. As a tone is the building block for the Fourier transform, chirp is the building block for the Mellin transform. At the end of Chapter 4 we study the chirp decomposition of Weierstrass-Mandelbrot function.

In conclusions we summarize main results obtained in this thesis.

CHAPTER 2

QUANTUM CALCULUS AND SELF-SIMILARITY

2.1. q-Calculus

The quantum calculus (q-calculus) is an old, classical branch of mathematics, which can be traced back to Euler and Gauss with important contributions of Jackson a century ago. In recent years there are many new developments and applications of the q-calculus in mathematical physics, especially concerning special functions and quantum mechanics. In this section, we shall give some definitions and properties of q-calculus.

Definition 2.1.0.1 *For any positive integer number n,*

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + ... + q^{n-1}, \tag{2.1}$$

is called q-basic number of n. As $q \to 1$, we have $[n]_q = 1 + ... + q^{n-1} = 1 + ... + 1 = n$. As we shall see $[n]_q$ plays the same role in q-calculus as the integer n does in ordinary calculus.

We can extend our definition of $[n]_q$;

Definition 2.1.0.2 *For any real number ζ,*

$$[\zeta]_q = \frac{q^\zeta - 1}{q - 1}. \tag{2.2}$$

Example 2.1 *Let us compute $[1]_q, [2]_{\sqrt{2}}, [\infty]_q$;*

1. $[1]_q = \frac{q^1 - 1}{q - 1} = 1.$
2. $[2]_{\sqrt{2}} = \frac{\sqrt{2}^2 - 1}{\sqrt{2} - 1} = \sqrt{2} + 1.$
3. $[\frac{1}{2}]_q = \frac{q^{1/2} - 1}{q - 1} = 1 - q^{1/2} + q^{1/4} + ..., \qquad q < 1.$
4. $[\infty]_q = 1 + q + q^2 + ... = \sum_{j=0}^{\infty} q^j = \frac{1}{(1-q)} \qquad |q| < 1.$

We can also extend the definition of q-number (2.1) to complex number;

Definition 2.1.0.3 *For any complex number z,*

$$[z]_q = \frac{q^z - 1}{q - 1}. \tag{2.3}$$

If $z = x + iy$ then we get the q-complex number as follows;

$$\begin{aligned} [z]_q = [x+iy]_q &= \frac{q^{x+iy} - 1}{q-1} \\ &= \frac{(q^x e^{iy \ln q}) - 1}{q-1} \\ &= \frac{(q^x(\cos(y \ln q) + i \sin(y \ln q))) - 1}{q-1} \\ &= \frac{q^x \cos(y \ln q) - 1}{q-1} + i \frac{q^x \sin(y \ln q)}{q-1}. \end{aligned} \tag{2.4}$$

As easy to see, the q-complex valued number is a complex function. In addition, this function is a holomorphic function, since,

$$\frac{\partial}{\partial \overline{z}}[z]_q = \frac{\partial}{\partial \overline{z}} \left(\frac{q^z - 1}{q-1} \right) = 0. \tag{2.5}$$

So the function

$$\begin{aligned} [z]_q = \frac{q^z - 1}{q-1} &= \frac{e^{z \ln q} - 1}{q-1} \\ &= \frac{1}{1-q} + \sum_{n=0}^{\infty} \frac{(\ln q)^n}{n!} z^n, \end{aligned} \tag{2.6}$$

is analytic in whole complex plane z, and it is an entire function of z. Therefore we can extend definition of q-number to q-operator.

Example 2.2 *We consider q-number operator of $h\frac{d}{dx}$ operator and if we choose $q = e$ then we get*

$$\left[h \frac{d}{dx} \right]_e = \frac{e^{h \frac{d}{dx}} - 1}{e - 1}, \tag{2.7}$$

and if we apply this operator to the function of f(x) we get

$$\left[h\frac{d}{dx}\right]_e f(x) = \frac{1}{1-e}(e^{h\frac{d}{dx}}f(x) - f(x)) = \frac{h}{1-e}\frac{f(x+h)-f(x)}{h} = \frac{h}{1-e}D_h f(x).$$

Thus $h\frac{d}{dx} \xrightarrow{[\,]_e} \frac{h}{1-e}D_h f(x)$ *where* D_h *is the h-derivative.*

Another examples are related to the exponential mapping, the angular momentum and the q-matrix operator.

Example 2.3 *We choose again* $q = e$ *and we consider the exponential mapping. The exponential mapping carries a Lie algebra to a Lie group. This means, that if* A *is the element of a Lie algebra and if we apply to this the operator,* $\exp$ *(exponential map), then we get* $\exp A$ *which is the element of the Lie group. By applying the q-number operator, we get*

$$[A]_e = \frac{\exp A - 1}{e-1} = \frac{1}{e-1}\exp A - \frac{1}{e-1}. \tag{2.8}$$

Thus we again get the element of Lie group.

Example 2.4 *We consider angular momentum operator* $\vec{L} = (\vec{r} \times \vec{p})$ *where* $\vec{r}$ *is the position operator and* $\vec{p}$ *is the momentum operator. In quantum mechanics* $\vec{L}$*,* $\vec{p}$*,* $\vec{r}$ *are operators having representation in cartesian coordinates* $\vec{L} = \{L_x, L_y, L_z\}$*,* $\vec{p} = \{p_x, p_y, p_z\}$ *and* $\vec{r} = (x, y, z)$*. Thus*

$$\begin{aligned} L_x &= y\,p_z - z\,p_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y &= z\,p_x - x\,p_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z &= x\,p_y - y\,p_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \end{aligned} \tag{2.9}$$

and also

$$L^2 = L_x^2 + L_y^2 + L_z^2. \tag{2.10}$$

In Cartesian coordinates the commutation relations between L_i ($i = x, y, z$) are

$$\begin{aligned}
[L_x, L_y] &= i\hbar L_z \\
[L_y, L_z] &= i\hbar L_x \\
[L_z, L_x] &= i\hbar L_y.
\end{aligned} \tag{2.11}$$

The commutation relations between x, y, z components of the angular momentum operator in quantum mechanics form a representation of a three-dimensional Lie algebra, which is the Lie algebra $so(3)$ of the three-dimensional rotation group. Applying the previous example, and choosing $A = i\varphi L_z = \varphi\hbar\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)$ and $q = e$ then we get

$$[i\varphi L_z]_e = \frac{e^{\varphi\hbar\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)} - 1}{e - 1}. \tag{2.12}$$

Therefore the result is: $i\varphi L_z \xrightarrow{[\,]_e} \frac{e^{i\varphi L_z} - 1}{e - 1}$.

Example 2.5 *The Pauli matrices form a set of three 2×2 complex matrices which are Hermitian and unitary:*

$$\sigma_{\mathbf{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_{\mathbf{2}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_{\mathbf{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

The Pauli matrices (after multiplication by i to make them anti-hermitian), also generate transformations in the sense of Lie algebras, and the matrices $i\sigma_1$,$i\sigma_2$,$i\sigma_3$ form a basis for $su(2)$, which exponentiates to the spin group $SU(2)$. Now if we apply $[\,]_e$ to $ia\sigma_2$ then we get

$$[ia\sigma_2]_e = \frac{e^{ia\sigma_2} - 1}{e - 1}, \tag{2.14}$$

and $e^{ia\sigma_2} = I\cos(a) + i\sigma_2 \sin(a)$ where I is the 2×2 identity matrix. Thus we get

$$[ia\sigma_2]_e = \frac{1}{e-1}[I\cos(a) + i\sigma_2\sin(a)] + \frac{I}{1-e}. \tag{2.15}$$

Definition 2.1.0.4 *Consider an arbitrary function $f(x)$. Its q- differential is defined as*

$$d_q f(x) = f(qx) - f(x). \tag{2.16}$$

Definition 2.1.0.5 *The following expression*

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \tag{2.17}$$

is called the q-derivative of the function $f(x)$.

Note that, for $f(x)$ differentiable

$$\lim_{q \to 1} D_q f(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{(q-1)x} = \lim_{q \to 1} f'(qx) = \frac{df(x)}{dx}. \tag{2.18}$$

Example 2.6 *Compute the q-derivative of $f(x) = x^n$ where n is positive integer. By definition,*

$$D_q x^n = \frac{qx^n - x^n}{(q-1)x} = \frac{q^n - 1}{(q-1)} x^{n-1}, \tag{2.19}$$

equation (2.19) becomes

$$D_q x^n = [n]_q x^{n-1}. \tag{2.20}$$

Proposition 2.1.0.6 *For any two functions f(x) and g(x) the following properties hold;*

1. $D_q(f(x) + g(x)) = D_q f(x) + D_q g(x),$ *(q-sum rule)*
2. $D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x),$ *(q-Leibniz rule)(*)*
3. $D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x),$ *(q-Leibniz rule)(**)*
4. $D_q(\frac{f(x)}{g(x)}) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)},$ $g(x) \neq 0$ *(q-quotient rule)*

Proof 2.1.0.7 *Firstly we consider q-sum rule;*

$$D_q(f(x)+g(x)) = \frac{d_q(f(x)+g(x))}{(q-1)x} = \frac{(f(qx)-f(x)+g(qx)-g(x))}{(q-1)x}, \tag{2.21}$$

and hence,

$$D_q(f(x)+g(x)) = D_q f(x) + D_q g(x). \tag{2.22}$$

Secondly we consider q-Leibniz rules;

$$D_q f(x)g(x) = \frac{d_q(f(x)g(x))}{(q-1)x} = \frac{f(qx)d_q g(x) + g(x)d_q f(x)}{(q-1)x}, \tag{2.23}$$

and hence,

$$D_q f(x)g(x) = f(qx)D_q g(x) + g(x)D_q f(x). \tag{2.24}$$

By the symmetry, we can interchange f and g and obtain

$$D_q f(x)g(x) = f(x)D_q g(x) + g(qx)D_q f(x), \tag{2.25}$$

which is equivalent to q-Leibniz rule.
Finally we consider q-quotient rule, if we apply q-Leibniz rule to derivative of

$$g(x)\frac{f(x)}{g(x)}, \qquad\qquad g(x) \neq 0, \tag{2.26}$$

we obtain

$$g(qx)D_q\Big(\frac{f(x)}{g(x)}\Big) + \frac{f(x)}{g(x)}D_q g(x) = D_q f(x), \tag{2.27}$$

and thus,

$$D_q\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(x)g(qx)}. \tag{2.28}$$

However if we use (2.25), we get

$$g(x)D_q\Big(\frac{f(x)}{g(x)}\Big) + \frac{f(qx)}{g(x)}D_qg(qx) = D_qf(x) \tag{2.29}$$

and

$$D_q\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(qx)D_qf(x) - f(qx)D_qg(x)}{g(x)g(qx)}.\blacksquare \tag{2.30}$$

2.2. q-Periodic Functions

Definition 2.2.0.8 *A function f is called a $q-periodic$ if it satisfies*

$$D_qf(x) \equiv 0 \qquad \textit{or} \qquad f(x) \equiv f(qx). \tag{2.31}$$

Example 2.7 *Consider first order q-difference equation*

$$D_qf(x) = 1, \tag{2.32}$$

the general solution is

$$f(x) = x + C_q(x), \tag{2.33}$$

where $C_q(x)$ is a q-periodic function.

In the limit $q \to 1$ the first order q-difference equation (2.32) reduces to the first order differential equation;

$$\frac{d}{dx} f(x) = 1, \tag{2.34}$$

with the general solution

$$f(x) = x + c, \tag{2.35}$$

where c is an arbitrary constant, $c = \lim_{q \to 1} C_q(x)$ and $D_q c = 0$. Therefore the constant c is a q-periodic function for an arbitrary values of q.

Example 2.8 *Consider $f(x) = \sin\left(\frac{2\pi}{\ln q} \ln x\right)$, for $q > 0$ and $q \neq 1$. This function is a q-periodic function, since*

$$\begin{aligned} f(qx) &= \sin\left(\frac{2\pi}{\ln q} \ln qx\right) = \sin\left(\frac{2\pi}{\ln q}(\ln q + \ln x)\right) \\ &= \sin\left(2\pi + \frac{2\pi}{\ln q} \ln x\right) = \sin\left(\frac{2\pi}{\ln q} \ln x\right) = f(x). \end{aligned} \tag{2.36}$$

Example 2.9 *Consider the Euler differential equation,*

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \omega^2 y = 0. \tag{2.37}$$

By substitution $x = e^t$ and using

$$x \frac{d}{dx} = \frac{d}{dt}, \tag{2.38}$$

then we obtain the harmonic oscillator equation,

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0. \tag{2.39}$$

The particular solutions of equation (2.39) are

$$y(t) = Ae^{\pm i\omega t}. \tag{2.40}$$

They imply the solutions of the Euler equation,

$$y(x) = Ae^{\pm i\omega \ln x}. \tag{2.41}$$

These functions are q-periodic $D_q y(x) = 0$ with $q = e^{\frac{2\pi}{\omega}}$. If we summarize above results then the Euler differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{4\pi^2}{(\ln q)^2} y = 0, \tag{2.42}$$

has the general solution

$$y(x) = A \cos\left(\frac{2\pi}{\ln q} \ln x\right) + B \sin\left(\frac{2\pi}{\ln q} \ln x\right), \tag{2.43}$$

which is q-periodic ($y(qx) = y(x)$).

Definition 2.2.0.9 *A q-dilatation operator M_q is defined as*

$$M_q \equiv q^{x\partial_x} \equiv e^{x\partial_x \ln q}, \qquad \partial_x = \frac{\partial}{\partial x}. \tag{2.44}$$

With using formal expansion in the Taylor series its action on the power-law function gives

$$M_q x^n = q^{x\partial_x} x^n = \sum_{m=0}^{\infty} \frac{[\ln q (x\partial_x)]^m}{m!} x^n = \sum_{m=0}^{\infty} \frac{(n \ln q)^m}{m!} x^n = (qx)^n, \tag{2.45}$$

and we get

$$M_q x^n = (qx)^n. \tag{2.46}$$

Then for an arbitrary analytical function f(x) one obtains in a similar manner ;

$$M_q f(x) = \sum_{m=0}^{\infty} \frac{[\partial_x f(x)|_{x=0}]^m}{m!} M_q x^m = \sum_{m=0}^{\infty} \frac{[\partial_x f(x)|_{x=0}]^m}{m!} (qx)^m = f(qx), \tag{2.47}$$

and we get

$$M_q f(x) = f(qx). \tag{2.48}$$

Definition 2.2.0.10 *The function F(x) is a q-antiderivative of f(x) if $D_q F(x) = f(x)$ and it is denoted by*

$$\int f(x)\, d_q x. \tag{2.49}$$

The question is if q-antiderivative is unique. From ordinary calculus we know the next;

Theorem 2.2.0.11 *(Mean-Value Theorem)(Thomas, 2009)*
Suppose f(x) is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{2.50}$$

Corollary 2.2.0.12 *If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.*

Corollary 2.2.0.13 *If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$. That is, $f - g$ is a constant on (a, b).*

In ordinary calculus the above theorem and corollaries show that an antiderivatives is unique up to constant. But the situation in quantum calculus is different. Given $D_q f(x) = 0$ if and only if $f(qx) = f(x)$, which does not necessarily imply that f is a constant. However if we take

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{2.51}$$

the condition $f(qx) = f(x)$ implies $q^n a_n = a_n$ for each n. It is possible only when $a_n = 0$ for any $n \geq 1$. So that f is a constant. Therefore, if f(x) is a formal power series, then f(x) has a unique q-antiderivative up to a constant term, which is

$$\int f(x)\, d_q x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{[n+1]_q} + C, \tag{2.52}$$

where C is an arbitrary constant. If f(x) is a general function, then we use the following proposition;

Proposition 2.2.0.14 *(Kac&Cheung, 2002) Let $0 < q < 1$. Then, up to adding a constant, any function f(x) has at most one q-antiderivative that is continuous at $x = 0$.*

The proposition tells us that the uniqueness of the q-antiderivative is required by continuity at $x = 0$.

Now we like to find q-antiderivative of a function in an explicit form. Suppose f(x) is an arbitrary function. To construct its q-antiderivative F(x), recall the operator M_q, defined by $M_q(F(x)) = F(qx)$. Then we have by the definition of a q-derivative:

$$\frac{1}{(q-1)x}(M_q - 1)F(x) = \frac{F(qx) - F(x)}{(q-1)x} = f(x). \tag{2.53}$$

Note that the order is important, because operators $(1 - M_q)$ and $\frac{1}{(q-1)x}$ do not commute. We can then formally write the q-antiderivative as

$$F(x) = \frac{1}{(1 - M_q)}((1-q)xf(x)) = (1-q)\sum_{j=0}^{\infty} {M_q}^j (xf(x)), \tag{2.54}$$

using the geometric series expansion, and thus we get;

Definition 2.2.0.15

$$\int f(x)\, d_q x = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x). \tag{2.55}$$

This series is called the q-integral (the Jackson integral) of $f(x)$.

From this definition one easily derives a more general formula:

$$\begin{aligned} \int f(x)\, D_q g(x)\, d_q x &= (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) D_q g(q^j x) \\ &= (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) \frac{g(q^j x) - g(q^{j+1} x)}{(1-q)q^j x} \\ &= \sum_{j=0}^{\infty} f(q^j x)(g(q^j x) - g(q^{j+1} x)). \end{aligned} \tag{2.56}$$

The integral given in (2.55) does not always converge to a real valued function F(x), even if the q-antiderivative exists. We now examine some of the cases under which the Jackson Integral converges to a q-antiderivative.

Theorem 2.2.0.16 *(Kac&Cheung, 2002) Suppose $0 < q < 1$. If $|f(x)x^\alpha|$ is bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then the Jackson Integral defined by (2.55) converges to a function F(x) on $(0, A]$, which is a q-antiderivative of f(x). Moreover, F(x) is continuous at $x = 0$ with $F(0) = 0$.*

Proof 2.2.0.17 *Suppose $|f(x)x^\alpha| < M$ on $(0, A]$. Fix $x \in (0, A]$. Then for all $j \geq 0$,*

$$|f(q^j x)| < M(q^j x)^{-\alpha}. \tag{2.57}$$

Multiplying by q^j, we have;

$$|q^j\, f(q^j x)| < M\, q^j\, (q^j x)^{-\alpha}. \tag{2.58}$$

Taking the sum from $j = 0$ to $j = \infty$ it follows that;

$$\sum_{j=0}^{\infty} \left|q^j\, f(q^j x)\right| < \sum_{j=0}^{\infty} M\, x^{-\alpha} (q^{1-\alpha})^j = \frac{M\, x^{-\alpha}}{1 - q^{1-\alpha}}, \tag{2.59}$$

since $1 - \alpha > 0$ and $0 < q < 1$. Thus, the sum in the Jackson Integral is majorized by a convergent geometric series, and so this sum converges to a function F(x). We observe from (2.55) that $F(0) = 0$. To prove that F(x) is continuous at $x = 0$, we observe that for $0 < x \leq A$,

$$(1-q)x \sum_{j=0}^{\infty} \left|q^j\, f(q^j x)\right| < \frac{M(1-q)\, x^{1-\alpha}}{1 - q^{1-\alpha}}, \tag{2.60}$$

which approaches 0 as $x \to 0$, since $1 - q > 0$. To verify the definition of the Jackson Integral given in (2.55) is a q-antiderivative of f(x), we can q-differentiate it;

$$\begin{aligned} D_q F(x) &= \frac{1}{(q-1)x}((1-q)qx \sum_{j=0}^{\infty} q^j f(q^{j+1}x) - (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x)) \\ &= -(\sum_{j=0}^{\infty} q^{j+1} f(q^{j+1}x) - \sum_{j=0}^{\infty} q^j f(q^j x)) \\ &= \sum_{j=0}^{\infty} q^j f(q^j x) - \sum_{j=1}^{\infty} q^j f(q^j x) \\ &= f(x). \blacksquare \end{aligned} \tag{2.61}$$

Example 2.10 *Let $f(x) = x^n$ where n is positive integer.*

$$\begin{aligned} \int x^n\, d_q x &= (1-q)x \sum_{j=0}^{\infty} q^j\, q^{jn}\, x^n \\ &= (1-q)x^{n+1} \sum_{j=0}^{\infty} q^{j(n+1)} \\ &= \frac{1-q}{1-q^{n+1}} x^{n+1} = \frac{x^{n+1}}{[n+1]_q}. \end{aligned} \tag{2.62}$$

Definition 2.2.0.18 *Suppose $0 < a < b$. The definite q-integral is defined as;*

$$\int_0^b f(x)\, d_q x = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b), \tag{2.63}$$

and

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x, \tag{2.64}$$

a more general formula:

$$\int_0^b f(x)\, d_q g(x) = \sum_{j=0}^{\infty} f(q^j b)(g(q^j b) - g(q^{j+1} b)). \tag{2.65}$$

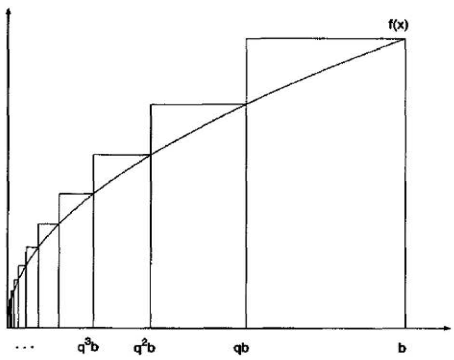

Figure 2.1. The definite q-integral correspond to the area of the union of an infinite number of rectangles.

On the interval $[\epsilon, b]$, where ϵ is a small positive number, the sum consists of finitely many terms, and is in fact a Riemann sum. Therefore, as $q \to 1$, the width of rectangles approaches zero in Fig.(2.1), and the sum tends to the Riemann integral on $[\epsilon, b]$. Since ϵ is arbitrary, we thus have, $f(x)$ is continuous in the interval $[0, b]$.

Example 2.11 *Let's evaluate the Jackson integral for* $f(x) = \ln x$ *and compare it with the standard integral when* $q \to 1$*;*

$$\begin{aligned}
\int_0^b f(x)\, d_qx &= (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b); \\
\int_0^b \ln x\, d_qx &= (1-q)b\sum_{j=0}^{\infty} q^j \ln(q^j b) \\
&= (1-q)b\sum_{j=0}^{\infty} q^j [j\ln q + \ln b] \\
&= (1-q)b(\sum_{j=0}^{\infty} q^j j\ln q + \sum_{j=0}^{\infty} q^j \ln b) \\
&= (1-q)b\ln q\sum_{j=0}^{\infty} q^j j + (1-q)b\ln b\sum_{j=0}^{\infty} q^j \\
&= (1-q)b\ln q\frac{q}{(1-q)^2} + (1-q)b\ln b\frac{1}{(1-q)}. \qquad (2.66)
\end{aligned}$$

Therefore we get;

$$\int_0^b \ln x\, d_qx = \frac{b\,q\ln q}{(1-q)} + b\ln b. \qquad (2.67)$$

Here we use

$$\sum_{j=0}^{\infty} q^j j = q\frac{d}{dq}\sum_{j=0}^{\infty} q^j = q\frac{d}{dq}\frac{1}{(1-q)} = \frac{q}{(1-q)^2} \qquad when \qquad |q| < 1. \qquad (2.68)$$

Let's check as equation (2.67) in the limit $q \to 1$ *reduces to the standard definite integral;*

$$\begin{aligned}
\lim_{q\to 1}(\int_0^b \ln x\, d_qx) &= \lim_{q\to 1}\big(\frac{b\,q\ln q}{(1-q)} + b\ln b\big) \\
&= b\,\ln b + \lim_{q\to 1}\Big(\frac{b\,q\ln q}{(1-q)}\Big) \\
&= b\,\ln b - b = \int_0^b \ln x\, dx. \qquad (2.69)
\end{aligned}$$

Definition 2.2.0.19 *The improper q-integral of f(x) on* $[0, +\infty)$ *is defined to be*

$$\int_0^\infty f(x)\, d_qx = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}}^{q^j} f(x)\, d_qx, \qquad 0 < q < 1 \tag{2.70}$$

or equivalently

$$\int_0^\infty f(x)\, d_qx = \sum_{j=-\infty}^{\infty} \int_{q^j}^{q^{j+1}} f(x)\, d_qx, \qquad q > 1. \tag{2.71}$$

Theorem 2.2.0.20 *(Kac&Cheung, 2002) The improper q-integral defined above converges if* $x^\alpha f(x)$ *is bounded when x is the neighborhood of* $x = 0$ *for some* $\alpha < 1$ *and for sufficiently large x for some* $\alpha > 1$.

Proof 2.2.0.21 *We have,*

$$\int f(x)\, d_qx = |1 - q| \sum_{j=-\infty}^{\infty} q^j f(q^j). \tag{2.72}$$

If we split up the summation

$$\sum_{j=-\infty}^{\infty} q^j f(q^j) = \sum_{j=0}^{\infty} q^j f(q^j) + \sum_{j=1}^{\infty} q^{-j} f(q^{-j}), \tag{2.73}$$

is the same whether we have q *or* q^{-1}, *we can consider, without loss of generality the case where* $q < 1$. *The first sum converges by the proof in Theorem* 2.2.0.15. *For the second sum, if we suppose that for large x,* $|f(x)x^\alpha| < M$ *for some* $\alpha > 1$ *and* $M > 0$. *Then for sufficiently large j,*

$$|q^{-j} f(q^{-j})| = q^{j(\alpha-1)} |q^{-j\alpha} f(q^{-j})| < M\, q^{j(\alpha-1)}. \tag{2.74}$$

So the second sum is smaller than a convergent geometric series, and thus converges as well, meaning the whole summation converges.■

Theorem 2.2.0.22 *(Fundamental Theorem of q-calculus) (Kac&Cheung, 2002)*
If F(x) is an antiderivative of f(x) and F(x) is continuous at $x = 0$ we have

$$\int_a^b f(x)d_qx = F(b) - F(a), \tag{2.75}$$

where $0 \leq a, b \leq \infty$.

2.3. q-Calculus on a Fractal Sets

In this section we apply q-calculus to the fractal sets, because it is dealing with re-scaling of functions, measured by q-derivative. In fractals, the self similarity property under re-scaling is crucial for definition of fractals.

2.3.1. Homogeneous Functions and Euler's Theorem

Definition 2.3.1.1 *For any $d \in \mathbb{R}$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree d if*

$$f(\lambda x) = \lambda^d f(x), \tag{2.76}$$

where $\forall\lambda > 0$ and $x \in \mathbb{R}^n$. A function is homogeneous of degree d for some $d \in \mathbb{R}$.

For example, the function $f(x) = 3x^2$ is a homogenous function of degree 2, $f(x, y, z) = xy^2 + z^3$ is a homogeneous function of degree 3, but $f(x, y) = e^{xy} - xy$ is not homogeneous function.

Proposition 2.3.1.2 *Let f be a differentiable function of n variables that is continuous of degree d. Then each of its partial derivatives f_i' (for i=1,...,n) is homogeneous of degree $d - 1$.*

Proof 2.3.1.3 *The homogeneity function means that*

$$f(\lambda x_1, ..., \lambda x_n) = \lambda^d f(x_1, ..., x_n), \qquad \forall\lambda > 0, \tag{2.77}$$

Now differentiate both sides of this equation with respect to x_i, to get

$$\lambda f_i'(\lambda x_1, ..., \lambda x_n) = \lambda^d f_i'(x_1, ..., x_n), \tag{2.78}$$

and divide both sides by λ to get

$$f_i'(\lambda x_1, ..., \lambda x_n) = \lambda^{d-1} f_i'(x_1, ..., x_n), \tag{2.79}$$

so that f_i' is homogeneous of degree $d-1$.■

Example 2.12 *This result can be used to demonstrate a nice result about the slope of the level curves of a homogeneous function. The slope of the level curve of the function f through (x_1, x_2) at this point is*

$$-\frac{\partial f/\partial x_1(x_1, x_2)}{\partial f/\partial x_2(x_1, x_2)} = -\frac{f_1'(x_1, x_2)}{f_2'(x_1, x_2)}, \tag{2.80}$$

assuming $f_2'(x_1, x_2) \neq 0$ and suppose f homogeneous function of degree d, and consider the level curve through (cx_1, cx_2) for some $c > 0$. At (cx_1, cx_2), the slope of this curve is

$$-\frac{f_1'(cx_1, cx_2)}{f_2'(cx_1, cx_2)}. \tag{2.81}$$

f_1' and f_2' are homogeneous of degree $d-1$, so this slope is equal to

$$-\frac{c^{d-1} f_1'(x_1, x_2)}{c^{d-1} f_2'(x_1, x_2)} = -\frac{f_1'(x_1, x_2)}{f_2'(x_1, x_2)}. \tag{2.82}$$

That is the slope of level curve through (cx_1, cx_2) at the point (cx_1, cx_2) is exactly the same as the slope of the level curve through (x_1, x_2) at the point (x_1, x_2). Let f be a differentiable function of two variables that is homogeneous of some degree. Then along any given ray from the origin, the slopes of the level curves of f are the same.

Theorem 2.3.1.4 *(Euler's Theorem)*
Let $f(x_1, ..., x_n)$ be a homogeneous function of degree d. That is

$$f(\lambda x_1, ..., \lambda x_n) = \lambda^d f(x_1, ..., x_n). \tag{2.83}$$

Then the following identity holds

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d\, f(x_1, ..., x_n). \tag{2.84}$$

Proof 2.3.1.5 *To prove's Euler's theorem, simply differentiate the homogeneity condition (2.83) with respect to λ;*

$$\frac{d}{d\lambda} f(\lambda x_1, ..., \lambda x_n) = \frac{d}{d\lambda}[\lambda^d f(x_1, ..., x_n)], \tag{2.85}$$

we get,

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial \lambda x_i} = d\lambda^{d-1} f(x_1, ..., x_n). \tag{2.86}$$

Then setting $\lambda = 1$, we have

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d\, f(x_1, ..., x_n).\blacksquare \tag{2.87}$$

Condition (2.84) may be written more compactly, using the notation $\vec{\nabla} f$ for the gradient vector of f and letting $x = (x_1, ..., x_n)$ as

$$x\vec{\nabla} f(x) = d.f(x), \qquad \text{for all } x. \tag{2.88}$$

Therefore the homogeneous functions are eigen-functions of Euler operator with degree of homogeneity as the eigen-value.

2.3.2. Mechanical Similarity and Scale Invariance

Mathematically any function $f(x)$ that satisfies equation (2.76) for an arbitrary λ is called homogeneous function. A homogeneous function is scale invariant for an arbitrary λ i.e., if we change the scale of measuring x so that $x \to x'(\equiv \lambda x)$, the new function $\hat{f}(x')(\equiv f(x))$ still has the same shape as the old one $f(x)$. This fact is guaranteed since $f(x) = \lambda^{-d} f(x')$ by equation (2.76) and hence $\hat{f}(x') \sim f(x)$. Now we give definition of scale invariance;

Definition 2.3.2.1 *A function f(x) is said to be scale-invariant if it satisfies the following property;*

$$f(\lambda x) = \lambda^d f(x), \tag{2.89}$$

for some choice of exponent $d \in \mathbb{R}$ and fixed scale factor $\lambda > 0$, which can be taken to be a length or size of re-scaling.

Here we like to stress the difference between homogeneous function and self-similar object. Homogeneous function satisfies the self-similarity condition for some fixed value of re-scaling λ, while homogeneous function is self-similar for all possible values of re-scaling λ.

Now we consider mechanical problems with potential energy in the form of homogeneous function. The potential energy is a homogeneous function of the coordinates and the problem is referred as the mechanical similarity (Landau & Lifshitz, 1960). This means potential energy

$$U(\lambda r_1, ..., \lambda r_n) = \lambda^d U(r_1, ..., r_n). \tag{2.90}$$

Here, d is the degree of homogeneity of U.

Now we re-scale equations of motion, suppose re-scaling of space and time is;

$$r_i = \alpha \hat{r}_i, \qquad t = \beta \hat{t} \tag{2.91}$$

Then

$$\frac{dr_i}{dt} = \frac{\alpha}{\beta}\frac{d\hat{r}_i}{d\hat{t}}, \qquad \frac{d^2 r_i}{dt^2} = \frac{\alpha}{\beta^2}\frac{d^2\hat{r}_i}{d\hat{t}^2} \tag{2.92}$$

The force F_i is given by

$$\begin{aligned} F_i &= -\frac{\partial}{\partial r_i}U(r_1, ..., r_n) \\ &= -\frac{\partial}{\partial \alpha \hat{r}_i}\alpha^d U(\hat{r}_1, ..., \hat{r}_n) \\ &= \alpha^{d-1}\hat{F}_i \end{aligned} \tag{2.93}$$

or equivalently

$$F_i = \alpha^{d-1}\hat{F}_i. \tag{2.94}$$

Thus, Newton's second law say,

$$\frac{\alpha}{\beta^2} m_i \frac{d^2\hat{r}_i}{d\hat{t}^2} = \alpha^{d-1}\hat{F}_i. \tag{2.95}$$

If we require

$$\frac{\alpha}{\beta^2} = \alpha^{d-1} \Rightarrow \beta = \alpha^{1-\frac{1}{2}d}, \tag{2.96}$$

then the equations of motion are invariant under the re-scaling transformation. This means that if $r(t)$ is a solution of the equations of motion, then so is $\alpha r(\alpha^{\frac{1}{2}d-1}t)$. If $r(t)$ is periodic with period T, then $r_i(t;\alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}d}T$. Thus

$$\frac{T'}{T} = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}d}. \tag{2.97}$$

Here, $\alpha = \frac{L'}{L}$ is the ratio of length scales. Velocities, energies and angular momenta are scaled accordingly;

$$v = \frac{L}{T} \qquad \Rightarrow \qquad \frac{v'}{v} = \frac{L'/L}{T'/T} = \alpha^{\frac{1}{2}d}, \tag{2.98}$$

$$E = \frac{ML^2}{T^2} \qquad \Rightarrow \qquad \frac{E'}{E} = \frac{(L'/L)^2}{(T'/T)^2} = \alpha^{d}, \tag{2.99}$$

$$\mathbf{L} = \frac{ML^2}{T} \qquad \Rightarrow \qquad \frac{\mathbf{L}'}{\mathbf{L}} = \frac{(L'/L)^2}{(T'/T)} = \alpha^{1+\frac{1}{2}d}. \tag{2.100}$$

Example 2.13 *Let us apply the mechanical similarity to the harmonic oscillator, the potential energy is $U = kx^2$ where k is the spring constant and x is the vibration amplitude. In this problem $d = 2$ and therefore,*

$$\frac{T'}{T} = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}d} = \left(\frac{L'}{L}\right)^{0} = 1, \tag{2.101}$$

the time-scale ratio is unity. It means that the frequencies for both re-scaled systems are the same. In other words, the frequencies of a lumped spring-mass system is unaffected by their vibration amplitudes.

$$x(t) \to x(t;\alpha) = \alpha x(t). \tag{2.102}$$

Thus, re-scaling lengths alone gives another solution.

Example 2.14 *Let us now consider the motion of two satellites around heavenly body. Newton's law of universal gravitational states*

$$F = \frac{G\,M\,m}{r^2} \qquad \Rightarrow \qquad U = -\frac{G\,M\,m}{r}, \tag{2.103}$$

where G is the universal gravitational constant, M and m are the masses of the heavenly body and the satellite and r is the distance between two bodies. Here $d = -1$. Thus

$$r(t) \to r(t;\alpha) = \alpha r(\alpha^{-\frac{3}{2}}t). \tag{2.104}$$

Thus, $r^3 \propto t^2$ *i.e.*

$$\frac{T'}{T} = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}d} = \left(\frac{L'}{L}\right)^{3/2} \tag{2.105}$$

or equivalently

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{L'}{L}\right)^3, \tag{2.106}$$

which states that square of the revolution ratio of the two satellites is proportional to the cube of the ratio of the orbital sizes (the Kepler third law).

2.3.3. Self-Similar Objects and Their Dimensions

The scale invariance means that if a part of a system is magnified to the size of the original system, this magnified part and the original system will look similar to each other. Therefore, scale invariant system must be self-similar and vice-versa. In this section we define self-similar objects and their dimensions.

Definition 2.3.3.1 *A self-similar object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts).*

Example 2.15 *We can consider the Logarithmic spiral for the self-similarity under rotation. Its defining equation in polar coordinates (r,θ) is :*

$$r(\theta) = r_0 e^{d\theta}, \tag{2.107}$$

where r_0 *and d are arbitrary real constants and* $r_0 > 0$.

A logarithmic spiral is given by equation parametric equation;

$$x(\theta) = r(\theta)\cos(\theta) = r_0 e^{d\theta}\cos(\theta) \tag{2.108}$$

$$y(\theta) = r(\theta)\sin(\theta) = r_0 e^{d\theta}\sin(\theta), \tag{2.109}$$

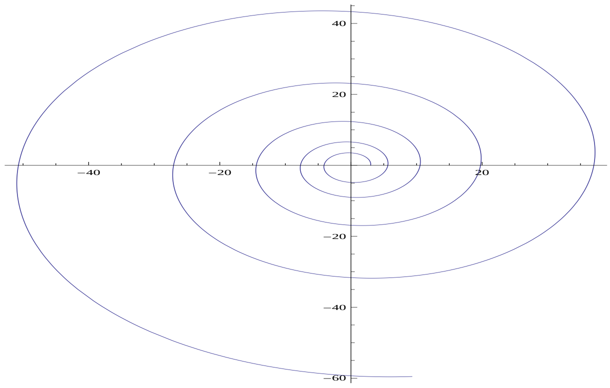

Figure 2.2. The logarithmic spiral, $r_0 = 3$, $d = 0.1$, $0 \le \theta \le 60$

.

and we rotate the curve an angle 2π and define a new curve $\hat{r}(\theta)$ of $r(\theta)$ as follows;

$$\hat{r}(\theta) = r(\theta + 2\pi) = e^{2\pi d}(r_0 e^{d\theta}) = e^{2\pi d} r(\theta). \tag{2.110}$$

As we see from the above equation, the curve $\hat{r}(\theta)$ can be obtained by scaling $r(\theta)$, by the factor $e^{2\pi d}$. Therefore the logarithmic spiral has the self-similarity under rotation.

At the beginning of this section we mentioned that the self-similar object must be self-similar. The following example shows the relation between scale-invariance and self-similar object.

Example 2.16 *Consider a straight line segment. Dividing the segment into N self-similar pieces by applying a ruler of length η, the length of the segment is then;*

$$L(\eta) = \eta N. \tag{2.111}$$

If $L(\eta) = 1$, then the ruler must have length $\eta = 1/N$ to exactly cover the line. Similarly the square of area $L^2(\eta)$ can be covered by N square elements, each of area η^2, so that

$$L^2(\eta) = \eta^2 N. \tag{2.112}$$

We have $\eta = (1/N)^{1/2}$ for $L^2(\eta) = 1$. In a similar way in three dimensions, a unit cube *is covered by N elementary cubes of side* $\eta = (1/N)^{1/3}$*. Generalizing for an arbitrary integer dimension* d *the volume is*

$$L^d(\eta) = \eta^d N, \tag{2.113}$$

and for $L^d(\eta) = 1$ *we have* $\eta^d N = 1$*. It implies* $\eta = (1/N)^{1/d}$ *and,*

$$N = \eta^{-d}. \tag{2.114}$$

Note that in each of these examples we constructed smaller objects of the same geometrical shape as the larger object in order to cover it. This geometrical equivalence is the basis of our notion of self-similarity. We can also see here the scale invariance law; For a line ($d = 1$*)*

$$\begin{aligned} L(\eta) &= \eta N \\ L(\lambda\eta) &= (\lambda)^d \eta N \\ &= \lambda\eta N = \lambda L(\eta). \end{aligned} \tag{2.115}$$

For a square ($d = 2$*),*

$$\begin{aligned} L^2(\eta) &= N\eta^2 \\ L^2(\lambda\eta) &= (\lambda)^d \eta N \\ &= (\lambda)^2 \eta N = (\lambda)^2 L^2(\eta). \end{aligned} \tag{2.116}$$

So we can write general formula for scale-invariance of the hyper-cubes volume in d*-dimensions,*

$$L^d(\lambda\eta) = \lambda^d L^d(\eta). \tag{2.117}$$

Here showing that it is the homogeneous function of degree d*.* λ *is an arbitrary scale* *factor and self-similar dimension is d. Explicit formula for the volume of hyper-cubes*

$$\begin{aligned}
V_1(x_1) &= x_1 \\
V_2(x_1, x_2) &= x_1.x_2 \\
\ldots & \quad \ldots\ldots. \\
V_n(x_1, x_2, \ldots, x_n) &= x_1.x_2 \ldots x_n,
\end{aligned} \tag{2.118}$$

shows that it is a homogeneous function degree n:

$$V(qx_1, qx_2, \ldots, qx_n) = q^n.V(x_1, x_2, \ldots, x_n). \tag{2.119}$$

In the above example we defined the dimension of objects. Now we give the formal definition of dimension;

Definition 2.3.3.2 *Equation (2.114) can be used to define the dimension d of a set in terms of the number N of elementary covering elements (of length,area,volume,etc.) that are constructed from basic intervals of length η. Taking the logarithm of both sides of (2.114) and rearranging yields*

$$d = \frac{\ln N}{\ln(1/\eta)}. \tag{2.120}$$

Example 2.17 *If we apply this formula for straight line in Fig.(2.3) $N = 3$, $\eta = 1/3$ then we get;*

$$d = \frac{\ln N}{\ln(1/\eta)} = \frac{\ln 3}{\ln 3} = 1. \tag{2.121}$$

If we apply this formula for square in Fig.(2.3) $N = 9$, $\eta = 1/3$ then we get;

$$d = \frac{\ln N}{\ln(1/\eta)} = \frac{\ln 9}{\ln 3} = 2. \tag{2.122}$$

Similar calculation for cube in Fig.(2.3), we get $d = 3$.

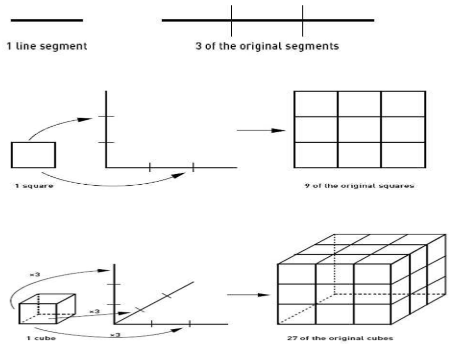

Figure 2.3. Geometrical objects for integer dimension.

As we expected, for smooth curve's (straight line), for surface's (square) and for volume's (cube) dimension d is integer valued. This take place for the objects which are smooth. If the dimension d is an integer then we call topological dimension. But in general the dimension d does not necessarily be integer as clear from (2.120). If the dimension d is non-integer then we will call it the self-similar dimension. A fractal is by definition a set for which the self-similar dimension strictly different from the topological dimension. Now we apply definition of self-similar dimension to non-integer valued object (a geometrical fractal).

Example 2.18 *Let us consider the Cantor set. This set is constructed by starting with the line segment of unit length and removing the middle third. This leaves two line segments,each of length* $\eta(1) = 1/3$ *at the first generation,* $k = 1$. *We then remove the middle third from each of these two line segments, leaving four line segments, each of length* $\eta(2) = 1/9$, *at the second generation,* $k = 2$. *Continuing this process, at the kth generation there are a total of* $N(k) = 2^k$ *line segments,each of length* $\eta(k) = 3^{-k}$.

Using the values of N and η *at the kth generation and then taking the limit as the number of generations goes to infinity,* $k \to \infty$, *we obtain for the self-similar dimension;*

$$d = \lim_{k\to\infty} \frac{\ln N(k)}{\ln(1/\eta(k))} = \lim_{k\to\infty} \frac{k \ln 2}{k \ln 3} = \frac{\ln 2}{\ln 3} \approx 0.6309. \tag{2.123}$$

Thus, this dimension classifies the set as being between a line $(d = 1)$ *and a point* $(d = 0)$.

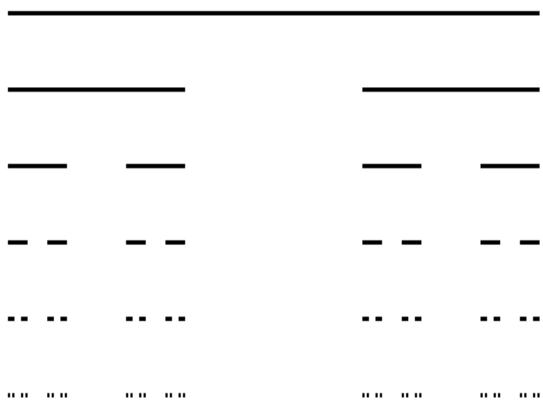

Figure 2.4. The steps of Cantor set.

The Cantor set is self-similar object with re-scaling parameter $\lambda = \frac{1}{3}$*. This means that the above recursion steps can be considered as images of the Cantor set at different scales. Number of these scales is infinite but countable.*

Example 2.19 *Another example is the Koch snowflake curve. This closed plane curve has an infinite length,but encloses a finite area. Starting with an equilateral triangle (the generator), the second stage is generated by replacing middle third of each line in the generator by a scaled down version of the generator. In Fig.*2.5 *the scaled-down version of the triangle is* $1/3$ *of the size of the generator in the preceding generation. Continuing this procedure result in a curve that is the limit of an infinite number of generations. Unlike the case of the middle-third Cantor set, where each line segment at the preceding stage, the Koch snowflake generates four new line segments for each line segment at the preceding stage.*

Thus, in the Koch snowflake, the length of a line segment at the kth stage is $\eta(k) = 3^{-k}$ *just as before, however the number of line segments is* $N(k) = 4^k$*. The dimensionality of the limiting set is therefore given by*

$$d = \lim_{k\to\infty} \frac{\ln N(k)}{\ln(1/\eta(k))} = \lim_{k\to\infty} \frac{k \ln 4}{k \ln 3} = \frac{\ln 4}{\ln 3} = 2\frac{\ln 2}{\ln 3} \approx 1.2618. \tag{2.124}$$

So that the self-similar dimension of the Koch snowflake is twice that of the middle-third Cantor set.

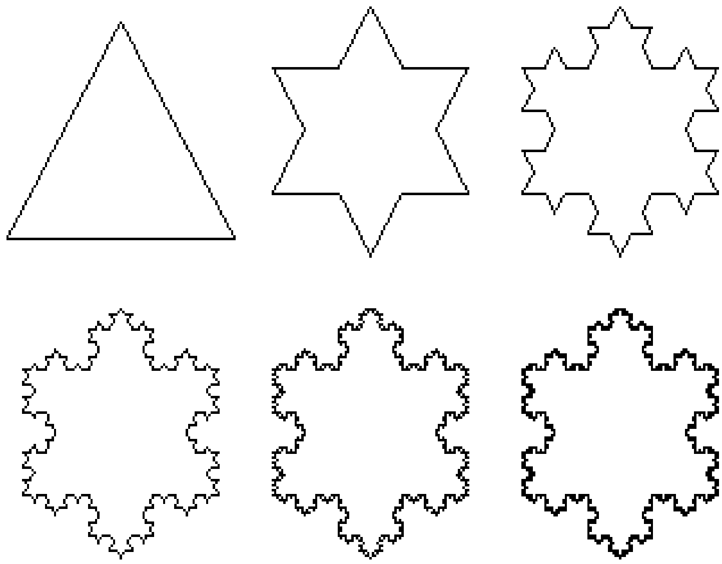

Figure 2.5. The steps of Koch snowflake.

Thus, this dimension classifies the set as being between a plane and a line.

2.3.4. Self-Similar Sets and q-calculus

In this section we show how to relate self-similar objects with q-calculus. By application of q-dilatation operator in (2.44) to the scale-invariant(as well as to homogenous) function f(x), satisfying;

$$f(qx) = q^d f(x), \tag{2.125}$$

where $\lambda = q$ in (2.89) and d is an arbitrary real number, we get the eigenvalue equation

$$M_q f(x) = f(qx) = q^d f(x), \tag{2.126}$$

for the q-dilatation operator M_q. This means that the scale invariant function is eigenfunction of the q-dilatation operator with eigen-value q^d.

On the other hand, the q-derivative of a function is defined as

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} = \frac{M_q - 1}{(q-1)x} f(x). \tag{2.127}$$

Now we are going to apply this definition to the scale invariant function. For this first we prove the next proposition.

Proposition 2.3.4.1 *The ordinary commutator of q-derivative and x gives the q-dilatation operator;*

$$D_q x - x D_q = [D_q, x] = M_q. \tag{2.128}$$

Proof 2.3.4.2 *Let's apply the* $[D_q, x]$ *to f(x);*

$$\begin{aligned} [D_q, x] f(x) &= D_q(x f(x)) - x(D_q f(x)) \\ &= f(qx) D_q x + x(D_q f(x)) - x(D_q f(x)) \\ &= f(qx) = M_q f(x). \blacksquare \end{aligned} \tag{2.129}$$

Next we have the following proposition;

Proposition 2.3.4.3 *A homogenous function f of degree* d *satisfies*

$$(x D_q) f(x) = [d]_q f(x). \tag{2.130}$$

Here $[d]_q$ *is the q-basic number*

$$[d]_q = \frac{q^d - 1}{q - 1}. \tag{2.131}$$

Proof 2.3.4.4 *From equation (3.73) we know that;*

$$M_q f(x) = q^d f(x). \tag{2.132}$$

Then applying the commutator

$$\begin{aligned}
[D_q, x]f(x) &= q^d f(x) \\
D_q(xf(x)) - x(D_q f(x)) &= q^d f(x) \\
qxD_q f(x) + f(x)(D_q x) - x(D_q f(x)) &= q^d f(x) \\
(q-1)xD_q f(x) &= (q^d - 1)f(x) \\
xD_q f(x) &= \frac{q^d - 1}{q - 1} f(x).
\end{aligned} \tag{2.133}$$

Finally for the self-similar function f we get the q-difference equation with fixed q;

$$(xD_q)f(x) = [d]_q f(x).\blacksquare \tag{2.134}$$

This equation is valid also for homogeneous function, but for any base q. Now we consider the general solution of this q-difference equation. We consider two cases. In the first case, d is positive integer number. Suppose that $f(x) = \sum_{k=0}^{\infty} a_k x^k$, is analytic in a disk, then

$$xD_q\Big(\sum_{k=0}^{\infty} a_k x^k\Big) = [d]_q\Big(\sum_{k=0}^{\infty} a_k x^k\Big) \tag{2.135}$$

or equivalently

$$x\Big(\sum_{k=1}^{\infty} a_k x^{k-1}[k]_q\Big) = [d]_q\Big(\sum_{k=0}^{\infty} a_k x^k\Big), \tag{2.136}$$

and hence

$$\Big(\sum_{k=1}^{\infty} a_k [k]_q x^k\Big) = [d]_q\Big(\sum_{k=0}^{\infty} a_k x^k\Big) \tag{2.137}$$

or equivalently

$$a_1[1]_q x + a_2[2]_q x^2 + ... = [d]_q(a_0 + a_1 x + a_2 x^2 + ...). \tag{2.138}$$

Comparing equal power terms, we find $a_0 = a_1 = ... = 0$ except $a_d \neq 0$ and only non-vanishing term is with $k = d$. It gives solution $f(x) = a_d x^d$ where a_d is a constant or a q-periodic function. In the second case, d is non-integer number. We have no power series solution of this equation.

Instead of this we consider an Ansatz $f(x) = ax^d$ where a is an arbitrary constant or a q-periodic function, so that equation is satisfied automatically.

As a result we found the general solution of the q-difference equation (2.130), in the following form,

$$f(x) = A_q(x)x^d, \tag{2.139}$$

where $A_q(x)$ is a q-periodic function.

As we can see, this solution is composed from the homogeneous function x^d and the q-periodic function $A_q(x)$. This solution is self-similar with scale factor q. Since a function q-periodic for all q is just a constant function, the solution for all q is just homogeneous function. Note that the following series ;

$$A_q(x) = x^{-\alpha} \sum_{n=-\infty}^{\infty} q^{-n\alpha} g(q^n x), \tag{2.140}$$

represents a q-periodic function $A_q(qx) = A_q(x)$, where function $g(x)$ is continuously differentiable at $x = 0$ and $\alpha > 0$, $q \neq 1$. Indeed,

$$\begin{aligned} A_q(qx) &= q^{\alpha} x^{-\alpha} \sum_{n=-\infty}^{\infty} q^{-n\alpha} g(q^{n+1}x) \\ &= x^{-\alpha} \sum_{n=-\infty}^{\infty} q^{-(n+1)\alpha} g(q^{n+1}x) \\ &= A_q(x). \end{aligned} \tag{2.141}$$

Therefore the general solution of equation (2.130) has in the following form;

$$\begin{aligned} f(x) &= A_q(x)x^d \\ &= x^{d-\alpha} \sum_{n=-\infty}^{\infty} q^{-nd} g(q^n x). \end{aligned} \tag{2.142}$$

Example 2.20 *Consider $g(x) = \sin x$, if we substitute equation (2.142) we get;*

$$A_q(x) = x^{-\alpha} \sum_{n=-\infty}^{\infty} \frac{\sin(q^n x)}{q^{n\alpha}}, \qquad 0 < \alpha < 1, \qquad q > 1. \tag{2.143}$$

This function is q-periodic since;

$$\begin{aligned} D_q A_q(x) &= \frac{A_q(qx) - A_q(x)}{(q-1)x} \\ &= \frac{(qx)^{-\alpha} \sum_{n=-\infty}^{\infty} q^{-n\alpha} \sin(q^{n+1}x) - x^{-\alpha} \sum_{n=-\infty}^{\infty} q^{-n\alpha} \sin(q^n x)}{(q-1)x} \\ &= \frac{x^{-\alpha}}{(q-1)x} \left(\sum_{n=-\infty}^{\infty} q^{-(n+1)\alpha} \sin(q^{n+1}x) - \sum_{n=-\infty}^{\infty} q^{-n\alpha} \sin(q^n x) \right) = 0. \end{aligned}$$

Example 2.21 *Consider $g(x) = 1 - e^{ix}$ then, if we substitute equation (2.142) we get;*

$$A_q(x) = x^{-\alpha} \sum_{n=-\infty}^{\infty} \frac{1 - e^{iq^n x}}{q^{n\alpha}}, \qquad 0 < \alpha < 1, \qquad q > 1. \tag{2.144}$$

This function is q-periodic since $D_q A_q(x) = 0$ and is called the q-periodic part of Weierstrass-Mandelbrot function.

Proposition 2.3.4.5 *The q-periodic function $A_q(x)$ is either a constant or a function that is periodic in $t = lnx$, with period $T = lnq$.*

Proof 2.3.4.6 *If $A_q(x)$ is q-periodic function then*

$$D_q A_q(x) = 0 \qquad or \qquad A_q(qx) = A_q(x). \tag{2.145}$$

If $A_q(x)$ is a constant function then (2.145) is satisfied automatically. In more general case we have by change of variables

$$A_q(qx) = A_q(x) \Rightarrow A_q(e^t e^T) = A_q(e^t) \Rightarrow F(t+T) = F(t). \tag{2.146}$$

where $t = \ln x$, $T = \ln q$ and $F(t) \equiv A_q(e^t)$. This implies that function $F(t)$ is periodic with period $T = \ln q$, and $A_q(x) = F(\ln x)$.■

Proposition 2.3.4.7 *If F has period T, $F(t + T) = F(t)$ and is integrable over $[-T, T]$, then it can be expanded to series*

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T}}, \tag{2.147}$$

with coefficients;

$$c_n = \frac{1}{T}\int_0^T F(t)e^{-\frac{i2\pi nt}{T}}dt, \tag{2.148}$$

is called the Fourier series for F; the numbers c_n are called the Fourier coefficients of F.

Proof 2.3.4.8 *If f(z) analytic in annular domain then it can be expanded to the Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \qquad c_n = \frac{1}{2\pi i}\oint_{|z|=1} \frac{f(z)}{z^{n+1}}dz. \tag{2.149}$$

If we apply this expansion to $|z| = 1$, so that $z = e^{it}$, then

$$f(e^{it}) \equiv F(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \tag{2.150}$$

and since $z = e^{it}$ with $0 \le t \le 2\pi$, one has $dz = izdt$ and the formula for c_n becomes

$$c_n = \frac{1}{2\pi i}\oint_{|z|=1} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{e^{i(n+1)t}}\,e^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}dt. \tag{2.151}$$

This function F(t) is periodic with $T = 2\pi$.

For function F(t) periodic with an arbitrary period T, $F(t + T) = F(t)$, by re-

scaling the argument we get

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T}}, \qquad c_n = \frac{1}{T}\int_0^T F(t)e^{-\frac{i2\pi nt}{T}}dt.\blacksquare \tag{2.152}$$

According to this result and proposition (2.3.4.5), an arbitrary q-periodic function analytic in an annular domain can be represented by complex series

$$A_q(x) = F(\ln x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi n}{T}t}, \tag{2.153}$$

where $t = \ln x, T = \ln q$. As a result we get next representation of q-periodic function;

$$A_q(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi n}{\ln q}\ln x} = \sum_{n=-\infty}^{\infty} c_n x^{i\frac{2\pi n}{\ln q}}, \tag{2.154}$$

where

$$c_n = \frac{1}{T}\int_1^{e^T} A_q(x)x^{-i\frac{2\pi n}{T}}\frac{dx}{x} = \frac{1}{\ln q}\int_1^q A_q(x)x^{-i\frac{2\pi n}{\ln q}}\frac{dx}{x}. \tag{2.155}$$

Then combining the above result we get;

Proposition 2.3.4.9 *The self-similar function f(x) as a solution of equation (2.130) can be represent in the following form;*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n x^{d_n}, \tag{2.156}$$

where $d_n = d + i\frac{2\pi n}{\ln q}$ *and*

$$c_n = \frac{1}{\ln q}\int_1^q f(x)x^{d_n}\frac{dx}{x}. \tag{2.157}$$

Proof 2.3.4.10 *The general solution of equation (2.130)*

$$f(x) = A_q(x)x^d, \tag{2.158}$$

and $A_q(x)$ *is a periodic in* $\ln x$*, with period* $\ln q$*.*
If we use Fourier expansion for $A_q(x)$ *then we get;*

$$\begin{aligned} f(x) &= A_q(x)x^d \\ &= x^d \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{\ln q}\ln x} \\ &= \sum_{n=-\infty}^{\infty} c_n x^{d+i\frac{2\pi}{\ln q}n} \\ &= \sum_{n=-\infty}^{\infty} c_n x^{d_n}. \end{aligned} \tag{2.159}$$

To find coefficients;

$$\begin{aligned} c_n &= \frac{1}{T}\int_0^T F(T)e^{-i\frac{2\pi nt}{T}}dt = \frac{1}{\ln q}\int_1^q A_q(x)x^{-i\frac{2\pi n}{\ln q}}\frac{dx}{x} \\ &= \frac{1}{\ln q}\int_1^q f(x)x^{-d-i\frac{2\pi n}{\ln q}}\frac{dx}{x} = \frac{1}{\ln q}\int_1^q f(x)x^{d_n}\frac{dx}{x}.\blacksquare \end{aligned} \tag{2.160}$$

Expansion (2.156) for function $f(x)$ is called the Mellin series. Convergency of this series require to study asymptotic formulas for special functions. In next Chapter we are going to introduce basic notations related with this analysis. And In Chapter 4 we return back to convergency of Mellin series.

CHAPTER 3

ASYMPTOTIC EXPANSIONS OF SPECIAL FUNCTIONS

In the previous chapter we considered q-periodic functions and their series representations such as Mellin series. To study convergency of these series in this chapter we are going to introduce basic notations of asymptotic expansions, Bernoulli polynomials and numbers, the gamma and the beta functions. On this basis we shall derive the Euler-Maclaurin formula and Stirling' s asymptotic formula. This allows as in Chapter 4 study convergency properties of q-periodic functions.

3.1. Asymptotic Expansions

In many problems of engineering and physical sciences we attempt to write the solutions as infinite series of functions. The simplest series representation is the power series. Given a function $f(x)$ of a real variable x containing a number x_0 in its domain of definition, we try to find a power series of the form

$$f(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j, \tag{3.1}$$

which provides a valid representation of $f(x)$ in the interval I of convergence of the power series. The so-called remainder term in the Taylor expansion plays a crucial role. When we write the above series as

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + R_n(x), \tag{3.2}$$

the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)^{n+1}. \tag{3.3}$$

If C denotes a uniform bound of $f^{(n+1)}(x)$ in I, that is, $|f^{(n+1)}(x)| \leq C$, $x \in I$, the error introduced by using the Taylor polynomial

$$f_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j, \tag{3.4}$$

for f(x) is the same order of magnitude as the first term which is neglected in the Taylor series. Also observe that in this case

$$\lim_{n\to\infty} |f(x) - f_n(x)| = 0. \tag{3.5}$$

The important feature of the Taylor polynomial $f_n(x)$ given by (3.4) is that it is a function $f_n(x) = g(n, x)$ of two independent variables. The convergent series approach is to consider x fixed and determine the behavior of $g(n, x)$ as n increases. Accordingly,the approximation is considered adequate if the error in using the Taylor polynomial can be made sufficiently small by choosing n appropriately large (Estrada & Kanwal 1994).

The concept of an asymptotic series reverses the role of n and x in $g(n, x)$. That is, the approximation is considered adequate if the error can be made sufficiently small,for any fixed number of terms, by using values of x sufficiently close to some value.

We devote this chapter to the basic notions of asymptotic analysis. We also present some simple methods for approximation of integrals and sums.

3.1.1. Order Symbols, Asymptotic Sequences and Series

Let M be a set of real or complex numbers with a limit point x_0. Let $f, g : M \to \mathbb{R}$ (or $f, g : M \to \mathbb{C}$) be some functions on M. In this section we introduce order symbols, asymptotic sequences and series for Stirling's asymptotic formula.

Definition 3.1.1.1 *Let f(x) and g(x) be functions defined in M. We say that f(x) is "big O" of g(x) as $x \to x_0$ and write*

$$f(x) = O(g(x)) \qquad \textit{as } x \to x_0, \tag{3.6}$$

if there exists a constant $C > 0$ such that

$$|f(x)| \leq C|g(x)|, \forall x \in M. \tag{3.7}$$

Observe that if g(x) does not vanish near x_0, then the relation $f(x) = O(g(x))$, as $x \to x_0$ is equivalent to the condition

$$\overline{\lim_{x \to x_0}} \Big| \frac{f(x)}{g(x)} \Big| < \infty. \tag{3.8}$$

Here $\overline{\lim}_{x \to x_0}$ denotes the limit superior as, $x \to x_0$.

Definition 3.1.1.2 *Let f(x) and g(x) be functions defined in M. We say that f(x) is "little o" of g(x) as $x \to x_0$ and write*

$$f(x) = o(g(x)) \qquad \text{as } x \to x_0, \tag{3.9}$$

if for each $\epsilon > 0$ such that

$$|f(x)| \leq \epsilon|g(x)|, \forall x \in M, \tag{3.10}$$

if g(x) does not vanish near x_0, the condition $f(x) = o(g(x))$, as $x \to x_0$ is equivalent to the vanishing of the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{3.11}$$

Example 3.1 *The function $f(x) = 3x^3 + 4x^2$ is $O(x^3)$ as $x \to \infty$. We have that $g(x) = x^3$.*

$$\overline{\lim_{x \to x_0}} \Big| \frac{f(x)}{g(x)} \Big| = \overline{\lim_{x \to \infty}} \Big| \frac{3x^3 + 4x^2}{x^3} \Big| = \overline{\lim_{x \to \infty}} \Big| 3 + \frac{4}{x} \Big| < \infty. \tag{3.12}$$

Example 3.2 *The function* $f(x) = 3x^3 + 4x^2$ *is* $o(x^4)$ *as* $x \to \infty$*. We have that* $g(x) = x^4$.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{3x^3 + 4x^2}{x^4} = \lim_{x \to \infty} \left(\frac{3}{x} + \frac{4}{x^2} \right) = 0. \tag{3.13}$$

Definition 3.1.1.3 *The functions f(x) and g(x) are called asymptotically equivalent as* $x \to x_0$ *if*

$$f(x) - g(x) = o(g(x)) \qquad \textit{as } x \to x_0. \tag{3.14}$$

In this case we write

$$f(x) \sim g(x) \qquad \textit{as } x \to x_0. \tag{3.15}$$

The relation $\sim$ is symmetric since actually $f \sim g$ as $x \to x_0$, iff in a neighborhood of x_0 the zeros of f and g coincide and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{3.16}$$

Example 3.3 *We consider some functions and their asymptotic equivalences;*

1. $\sin z \sim z \qquad (z \to 0)$.
2. $n! \sim \sqrt{2\pi n} e^{-n} n^n \qquad (n \to \infty)$.

Definition 3.1.1.4 *Let* $\varphi_n : M \to \mathbb{R}$*,* $n \in \mathbb{N}$*, and* x_0 *be a limit point of M. Let* $\varphi_n(x) \neq 0$ *in neighborhood* U_n *of* x_0*. The sequence* $\{\varphi_n\}$ *is called asymptotic sequence at* $x \to x_0$*,* $x \in M$*, if* $\forall n \in \mathbb{N}$*;*

$$\varphi_{n+1}(x) = o(\varphi_n(x)) \qquad (x \to x_0, x \in M). \tag{3.17}$$

Example 3.4 *We consider power asymptotic sequences;*

1. $\{(x-x_0)^n\}$, *as* $x \to x_0$.
2. $\{x^{-n}\}$, *as* $x \to \infty$.
3. *Let* $\{\alpha_n\}$ *be a decreasing sequence of real numbers, i.e.* $\alpha_{n+1} < \alpha_n$, *and let* $0 < \epsilon \leq \frac{\pi}{2}$. *Then the following equation is an asymptotic sequence*

$$\varphi_n(z) = e^{\alpha_n z}, \qquad z \to \infty, \qquad |arg\, z| \leq \frac{\pi}{2} - \epsilon. \tag{3.18}$$

Definition 3.1.1.5 *Let* $\{\varphi_n\}$ *be an asymptotic sequence as* $x \to x_0$, $x \in M$. *We say that the function f is expanded in an asymptotic series;*

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \qquad (x \to x_0, x \in M), \tag{3.19}$$

where a_n *are constants, if* $\forall N \geq 0$

$$R_N(x) \equiv f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N(x)), \qquad (x \to x_0, x \in M). \tag{3.20}$$

This series is called asymptotic expansion of the function f with respect to the asymptotic sequence $\{\varphi_n\}$. $R_N(x)$ *is called the rest term of the asymptotic series.*

Remark 3.1

1. *The condition* $R_N(x) = o(\varphi_n)$, *means, in particular, that*

$$\lim_{x \to x_0} R_N(x) = 0 \qquad \text{for any fixed N.} \tag{3.21}$$

2. *Asymptotic series could diverge. This happens if*

$$\lim_{N \to \infty} R_N(x) \neq 0 \qquad \text{for some fixed x.} \tag{3.22}$$

3.1.2. Bernoulli Polynomials and Bernoulli Numbers

In this subsection we introduce Bernoulli polynomials and the Bernoulli numbers. They play important role in asymptotic formulas (Euler-Maclaurin formula), (Kac&Cheung, 2002).

Definition 3.1.2.1 *In the Taylor expansion,*

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{ze^{zx}}{e^z - 1}. \tag{3.23}$$

$B_n(x)$ are polynomials in x, for each nonnegative integer n. They are known as Bernoulli polynomials.

Remark 3.2 *If we differentiate both sides of (3.23) with respect to x, we get*

$$\sum_{n=0}^{\infty} \frac{B_n'(x)}{n!} z^n = z\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^{n+1}. \tag{3.24}$$

Equating coefficients z^n, where $n \geq 1$, yields

$$B_n'(x) = nB_{n-1}(x). \tag{3.25}$$

Together with the fact that $B_0(x) = 1$, which may be obtained by letting z tend to zero on both sides of (3.23), it follows that the degree of $B_n(x)$ is n and its leading coefficient is unity. Using (3.25), we can determine $B_n(x)$ one by one, provided that their constant terms are known.

Definition 3.1.2.2 *For $n \geq 0$,$B_n = B_n(0)$ are called $Bernoulli$ numbers.*

If we use definition of $Bernoulli$ polynomials as $x = 0$ then we get

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. \tag{3.26}$$

Since using Taylor's expansion we have

$$\frac{z}{e^z-1} = \frac{1}{1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+....}, \tag{3.27}$$

we can use long division to find the Bernoulli numbers. However,we would like to determine B_n and $B_n(x)$ in an easier and more systematic way.

Proposition 3.1.2.3 *For any $n \geq 1$,*

$$B_n(x+1) - B_n(x) = nx^{n-1}. \tag{3.28}$$

Proof 3.1.2.4 *Comparing the coefficient of z^n in*

$$\sum_{n=0}^{\infty} \frac{B_n(x+1)}{n!} z^n - \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{ze^{z(x+1)} - ze^{zx}}{e^z - 1} = ze^{zx} = \frac{d}{dx} e^{zx}, \tag{3.29}$$

where

$$e^{zx} = \sum_{n=0}^{\infty} \frac{x^n z^n}{n!}, \tag{3.30}$$

we have the following equality,

$$B_n(x+1) - B_n(x) = \frac{d}{dx} x^n = nx^{n-1}. \tag{3.31}$$

as desired.■

Proposition 3.1.2.5 *For any $n \geq 0$,*

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}. \tag{3.32}$$

Proof 3.1.2.6 *Let*

$$F_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}. \tag{3.33}$$

It suffices to show that

1. $F_n(0) = B_n$ *for* $n \geq 0$.

2. $F_n^{'}(x) = nF_{n-1}(x)$ *for any* $n \geq 1$.

Since these two properties uniquely characterize $B_n(x)$. *The first property is obvious. As for the second property, using the fact that for* $n > j \geq 0$,

$$(n-j)\binom{n}{j} = \frac{n!}{j!(n-j-1)!} = n\binom{n-1}{j}, \tag{3.34}$$

we have for $n \geq 1$

$$\frac{d}{dx}F_n(x) = \sum_{j=0}^{n-1} \binom{n}{j}(n-j)B_j x^{n-j-1} = n\sum_{j=0}^{n-1} \binom{n-1}{j} B_j x^{n-j-1}, \tag{3.35}$$

as desired.■

Putting $x = 1$ in (3.32), we have

$$B_n(1) = \sum_{j=0}^{n} \binom{n}{j} B_j = B_n + \sum_{j=0}^{n-1} \binom{n}{j} B_j \qquad n \geq 1. \tag{3.36}$$

But, for any $n \geq 2$, we have $B_n(1) = B_n$, which follows from (3.28) with $x = 0$. Therefore, we obtain the obtain the formula

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j = 0 \qquad n \geq 2. \tag{3.37}$$

This formula allows us to compute the $Bernoulli$ numbers inductively. The first few of them are

$$B_0 = 1,\ B_1 = \frac{-1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}. \tag{3.38}$$

Proposition 3.1.2.7 *For any $n \geq 1$,*

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j(x) = nx^{n-1}. \tag{3.39}$$

Proof 3.1.2.8 *We will use mathematical induction. The case where $n = 1$ is obvious. If we assume that (3.39) is true for some $k \geq 1$, we have, by (3.25)*

$$\begin{aligned}
\frac{d}{dx}\sum_{j=0}^{k}\binom{k+1}{j}B_j(x) &= \sum_{j=1}^{k} j\binom{k+1}{j}B_{j-1}(x) \\
&= (k+1)\sum_{j=1}^{k}\binom{k}{j-1}B_{j-1}(x) \\
&= (k+1)\sum_{j=0}^{k-1}\binom{k}{j}B_j(x) \\
&= (k+1)kx^{k-1} = (k+1)\frac{d}{dx}x^k
\end{aligned} \tag{3.40}$$

or equivalently

$$\sum_{j=0}^{k}\binom{k+1}{j}B_j(x) = (k+1)x^k + C, \tag{3.41}$$

for some constant C. Putting $x = 0$ and using (3.37) show that C=0. Hence, by induction, (3.39) is true for any positive integer.■

As has been mentioned above , formula (3.25) and the knowledge of Bernoulli numbers allow us to determine the Bernoulli polynomials inductively.

The first six of them are listed below;

$$
\begin{aligned}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}. \qquad (3.42)
\end{aligned}
$$

3.1.3. The Gamma and the Beta Functions

Definition 3.1.3.1 *The gamma function is defined as,*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \textit{for } z \in \mathbb{C},\ \Re z > 0. \qquad (3.43)$$

Theorem 3.1.3.2

$$\Gamma(z+1) = z\Gamma(z), \qquad \Re z > 0. \qquad (3.44)$$

Proof 3.1.3.3 *From definition of gamma function and using integration by part, we obtain*

$$
\begin{aligned}
\Gamma(z+1) &= \int_0^\infty e^{-t} t^z dt = -\int_0^\infty t^z\, d(e^{-t}) = -e^{-t} t^z|_0^\infty + \int_0^\infty e^{-t}\, d(t^z) \\
&= z\int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z), \qquad \Re z > 0. \blacksquare \qquad (3.45)
\end{aligned}
$$

Further we have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}|_0^\infty = 1. \qquad (3.46)$$

Combining (3.44) and (3.46), this leads to

$$\Gamma(n+1) = n!, \qquad n = 0, 1, 2, ... \tag{3.47}$$

The definition of gamma function in eqn.(3.87) give us the $\Gamma(z)$ is analytic for $\Re z > 0$. The functional relation (3.44) also holds for $\Re z > 0$.

Let $-1 < \Re z \leq 0$, then we have $\Re(z+1) > 0$. Hence, $\Gamma(z+1)$ is defined by the integral representation (3.45). Now we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad -1 < \Re z \leq 0, z \neq 0. \tag{3.48}$$

Then the gamma function $\Gamma(z)$ is analytic for $\Re z > -1$ except $z = 0$. For $z = 0$ we have

$$\lim_{z \to 0} z\Gamma(z) = \lim_{z \to 0} \Gamma(z+1) = \Gamma(1) = 1. \tag{3.49}$$

This implies that $\Gamma(z)$ has a single pole at $z = 0$ with residue 1. This process can be repeated for $-2 < \Re z \leq -1$, $-3 < \Re z \leq -2$, etcetera. Then the gamma function turns out to be an analytic function on $\mathbb{C}$ except for single poles at $z = 0, -1, -2, -3, ...$
The residue at $z = -n$ equals

$$\begin{aligned} \lim_{z \to -n} (z+n)\Gamma(z) &= \frac{\Gamma(1)}{(-n)(-n+1)....(-1)} \\ &= \frac{(-1)^n}{n!}, \qquad n = 0, 1, 2, ... \end{aligned} \tag{3.50}$$

Alternatively we can define the gamma function as follows.

Definition 3.1.3.4 *For all complex numbers $z \neq 0, -1, -2, ..$ the gamma function is defined by*

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{(z)_{n+1}}, \tag{3.51}$$

where $(z)_n = \prod_{k=1}^{n-1}(z+k)$ and $(z)_0 = 1$.

This definition comes from as the following integral;

$$\int_0^1 (1-t)^n t^{z-1} dt = \frac{n!}{(z)_{n+1}}, \tag{3.52}$$

for $\Re z > 0$ and $n = 0, 1, 2, ...$ In order to prove (3.52) by induction we first take $n = 0$ to obtain for $\Re z > 0$

$$\int_0^1 t^{z-1} dt = \frac{1}{z} = \frac{0!}{(z)_1}. \tag{3.53}$$

Now we assume that (3.52) holds for $n = k$. Then we have

$$\begin{aligned}\int_0^1 (1-t)^{k+1} t^{z-1} dt &= \int_0^1 (1-t)(1-t)^k t^{z-1} dt = \int_0^1 (1-t)^k t^{z-1} dt - \int_0^1 (1-t) t^z dt \\ &= \frac{k!}{(z)_{k+1}} - \frac{k!}{(z+1)_{k+1}} = \frac{(k+1)!}{(z)_{k+2}},\end{aligned} \tag{3.54}$$

which is (3.52) for $n = k + 1$. This proves that (3.52) holds for all $n = 0, 1, 2, ...$
Now we set $t = u/n$ into (3.52) to find that

$$\frac{1}{n^z} \int_0^n \left(1 - \frac{u}{n}\right)^n u^{z-1}\, du = \frac{n!}{(z)_{n+1}} \Rightarrow \int_0^n \left(1 - \frac{u}{n}\right)^n u^{z-1}\, du = \frac{n!\, n^z}{(z)_{n+1}}. \tag{3.55}$$

Since we have

$$\lim_{n\to\infty} \left(1 - \frac{u}{n}\right)^n = e^{-u}, \tag{3.56}$$

we conclude that

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du = \lim_{n\to\infty} \frac{n! n^z}{(z)_{n+1}}. \tag{3.57}$$

Definition 3.1.3.5 *The beta function is defined as,*

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt, \qquad \Re u, \Re v > 0. \tag{3.58}$$

The connection between the beta and the gamma function is given by the following theorem;

Theorem 3.1.3.6

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \qquad \Re u, \Re v > 0. \tag{3.59}$$

Proof 3.1.3.7 *From the definition of gamma function, we get*

$$\Gamma(u)\Gamma(v) = \int_0^\infty e^{-t}t^{u-1}dt \int_0^\infty e^{-s}s^{v-1}ds = \int_0^\infty \int_0^\infty e^{-(s+t)}t^{u-1}s^{v-1}dtds. \tag{3.60}$$

Now we apply change of variables $t = xy$ and $s = x(1-y)$ to this double integral. Note that $t+s = x$ and that $0 < t < \infty$ and $0 < s < \infty$ imply that $0 < x < \infty$ and $0 < y < 1$. The Jacobian transformation is

$$\frac{\partial(t,s)}{\partial(x,y)} = -x. \tag{3.61}$$

Since $x > 0$ we conclude that $dt\,ds = \left|\frac{\partial(t,s)}{\partial(x,y)}\right| dx\,dy = x\,dx\,dy$. Hence we have

$$\begin{aligned}
\Gamma(u)\Gamma(v) &= \int_0^1 \int_0^\infty e^{-x}\, x^{u-1}\, y^{u-1}\, x^{v-1}\, (1-y)^{v-1}\, x\, dx\, dy \qquad (3.62)\\
&= \int_0^\infty e^{-x}\, x^{u+v-1}\, dx \int_0^1 y^{u-1}\, (1-y)^{v-1}\, dy = \Gamma(u+v)\, B(u,v).
\end{aligned}$$

This proves (3.59).■

3.1.4. The Euler-Maclaurin Formula and the Stirling's Asymptotic Formula

In this subsection we give a precise formula for the approximation of sums by integrals, the celebrated Euler-Maclaurin formula.

Theorem 3.1.4.1 *(Euler-Maclaurin Formula) (Olver, 1974)*
If $f \in C^{2m}[a,b]$, a and b integers,then

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \frac{1}{2}f(a) + \frac{1}{2}f(b)$$
$$+\sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\{f^{(2k-1)(b)} - f^{(2k-1)(a)}\} - \int_a^b \frac{B_{2m}(x-[x])}{(2m)!} f^{(2m)}(x)dx. \quad (3.63)$$

where $B_n(x)$ is the nth Bernoulli polynomial, B_n is nth Bernoulli numbers, and m is any positive integer. The symbol $[x]$ for a real number x denotes the fractional part of x.

The Euler-Maclaurin make the connection between the sum and the integral explicit for sufficiently smooth functions. We will use these formulas to get asymptotic expansions of gamma function known as Stirling's asymptotic formula. This formula play important role in the convergence of infinite sums.
Now we apply this formula to get Stirling's asymptotic formula.

Theorem 3.1.4.2 *(Andrew, 1999) Let $z \in \mathbb{C} - (-\infty, 0]$,*

$$\begin{aligned} \log\Gamma(z) &= (z-\frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)} z^{1-2j} \\ &- \frac{1}{2m}\int_0^\infty \frac{B_{2m}(t-[t])}{(z+t)^{2m}}dt. \end{aligned} \quad (3.64)$$

Proof 3.1.4.3 *Start with the following expression of the gamma function,*

$$\Gamma(z) = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{k}{z+k-1}\left(\frac{k+1}{k}\right)^{z-1}. \quad (3.65)$$

Then

$$\log\Gamma(z) = \lim_{n\to\infty}\left[(z-1)\log(n+1) - \sum_{k=1}^{n}\log(\frac{z+k-1}{k})\right], \tag{3.66}$$

where the principal branch of the $\log$ *function in* $z \in \mathbb{C} - (-\infty, 0]$ *is chosen. In Euler-Maclaurin formula, take*

$$f(t) = \log\left(\frac{t+z-1}{t}\right) = \log(t+z-1) - \log t, \tag{3.67}$$

to get the following equality,

$$\begin{aligned}\sum_{k=1}^{n}\log(\frac{z+k-1}{k}) &= \log z + \sum_{k=2}^{n}\log(\frac{z+k-1}{k}) \\ &= \log(z) + \int_1^n [\log(t+z-1) - \log(t)]dt \\ &+ \sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(n+z-1)^{2j-1}} - \frac{1}{n^{2j-1}} - \frac{1}{z^{2j-1}} + 1\right] \\ &+ \frac{1}{2}[\log(n+z-1) - \log n - \log z] \\ &+ \frac{1}{2m}\int_0^n B_{2m}(t-[t])\left[\frac{1}{(z+t-1)^{2m}} - \frac{1}{t^{2m}}\right]dt. \end{aligned} \tag{3.68}$$

Here we have used the fact that $B_1 = \frac{-1}{2}$ *and* $B_{2j+1} = 0$ *for* $j \geq 0$*. Compute the first of the above integrals and observe that some cancellation the terms involve* n*.*

$$\begin{aligned}(n+z-1)\log(n+z-1) &- n\log n + \frac{1}{2}\left[\frac{\log(n+z-1)}{n}\right] \\ &+ \sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(n+z-1)^{2j-1}} - \frac{1}{n^{2j-1}}\right].\end{aligned} \tag{3.69}$$

Subtract this from $(z-1)\log(n+1)$ *and let* $n \to \infty$ *to compute the limit in (3.65).*

The result is

$$\log\Gamma(z) = (z-\frac{1}{2})\log z - z + 1 + \sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}(z^{1-2j}-1) \tag{3.70}$$

$$- \frac{1}{2m}\int_0^\infty B_{2m}(t-[t])\left[\frac{1}{(t+z-1)^{2m}} - \frac{1}{t^{2m}}\right]dt. \tag{3.71}$$

We know that from Euler-Maclaurin formula for $f(t) = \log t$ and $m = 1$ as $n \to \infty$,

$$\log n! - \left(n+\frac{1}{2}\right)\log n + n = 1 + \int_0^n \frac{B_1(t-[t])}{t}dt. \tag{3.72}$$

If we use Riemann-zeta function (Andrew, 1999) then we get,

$$\zeta'(0) = -1 - \int_0^n \frac{B_1(t-[t])}{t}dt = -\frac{1}{2}\log(2\pi). \tag{3.73}$$

If we use equations (3.72) and (3.73) then we get

$$\lim_{z\to\infty}[\log\Gamma(z) - \left(z+\frac{1}{2}\right)\log z + z] = \frac{1}{2}\log(2\pi). \tag{3.74}$$

So let $z \to \infty$ in (3.74) to see that

$$1 - \sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)} + \frac{1}{2m}\int_0^\infty \frac{B_{2m}(t-[t])}{t^{2m}} = \frac{1}{2}\log(2\pi). \tag{3.75}$$

The result combined with (3.71) gives the formula in theorem (3.1.4.2).■

Now give an asymptotic formula for $\Gamma(z)$ for $\Re z$ large, when $\Im z$ is fixed.

Theorem 3.1.4.4 *(Andrew, 1999)*

$$\Gamma(z) \sim \sqrt{2\pi}z^{z-1/2}e^{-z} \qquad \text{as } \Re z \to \infty. \tag{3.76}$$

Proof 3.1.4.5 *Denote the right side of the equation*

$$\Gamma(z+n) = \sum_{k=1}^{n-1} \log(k+n) + \Gamma(z+1), \tag{3.77}$$

by c_n, so that

$$c_{n+1} + c_n = \log(z+n). \tag{3.78}$$

By the analogy between the derivative and the finite difference we consider c_n to be approximately the integral of $\log(z+n)$ and set

$$c_n = (n+z)\log(z+n) - (n+z) + d_n, \tag{3.79}$$

Substitute this in the previous equation to obtain

$$\begin{aligned}\log(z+n) &= (n+1+z)\log(n+1+z) \\ &- (n+z)\log(z+n) + d_{n+1} - d_n - 1.\end{aligned} \tag{3.80}$$

Thus we have,

$$\begin{aligned}d_{n+1} - d_n &= 1 - (n+z+1)\log\big(1 + \frac{1}{n+z}\big) \\ &= 1 - (n+z+1)\Big[\frac{1}{n+z} + \frac{1}{2(n+z)^2} + \frac{1}{3(n+z)^3} + ...\Big] \\ &= -\frac{1}{2(n+z)} + \frac{1}{6(n+z)^2} + ...\end{aligned} \tag{3.81}$$

Proceeding as before, take

$$d_n = e_n - \frac{1}{2}\log(n+z), \tag{3.82}$$

and substitute in the previous equation to get

$$
\begin{aligned}
e_{n+1} - e_n &= \frac{1}{2}\log(1 + \frac{1}{n+z}) - \frac{1}{2(n+z)} + \frac{1}{6(n+z)^2} + ... \\
&= -\frac{1}{12(n+z)^2} + O\Big(\frac{1}{(n+z)^3}\Big). \qquad (3.83)
\end{aligned}
$$

Now

$$
e_n - e_0 = \sum_{k=0}^{n-1}(e_{k+1} - e_k) = \sum_{k=0}^{n-1}\Big[-\frac{1}{12(k+z)^2} + O\Big(\frac{1}{(n+z)^3}\Big)\Big]. \qquad (3.84)
$$

Therefore, $\lim_{n\to\infty}(e_n - e_0) = K_1(z)$ *exist. If we set the following equation then we obtain,*

$$
e_n = K(z) + \frac{1}{12(k+z)} + O\Big(\frac{1}{(n+z)^2}\Big), \qquad (3.85)
$$

where $K(z) = K_1(z) + e_0$*. The term* $(n+z)^{-1}$ *comes from completing the sum in (3.84) to infinity and approximating the added sum by an integral. So we can write*

$$
\begin{aligned}
c_n &= (n+z)\log(n+z) - (n+z) - \frac{1}{2}\log(n+z) \\
&+ \log C(z) + \frac{1}{12(n+z)} + O\Big(\frac{1}{(n+z)^2}\Big), \qquad (3.86)
\end{aligned}
$$

where $K(z) = \log C(z)$*. This implies that*

$$
\Gamma(z+n) = C(z)(n+z)^{n+z-\frac{1}{2}}\exp\Big[-(n+z) + \frac{1}{12(n+z)} + O\Big(\frac{1}{(n+z)^2}\Big)\Big]. \qquad (3.87)
$$

We claim that $C(z)$ *is independent of z. By the definition of the gamma function*

$$
\lim_{n\to\infty}\frac{\Gamma(z_1+n)}{\Gamma(z_2+n)}n^{z_2-z_1} = \frac{\Gamma(z_1)}{\Gamma(z_2)}\lim_{n\to\infty}\frac{(z_1)_n}{(z_2)_n}n^{z_2-z_1} = \frac{\Gamma(z_1)}{\Gamma(z_2)}.\frac{\Gamma(z_2)}{\Gamma(z_1)} = 1. \qquad (3.88)
$$

Now, from (3.87) and (3.88) we can conclude that

$$1 = \lim_{n\to\infty} n^{-z}\frac{\Gamma(z+n)}{\Gamma(n)} = \frac{C(z)}{C(0)}\lim_{n\to\infty}(1+\frac{z}{n})^n e^{-z} = \frac{C(z)}{C(0)}. \tag{3.89}$$

Thus $C(z)$ is a constant and

$$\Gamma(z) \sim C z^{z-1/2}e^{-z} \qquad \textit{as } \Re z \to \infty. \tag{3.90}$$

To find C, we use the Wallis formula (Andrew, 1999);

$$\begin{aligned}\sqrt{\pi} &= \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!}\frac{1}{\sqrt{n}} \\ &= \lim_{n\to\infty}\frac{2^{2n}C^2 n^{2n+1}e^{-2n+O(\frac{1}{n})}}{C(2n)^{2n+\frac{1}{2}}e^{-2n+O(\frac{1}{n})}} \\ &= \frac{C}{\sqrt{2}}. \end{aligned} \tag{3.91}$$

This gives $C = \sqrt{2\pi}$ and proves the theorem.■

From theorem (3.1.4.2) the following corollary is immediately obtained.

Corollary 3.1.4.6 *For $\delta > 0$ and $|argz| \leq \pi - \delta$,*

$$\Gamma(z) \sim \sqrt{2\pi}z^{z-1/2}e^{-z} \qquad \textit{as } \Re z \to \infty. \tag{3.92}$$

Corollary 3.1.4.7 *When $z = a + ib$, $a_1 \leq a \leq a_2$ and $|b| \to \infty$, then*

$$|\Gamma(a+ib)| = \sqrt{2\pi}|b|^{a-1/2}e^{-\pi|b|/2}[1 + O(1/|b|)]. \tag{3.93}$$

Proof 3.1.4.8 *Take $|b| > 1$, $a > 0$. We use Bernoulli polynomial $B_2 - B_2(t) = t - t^2$. Thus $\frac{1}{2}|B_2 - B_2(t)| \leq \frac{1}{2}|t(1-t)| \leq \frac{1}{8}$ for $0 \leq t \leq 1$. So (3.63) with $m = 1$ is*

$$\Gamma(a+ib) = (a+ib-\frac{1}{2})\log(a+ib) - (a+ib) + \frac{1}{2}\log 2\pi + R(x), \tag{3.94}$$

and

$$|R(x)| \leq \frac{1}{8}\int_0^\infty \frac{dt}{|t+z|^2} = \frac{1}{8}\int_0^\infty \frac{dt}{(a+t)^2+b^2} = \frac{1}{8|b|}\tan^{-1}\frac{|b|}{a}, \qquad b \neq 0. \quad (3.95)$$

Now we consider the following equation,

$$\Re[(a+ib-\frac{1}{2})\log(a+ib)] = (a-\frac{1}{2})\log(a^2+b^2)^{1/2} - b\arctan\frac{b}{a}. \qquad (3.96)$$

Also,

$$\log(a^2+b^2)^{1/2} = \frac{1}{2}\log b^2 + \log(1+\frac{a^2}{b^2}) = \log|b| + O\Big(\frac{1}{b^2}\Big). \qquad (3.97)$$

Moreover,

$$\arctan\frac{b}{a} + \arctan\frac{a}{b} = \begin{cases} \frac{\pi}{2} & \textit{if } b > 0 \\ -\frac{\pi}{2} & \textit{if } b < 0. \end{cases} \qquad (3.98)$$

This gives

$$\begin{aligned} -b\arctan\frac{b}{a} &= -b\Big[\pm\frac{\pi}{2} - \frac{a}{b} + O\Big(\frac{1}{b^2}\Big)\Big] \\ &= -\frac{\pi}{2}|b| + a + O\Big(\frac{1}{b^2}\Big). \end{aligned} \qquad (3.99)$$

Putting all together gives

$$\log|\Gamma(a+ib)| = (a-\frac{1}{2})\log|b| - \frac{\pi}{2}|b| + \frac{1}{2}\log 2\pi + O\Big(\frac{1}{|b|}\Big). \blacksquare \qquad (3.100)$$

CHAPTER 4

WEIERSTRASS-MANDELBROT FUNCTIONS AND THE CHIRP DECOMPOSITION

In this Chapter we are going to apply q-periodic functions from Chapter 2 and asymptotic formulas from chapter 3 to fractal functions and expansion in their harmonics.

4.1. The Weierstrass-Mandelbrot Function

In 1872, Karl Weierstrass introduced a function defined by

$$\widehat{W}(t) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi t), \tag{4.1}$$

with $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$ and $b > 1$ an odd integer, $t \in \mathbb{R}$, as an example of a continuous and nowhere differentiable function (Edgar, 1993). The basic idea is to construct a function as an infinite convergent series of continuous functions, but with divergent derivative. This example created a sensation in the mathematical world because it challenged the then widespread belief among mathematicians that continuous functions had to be differentiable everywhere except possibly at isolated "singular" points. A. Ampere had even published a proof of this fact in 1806. Weierstrass's example thus brought to an end of a long string of futile attempts to show that differentiability somehow follows from continuity.

Weierstrass was not the first to claim the existence of an everywhere continuous, nowhere differentiable function, but he was the first provide a rigorous proof. About 1830, Bolzano had constructed a similar example but was unable to prove that it was nowhere differentiable (Jarnik, 1922).

In 1860, the Swiss mathematicians Charles Cellerier gave another example,

$$C(t) = \sum_{k=1}^{\infty} \frac{\sin(a^k t)}{a^k}, \tag{4.2}$$

which is nowhere differentiable when a is a sufficiently large positive integer. His result was not published, however until 1890, whereas ”Weierstrass” result was published in 1875 (Cellerier, 1890).

According to Weierstrass, Riemann claimed in his lectures in 1861 that the function

$$R(t) = \sum_{k=1}^{\infty} \frac{\sin(k^2 t)}{k^2}, \tag{4.3}$$

is nowhere differentiable, or at least nondifferentiable on a dense subset of $\mathbb{R}$ (Riemann, 1854). Weierstrass was unable to verify nondifferentiability of Riemann's function (4.3) and subsequently constructed his own example, the function (4.1) above. In 1970, J.Gerver proved that in fact Riemann's function (4.3) is differentiable at infinitely many points namely when $x = a\pi$, where $a = \frac{2p+1}{2q+1}$ for $p, q \in \mathbb{Z}$ (Gerver, 1970).

In 1878 Italian mathematician Ulisse Dini proposed more general class of continuous nowhere differentiable functions. He had generalized Weierstrass function (4.1) as the following;

$$\widehat{W}_{D_1}(t) = \sum_{k=1}^{\infty} \frac{a^k}{1.3.5.....(2k-1)} \cos(1.3.5....(2k-1)\pi t), \tag{4.4}$$

and

$$\widehat{W}_{D_2}(t) = \sum_{k=1}^{\infty} \frac{a^k}{1.5.9.....(4k+1)} \cos(1.5.9....(4k+1)\pi t), \tag{4.5}$$

where $|a| > 1 + \frac{3}{2}\pi$ (Dini, 1877).

Polish mathematicians Karol Hertz gave in his paper from 1879 another generalization of Weierstrass function (4.1) namely,

$$\widehat{W}_H(t) = \sum_{k=1}^{\infty} a^k \cos^p(b^k \pi t), \tag{4.6}$$

where $a > 1$, $p \in \mathbb{N}$ is odd, b odd integer and $ab > 1 + \frac{2}{3}p\pi$ (Hertz, 1879).

In 1916 Hardy proved that the function $\widehat{W}$ defined above is continuous and nowhere differentiable if $0 < a < 1$, $ab \geq 1$ and $b > 1$ (not necessarily odd integer), $t \in \mathbb{R}$.

Theorem 4.1.0.9 *(Hardy, 1916) The Weierstrass function,*

$$\widehat{W}(t) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi t), \tag{4.7}$$

for $0 < a < 1$, $ab \geq 1$ and $b > 1$, $t \in \mathbb{R}$, is continuous and nowhere differentiable function on $\mathbb{R}$.

Proof 4.1.0.10 *We start with establishing continuity of this function and, observe that $0 < a < 1$ implies geometric series $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \infty$. This together with inequality $\sup_{t\in\mathbb{R}} |a^k \cos(b^k \pi t)| \leq a^k$ gives, using the Weierstrass M-test (the comparison test), that $\sum_{k=0}^{\infty} a^k \cos(b^k \pi t)$ converges uniformly to $\widehat{W}$ on $\mathbb{R}$. The continuity of $\widehat{W}$ now follows from the uniform convergence of the series just established and from corollary $(A.0.0.10)$ in appendix A.*

Now we are going to show nowhere differentiability of $\widehat{W}(t)$. We will calculate and compare derivative of $\widehat{W}$ from the left and right hand sides. During rest of this proof we assume that Weierstrass original assumptions hold, i.e. $ab > 1 + 3\pi/2$ and $b > 1$ an odd integer. For a more general proof with $ab \geq 1$ and $b > 1$ we refer to (Hardy, 1916).

Let $t_0 \in \mathbb{R}$ be arbitrary but fixed and let $m \in \mathbb{N}$ be arbitrary. Choose $\alpha_m \in \mathbb{Z}$ such that $b^m t_0 - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and define $t_{m+1} = b^m t_0 - \alpha_m$. Put

$$y_m = \frac{\alpha_m - 1}{b^m} \qquad \textit{and} \qquad z_m = \frac{\alpha_m + 1}{b^m}. \tag{4.8}$$

This gives the inequality

$$y_m - t_0 = -\frac{1 + t_{m+1}}{b^m} < 0 < \frac{1 - t_{m+1}}{b^m} = z_m - t_0, \tag{4.9}$$

and therefore $y_m < t_0 < z_m$. As $m \to \infty$, $y_m \to t_0$ from the left and $z_m \to t_0$ from the right.

a) First consider the left-hand difference quotient;

$$\begin{aligned}
\frac{\widehat{W}(y_m)-\widehat{W}(t_0)}{y_m-t_0} &= \sum_{k=0}^{\infty}\left(a^k\frac{\cos(b^k\pi y_m)-\cos(b^k\pi t_0)}{y_m-t_0}\right)\\
&= \sum_{k=0}^{m-1}\left((a\,b)^k\frac{\cos(b^k\pi y_m)-\cos(b^k\pi t_0)}{b^k(y_m-t_0)}\right) \qquad (4.10)\\
&+ \sum_{k=0}^{\infty}\left(a^{m+k}\frac{\cos(b^{m+k}\pi y_m)-\cos(b^{m+k}\pi t_0)}{y_m-t_0}\right)=S_1+S_2.
\end{aligned}$$

We treat this sum separately, starting with S_1. Since $\left|\frac{\sin(t)}{t}\right| \leq 1$ we can, by using the trigonometric identity, bound the sum by

$$\begin{aligned}
|S_1| &= \left|\sum_{k=0}^{m-1}(a\,b)^k(-\pi)\sin\Big(\frac{b^k\pi(y_m+t_0)}{2}\Big)\frac{\sin\big(\frac{b^k\pi(y_m-t_0)}{2}\big)}{b^k\pi\frac{(y_m-t_0)}{2}}\right|\\
&\leq \sum_{k=0}^{m-1}\pi(a\,b)^k=\frac{\pi((a\,b)^m-1)}{(a\,b)-1}\leq\frac{\pi((a\,b)^m)}{(a\,b)-1}. \qquad (4.11)
\end{aligned}$$

Considering the sum S_2 we can use $b>1$ as an odd integer and $\alpha_m \in \mathbb{Z}$,

$$\begin{aligned}
\cos(b^{m+k}\pi y_m) &= \cos\left(b^{m+k}\pi\frac{\alpha_m-1}{b^m}\right)=\cos(b^k\pi(\alpha_m-1))\\
&= [(-1)^{b^k}]^{(\alpha_m-1)}=-(-1)^{\alpha_m}, \qquad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
\cos(b^{m+k}\pi t_0) &= \cos\left(b^{m+k}\pi\frac{\alpha_m+t_{m+1}}{b^m}\right)\\
&= \cos(b^k\pi\alpha_m)\cos(b^k\pi t_{m+1})-\sin(b^k\pi\alpha_m)\sin(b^k\pi t_{m+1})\\
&= [(-1)^{b^k}]^{\alpha_m}\cos(b^k\pi t_{m+1})-0=(-1)^{\alpha_m}\cos(b^k\pi t_{m+1}), \quad (4.13)
\end{aligned}$$

to express the sum as

$$
\begin{aligned}
S_2 &= \sum_{k=0}^{\infty} a^{m+k} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^k \pi t_{m+1})}{-\frac{1+t_{m+1}}{b^m}} \\
&= (a\,b)^m (-1)^{\alpha_m} \sum_{k=0}^{\infty} a^k \frac{1+\cos(b^k\pi t_{m+1})}{1+t_{m+1}}. \qquad (4.14)
\end{aligned}
$$

Each term in the series above is non-negative and $t_{m+1} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ *so we can find a lower bound by*

$$
\sum_{k=0}^{\infty} a^k \frac{1+\cos(b^k\pi t_{m+1})}{1+t_{m+1}} \geq \frac{1+\cos(\pi t_{m+1})}{1+t_{m+1}} \geq \frac{1}{1+\frac{1}{2}} = \frac{2}{3}. \qquad (4.15)
$$

The inequalities (4.11) and (4.15) ensures the existence of an $\epsilon_1 \in [-1,1]$ *and an* $\eta_1 > 1$ *such that*

$$
\frac{\widehat{W}(y_m) - \widehat{W}(t_0)}{y_m - t_0} = (-1)^{\alpha_m}(a\,b)^m \eta_1 \left(\frac{2}{3} + \epsilon_1 \frac{\pi}{(a\,b)-1}\right). \qquad (4.16)
$$

b) As with the left-hand difference quotient, for the right-hand quotient we do pretty much the same, starting by expressing the said faction as

$$
\frac{\widehat{W}(z_m) - \widehat{W}(t_0)}{z_m - t_0} = S_1' + S_2'. \qquad (4.17)
$$

As before it can be deduced that

$$
|S_1'| \leq \frac{\pi((a\,b)^m)}{(a\,b)-1}. \qquad (4.18)
$$

The cosine term containing z_m *can be simplified again since* $b > 1$ *is an odd integer.*

$$
\begin{aligned}
\cos(b^{m+k}\pi z_m) &= \cos\left(b^{m+k}\pi \frac{\alpha_m - 1}{b^m}\right) = \cos(b^k \pi(\alpha_m - 1)) \\
&= [(-1)^{b^k}]^{(\alpha_m - 1)} = -(-1)^{\alpha_m}, \qquad \alpha_m \in \mathbb{Z}, \qquad (4.19)
\end{aligned}
$$

which gives

$$\begin{aligned} S_2 &= \sum_{k=0}^{\infty} a^{m+k} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m} \cos(b^k \pi t_{m+1})}{\frac{1-t_{m+1}}{b^m}} \\ &= -(a\,b)^m (-1)^{\alpha_m} \sum_{k=0}^{\infty} a^k \frac{1+\cos(b^k \pi t_{m+1})}{1-t_{m+1}}. \end{aligned} \tag{4.20}$$

as before we can find a lower bound for the series by

$$\sum_{k=0}^{\infty} a^k \frac{1+\cos(b^k \pi t_{m+1})}{1-t_{m+1}} \geq \frac{1+\cos(\pi t_{m+1})}{1-t_{m+1}} \geq \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}. \tag{4.21}$$

By the same argument as for the left-hand difference quotient (but by using the inequalities (4.18) and (4.21) instead), there exists an $\epsilon_2 \in [-1, 1]$ and an $\eta_2 > 1$ such that

$$\frac{\widehat{W}(z_m) - \widehat{W}(t_0)}{z_m - t_0} = -(-1)^{\alpha_m} (a\,b)^m \eta_2 \left(\frac{2}{3} + \epsilon_2 \frac{\pi}{(a\,b) - 1} \right). \tag{4.22}$$

By assumption $ab > 1 + \frac{3}{2}\pi$, which is equivalent to $\frac{\pi}{ab-1} < \frac{3}{2}$, the left-hand and right-hand difference quotients have different signs. Since also $(a\,b)^m \to \infty$ as $m \to \infty$ it is clear that $\widehat{W}$ has no derivative at t_0. The choice of $t_0 \in \mathbb{R}$ was arbitrary so it follows that $\widehat{W}(t)$ is nowhere differentiable.■

4.1.1. Self-Similarity of Weierstrass-Mandelbrot Function

Now we are going to study self-similarity property of function $\widehat{W}(t)$. Before we check the self-similarity, we choose special values of a and b as $a = q^{-d}$ and $b = q$ and $\pi t \to t$ in equation (4.1). So we get

$$\widehat{W}(t) = \sum_{n=0}^{\infty} q^{-nd} \cos(q^n t), \qquad 0 < d < 1, q > 1. \tag{4.23}$$

It is easy to see that this function is not truely self-similar since,

$$\widehat{W}(qt) = q^d[\widehat{W}(t) - \cos t] \neq q^d\widehat{W}(t). \tag{4.24}$$

To find the self-similar of $\widehat{W}(t)$, Mandelbrot proposed the natural generalization of (4.23) by extension of summation to all integer numbers,

$$W(t) = \sum_{n=-\infty}^{\infty} q^{-nd}(1 - e^{iq^n t})e^{i\varphi_n}, \tag{4.25}$$

where an extra degree of arbitraries is determined by phases φ_n. Function (4.25) called the Weierstrass-Mandelbrot function has been widely used as an example of fractal with dimension $2 - d$ (Barros & Bevilacqua, 2001). Below we consider the case, where the phases are $\varphi_n = \varphi_1 n$, $n = 0, \pm 1, \pm 2, ...$
Then $W(t)$ obeys the following equation;

$$\begin{aligned} W(qt) &= \sum_{n=-\infty}^{\infty} q^{-nd}(1 - e^{iq^{n+1}t})e^{i\varphi_1 n} \\ &= e^{i\varphi_1} q^d \sum_{n=-\infty}^{\infty} q^{-nd}(1 - e^{iq^n t})e^{i\varphi_1 n} \\ &= e^{i\varphi_1} q^d W(t) \end{aligned} \tag{4.26}$$

or equivalently

$$W(qt) = e^{i\varphi_1} q^d W(t). \tag{4.27}$$

For $\varphi_1 = 0$ it means that function $W(t)$ is self-similar function. Moreover this equation implies that the whole function W can be reconstructed from its value in the range $t_0 \leq t < qt_0$, $t_0 \neq 0$. Indeed, according to this formula it is determined in intervals,

$$... \cup [\frac{1}{q}t_0, t_0) \cup [t_0, qt_0) \cup [qt_0, q^2 t_0) \cup [q^2 t_0, q^3 t_0) \cup ... \tag{4.28}$$

or equivalently

$$\cup_{n=-\infty}^{\infty}[q^n t_0, q^{n+1}t_0). \tag{4.29}$$

Then for $t_0 > 0$ it determines $W(t)$ on $\mathbb{R}^+$ and for $t_0 < 0$ it determines $W(t)$ on $\mathbb{R}^-$. If we apply q-difference operator on $W(t)$ then we get

$$\begin{aligned} D_q W(t) &= \frac{W(qt) - W(t)}{(q-1)t} = \frac{e^{i\varphi_1} q^d W(t) - W(t)}{(q-1)t} \\ &= \frac{q^{d+i\frac{\varphi_1}{\ln q}} - 1}{(q-1)t} W(t) \end{aligned} \tag{4.30}$$

or equivalently,

$$(tD_q)W(t) = \left[d + i\frac{\varphi_1}{\ln q}\right]_q W(t). \tag{4.31}$$

where $\left[d + i\frac{\varphi_1}{\ln q}\right]_q$ is complex q-number.

Note that although $\lim_{q\to 1} \frac{tW'(t)}{W(t)}$ is undefined and the function is not differentiable, the q-derivative of $W(t)$ is well defined.

$$\lim_{q\to 1} \frac{(tD_q)W(t)}{W(t)} = \lim_{q\to 1} \left[d + i\frac{\varphi_1}{\ln q}\right]_q. \tag{4.32}$$

In the limit $q \to 1$ and $\varphi_1 \equiv 0$, $D_q \to \frac{d}{dt}$ and function $W(t)$ is divergent, but this expression is finite and equal d.

4.1.2. Relation with q-periodic Function

In this subsection we are going to find relation between $W(t)$ and the q-periodic functions. Function $W(t)$ satisfies equation (4.31) which is the complex version of equation (2.130). Following the similar arguments as for of equation (2.130) we can see that

$W(t)$ can be represented in the form;

$$W(t) = t^{d+i\frac{\varphi_1}{\ln q}} A_q(t), \tag{4.33}$$

where $A_q(t)$ is q-periodic function. Since $A_q(t)$ is q-periodic, it can be expressed by

$$A_q(t) = \sum_{m=-\infty}^{\infty} c_m e^{\frac{i2\pi m}{\ln q}\ln t}, \tag{4.34}$$

from equation (2.153). So we can write W(t) in the following form;

$$\begin{aligned} W(t) &= t^{d+i\frac{\varphi_1}{\ln q}} \sum_{m=-\infty}^{\infty} c_m e^{\frac{i2\pi m}{\ln q}\ln t} \\ &= \sum_{m=-\infty}^{\infty} c_m t^d \exp\left[i(\varphi_1 + 2\pi m)\frac{\ln t}{\ln q}\right] \\ &= \sum_{m=-\infty}^{\infty} c_m f_m(t). \end{aligned} \tag{4.35}$$

or equivalently

$$W(t) = \sum_{m=-\infty}^{\infty} c_m f_m(t), \tag{4.36}$$

where

$$f_m(t) = t^d \exp\left[i(\varphi_1 + 2\pi m)\frac{\ln t}{\ln q}\right]. \tag{4.37}$$

To find coefficients c_m in this expansion, we start from definition of $W(t)$ by infinite sum (4.25). By using the Poisson summation formula (Berry&Lewis, 1979)

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} f(t) e^{i2\pi k t} dt, \tag{4.38}$$

we can transform the sum over n in (4.25) to the integral

$$W(t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{+\infty} dn \left(\frac{1 - e^{iq^n t}}{q^{d\,n}} \right) e^{i(\varphi_1 + 2\pi m)n}. \tag{4.39}$$

Now we calculate this integral, by substitution $q^n = e^{n \ln q}$ and $e^n = z$ then we get

$$\int_0^{+\infty} \frac{dz}{z} \left(\frac{1 - e^{i\,t\,z^{\ln q}}}{z^{d\,\ln q}} \right) z^{i(\varphi_1 + 2\pi m)} = I_1 + I_2. \tag{4.40}$$

where

$$I_1 = \int_0^{+\infty} z^{i(\varphi_1 + 2\pi m) - d\,\ln q - 1}\, dz = 0, \tag{4.41}$$

and

$$I_2 = -\int_0^{+\infty} e^{i\,t\,z^{\ln q}} z^{i(\varphi_1 + 2\pi m) - d\,\ln q - 1} dz. \tag{4.42}$$

If we choose $t\,z^{\ln q} = i\tau$ then we get $i\,d\tau = t\,\ln q\, z^{\ln q - 1}\, dz$ and $z = (\frac{i\tau}{t})^{\frac{1}{\ln q}}$ and if we substitute these equations in the above integral we get

$$I_2 = -\int_0^{+\infty} e^{-\tau} \left(\frac{i\,\tau}{t} \right)^{\frac{1}{\ln q}(i(\varphi_1 + 2\pi m) - d\,\ln q - \ln q)} i \frac{d\tau}{t\,\ln q} \tag{4.43}$$

or equivalently

$$I_2 = \frac{-i}{t\,\ln q} \left(\frac{i}{t} \right)^{\frac{i(\varphi_1 + 2\pi m)}{\ln q} - d - 1} \int_0^{+\infty} e^{-\tau} \tau^{\frac{i(\varphi_1 + 2\pi m)}{\ln q} - d - 1} d\tau, \tag{4.44}$$

and last integral can be expressed as Γ function, with the result

$$W(t)=\frac{e^{i\frac{\pi}{2}(d+2)}e^{-\frac{\pi}{2}\varphi_1\ln q}}{\ln q}\sum_{m=-\infty}^{\infty}f_m(t)e^{\frac{-\pi^2 m}{\ln q}}\Gamma\left(-d+i\frac{\varphi_1+2\pi m}{\ln q}\right), \qquad (4.45)$$

where $f_m(t)=t^d\exp\left[i(\varphi_1+2\pi m)\frac{\ln t}{\ln q}\right]$. This expression is explicit realization of series expansion (4.36) in terms of functions (4.37).

4.1.3. Convergency of Weierstrass-Mandelbrot Function

To study convergency property of Weierstrass-Mandelbrot function in representation (4.45) we apply Stirling's asymptotic formula;

$$|\Gamma(a+ib)|\sim\sqrt{2\pi}|b|^{a-1/2}e^{-\pi|b|/2}, \qquad |b|\to\infty, \qquad (4.46)$$

$$\begin{aligned}\Gamma\left(-d+i\frac{\varphi_1+2\pi m}{\ln q}\right) &\sim \sqrt{2\pi}\left|\frac{\varphi_1+2\pi m}{\ln q}\right|^{(-d-\frac{1}{2})}e^{-\left|\frac{\varphi_1+2\pi m}{\ln q}\right|\frac{\pi}{2}} \qquad (4.47)\\ &= \sqrt{2\pi}\left(\frac{2\pi|m|}{\ln q}\right)^{-\left(d+\frac{1}{2}\right)}\left|1+\frac{\varphi_1}{2\pi m}\right|^{-\left(d+\frac{1}{2}\right)}e^{-\frac{\pi}{2\ln q}2\pi|m|\left|1+\frac{\varphi_1}{2\pi m}\right|}.\end{aligned}$$

Since $|m|\to\infty$, the term $\frac{\varphi_1}{2\pi m}\to 0$ and we get

$$\Gamma\left(-d+i\frac{\varphi_1+2\pi m}{\ln q}\right)\sim(2\pi)^{-d}(\ln q)^{\left(d+\frac{1}{2}\right)}|m|^{-\left(d+\frac{1}{2}\right)}e^{-\frac{\pi^2|m|}{\ln q}}. \qquad (4.48)$$

If we substitute this formula in equation (4.45) then we get

$$W(t)\approx\frac{e^{i\frac{\pi}{2}(d+2)}e^{-\frac{\pi}{2}\varphi_1\ln q}}{\ln q}\sum_{m=-\infty}^{\infty}f_m(t)e^{-\frac{\pi^2 m}{\ln q}}(2\pi)^{-d}(\ln q)^{\left(d+\frac{1}{2}\right)}|m|^{-\left(d+\frac{1}{2}\right)}e^{-\frac{\pi^2|m|}{\ln q}}. \qquad (4.49)$$

We have found $f_m(t) = t^d \exp\left[i(\varphi_1 + 2\pi m)\frac{\ln t}{\ln q}\right]$, if we arrange this equation for $m \gg 1$ then we get

$$\begin{aligned} f_m(t) &= t^d e^{i2\pi m\left(1+\frac{\varphi_1}{2\pi m}\right)\frac{\ln t}{\ln q}} \\ &\sim t^d e^{i2\pi m\frac{\ln t}{\ln q}} \end{aligned} \tag{4.50}$$

or equivalently

$$f_m(t) = t^d e^{i\,c\,m}, \tag{4.51}$$

where $c = 2\pi\frac{\ln t}{\ln q}$ and $m \gg 1$. Therefore we obtain;

$$W(t) \approx -(2\pi)^{-d}(\ln q)^{\left(d-\frac{1}{2}\right)}e^{-\frac{\pi}{2}(\varphi_1 \ln q - i\,d)}t^d \sum_{m=-\infty}^{\infty} e^{i\,c\,m}e^{-\frac{\pi^2 m}{\ln q}}|m|^{-\left(d+\frac{1}{2}\right)}e^{-\frac{\pi^2|m|}{\ln q}}. \tag{4.52}$$

Using Stirling's formula and ratio test, we show that the series is convergent.

4.1.4. Mellin Expansion for q-periodic function

Now we consider the real part of the Weierstrass-Mandelbrot function when all the phases are chosen zero,

$$f(t) = \Re(W(t))|_{\varphi_n=0} = \sum_{n=-\infty}^{\infty} q^{-nd}(1-\cos(q^n t)). \tag{4.53}$$

If we scale the time in (4.53) by the parameter q we obtain;

$$f(qt) = \sum_{n=-\infty}^{\infty} q^{-nd}(1-\cos(q^{n+1}t)) = q^d \sum_{n=-\infty}^{\infty} q^{-nd}(1-\cos(q^n t)) = q^d f(t). \tag{4.54}$$

or equivalently

$$f(qt) = q^d f(t). \tag{4.55}$$

So that the function $f(t)$ is scale-invariant and from equation (2.139) the solution to the scaling equation, (4.55), is given by the functional form;

$$f(t) = t^d A_q(t), \tag{4.56}$$

where $A_q(t)$ q-periodic function. From (2.153), the general form of the q-periodic function is given by

$$A_q(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{\ln q}\ln t}. \tag{4.57}$$

Note that we can write the solution of the scaling equation (4.56) in terms of a complex exponent as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n t^{d_n}, \tag{4.58}$$

where the exponent is indexed by the integer n,

$$d_n = d + i\frac{2\pi n}{\ln q}. \tag{4.59}$$

In Chapter 2 we have seen the scale invariance and how to related it to the q-periodic functions. In equation (4.58) we get the Mellin form of the function as equation (2.159). This form is very important. Because section 4.2 we show how to relate self-similarity and Mellin representation.

4.1.5. Graphs of Weierstrass-Mandelbrot Function

Each graph plots $-10 \leq n \leq 10$ terms in equation (4.53) and $D = 2 - d$ is the self-similar dimension of the Weierstrass-Mandelbrot function.

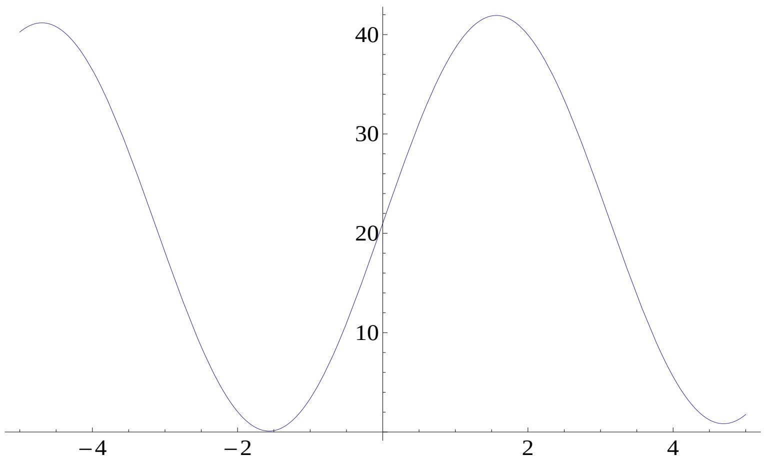

Figure 4.1. Weierstrass-Mandelbrot fractal function; $q = 1.01$, $D = 1.5$, $\varphi_n = \frac{\pi}{2}$, $-5 \leq t \leq 5$.

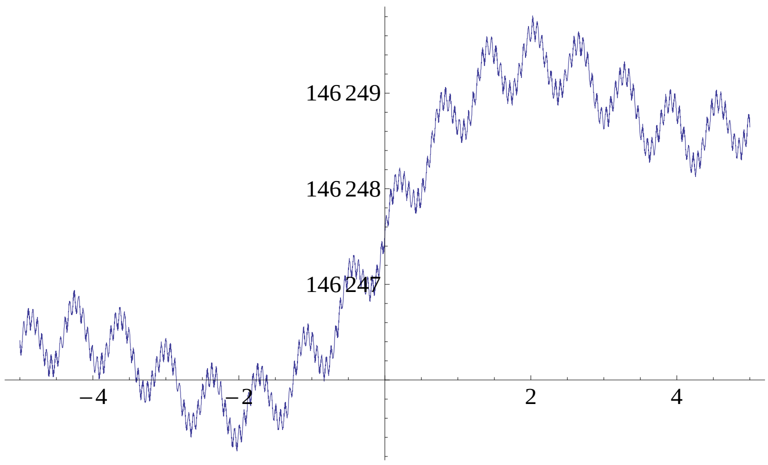

Figure 4.2. Weierstrass-Mandelbrot fractal function; $q = 10$, $D = 1.5$, $\varphi_n = \frac{\pi}{2}$, $-5 \leq t \leq 5$.

Figures (4.1) and (4.2) show how almost smooth curve with $q = 1.01$ Fig.(4.1) becomes a curve with fractal microstructure for $q = 10$ Fig.(4.2).

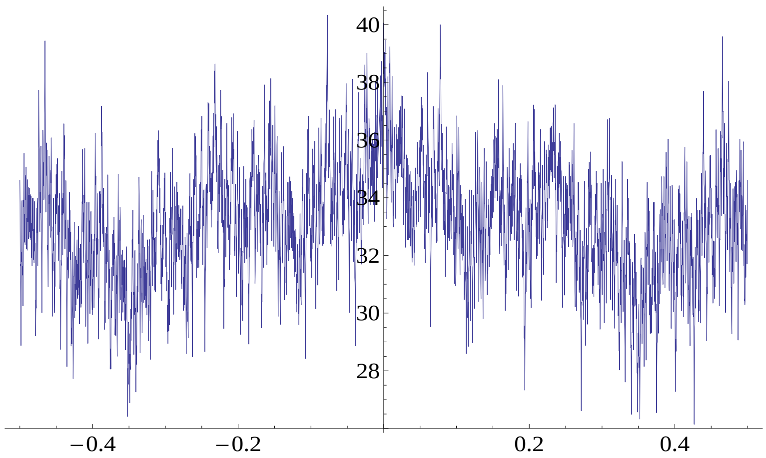

Figure 4.3. Weierstrass-Mandelbrot fractal function; $q = 3$, $D = 1.99$, $\varphi_n = \pi$, $-0.5 \le t \le 0.5$.

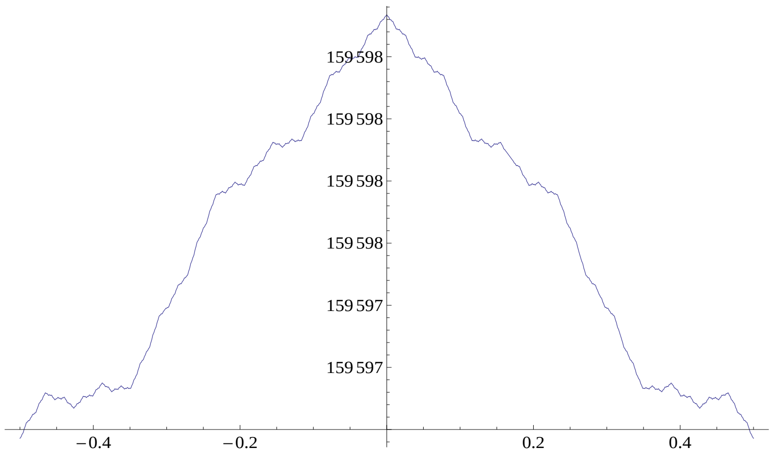

Figure 4.4. Weierstrass-Mandelbrot fractal function; $q = 3$, $D = 1.01$, $\varphi_n = \pi$, $-0.5 \le t \le 0.5$.

The dimension of Weierstrass-Mandelbrot fractal function change between $1 < D < 2$ and D is non-integer number. As we found in Chapter 2, the non-integer dimension is related with nowhere differentiability of the function. This means that geometrically the slope between any two points on this curves is undefined because the curves are never stop themselves. In addition, the fractal dimension characterizes the measure of space-filling. For $D = 1.99$, which is close to the dimension of plane $D = 2$, the curve cover the big portion (almost full) of plane Fig.(4.3). For D=1.01 which is close to dimension of a smooth curve looks like one dimensional continuous curve, though not so smooth.

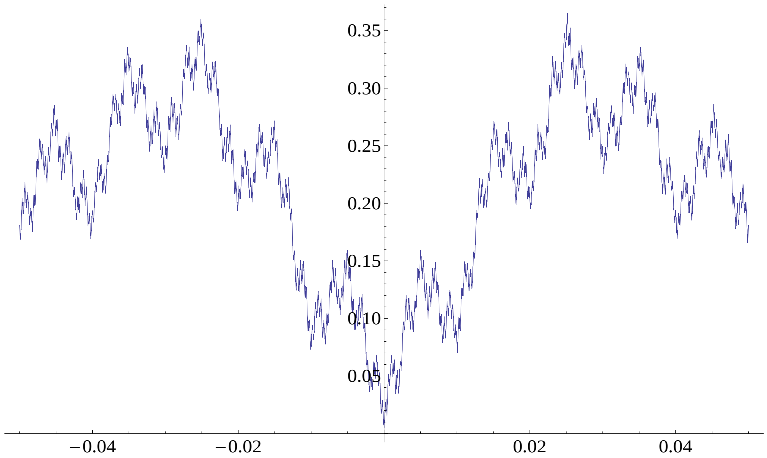

Figure 4.5. Weierstrass-Mandelbrot fractal function; $q = 5$, $D = 1.5$, $\varphi_n = 0$, $-0.5 \leq t \leq 0.5$.

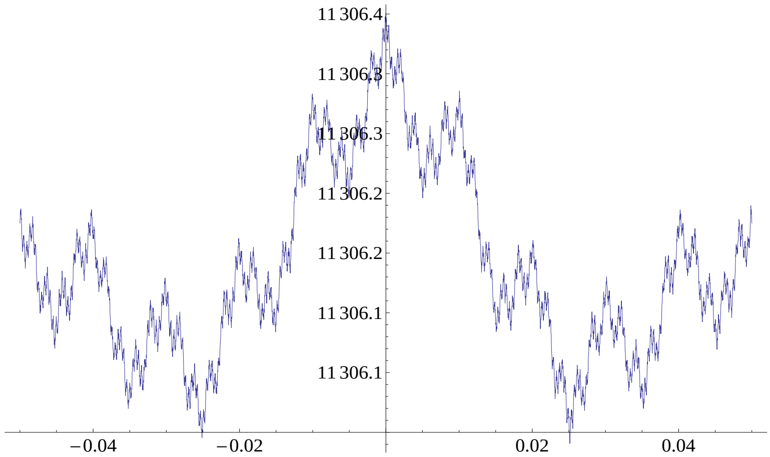

Figure 4.6. Weierstrass-Mandelbrot fractal function; $q = 5$, $D = 1.5$, $\varphi_n = \pi$, $-0.5 \leq t \leq 0.5$.

Differences in the function plots related to different phase are shown in Fig.(4.5) ($\varphi_n = 0$) and in Fig.(4.6) ($\varphi_n = \pi$).

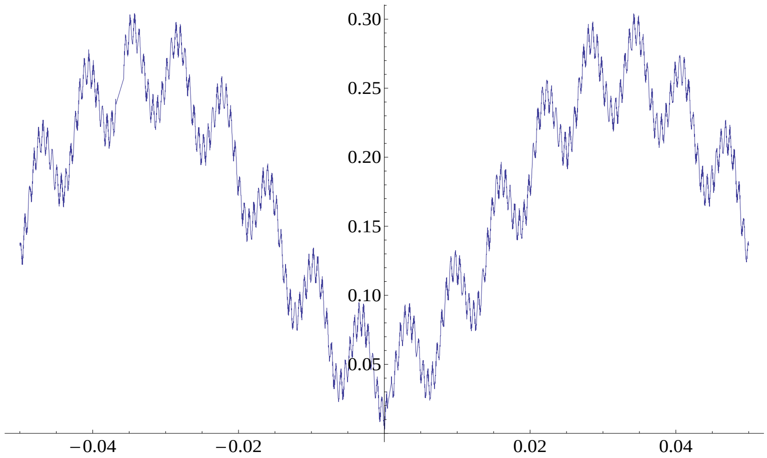

Figure 4.7. Weierstrass-Mandelbrot fractal function; $q = 10$, $D = 1.5$, $\varphi_n = 0$, $-0.05 \leq t \leq 0.05$.

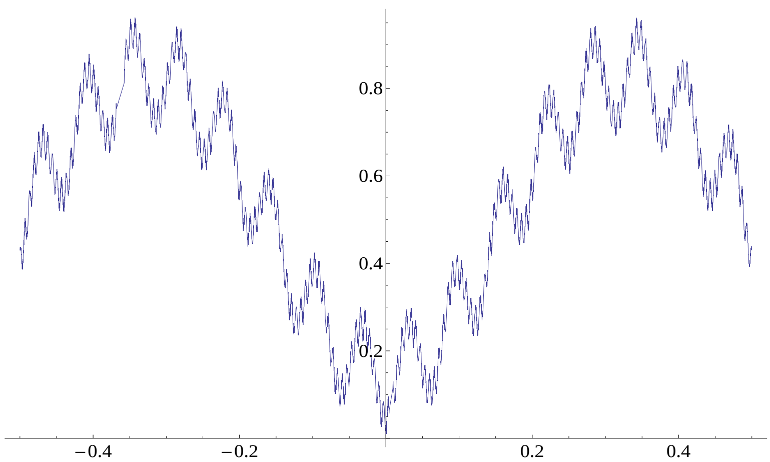

Figure 4.8. Weierstrass-Mandelbrot fractal function; $q = 10$, $D = 1.5$, $\varphi_n = 0$, $-0.5 \leq t \leq 0.5$.

The self-similarity of f(t) defined by (4.55) is shown in Fig.(4.7), (4.8), (4.9). Change in the scale by $q = 10$ in Fig.(4.8) and by $q^{-1} = 10^{-1}$ in Fig.(4.9) does not change the shape of the curve.

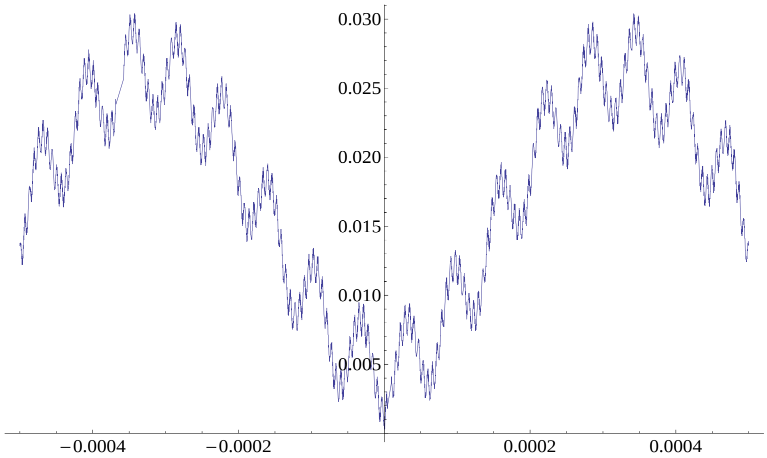

Figure 4.9. Weierstrass-Mandelbrot fractal function; $q = 10$, $D = 1.5$, $\varphi_n = 0$, $-0.0005 \leq t \leq 0.0005$.

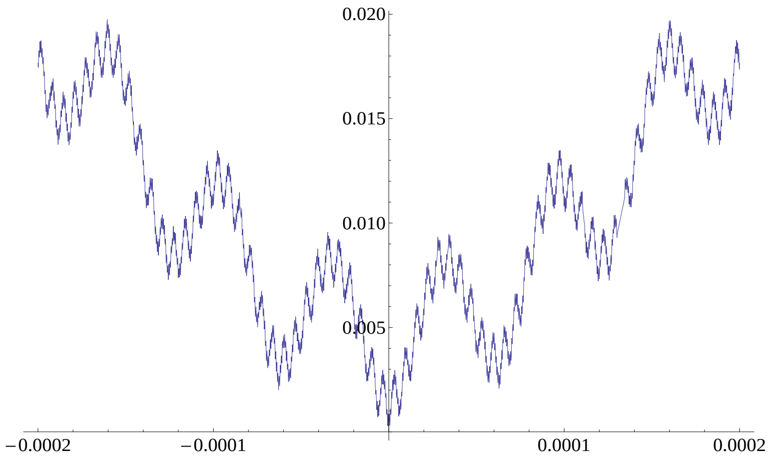

Figure 4.10. Weierstrass-Mandelbrot fractal function; $q = 10$, $D = 1.5$, $\varphi_n = 0$, $-0.0002 \leq t \leq 0.0002$.

However if the scale change in not by $q^{\pm n} = 10^{\pm n}$, then the figure change the shape, as shown in Fig.(4.10) for scale 250.

4.2. Tones and Chirps

In the previous section we discussed the scale invariance (self-similarity) and relation with q-periodicity for the Weierstrass-Mandelbrot function. By using Fourier series and proper transformation of the argument we derived the Mellin series for Weierstrass-Mandelbrot function. In this section we define Fourier transformation and Mellin transformation in general. These two transformations are both characterized by stationarity and self-similarity respectively. The stationarity (shift invariance) of a function will be called tone (Fourier mode) and the self-similarity (scale invariance) will be called chirp (Mellin mode). We show this relation by the Lamperti transformation. The Lamperti transformation defines one to one correspondence between stationary processes on the real line (Fourier Mode) and self-similar processes on the real half line (Mellin Mode).

4.2.1. Stationarity and Self-Similarity

The idea of stationarity is equivalent to the shift or translation invariance and this concept is related to Fourier transformation. Since the Fourier transformation is shift or translation invariant. The self similarity is equivalent to the scale invariance and this concept is related to Mellin transformation. Since the Mellin transformation is scale invariant. In this subsection we define stationarity and self-similarity processes.

Definition 4.2.1.1 *Given $\tau \in \mathbb{R}$, the shift or translation operator S_τ operates on processes $\{Y(t), t \in \mathbb{R}\}$ according to;*

$$(S_\tau Y)(t) \equiv Y(t+\tau). \tag{4.60}$$

Definition 4.2.1.2 *Given $H > 0$ and $\lambda > 0$, the renormalized dilation operator $M_{H,\lambda}$, operates on processes $\{X(t), t > 0\}$ according to;*

$$(M_{H,\lambda} X)(t) \equiv \lambda^{-H} X(\lambda t). \tag{4.61}$$

Definition 4.2.1.3 *A process $\{Y(t), t \in \mathbb{R}\}$ is said to be stationary if*

$$\{(S_\tau Y)(t), t \in \mathbb{R}\} \equiv \{Y(t), t \in \mathbb{R}\}. \tag{4.62}$$

Definition 4.2.1.4 *A process $\{X(t), t > 0\}$ is said to be self-similar of index H if*

$$\{(M_{H,\lambda} X)(t), t > 0\} \equiv \{X(t), t > 0\}, \tag{4.63}$$

for any $\lambda > 0$.

4.2.2. Transformations for Chirps

Definition 4.2.2.1 *(Borgnat&Flandrin, 2002) We will call 'chirps' any complex signals of the form $a(t)exp\{i\psi(t)\}$, with $\psi(t) = 2\pi n \log t$ and $a(t) > 0$.*

We can consider a q-periodic function as an example of chirp. In Chapter 2 We defined q-periodic function expansion as;

$$A_q(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi n}{\ln q} \ln t}, \tag{4.64}$$

It shows the q-periodic function representation in the chirp form. For example;

$$A_q(t) = \cos\left(2\pi \frac{\ln t}{\ln q}\right) = \frac{1}{2} e^{i\frac{2\pi}{\ln q} \ln t} + \frac{1}{2} e^{-i\frac{2\pi}{\ln q} \ln t}, \tag{4.65}$$

where $q > 0$ and $q \neq 1$. Therefore any q-periodic function is super-position of chirp. And we know that q-periodic function is self-similar. So any (convergent) super-position of chirp is given by self-similarity.

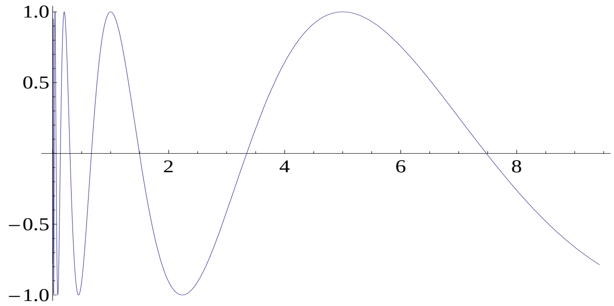

Figure 4.11. The graph of $\cos\left(2\pi\frac{\ln t}{\ln q}\right)$, $q = 5$, $0 < t < 3\pi$

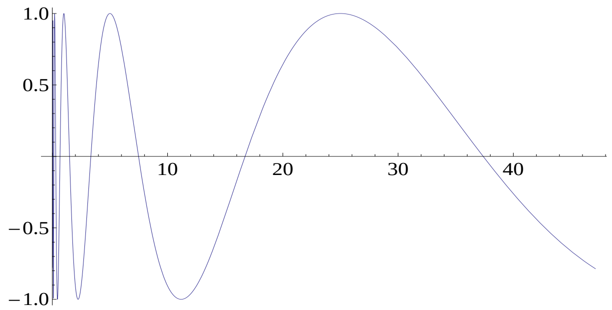

Figure 4.12. The graph of $\cos\left(2\pi\frac{\ln t}{\ln q}\right)$, $q = 5$, $0 < t < 15\pi$

It is illustrated in Fig.(4.11), (4.12) with $q = 5$ the scale difference in these two figures is by coefficient 5. And we can see complete identity of figures at these different scales.

Definition 4.2.2.2 *Given $H > 0$, the Lamperti transform $\mathcal{L}_H$ operates on processes $\{Y(t), t \in \mathbb{R}\}$ according to:*

$$(\mathcal{L}_H Y)(t) = t^H Y(\log t), \qquad t > 0, \tag{4.66}$$

and corresponding inverse Lamperti transform $\mathcal{L}_H^{-1}$ operates on processes $\{X(t), t > 0\}$ according to:

$$(\mathcal{L}_H^{-1} X)(t) = e^{-Ht} X(e^t), \qquad t \in \mathbb{R}. \tag{4.67}$$

The Lamperti transform is invertible, which guarantees that that $(\mathcal{L}_H^{-1}\mathcal{L}_H Y)(t) = Y(t)$ for any process $\{Y(t), t \in \mathbb{R}\}$ and $(\mathcal{L}_H \mathcal{L}_H^{-1} X)(t) = X(t)$ for any process $\{X(t), t > 0\}$. We can however remark that, given two different parameters H_1 and H_2, we only have

$$(\mathcal{L}_{H_2}^{-1}\mathcal{L}_{H_1} Y)(t) = e^{H_1 - H_2} Y(t), \tag{4.68}$$

and, in a similar way

$$(\mathcal{L}_{H_2}\mathcal{L}_{H_1}^{-1} X)(t) = t^{H_2 - H_1} X(t). \tag{4.69}$$

Lemma 4.2.2.3 *(Borgnat&Flandin, 2002) The Lamperti transform (4.66)-(4.67) guarantees an equivalence between the shift operator (4.60) and the renormalized dilation operator (4.61) in the sense that for any $\lambda > 0$*

$$\mathcal{L}_H^{-1} M_{H,\lambda} \mathcal{L}_H = S_{\log\lambda}. \tag{4.70}$$

Proof 4.2.2.4 *Assuming that $\{Y(t), t \in \mathbb{R}\}$ is stationary and using the equations (4.60), (4.61) and (4.66) we may write*

$$\begin{aligned}
(\mathcal{L}_H^{-1} M_{H,\lambda} \mathcal{L}_H Y)(t) &= (\mathcal{L}_H^{-1} M_{H,\lambda})(t^H Y(\log t)) \\
&= \mathcal{L}_H^{-1}(\lambda^{-H}(\lambda t)^H Y(\log \lambda t)) \\
&= e^{-Ht}(s^H Y(\log \lambda s))_{s=e^t} \\
&= Y(t + \log\lambda) \\
&= (S_{\log\lambda} Y)(t). \blacksquare
\end{aligned} \tag{4.71}$$

This lemma is the key ingredient for establishing a one to one connection between stationarity and self-similarity. This fact is referred to as Lamperti's theorem.

Theorem 4.2.2.5 *(Lamperti, 1962) If $\{Y(t), t \in \mathbb{R}\}$ is stationary, its Lamperti transform $\{(\mathcal{L}_H Y)(t), t \in \mathbb{R}\}$ is self-similar with index H. Conversely, $\{X(t), t > 0\}$ is self-similar with index H, its inverse Lamperti transform $\{(\mathcal{L}_H^{-1} X)(t), t > 0\}$ is stationary.*

Proof 4.2.2.6 *Let $\{Y(t), t \in \mathbb{R}\}$ be stationary process. Using equation (4.62) and lemma (4.2.2.3), we have for any $\lambda > 0$,*

$$\{Y(t), t \in \mathbb{R}\} \equiv \{(S_{\log\lambda}Y)(t) = (\mathcal{L}_H^{-1} M_{H,\lambda} \mathcal{L}_H Y)(t), t \in \mathbb{R}\}, \tag{4.72}$$

and it follows from equation (4.63) that the Lamperti transform $X(t) \equiv (\mathcal{L}_H Y)(t)$ is self-similar with index H. Since

$$\{(M_{H,\lambda}X)(t), t > 0\} \equiv \{X(t), t > 0\}, \tag{4.73}$$

for any $\lambda > 0$.

Conversely, let $\{X(t), t > 0\}$ be a self-similar process with index H. Using equation (4.63) and lemma (4.2.2.3) we have for any $\lambda > 0$,

$$\{X(t), t > 0\} \equiv \{(M_{H,\lambda}X)(t) = (\mathcal{L}_H S_{\log\lambda} \mathcal{L}_H^{-1} X)(t), t > 0\}, \tag{4.74}$$

and it follows from equation (4.62) that inverse Lamperti transform $Y(t) \equiv (\mathcal{L}_H^{-1}X)(t)$ is stationary since

$$\{Y(t), t \in \mathbb{R}\} \equiv \{(S_{\log\lambda}Y)(t), t \in \mathbb{R}\}, \tag{4.75}$$

for any $\lambda > 0$.$\blacksquare$

Self-similar process can be obtained by ”lampertizing” corresponding stationary process. A transformation F on a self-similar process with index H, $\{X(t), t > 0\}$ can be equivalently achieved as $F = \mathcal{L}_H \overline{F} \mathcal{L}_H^{-1}$, according to the commutative diagram:

$$\begin{tikzcd}
X(t) \arrow[r, "F"] \arrow[d, "\text{inverse Lamperti}"'] & (F\,X)(t) \arrow[r, rightsquigarrow] & \text{(self-similarity world)} \\
(\mathcal{L}_H^{-1}X)(t) \arrow[r, "\overline{F}"'] & (\overline{F}\mathcal{L}_H^{-1}X)(t) \arrow[u, "\text{Lamperti}"'] \arrow[r, rightsquigarrow] & \text{(stationarity world)} \arrow[u]
\end{tikzcd}$$

Example 4.1 *The stationary random phase "tone"*

$$Y_0(t) \equiv a\cos(2\pi f_0 t + \varphi), \qquad t \in \mathbb{R}, \tag{4.76}$$

with $a, f_0 > 0$ *and* $\varphi \in [0, 2\pi]$*, is "lampertized" into the (self-similar) random phase "chirp"*

$$X_0(t) \equiv (\mathcal{L}_H Y_0)(t) = at^H \cos(2\pi f_0 \log t + \varphi), \qquad t > 0. \tag{4.77}$$

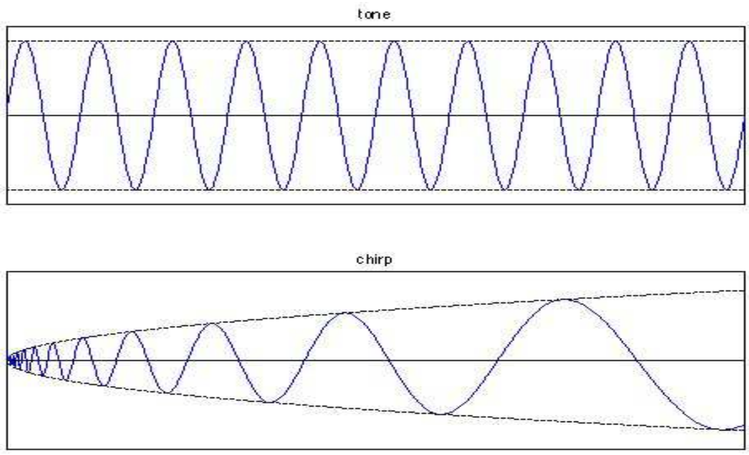

Figure 4.13. Tones and Chirps

Remark 4.1 $X_0(t) = \Re\{ae^{i\varphi}m_s(t)\}$ *with* $s = H + i2\pi f_0$ *and* $m_s(t) = t^s$ *the basic building block of the Mellin transform.*

The Lamperti transform of a tone is a "chirp" with a power law amplitude modulation and logarithmic frequency modulation. Said in other words, the Lamperti transform maps the Fourier basis onto a Mellin basis.

Lamperti transform allows for a one-to-one correspondence between periodic and self-similar functions. Periodic functions can be expanded on "tones"(or Fourier modes):

$$e_n(t) = e^{i2\pi nt}, \tag{4.78}$$

whose Lamperti transform expresses straightforwardly as:

$$C_{H,n}(t) = (\mathcal{L}_H e_n)(t) = t^{H+i2\pi n}, \qquad t > 0. \tag{4.79}$$

Definition 4.2.2.7 *Given $s \in \mathbb{R}$ and $e_s(t)$ as in (4.78), the Fourier transform of a function $\{Y(t), t \in \mathbb{R}\}$ is defined by:*

$$(\mathcal{F}Y)(s) = \int_{-\infty}^{+\infty} Y(t)\overline{e_s(t)}dt, \tag{4.80}$$

with the corresponding reconstruction formula:

$$Y(t) = \int_{-\infty}^{+\infty} (\mathcal{F}Y)(s)e_s(t)ds. \tag{4.81}$$

Definition 4.2.2.8 *Given $H > 0$, $s \in \mathbb{R}$ and $C_{H,s}(t)$ as in (4.79), the Mellin transform of a function $\{X(t), t > 0\}$ is defined by:*

$$(\mathcal{M}_H X)(s) = \int_0^{+\infty} X(t)\overline{C_{H,s}(t)}dt/t^{2H+1}, \tag{4.82}$$

with the corresponding reconstruction formula:

$$X(t) = \int_{-\infty}^{+\infty} (\mathcal{M}_H X)(s)C_{H,s}(t)ds. \tag{4.83}$$

Now we consider the self-similarity (scale invariant) of Mellin transform and stationarity (shift invariant) of Fourier transform.

The scale invariance of Mellin transform is shown by considering a function $X(\lambda t)$. Then

$$(\mathcal{M}_{H,\lambda} X)(s) = \int_0^{+\infty} X(\lambda t)\, t^{-\alpha-1}\, dt = \lambda^{-\alpha}(\mathcal{M}_H X)(s) \tag{4.84}$$

or equivalently

$$(\mathcal{M}_{H,\lambda}X)(s) = \lambda^{-\alpha}(\mathcal{M}_H X)(s), \tag{4.85}$$

where $\alpha = H + i2\pi s$.

In a similar way the Fourier transform

$$(\mathcal{F}X)(s) = \int_{-\infty}^{+\infty} X(t)\, e^{-i2\pi st}\, dt, \tag{4.86}$$

can be shown to be shift or translation invariant by considering a function $X(t-a)$

$$(\mathcal{F}X)(s,a) = \int_{-\infty}^{+\infty} X(t-a)\, e^{-i2\pi st}\, dt = e^{-i2\pi sa}(\mathcal{F}X)(s) \tag{4.87}$$

or equivalently

$$(\mathcal{F}X)(s,a) = e^{-i2\pi sa}(\mathcal{F}X)(s). \tag{4.88}$$

Clearly, for the Fourier transform under translation, we have

$$|(\mathcal{F}X)(s,a)| = |(\mathcal{F}X)(s)|, \tag{4.89}$$

while for the Mellin transform under re-scale

$$|(\mathcal{M}_{H,\lambda}X)(s)| = \lambda^{-H}|(\mathcal{M}_H X)(s)|. \tag{4.90}$$

Therefore the Mellin transform is scale invariant and the Fourier transform is shift invariant. In the next section we define chirp decomposition of the generalized Weierstrass-Mandelbrot function and we will see that this function represents, the Mellin transform.

4.2.3. Chirp Form of the Generalized Weierstrass-Mandelbrot Function

In this subsection we shall define the chirp decomposition of the generalized Weierstrass-Mandelbrot function. For this purpose firstly we must write the general form of Weierstrass-Mandelbrot function.

We know the Weierstrass-Mandelbrot function representation in the q-periodic form from equation (4.33),

$$W(t) = t^{d+i\frac{\varphi_1}{\ln q}} A_q(t), \tag{4.91}$$

where $A_q(t)$ is q-periodic function and φ_1 is an arbitrary phase. If we choose $\varphi_1 = 0$ then we get

$$W(t) = t^d A_q(t). \tag{4.92}$$

We have written q-periodic function in equation (2.140);

$$A_q(t) = t^{-d} \sum_{n=-\infty}^{\infty} q^{-nd} g(q^n t), \tag{4.93}$$

where g(t) can be periodic function, provided that it is continuously differentiable at $t = 0$. Now if we substitute equation (4.93) to (4.92) then we obtain

$$W(t) = \sum_{n=-\infty}^{\infty} q^{-nd} g(q^n t). \tag{4.94}$$

The specific form of the Weierstrass-Mandelbrot function given in (4.94) can itself be generalized to;

$$W_g(t) = \sum_{n=-\infty}^{\infty} (q^{-nd} g(q^n t)) e^{i\varphi_n}, \tag{4.95}$$

where φ_n is an arbitrary phase and g(t) can be any periodic function, provided that it is continuously differentiable at $t = 0$. This function is called generalized Weierstrass-Mandelbrot function.

Proposition 4.2.3.1 *(Borgnat&Flandrin, 2002) The scale-invariant generalized Weierstrass-Mandelbrot function admits the chirp decomposition:*

$$W_g(t) = \sum_{m=-\infty}^{\infty} a_m C_{d,m/\log q}(t), \tag{4.96}$$

where

$$a_m = \frac{(\mathcal{M}_d g)(m/\log q)}{\log q}, \qquad C_{d,m/\log q}(t) = t^{d+i2\pi \frac{m}{\log q}}, \tag{4.97}$$

with $(\mathcal{M}_d g)(.)$ the Mellin transform of $g(t)$.

Proof 4.2.3.2 *Suppose that $\varphi_n = 0$ in equation (4.95). We obtain;*

$$(\mathcal{L}_d^{-1} W_g)(t) = (\mathcal{L}_d^{-1} W_g)(t + k \log q), \tag{4.98}$$

$k \in \mathbb{Z}$, thus proving that the inverse Lamperti transform of a scale-invariant generalized Weierstrass-Mandelbrot function is periodic of period $\log q$.
As a periodic function, it can be expanded in a Fourier series;

$$(\mathcal{L}_d^{-1} W_g)(t) = \sum_{m=-\infty}^{\infty} a_m e_{m/\log q}(t), \tag{4.99}$$

with:

$$a_m = \frac{1}{\log q} \int_0^{\log q} (\mathcal{L}_d^{-1} W_g)(t) \overline{e_{m/\log q}(t)} dt. \tag{4.100}$$

Inverting (4.99) and using the fact that the Lamperti transform of a Fourier tone is a chirp, *we get*

$$W_g(t) = \sum_{m=-\infty}^{\infty} a_m C_{d,m/\log q}(t), \tag{4.101}$$

with:

$$\begin{aligned} a_m &= \frac{1}{\log q} \int_0^{\log q} [e^{-d\theta} \sum_{n=-\infty}^{\infty} q^{-nd} g(q^n e^{\theta})] \overline{e_{m/\log q}(\theta)} d\theta \\ &= \frac{1}{\log q} \sum_{n=-\infty}^{\infty} q^{-nd} \int_{q^n}^{q^{n+1}} g(u)(q^{-n}u)^{-d} \overline{e_{m/\log q}(\log u - n\log q)} du/u \\ &= \frac{1}{\log q} \sum_{n=-\infty}^{\infty} \int_{q^n}^{q^{n+1}} g(u) \overline{C_{-d,m/\log q}(u)} du/u \\ &= \frac{1}{\log q} \int_0^{\infty} g(u) \overline{C_{d,m/\log q}(u)} du/u^{2d+1} \\ &= \frac{(\mathcal{M}_d g)(m/\log q)}{\log q}. \blacksquare \end{aligned} \tag{4.102}$$

Example 4.2 *Let us consider the standard Weierstrass-Mandelbrot function (4.94) with $\varphi_n = 0$. We have in this case $g(t) = 1 - e^{it}$ and*

$$a_m = \frac{1}{\log q} \int_0^{\infty} (1 - e^{iu}) u^{-\alpha - 1} du, \tag{4.103}$$

with $\alpha = d + \frac{i2\pi m}{\log q}$. An integration by parts leads to

$$a_m = \frac{1}{\log q} \left[\frac{e^{-i\pi/2}}{\alpha} \int_0^{\infty} e^{iu} u^{(1-\alpha)-1} du \right], \tag{4.104}$$

with $\Re(1-\alpha) = 1 - d > 0$, since $0 < d < 1$, thus guaranteeing the convergence of the integral. Making the change of variable $v = ue^{-i\pi/2}$, we finally find the result,

$$a_m = -\frac{1}{\log q} \exp\left\{ -i\frac{\pi}{2} \left(d + \frac{i2\pi m}{\log q} \right) \right\} \Gamma(-d - \frac{i2\pi m}{\log q}). \tag{4.105}$$

CHAPTER 5

CONCLUSIONS

In the present thesis we studied self-similar objects by methods of the q-calculus. We have introduced the basic notations of q-calculus, q-numbers, q-derivative and q-integral.

Then we introduced q-periodic functions and their relation with periodic functions by the Mellin transform. We applied our technique to the self-similar objects. We introduced self-similarity in connection with homogeneous functions and studied some properties of these functions. Then we considered some applications of these functions in geometry and theory of ordinary differential equations.

Self-similar object of fractal type were introduced and dimension of these objects calculated. Relation with q-calculus was explicitly demonstrated.

We have reviewed some basic formulas of asymptotic analysis with the goal to study special type of fractal curves as the Weierstrass-Mandelbrot function. Several results on convergency and asymptotics of this function were derived.

For self-similarity is the special transform to the logarithm scale like the Lamperti transform were derived. Chirp decomposition of the generalized Weierstrass-Mandelbrot function was found.

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APPENDIX A

CONVERGENCY OF SERIES

Many constructions of nowhere differentiable continuous functions are based on infinite series of functions. Therefore we will give a few general theorems about series and sequences.

Definition A.0.0.3 *A sequence S_n of functions on the interval I is said to converges pointwise to a function S on I if for every $x \in I$*

$$\lim_{n \to \infty} S_n(x) = S(x), \tag{A.1}$$

that is

$$\forall x \in I, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |S_n(x) - S(x)| < \epsilon. \tag{A.2}$$

The convergence is said to be uniformly on I if

$$\lim_{n \to \infty} \sup_{x \in I} |S_n(x) - S(x)| = 0 \tag{A.3}$$

that is

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \sup_{x \in I} |S_n(x) - S(x)| < \epsilon. \tag{A.4}$$

Theorem A.0.0.4 *The sequence S_n converges uniformly on I if and only if an uniformly Cauchy sequence on I, that is*

$$\lim_{m,n \to \infty} \sup_{x \in I} |S_n(x) - S_m(x)| = 0 \tag{A.5}$$

or equivalently

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \sup_{x \in I} |S_n(x) - S_m(x)| < \epsilon. \tag{A.6}$$

Proof A.0.0.5 *First, assume that S_n converges uniformly to S on I, that is*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \sup_{x \in I} |S_n(x) - S(x)| < \frac{\epsilon}{2}. \tag{A.7}$$

For such $\epsilon > 0$ and for $m, n \in \mathbb{N}$ with $m, n \geq N$ we have

$$\begin{aligned} \sup_{x \in I} |S_n(x) - S_m(x)| &\leq \sup_{x \in I}(|S_n(x) - S(x)| + |S(x) - S_m(x)|) \qquad \text{(A.8)} \\ &\leq \sup_{x \in I} |S_n(x) - S(x)| + \sup_{x \in I} |S(x) - S_m(x)| < 2\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Conversely, assume that $\{S_n\}$ is a uniformly Cauchy sequence i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \sup_{x \in I} |S_n(x) - S_m(x)| < \frac{\epsilon}{2}. \tag{A.9}$$

For any fixed $x \in I$, the sequence $\{S_n(x)\}$ is clearly Cauchy sequence of real numbers. Hence the converges to a real number, say $S(x)$. From the assumption and the pointwise convergence just established we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \sup_{x \in I} |S_n(x) - S_m(x)| < \frac{\epsilon}{2}. \tag{A.10}$$

and

$$\forall \epsilon > 0, \forall x \in I, \exists m_x > N, |S_{m_x}(x) - S(x)| < \frac{\epsilon}{2}. \tag{A.11}$$

If $\epsilon > 0$ is arbitrary and $n > N$, then

$$\begin{aligned}\sup_{x\in I}|S_n(x) - S(x)| &\leq \sup_{x\in I}(|S_n(x) - S_{m_x}(x)| \\ &+ |S_{m_x}(x) - S_m(x)|) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{A.12}$$

Hence the convergence of S_n to S is uniform on I.■

Theorem A.0.0.6 *(Weierstrass M-test) Let $f_k : I \to \mathbb{R}$ be a sequence of functions such that $\sup_{x\in I}|f_k(x)| \leq M_k$ for every $k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} M_k < \infty$ then the series $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on I.*

Proof A.0.0.7 *Let $m, n \in \mathbb{N}$ with $m > n$. Then*

$$\begin{aligned}\sup_{x\in I}|S_n(x) - S_m(x)| &= \sup_{x\in I}\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| \\ &= \sup_{x\in I}\left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} \sup_{x\in I}|f_k(x)| \\ &\leq \sum_{k=m+1}^{n} M_k = \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k. \end{aligned} \tag{A.13}$$

Since $M = \sum_{k=1}^{\infty} M_k < \infty$ it follows that

$$\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k = M - M = 0 \qquad \text{as } m, n \to \infty, \tag{A.14}$$

which gives that $\{S_n\}$ is uniformly Cauchy sequence on I. Using above theorem we obtain the series $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on I.■

Theorem A.0.0.8 *If $\{S_n\}$ is a sequence of continuous functions on I and S_n converges uniformly to S on I, then S is continuous function on I.*

Proof A.0.0.9 *Let $x_0 \in I$ be arbitrary. By assumption we have*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \sup_{x\in I}|S_n(x) - S(x)| < \frac{\epsilon}{3}. \tag{A.15}$$

and

$$\forall\epsilon > 0, \exists\delta \;\; \textit{such that} \;\; |x - x_0| < \delta \Rightarrow |S_n(x) - S(x_0)| < \frac{\epsilon}{3}. \tag{A.16}$$

Let $\epsilon > 0$ *be given,* $x \in I$*,* $n \in \mathbb{N}$ *with* $n > N$ *and* $|x - x_0| < \delta$*. Then*

$$\begin{aligned} |S(x) - S(x_0)| &\leq |S(x) - S_n(x)| + |S_n(x) - S_n(x_0)| \\ &+ |S_n(x_0) - S(x_0)| < 3\frac{\epsilon}{3} = \epsilon, \end{aligned} \tag{A.17}$$

and therefore S *is continuous at* x_0*. Since* x_0 *was arbitrary,* S *continuous on* I*.*■

Corollary A.0.0.10 *If* $f_k : I \to \mathbb{R}$ *is continuous function for every* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} f_k(x)$ *converges uniformly to* $S(x)$ *on* I*, then* S *is continuous function on* I*.*