CLASSICAL AND QUANTUM EULER EQUATION

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by
Neslihan ETİ

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İZMİR

We approve the thesis of **Neslihan ETİ**

Date of Signature

. . . . . **27 June 2007**
Prof. Dr. Oktay PASHAEV
Supervisor
Department of Mathematics
İzmir Institute of Technology

. . . . . **27 June 2007**
Prof. Dr. Recai ERDEM
Department of Physics
İzmir Institute of Technology

. . . . . **27 June 2007**
Assist. Prof H. Sami SÖZÜER
Department of Physics
İzmir Institute of Technology

. . . . . **27 June 2007**

Prof. Dr. Oğuz YILMAZ
Head of Department
Department of Mathematics
İzmir Institute of Technology

. . . . . .
Prof. Dr. M. Barış ÖZERDEM
Head of the Graduate School

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ABSTRACT

CLASSICAL AND QUANTUM EULER EQUATION

In the present thesis we give generalization of analytical mechanics to describe dynamical systems with dissipation. The Lagrangian function in this case is determined by non-stationary pseudo-Riemannian metric for the kinetic energy, and by general quadratic form, non-diagonal in the generalized coordinates and velocities. Skew symmetric non-diagonal terms in our approach play the role of dissipation coefficients. As an application we study in details the classical damped harmonic oscillator. We show that two known formulations of this oscillator, the Bateman dual and the Caldirola-Kanai formulations are particular realizations of our general approach. The Hamiltonian formulation and quantization of the model in both representations are given. Moreover Ostrogradsky generalization of Lagrangian and Hamiltonian formalism for description of systems with higher order derivatives and its application to the constant coefficient equations of an arbitrary order are considered. We construct related with the last one the Euler differential equation of an arbitrary order and its Lagrangian and Hamiltonian structure. Quantum Euler systems are introduced and solved for the stationary Schrödinger picture. Non-stationary nonlinear quantum models corresponding to arbitrary Euler Hamiltonian are solved exactly in the Heisenberg picture.

ÖZET

KLASİK VE KUANTUM EULER DENKLEMİ

Bu tezde, dağılımlı dinamik sistemlerin tarifi için analitik mekanikte bir genelleme verilmiştir. Bu durumda, Lagrangianın kinetik kısmı durağan ve köşegen olmayan Riemannianımsı (pseudo-Riemannian) metrik ile genelleştirilmiş koordinatlarda ve hızlarda belirlenmiştir. Yaklaşımımızda simetrik ve köşegen olmayan terimler dağılma katsayısı rolü oynamıştır. Bir uygulama olarak klasik sönümlü harmonik salınıcı detaylıca çalışılmıştır. Bu salınıcının iki bilinen formulasyonunun; Bateman dual formulasyonu ve Cardirola-Kanai formulasyonu, genel yaklaşımımızın özel durumları olarak ortaya konulmuştur. Her iki modelin temsilinde de Hamiltonian formulasyonu verilmiş, ayrıca kuantizasyon yapılmıştır. Yüksek dereceden türevli sistemlerin tanımı için Lagrangian ve Hamiltonian formalisimlerinin Ostogradski genellemesi verilmiş ve bunun gelişigüzel sabit katsayılı denklemlere uygulaması yapılmıştır. Son olarak gelişigüzel dereceli Euler differansiyel denklemini veren Lagrangian ve Hamiltonian yapıları çalışılmıştır. Kuantum Euler sistemleri tanıtılmış ve durağan Schröndinger resmi için çözümlenmiştir. Gelişigüzel Euler Hamiltoniana karşılık gelen durağan olmayan lineer olmayan kuantum modelleri Heisenberg resminde tam olarak çözülmüştür.

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CHAPTER 1

INTRODUCTION

The quantum damped oscillator has been studied to understand dissipation in quantum theory since the damped harmonic oscillator is one of the simplest quantum systems displaying the dissipation of energy.

Quantum damped harmonic oscillator is studied within two model oscillators. One representation is the Bateman-Feshbach-Tikochinsky oscillator (Bateman oscillator) as a closed system with two degrees of freedom. The other representation is Caldirola-Kanai oscillator as an open system with one degree of freedom.

The Bateman-Feshbach-Tikochinsky damped harmonic oscillator described as an open system in which energy is dissipated by interaction with a heat bath. This is when quantization of a damped harmonic oscillator leads to so called Bateman's dual system. Bateman has shown that to apply the standard canonical formalism of classical mechanics to dissipative systems, one can double the numbers of degrees of freedom. The new degrees of freedom are assumed to represent a reservoir, also called heat bath. Applying this idea to damped harmonic oscillator one obtains a pair of damped oscillators ,so called Bateman's dual system (Bateman 1931). This system includes a primary one and its time reversed image. The Bateman dual Hamiltonian has been rediscovered by Morse and Feshbach and the detailed quantum mechanical analysis was performed by Feshbach and Tikochinski (Feshbach and Tikochinsky 1977). The quantum Bateman system was then analyzed by many authors, for review and references (Dekker 1981, Um et al. 2001).

Caldirola - Kanai model is a one-dimensional system with an exponentially increasing mass (Caldirola 1941, Kanai 1948) and the corresponding Lagrangian given by Bateman (Bateman 1931). This oscillator is an open system because its parameters such as mass or frequency depend explicitly on time. The fact that it's classical motion describes a damping motion has motivated the investigation of the Caldirola-Kanai Hamiltonian as a quantum damped system. The connection between two models has been found (Dekker 1981).

The goal of the present thesis is to study damped harmonic oscillator in the classical and quantum mechanical frames and then to show the relation with Euler equation

again classical and quantum cases. This is why we divide this thesis into two parts.

In the first part, we consider dissipation in analytic mechanics and quantum mechanics. In chapter 2 we give the basic notions in analytic mechanics. In section 2.1 we introduce the basic postulates of analytic mechanics. In section 2.2 Lagrangian formulation is shortly reviewed.The Legendre transformation, Hamiltonian formalism, Poisson brackets and canonical transformations are studied in section 2.3.

In chapter 3 we give a generalization of analytical mechanics and dissipation. In section 3.1 we consider the standard approach to dissipation and show it is not suitable for Hamiltonian formalism. In section 3.2 we give generalized Lagrangian formalism for dissipative linear oscillations which allows to study dissipative systems. This is why we start from linear oscillations in quadratic form, then give a formulation with dissipation. After that, some examples are considered for dissipative Lagrangian. Also Hamiltonian formulation of generalized analytical mechanics is discussed.

In chapter 4 we study the classical damped harmonic oscillator. In section 4.1 Damped harmonic oscillator problem is solved. Then we show reduction of the damped harmonic oscillator equation from the telegraph equation. Bateman's formalism for damped harmonic oscillator is given in section 4.2 and Caldirola-Kanai formalism is given in section 4.3. We show in section 4.4 how to pass from Bateman's formalism to the Caldirola-Kanai formalism. In section 4.5, the hyperbolic form of Bateman's damped harmonic oscillator is discussed.

Chapter 5 is devoted to the quantum damped harmonic oscillator. In section 5.1 Hamiltonian formulation and canonical quantization are constructed. Then in sections 5.2 and 5.3 quantization of the Batemans and Caldirola-Kanai's damped harmonic oscillators are shown.

In the second part, we discuss the Euler equation as follows: In chapter 6 we consider generalized analytical mechanics with higher derivatives. In section 6.1 we give the generalization of analytical mechanics for higher derivatives. In section 6.2 the generalized Lagrange and Hamiltonian formalisms for higher degrees are shown. In section 6.3 the generalized form of Bateman's Lagrangian is given.

In chapter 7 we deal with the Euler equation. In section 7.1 we discuss how the Euler equation appears in physical and mathematical problems. We solve Dirichlet problem for a circular disk as a mathematical example and a particle in a sphere is considered

as a physical example as a third example we gave the Euler equation for radiation damping. In section 7.2 the Lagrangian formalism of the Euler equation for $N = 2$ is discussed with the concept of self-adjoint differential operators . In section 7.3 the Lagrangian formulation of the Euler equation for the special case, $N = 2l$ is given and then several examples of the Euler equation are studied.

In chapter 8 quantum Euler equation is studied. In section 8.1 quantum Euler equation is discussed in the stationary case. Furthermore in the subsections, the Euler Equation in Heisenberg-Weyl and Fock-Bargmann representations are given and Euler Hamiltonian in terms of the standard harmonic oscillator Hamiltonian is formalized. In section 8.2 quantum Euler equation in the non-stationary case is studied with the exact solutions for non-linear Heisenberg equations. In section 8.3 quantum Euler equation for $N = 2$ is given in Schrödinger picture.

In conclusion we discuss the classical and quantum aspects of the Euler equation and its relation with damped harmonic oscillator. In Appendix A solution of the constant coefficient equation is discussed in detail. In Appendix B the integrating factor is found for self-adjoint differential equation . In Appendix C quantum fundamentals including postulates of quantum mechanics, Heisenberg uncertainty principle, Hilbert space, linear operators etc. are given. In Appendix D the stationary Schrödinger equation for the standard harmonic oscillator is solved and Hermite polynomials are presented. In Appendix E, constant relations of the Euler equation are given in two different ways. In Appendix F quantum dynamics is discussed. Dynamical behavior of a quantum system is given by the the Schrödinger picture and the Heisenberg picture.

Part I

DISSIPATION IN ANALYTICAL MECHANICS AND QUANTUM MECHANICS

CHAPTER 2

FUNDAMENTAL CONCEPTS OF ANALYTICAL MECHANICS

2.1. Basic Postulates of Analytical Mechanics

Definition 2.1 *The number of independent quantities which must be specified in order to define uniquely the position of any system is called the number of degrees of freedom.*

Definition 2.2 *Any n quantities $q_1, ..., q_n$ which completely define the position of a system with n degrees of freedom are called generalized coordinates of the system. The derivatives $\dot{q}_1, ..., \dot{q}_n$ are called generalized velocities of the system.*

Postulate 2.1 *The state of a mechanical system with n degrees of freedom at time t is characterized by the point $(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$ in $2n$ dimensional space.*

Postulate 2.2 *Every mechanical system is characterized by a defined function $L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t)$ called the Lagrangian function (functional) of the system.*

The Lagrangian formulation of mechanics is based on the observation that there are variational principles behind the fundamental laws of force balance as given by Newton's law $F = ma$. One chooses a configuration space Q with coordinates q_i, $(i = 1, ..., n)$ that describe the configuration of the system under study (Marsden 2002). Then one introduces the Lagrangian $L(q_i, \dot{q}_i, t)$. Its definition is

$$L \equiv T - U \tag{2.1}$$

where T is the kinetic energy and U is the potential energy of the system.

Postulate 2.3 *(Hamilton's least action principle), the system with Lagrangian function L moves between positions $q_i(t_1)$ and $q_i(t_2)$ in such a way that the integral*

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt \tag{2.2}$$

takes the least possible value. Integral S is called the *action*.

2.2. Lagrangian Formulation

The variational principle of Hamilton states

$$\delta \int_a^b L(q_i, \dot{q}_i, t)dt = 0 \tag{2.3}$$

In this principle, we choose curves $q_i(t)$ joining two fixed points in Q over a fixed time interval $[a, b]$ and calculate the integral regarded as a function of this curve. Hamilton's principle states that this function (2.2) has a critical point at a solution within the space of curves. If we let δq_i be a variation, that is, the derivative of a family of curves with respect to a parameter, then by the chain rule, (2.3) is equivalent to

$$\sum_{i=1}^{n} \int_a^b \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0 \tag{2.4}$$

for all variations of δq_i. Since

$$\delta \dot{q}_i = \frac{d}{dt} \delta q_i$$

one can integrate the second term of (2.4) by parts, and employing the boundary conditions $\delta q_i = 0$ at $t = a$ and b, (2.4) becomes

$$\sum_{i=1}^{n} \int_a^b \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt = 0 \tag{2.5}$$

Since δq_i is arbitrary (being zero at the endpoints), (2.5) is equivalent to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \qquad (i = 1, 2, ..., n). \tag{2.6}$$

Remark 2.4 *If $L(q, \dot{q}, t)$ is Lagrangian and $F(q, t)$ is any differentiable function of the generalized coordinates and time, then*

$$\underline{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \tag{2.7}$$

is a Lagrangian resulting in the same equations of motion.

2.3. Hamiltonian Formalism

In the Hamiltonian formalism the state of the mechanical system with n degrees of freedom, is determined by n coordinates $q_1, q_2, ..., q_n$ and n momenta $p_1, p_2, ..., p_n$. Evaluation of the system is determined by a function $H(q, p)$ which is called the *Hamiltonian function*. Transformation from old variables $(q, \dot{q}, L)$ to the new one (q, p, H) is implemented by the *Legendre transformation*.

2.3.1. Legendre Transformation

Definition 2.3 *The Legendre transformation changes a given function of a given set of variables into a new function of a new set of variables. The old and new variables are related to each other by a point transformation. (Lanczos 1957)*

Let F be a given function of n variables $u_1, ..., u_n$,

$$F = F(u_1, ..., u_n). \tag{2.8}$$

We introduce a new set of variables $v_1, ..., v_n$, by means of the following transformation:

$$v_i = \frac{\partial F}{\partial u_i} \quad i = 1, ..., n. \tag{2.9}$$

The previous transformation is only solvable for the u_i as functions of the v_i when Hessian condition is satisfied. According to the condition the determinant formed by the second partial derivatives of F, assumed to be different from zero,

$$\begin{aligned} \frac{\partial v_i}{\partial u_j} &= \frac{\partial^2 F}{\partial u_i \partial u_j} = \frac{\partial v_j}{\partial u_i}, \\ &= \frac{\partial v_j}{\partial u_i} = h_{ij} = h_{ji} \\ \det h_{ij} &= h \neq 0 \end{aligned} \tag{2.10}$$

guaranteeing the independence of the n variables v_i.
We now define a new function G as follows:

$$G = \sum_{i=1}^{n} u_i v_i - F. \tag{2.11}$$

Expressing u_i as $u_i = u_i(v_1, ..., v_n)$ and substituting in (2.11) we have

$$G = G(v_1, ..., v_n) \tag{2.12}$$

We now consider the infinitesimal variation of G produced by arbitrary infinitesimal variations of the v_i. The combination of Eq.(2.11) and Eq. (2.12) gives

$$\begin{aligned}\delta G &= \sum_{i=1}^{n} \frac{\partial G}{\partial v_i}\delta v_i \\ &= \sum_{i=1}^{n}(u_i\delta v_i + v_i\delta u_i) - \delta F \\ &= \sum_{i=1}^{n}\left[(u_i\delta v_i + \left(v_i - \frac{\partial F}{\partial u_i}\right)\delta u_i\right].\end{aligned}$$

Because of Eq.(2.9) the coefficient of δu_i is zero, so u_i can be expressed in terms of the variations of the v_i,

$$u_i = \frac{\partial G}{\partial v_i} \tag{2.13}$$

This result express duality of Legendre transformations. The following scheme brings out this duality:

Old system	*New system*
variables: $u_1, ..., u_n$;	$v_1, ..., v_n$.
function: $F = F(u_1, ..., u_n)$;	$G = G(v_1, ..., v_n)$.

Transformation

$v_i = \frac{\partial F}{\partial u_i}$,	$u_i = \frac{\partial G}{\partial v_i}$,
$G = \sum u_i v_i - F$,	$F = \sum u_i v_i - G$,
$G = G(v_1, ..., v_n)$,	$F = F(u_1, ..., u_n)$.

This transformation is symmetrical in both systems and the same transformation that leads from the old to the new system leads back from the new to the old system.

2.3.2. Legendre Transformation Applied to the Lagrangian Function

Legendre transformation can be applied to the Lagrangian function L. By this transformation the velocities are transformed into the momenta; the Lagrangian function is transformed into the Hamiltonian function.

Legendre transformation is proceeded in three steps.

Step 1**.** We introduce new variables, momenta, denoted by p_i:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}. \tag{2.14}$$

Step 2**.** We introduce a new function, denoted by H:

$$H \equiv \sum_{i=1}^{n} p_i \dot{q}_i - L. \tag{2.15}$$

where H is the energy function.

Step 3**.** We express new function H in terms of new variables p_i by solving (2.14) for $\dot{q}$ and substituting in (2.15). We thus obtain

$$H = H(q_1, ..., q_n; p_1, ..., p_n; t) \tag{2.16}$$

H is now called "Hamiltonian function" on the phase space $(q_1, ..., q_n; p_1, ..., p_n)$. The basic features of the transformation are as follows:

Old system	*New system*
variables: $velocities$;	$momenta.$
function: $Lagrangian funcion\ L$;	$Hamiltonian function\ H$

Transformation

$\dot{q}_i = \frac{\partial H}{\partial p_i},$	$p_i = \frac{\partial L}{\partial \dot{q}_i},$
$L = \sum p_i \dot{q}_i - H,$	$H = \sum p_i \dot{q}_i - L,$
$L = L(q_1, ..., q_n; \dot{q}_1, ..., \dot{q}_n; t).$	$H = H(q_1, ..., q_n; p_1, ..., p_n; t),$

Transformation of Lagrangian Equations of Motion

The Lagrangian equations of motion are differential equations of the second order in time. However, they can be written as differential equations of the first order if we introduce momenta p_i, defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.17}$$

The Lagrange equations can then be written in the form

$$\dot{p} = \frac{\partial L}{\partial q_i} \tag{2.18}$$

To construct canonical equations of Hamilton. We now we apply Legendre transformation

$$H(q_i, p_i, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q_i, \dot{q}_i, t). \tag{2.19}$$

Writing the total differentials forms of H and L we get

$$dH = \sum p_i d\dot{q}_i + \sum \dot{q}_i dp_i - dL. \tag{2.20}$$

$$dL = \sum \frac{\partial L}{\partial q_i} dq_i + \sum \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{2.21}$$

Inserting equations (2.17) and (2.18) into equation (2.21) yields

$$dL = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{2.22}$$

Substitution of dL into equation (2.20) follows that

$$dH = \sum \dot{q}_i dp_i - \sum \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt. \tag{2.23}$$

Therefore $H = H(q_i, p_i, t)$, and we have

$$dH = \sum \frac{\partial H}{\partial q_i} dq_i + \sum \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt. \tag{2.24}$$

Comparing the last two equations we can write Hamilton equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{2.25}$$

2.3.3. Poisson Brackets

Definition 2.4 *Poison bracket of two functions $u = u(p, q)$, $v = v(p, q)$ with respect to the canonical variables q, p is introduced as,*

$$\{u, v\} \equiv \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right). \tag{2.26}$$

Properties of The Poisson Bracket:

Let u, v, and w be functions of position and momentum, and c is independent of position and momentum. the poisson brackets satisfies following conditions

i) Antisymmetry:

$$\{u, v\} = -\{v, u\}$$

ii)Bilinearity:

$$\begin{aligned} \{u, v + w\} &= \{u, v\} + \{u, w\} \\ \{u + v, w\} &= \{u, w\} + \{v, w\} \\ \{u, cv\} &= c\{u, v\} \\ \{cu, v\} &= c\{u, v\} \end{aligned}$$

iii)The Leibniz rule:

$$\{uv, w\} = u\{v, w\} + v\{u, w\}$$

iv) Jacobi's identity:

$$0 = \{u, \{v, w\}\} + \{w, \{u, v\}\} + \{v, \{w, u\}\}$$

Lemma 2.5 *Poisson brackets are invariant under canonical transformation.(Goldstein 1980)*

Poisson Brackets Formalism and Hamilton's Equations:

Hamiltonian mechanics can be restated in terms of Poisson brackets. As a result of canonical invariance of the Poisson brackets, the relations so obtained will also be invariant in form under a canonical transformation. Suppose we look for the total time derivative of some function of the canonical variables and time , $u(q, p, t)$ by use of Hamilton's equations of motion:

$$\frac{du}{dt} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{p}_i \right) + \frac{\partial u}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial u}{\partial t}, \tag{2.27}$$

or

$$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t}. \tag{2.28}$$

Equation (2.28) may be looked upon as the generalized equation of motion for an arbitrary function u in the Poisson bracket formulation. It contains Hamilton's equations as a special case when we substitute one of the canonical variables, (Goldstein 1980)

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\} \tag{2.29}$$

Poisson Brackets Formalism and Constants of the Motion:

Theorem 2.6 *A constant of motion is a quantity that is conserved throughout the motion, imposing in effect a constraint on the motion.(Goldstein 1980)*

Theorem 2.7 *If u is a constant of the motion, then Eq. (2.28) says it must have the property (Goldstein 1980)*

$$\{H, u\} = \frac{\partial u}{\partial t}. \tag{2.30}$$

All functions that obey (2.30) are constants of the motion, and conversely the Poisson bracket of H with any constant of the motion must be equal to partial time derivative of the constant function.(Goldstein 1980)

Remark 2.8 *(Poisson theorem) The Poisson bracket of any two constants of the motion is also a constant of the motion. (Goldstein 1980)*

2.3.4. Canonical Transformations

A canonical transformation is a change of canonical coordinates $(q, p, t) \longrightarrow (Q, P, t)$ that preserves the form of Hamilton's equations, although it might not preserve the Hamiltonian itself.

Let there exist some function $K(Q, P, t)$ such that the equations of motion in the new set are in the Hamiltonian form

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{2.31}$$

Here function K plays the role of the Hamiltonian in the new coordinate set. If Q_i and P_i are to be canonical coordinates, we can write for them the Hamilton's principle so we can put them in the following form

$$\delta \int_{t_2}^{t_1} \sum_i (P_i \dot{Q}_i - K(Q_i, P_i, t))dt = 0. \tag{2.32}$$

From another side the old canonical coordinates satisfy a similar principle:

$$\delta \int_{t_2}^{t_1} \sum_i (p_i \dot{q}_i - H(q_i, p_i, t))dt = 0. \tag{2.33}$$

Since the two expressions (2.32) and (2.33) are equal both statements will be satisfied. We can say if the integrands are connected by a relation of the form

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{dF}{dt} \tag{2.34}$$

where λ is a constant and F is any function, with continuous second derivatives, called the generating function that acts as a bridge between the two sets of canonical variables.

A transformation of canonical coordinates for which $\lambda \neq 1$ will be called an *extended canonical transformation*. If $\lambda = 1$ and Eq.(2.34) holds then it is *a canonical transformation*.

Given a canonical transformation which maps (q, p) into (Q, P), we define four kinds of generating functions:

- The generating function of the first kind, $F_1(q, Q, t)$ which satisfies:

$$\begin{aligned} p &= \frac{\partial F_1}{\partial q}(q, Q, t), \\ P &= -\frac{\partial F_1}{\partial Q}(q, Q, t). \end{aligned}$$

- The generating function of the second kind, $F_2(q, P, t)$ which satisfies:

$$\begin{aligned} p &= \frac{\partial F_2}{\partial q}(q, P, t), \\ Q &= -\frac{\partial F_2}{\partial P}(q, P, t). \end{aligned}$$

- The generating function of the third kind, $F_3(p, Q, t)$ which satisfies:

$$\begin{aligned} q &= -\frac{\partial F_3}{\partial p}(p, Q, t), \\ P &= -\frac{\partial F_3}{\partial Q}(p, Q, t). \end{aligned}$$

- The generating function of the fourth kind, $F_4(p, P, t)$ which satisfies:

$$\begin{aligned} q &= -\frac{\partial F_4}{\partial p}(p, P, t), \\ Q &= \frac{\partial F_4}{\partial P}(p, P, t). \end{aligned}$$

CHAPTER 3

GENERALIZATION OF ANALYTICAL MECHANICS AND DISSIPATION

At this chapter first we briefly review approach to dissipation and then will give generalized Lagrangian and Hamiltonian formulations for dissipative oscillations.

3.1. Standard Approach to Dissipation

Velocity Dependent Potentials and Conservative Forces

Definition 3.1 *Conservative forces F can be derived from a potential $U(q,t)$ as*

$$F(q,t) = -\nabla U(q,t), \tag{3.1}$$

where potential $U(q,t)$ is a function of the position and time.

In some physical systems F and U can be functions of $\dot{q}$. For example the Lorentz force

$$F = q\left(E + \frac{1}{c}(v \times B)\right)$$

which acts on a charged particle in the electromagnetic field is velocity-dependent . When F is a function of $\dot{q}$, F can be expressed by a potential U that depends on the generalized coordinates q_j, the generalized velocities $\dot{q}_j$ and the time t, according to

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} \tag{3.2}$$

with $U = U(q_j, \dot{q}_j, t)$. Also Q_j is called generalized force (Greiner 2003).

Remark 3.1 *This velocity-dependent potential is sometimes called the generalized potential .*

Non-conservative Forces

We now consider systems with nonconservative forces. Such systems are systems with friction. Their Lagrange equations take the form

$$Q_j = -\frac{\partial T}{\partial q_j} + \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}. \tag{3.3}$$

We can split the generalized forces Q_j in a conservative part $Q_j^{(c)}$ and a nonconservative or friction part $Q_j^{(f)}$, so that

$$Q_j = Q_j^{(c)} + Q_j^{(f)} \tag{3.4}$$

Using Eq. (3.2) for $Q_j^{(c)}$ and the definition of the Lagrangian $L = T - U$, Eq. (3.3) leads the form ,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{(f)}, \quad j = 1, 2, ..., n. \tag{3.5}$$

We can write dissipative forces as

$$Q_j^{(f)} = -\sum_{k=1}^{n} f_{jk}\dot{q}_k \tag{3.6}$$

where the f_{jk} are the friction coefficients. If the friction tensor f_{jk} is symmetric, $f_{jk} = f_{kj}$, the friction forces $Q_j^{(f)}$ can be obtained by partial derivation with respect to the generalized velocities $\dot{q}_j$ from the function (Greiner, 2003)

$$D \equiv \frac{1}{2}\sum_{k,l=1}^{n} f_{jk}\dot{q}_k\dot{q}_l \tag{3.7}$$

according to

$$Q_j^{(f)} = -\frac{\partial D}{\partial \dot{q}_j} \tag{3.8}$$

D is called the dissipation function (friction function). The Lagrange equations (2.6) can now be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial D}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{3.9}$$

This formalism is the standard approach for dissipative systems in analytic mechanics. But this approach, based on dissipation function (3.7) is not suitable. The reason is that, Eq. (3.9) results by adding a dissipative term to Euler-Lagrange equation not from Lagrangian function. So it violates the postulates of analytic mechanics and this is why it can't be considered as the proper solution of the problem. The Lagrangian formalism is important in connection with Hamiltonian formalism and canonical quantization of the system. In the next section we consider generalized approach to the dissipation into Lagrangian formalism.

3.2. Generalized Lagrangian Formalism for Dissipative Linear Oscillations

In this section we consider the generalization of the Lagrange function which allows to a treatment dissipative systems. For this aim we start from the linear oscillations in quadratic form, then we formulate linear oscillations with dissipation. After this formulation we give specific examples of dissipative Lagrangian. At the last section we introduce corresponding Hamiltonian formulation for the system.

3.2.1. Linear Oscillations

At analytical mechanics to describe Linear oscillations with s degrees of freedom and generalized coordinates $q_1, ..., q_s$, Lagrangian function is given by the quadratic form

$$L = \frac{1}{2}\sum_{ij}^{s} a_{ij}(q)\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij}^{s} k_{ij}q_iq_j, \tag{3.10}$$

where symmetrical matrix $a_{ij} = a_{ji}$ is positive definite (signature $a_{ij} = (+, +, ..., +)$). By diagonalization, this system can be reduced to the diagonal form (Landau and Lifshitz 1980)

$$L = \frac{1}{2}\sum_{i} m_i(\dot{Q}_i^2 - \omega_i^2 Q_i^2), \tag{3.11}$$

with equations of motion

$$\ddot{Q}_i + \omega_i^2 Q_i^2 = 0, \tag{3.12}$$

where ω_i are called characteristic frequencies or eigen-frequencies of the system and Q_i called normal coordinates of the system.

To generalize this Lagrangian to dissipative linear systems, first of all we will specify the geometrical character of the kinetic term as a metric in a Riemannian space.

3.2.2. Riemannian Metric and Kinetic Energy

To represent kinetic energy in terms of Riemannian metric first we introduce one single differential quantity called "*the line element*" ,ds.

Definition 3.2 *The line element, expressed in terms of the coordinates and their differentials, is the distance between two neighboring points of space, where the square of the*

line element is given by the expression

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j \tag{3.13}$$

where g_{ij} is a symmetric tensor ($g_{ji} = g_{ij}$), called the Riemannian metric .

As an example, we can consider the line element in cartesian coordinates (Dubrovin et al. 1984)

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Definition 3.3 *Kinetic energy in non-relativistic classical mechanics for a particle of mass m, is defined as a positive, diagonal quadratic form in n dimensional space*

$$T = \frac{m}{2} \sum_{i=1}^{n} v_i^2. \tag{3.14}$$

Since, velocity $v_i = \frac{dx^i}{dt}$, this form can be re-written in terms of the line element

$$\begin{aligned} T &= \frac{m}{2} \sum_{i=1}^{n} \frac{dx_i}{dt} \frac{dx_i}{dt} \\ &= \frac{m}{2} \left(\frac{ds}{dt} \right)^2 \end{aligned} \tag{3.15}$$

where

$$ds^2 = \sum_{i=1}^{n} dx_i dx_i = dx_1^2 + ... + dx_n^2. \tag{3.16}$$

Representing the system in terms of generalized coordinates $q_1, ..., q_s$ and velocities, according to

$$x_i = x_i(q_1, ...q_s), \quad i = 1, ..., n \tag{3.17}$$

this kinetic energy becomes the symmetric quadratic form

$$T = \frac{1}{2} \sum_{i,j=1}^{s} a_{ij}(q) \dot{q}_i \dot{q}_j \tag{3.18}$$

and for $m = 1$, a_{ij} are functions of q which determine the line element

$$ds^2 = \sum_{ij}^{s} a_{ij}(q) dq_i dq_j. \tag{3.19}$$

Comparing quadratic forms for the line elements Eq.(3.16) and Eq.(3.19), we notice that in cartesian coordinates $(x_1...x_n)$, it is Euclidian metric in R^n while in terms

of generalized coordinates in general it is a Riemannian metric. (For $s < n$ it is the induced metric on hyper-surfaces determined by Eqs.(3.17)).

In summary, the kinetic energy in analytic mechanics is a quadratic form in velocities having the properties: (a) symmetry property, $a_{ij} = a_{ji}$ (b) positive definiteness, $a > 0$ which allows us to interpret a_{ij} as a second rank metric tensor in Riemannian space with coordinates $(q_1, ..., q_s)$.

3.3. Dissipation and Generalized Analytical Mechanics for Linear Oscillations

For the linear systems with generalized coordinates $q_1 ... q_s$ the Lagrangian function is given by the quadratic form (3.10). However it is not sufficiently general to include dissipation. We propose to generalize this function in the next quadratic form

$$L = \sum_{ij}^{s} \left(\frac{1}{2} a_{ij}(q,t)\dot{q}_i\dot{q}_j + b_{ij}\dot{q}_i q_j - \frac{1}{2} k_{ij}(t) q_i q_j \right) \tag{3.20}$$

where matrices $a_{ij} = a_{ji}$ and $k_{ij} = k_{ji}$ are symmetrical functions of $q_1, ..., q_s$ and time t, which are not in general positive definite.

Comment: Most general case includes $b_{ij} = b_{ij}(q,t)$. But for our purpose it is sufficient to consider b_{ij} as a constant metric.

3.3.1. Kinetic Energy as Non-Stationary Pseudo Riemannian Metric

Due to the properties;

1) Symmetry property, $a_{ij} = a_{ji}$

2) Signature, $a_{ij} = (\underbrace{+, ..., +}_{l}, \underbrace{-, ..., -}_{m})$; $l + m = n$

a_{ij} determine the line element $ds^2 = a_{ij} dq_i dq_j$ in non-stationary pseudo-Riemanian space with signature $(\underbrace{+, ..., +}_{l}, \underbrace{-, ..., -}_{m})$ and coordinates $q_1, ..., q_n$.

According to the inertia theorem from the linear algebra we can chose new coordinates Q_i, to diagonalize a_{ij} in the canonical form,

$$\begin{aligned} T &= \frac{1}{2} \sum a_{ij}(q,t)\dot{q}_i\dot{q}_j \\ &= \frac{1}{2} \left(m_1\dot{Q}_1^2 + m_2\dot{Q}_2^2 + ... + m_l\dot{Q}_l^2 - m_{l+1}\dot{Q}_{l+1}^2 - ... - m_n\dot{Q}_n^2 \right). \end{aligned}$$

Proposition 3.1 *In generalized Lagrangian* (3.20)*, we can restrict the matrix* b_{ij} *to be skew-symmetric* $b_{ij} = -b_{ji}$*.*

Proof. $\forall\ b_{ij} = b^s_{ij} + b^{ss}_{ij}$, where

$$\begin{aligned} b^s_{ij} &= \frac{b_{ij} + b_{ji}}{2} = b_{ji}, \\ b^{ss}_{ij} &= \frac{b_{ij} - b_{ji}}{2} = -b_{ji} \end{aligned} \tag{3.21}$$

and

$$\sum b^s_{ij}\dot{q}_i q_j = \sum b^s_{ji}\dot{q}_j q_i, \tag{3.22}$$

$$= \frac{1}{2}\sum b^s_{ij}(\dot{q}_j q_i + q_i\dot{q}_j), \tag{3.23}$$

since

$$\frac{1}{2}\sum b^s_{ij}(\dot{q}_j q_i + q_i\dot{q}_j) = \frac{d}{dt}\left(\frac{1}{2}\sum_{ij} b^s_{ij} q_i q_j\right), \tag{3.24}$$

the last term is total time derivative and can be omitted in Lagrange function. This shows that only skew-symmetric part of metric b_{ij} contributes to the Euler-Lagrange equation. According to above proposition we can write our generalized Lagrangian in the form,

$$L = \frac{1}{2}\sum_{ij} a_{ij}(q,t)\dot{q}_i\dot{q}_j + \frac{1}{2}\sum_{ij} b_{ij}(\dot{q}_i q_j - q_i\dot{q}_j) - \frac{1}{2}\sum_{ij} k_{ij}(t) q_i q_j. \tag{3.25}$$

Examples :

Systems with one degree of freedom.

$1)$ In this example $n = 1$ and the coefficients $a_{i,j}(q,t), k_{i,j}(t)$ are considered as constant,

$$L = a\dot{q}^2 - kq^2, \tag{3.26}$$

the middle term $\frac{1}{2}\sum b_{ij}(\dot{q}_i q_j - q_i\dot{q}_j)$ in Eq. (3.25) is absent. Eq.(3.26) corresponds to classical non-dissipative Lagrangian.

$2)$ Let's consider $n = 1$ case when $b_{i,j} = 0$ but $a(q,t), k(t)$ depend on time,

$$a(t) = \frac{m}{2}f(t), \quad k(t) = \frac{mw^2}{2}f(t).$$

The Lagrangian,

$$L = f(t)\left(\frac{m}{2}\dot{q}^2 - \frac{mw^2}{2}q^2\right). \tag{3.27}$$

describes non-relativistic oscillator with time dependent mass $m(t) = mf(t)$.

Substituting (3.27) into the Euler-Lagrange equations we get the equation of motion for a damped harmonic oscillator

$$\ddot{q} + \Gamma\dot{q} + w^2 q = 0 \tag{3.28}$$

where $\Gamma(t) = \frac{\dot{f}(t)}{f(t)}$ characterizes the dissipation rate.
If Γ is a constant, then

$$f(t) = f_0 e^{\Gamma t}, \quad \Gamma = \frac{\gamma}{m} \tag{3.29}$$

This case will be considered in the following chapters as the Caldirola-Kanai formalism for damped oscillator. The Caldirola-Kanai Lagrangian is given by,

$$L = e^{\frac{\gamma}{m}t}\left(\frac{m}{2}\dot{q}^2 - \frac{mw^2}{2}q^2\right).$$

Systems with two degrees of freedom.
3)In this example to treat dissipation we extend the system to two degrees of freedom, described by generalized coordinates q_1 , q_2 and generalized velocities $\dot{q}_1$, $\dot{q}_2$, where coefficients $a_{ij}(q,t) = a_{ij}, k_{ij}(t) = k_{ij}$ are considered as a constant . Then Lagrangian is given by

$$\begin{aligned} L &= \frac{1}{2}a_{11}\dot{q}_1^2 + a_{12}\dot{q}_1\dot{q}_2 + \frac{1}{2}a_{22}\dot{q}_2^2 + \\ &+\frac{1}{2}b(\dot{q}_1 q_2 - q_1\dot{q}_2) - \left(\frac{1}{2}k_{11}q_1^2 + k_{12}q_1 q_2 + \frac{1}{2}k_{22}q_2^2\right) \end{aligned} \tag{3.30}$$

In particular case when diagonal elements $a_{ii} = 0$, $k_{ii} = 0$ vanish

$$L = a_{12}\dot{q}_1\dot{q}_2 + b(q_1\dot{q}_2 - \dot{q}_1 q_2) - k_{12}q_1 q_2 \tag{3.31}$$

From the Euler-Lagrange equations we get equations of motion,

$$a_{12}\ddot{q}_1 - 2b_2\dot{q}_1 + k_{12}q_1 = 0 \tag{3.32}$$

$$a_{12}\ddot{q}_2 + 2b_2\dot{q}_2 + k_{12}q_2 = 0 \tag{3.33}$$

The first equation is damped harmonic oscillator, while the second one can be considered as its time-reversed image.

From examples 2. and 3. we have two possibilities for description of damped oscillator. In the first case we have time dependent parameters, while in the second case extended system with additional degrees of freedom.

In the next section we give Hamiltonian formulation for generalized analytical mechanics.

3.3.2. Hamiltonian Formulation of Generalized Analytical Mechanics

In the previous section we have formulated generalized analytical mechanics by Lagrangian (3.25) with non-stationary pseudo-Riemannian metric. Now we introduce corresponding Hamiltonian formulation of the system. By using the Legendre transformation to pass from $(q, \dot{q})$ to (q, p)

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = a_{ij}\dot{q}_j + b_{ij}q_j \tag{3.34}$$

we get the linear system of equations for $\dot{q}_j$;

$$a_{ij}\dot{q}_j = p_i - b_{ij}q_j. \tag{3.35}$$

When $\det a_{ij} \neq 0$ (the non-degenerate case), we can write $\dot{q}_j$ as

$$\dot{q}_j = f_j(q, p, t). \tag{3.36}$$

Then the Hamiltonian is

$$\begin{aligned} H &= \sum \dot{q}_i p_i - L, \\ &= H(q, p, t) \end{aligned}$$

and Hamilton equations of motion can be written as

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{3.37}$$

Concrete applications will be in the next chapter

CHAPTER 4

CLASSICAL DAMPED HARMONIC OSCILLATOR

4.1. Damped Harmonic Oscillator

The Classical Damped Harmonic Oscillator is described by $x(t)$ subject to the 2nd order linear ordinary differential equation with constant coefficients, where coefficient $\gamma > 0$ for the first derivative term is called damping coefficient and k is the harmonic coefficient

$$m\ddot{x} + \gamma\,\dot{x} + kx = 0 \tag{4.1}$$

The characteristic equation of the differential equation is

$$m\lambda^2 + \gamma\lambda + k = 0.$$

Depending on the relation between damping and harmonic coefficients we have three different cases as shown in Figure 4.1.

Case I : The under-damping case; $\frac{\gamma^2}{4m^2} < \frac{k}{m}$

$$\Omega_u^2 = \frac{k}{m} - \frac{\gamma^2}{4m^2} \tag{4.2}$$

$$x(t) = e^{-\frac{\gamma t}{2}}[A\cos\Omega_u t + B\sin\Omega_u t]$$

CaseII : The over-damping case; $\frac{\gamma^2}{4m^2} > \frac{k}{m}$

$$\Omega_o^2 = \frac{\gamma^2}{4m^2} - \frac{k}{m} \tag{4.3}$$

$$x(t) = e^{-\frac{\gamma t}{2}}[A\cosh\Omega_o t + B\sinh\Omega_o t]$$

Case III : The critical-damping case; $\frac{\gamma^2}{4m^2} = \frac{k}{m}$

$$x(t) = Ae^{\frac{-\gamma t}{2}} + Bte^{\frac{-\gamma t}{2}} \tag{4.4}$$

When $\gamma = 0$ we have the standard harmonic oscillator equation. Classical damped harmonic oscillator is a dissipative system where the energy is not conserved. For such systems the variational principle leading to the Euler-Lagrange equations fails. But to get the canonical formalism for dissipative systems we need Lagrangian-Hamiltonian forms. Batemans formulation solves this problem for the classical theory.

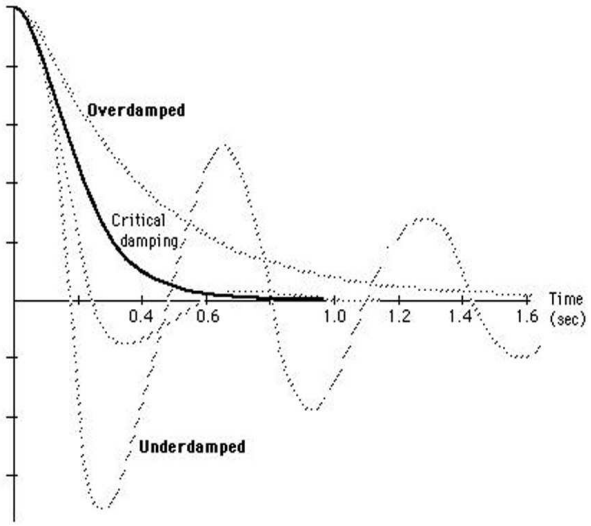

Figure 4.1. Damped Harmonic Oscillator

4.2. Bateman Dual Oscillator Formalism

In 1931, Bateman presented (Bateman 1931) the so-called dual or mirror image formalism for damped oscillator. According to this, the energy dissipated by the oscillator is completely absorbed at the same time by the mirror image oscillator, and thus the energy of the total system is conserved. The dual system is

$$m\ddot{x} + \gamma\,\dot{x} + kx = 0, \tag{4.5}$$

$$m\ddot{y} - \gamma\,\dot{y} + ky = 0. \tag{4.6}$$

Here the second equation describes time reversal process to the first equation and together they represent so called dual damped oscillator system. Since the energy of the total system is constant, the system of damped harmonic oscillator and its time-reversed image is then a closed system described by the Lagrangian

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) - kxy \tag{4.7}$$

The canonical momenta for this dual system can be obtained from Eq. (4.7) by using the following formulas

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}y, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}x. \tag{4.8}$$

Clearly, these differ from the oscillator's mechanical momenta.

To reach The Bateman's dual Hamiltonian we can use Legendre transformation;

$$H = \dot{x}p_x + \dot{y}p_y - L,$$

then we get the Bateman Hamiltonian in the following form

$$H = \frac{p_x p_y}{m} + \frac{\gamma}{2m}(yp_y - xp_x) + (k - \frac{\gamma^2}{4m})xy. \tag{4.9}$$

We can write the canonical equations of Hamilton as follows,

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_y}{m} - \frac{\gamma}{2m}x, \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_x}{m} + \frac{\gamma}{2m}y, \tag{4.10}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{\gamma}{2m}p_x - (k - \frac{\gamma^2}{4m})y, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{\gamma}{2m}p_y - (k - \frac{\gamma^2}{4m})x. \tag{4.11}$$

Hamilton equations of motion reproduce correctly the doubled system.

The brackets of the system are

$$\{x, p_x\} = 1, \quad \{y, p_y\} = 1, \quad \{x, p_y\} = 0, \quad \{y, p_x\} = 0, \tag{4.12}$$

$$\{x, y\} = 0, \quad \{p_x, p_y\} = 0. \tag{4.13}$$

The Poisson bracket formulation of Hamilton's equations is

$$\dot{x} = \{x, H\}, \quad \dot{y} = \{y, H\}, \quad \dot{p}_x = \{p_x, H\}, \quad \dot{p}_y = \{p_y, H\}. \tag{4.14}$$

If a Poisson bracket of a quantity with H is equal to zero, then we call this quantity a constant of the motion, so we can say that the Hamiltonian is a constant of the motion since $\{H, H\} = \frac{\partial H}{\partial t} = 0$.

As a conclusion, the energy dissipated by the original oscillator is completely absorbed by the dual of the system.

In the next section we consider how damped oscillator appears from the telegraph equation.

4.3. Reduction of the Damped Harmonic Oscillator Equation from the Telegraph Equation

The linear telegraph equation is

$$U_{xx} = aU_{tt} + bU_t + cU. \tag{4.15}$$

Using variational principle for the telegraph equation, in which both U and V are to be varied,

$$\delta \int V(aU_{tt} - U_{xx} + bU_t + cU)dxdt = 0 \tag{4.16}$$

one can find it's complementary equation;

$$V_{xx} = aV_{tt} - bV_t + cV. \tag{4.17}$$

Using traveling wave substitution one can reduce the telegraph equation to the damped harmonic oscillator. Let

$$U(x,t) = U(x - vt) = U(\xi), \quad V(x,t) = V(x - vt) = V(\xi). \tag{4.18}$$

Differentiating U with respect to $\xi = x - vt$

$$U_t = (-v)U^{'}, \quad U_{tt} = (v^2)U^{''}, \quad U_{xx} = U^{''}, \tag{4.19}$$

where $U^{'} = dU/d\xi$. Differentiating V in the same way

$$V_t = (-v)V^{'}, \quad V_{tt} = (v^2)V^{''}, \quad V_{xx} = V^{''}, \tag{4.20}$$

From the telegraph equation ant it's mirror image one may get;

$$U'' = av^2U'' - bvU' + cU, \tag{4.21}$$

$$V'' = av^2V'' + bvV' + cV. \tag{4.22}$$

Rewriting

$$(av^2 - 1)U'' - bvU' + cU = 0 \tag{4.23}$$

$$(av^2 - 1)V'' + bvV' + cV = 0 \tag{4.24}$$

and setting the correspondence;

$$(av^2 - 1) \Rightarrow mass, \ \ bv \Rightarrow \gamma, \ \ c \Rightarrow k$$

one can find damped harmonic oscillator equation and its mirror image;

$$mU'' - \gamma U' + kU = 0, \tag{4.25}$$

$$mV'' + \gamma V' + kV = 0. \tag{4.26}$$

To avoid extension of the damped system by time-reversal image in 1941 Caldirola developed (Caldirola 1941) another alternative approach. In his approach mass and frequency are time dependent.

4.4. Caldirola-Kanai Formalism

Caldirola (Caldirola 1941) and Kanai (Kanai 1948) introduced a new Lagrangian function. That is known as Caldirola-Kanai Lagrangian. According to Caldirola-Kanai approach; a harmonic oscillator can be considered as a standard oscillator with exponentially growing time dependent mass.

$$L = e^{\frac{\gamma t}{m}} \frac{m}{2}(\dot{x}^2 - \omega^2 x^2), \tag{4.27}$$

introducing $m(t) = me^{\frac{\gamma t}{m}}$,

$$L = m(t)\left(\frac{\dot{x}^2}{2} - \omega^2 \frac{x^2}{2}\right). \tag{4.28}$$

Physically, the situation described by the Lagrangian (4.27) might be realized for example by a frictionless pendulum consisting of a pail collecting rain.

Using Legendre transformation Caldirola-Kanai Hamiltonian is

$$H = e^{-\frac{\gamma t}{m}} \frac{p^2}{2m} + e^{\frac{\gamma t}{m}} \frac{1}{2} m\omega^2 x^2 \tag{4.29}$$

It leads to classical equation of motion for damped oscillator;

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \omega^2 x = 0.$$

4.5. From Bateman to Caldirola-Kanai

We are going to establish the relation between Bateman's and Caldirola's- Kanai's form (Dekker 1981). We expect that proper exclusion of the y component will lead doubled Bateman Hamiltonian to the Caldirola Hamiltonian formalism.

The explicit solution of Hamilton's equations (4.10), (4.11) for the under-damped case are

$$\begin{aligned}
x(t) &= e^{-\frac{\gamma}{2m}t}\left\{x(0)\cos\Omega t + \frac{p_y(0)}{m\Omega}\sin\Omega t\right\}, \\
y(t) &= e^{\frac{\gamma}{2m}t}\left\{y(0)\cos\Omega t + \frac{p_x(0)}{m\Omega}\sin\Omega t\right\}, \\
p_x(t) &= e^{\frac{\gamma}{2m}t}\left\{p_x(0)\cos\Omega t - m\Omega y(0)\sin\Omega t\right\}, \\
p_y(t) &= e^{-\frac{\gamma}{2m}t}\left\{p_y(0)\cos\Omega t - m\Omega x(0)\sin\Omega t\right\},
\end{aligned} \tag{4.30}$$

where $\Omega^2 = \frac{k}{m} - \frac{\gamma^2}{4m^2}$.

Now we apply a canonical transformation. Generating function of the transformation is $F = F(x, y, P_x, P_y)$, where

$$F = xP_x e^{\frac{\gamma}{2m}t} + yP_y e^{-\frac{\gamma}{2m}t}.$$

We find

$$X = \frac{\partial F}{\partial P_x} = xe^{\frac{\gamma}{2m}t}, \quad p_x = \frac{\partial F}{\partial x} = P_x e^{\frac{\gamma}{2m}t}, \tag{4.31}$$

$$Y = \frac{\partial F}{\partial P_y} = ye^{-\frac{\gamma}{2m}t}, \quad p_y = \frac{\partial F}{\partial y} = P_y e^{-\frac{\gamma}{2m}t}. \tag{4.32}$$

Substituting new variables into the Hamiltonian at Eq. (4.9) we get

$$H = \frac{P_x P_y}{m} + \frac{\gamma}{2m}(YP_y - XP_x) + (k - \frac{\gamma^2}{4m})XY.$$

New hamiltonian, let's call it $H' = H + \frac{\partial F}{\partial t}$ is,

$$H' = \frac{P_x P_y}{m} + m\Omega^2 XY \tag{4.33}$$

This Hamiltonian describes undamped oscillations at the frequency $m\Omega^2 = (k - \frac{\gamma^2}{4m})$. We can see this also if we substitute Eq. (4.31) into Eq. (4.30). The $e^{\pm\frac{\gamma}{2m}t}$ terms at Eq. (4.33) disappear and they reduce to undamped oscillator. Also H is a constant of the motion, equal to its initial value $H'(0) = \frac{P_x(0)P_y(0)}{m} + m\Omega^2 X(0)Y(0)$, satisfying $P_x(0) = P_y(0)$ and $X(0) = Y(0)$. Because of the time-reversible nature of the new solutions, this suggests $P_x(t) = P_y(t)$ and $X(t) = Y(t)$.

Now we want to extend the real coordinates and momenta into the complex plane and to introduce a canonical transformation from X, Y, P_x, P_y to $Q, \bar{Q}, P, \bar{P}$

$$X = \frac{1}{2}\left(Q + \bar{Q}\right) + \frac{i}{2\Omega m}\left(P - \bar{P}\right), \quad Y = \frac{1}{2}\left(Q + \bar{Q}\right) - \frac{i}{2\Omega m}\left(P - \bar{P}\right), \tag{4.34}$$

$$P_x = \frac{1}{2}\left(P + \bar{P}\right) + \frac{i\Omega m}{2}\left(Q - \bar{Q}\right), \quad P_y = \frac{1}{2}\left(P + \bar{P}\right) - \frac{i\Omega m}{2}\left(Q - \bar{Q}\right). \tag{4.35}$$

where the new variables can be taken to be real. In the end one only considers those solutions for which $\bar{Q}_0 = Q_0$ and $\bar{P}_0 = P_0$, which confines the original variables again to real phase space.

Writing Eq. (4.33) in terms of the new coordinates we get

$$H' = \left(\frac{P^2}{2m} + \frac{m\Omega^2}{2}Q^2\right) + \left(\frac{\bar{P}^2}{2m} + \frac{m\Omega^2}{2}\bar{Q}^2\right) \tag{4.36}$$

This Hamiltonian represents the energy of two independent identical oscillators, because of this we will only concentrate on one oscillator, the $Q - P$ system.

The canonical generator of the system is,

$$F = \Pi Q + \frac{1}{2}\frac{\gamma}{2}Q^2. \tag{4.37}$$

Then transforms Q, P into Z, Π, we find

$$P = \frac{\partial F}{\partial Q} = \Pi + \frac{\gamma}{2}Q, \quad Z = \frac{\partial F}{\partial \Pi} = Q \tag{4.38}$$

We can write H' in terms of new variables,

$$H' = \frac{\Pi^2}{2m} + \frac{\gamma}{2m}\Pi Z + \frac{1}{2}kZ^2 \tag{4.39}$$

We will make one more transformation from $Z\Pi$ to $z\pi$ generated by

$$F = \pi Z e^{-\frac{\gamma}{2m}t}, \tag{4.40}$$

where

$$\Pi = \frac{\partial F}{\partial Z} = \pi e^{-\frac{\gamma}{2m}t}, \quad z = \frac{\partial F}{\partial \pi} = Z e^{-\frac{\gamma}{2m}t} \tag{4.41}$$

Since we can write $H'' = H' + \frac{\partial F}{\partial t}$, we get our new Hamiltonian,

$$H'' = \frac{\pi^2}{2m}e^{-\frac{\gamma}{m}t} + \frac{k}{2}z^2 e^{\frac{\gamma}{m}t} \tag{4.42}$$

This Hamiltonian is the Caldirola-Kanai Hamiltonian.

4.6. Hyperbolic Form Of Bateman's Damped Harmonic Oscillator

In doubled formalism H has an unusual form for quantization which is why we will re-write it in the so called hyperbolic form (Blasone et al. 1996), (Pashaev unpublished). We use hyperbolic coordinates $x_1(t)$ and $x_2(t)$:

$$x(t) = \frac{x_1(t) + x_2(t)}{\sqrt{2}} \ , y(t) = \frac{x_1(t) - x_2(t)}{\sqrt{2}},$$

in terms of x_1 and x_2 the Lagrangian (4.7) can be written as

$$L = (\frac{m}{2}\dot{x_1}^2 - \frac{k}{2}{x_1}^2) - (\frac{m}{2}\dot{x_2}^2 - \frac{k}{2}{x_2}^2) + \frac{\gamma}{2}(\dot{x_1}x_2 - \dot{x_2}x_1)$$

Here the dissipative term plays a role of a coupling between oscillators x_1 and x_2 on the hyperbolic plane. The momenta are

$$p_1 = m\dot{x}_1 + \frac{\gamma}{2}x_2, \quad p_2 = -m\dot{x}_2 - \frac{\gamma}{2}x_1. \tag{4.43}$$

The equations of motion corresponding to the set are

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0, \tag{4.44}$$

$$m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0. \tag{4.45}$$

In hyperbolic coordinates, the Hamiltonian at (4.9) becomes

$$H = \frac{1}{2m}(p_1 - \frac{\gamma}{2}x_2)^2 + \frac{k}{2}{x_1}^2 - \frac{1}{2m}(p_2 + \frac{\gamma}{2}x_1)^2 - \frac{k}{2}{x_2}^2.$$

In this form it looks like as the difference of two oscillators $H = H_1 - H_2$. Also the damping term acts as a correction in the kinetic energy for both oscillators.

Now we will write the canonical brackets of the system. The canonical commutation relations are

$$\{x_i, p_j\} = \delta_{ij},\ \{x_i, x_j\} = \{p_i, p_j\} = 0. \quad (i, j = 1, 2)$$

The Hamilton equations are

$$\dot{p}_i = \{p_i, H\}, \quad \dot{x}_i = \{x_i, H\}, \quad (i = 1, 2)$$

or

$$\dot{p}_1 = \frac{\gamma}{2m}(p_2 + \frac{\gamma}{2}x_1) - kx_1,\ \ \dot{p}_2 = \frac{\gamma}{2m}(p_1 - \frac{\gamma}{2}x_2) + kx_2, \tag{4.46}$$

$$\dot{x}_1 = \frac{1}{m}(p_1 - \frac{\gamma}{2}x_2),\ \ \dot{x}_2 = -\frac{1}{m}(p_2 + \frac{\gamma}{2}x_1). \tag{4.47}$$

If we write the equations in terms of x^+ and x^- , where

$$x^{\pm} = x^1 \pm x^2,\ \ p^{\pm} = p^1 \pm p^2, \tag{4.48}$$

we get the form

$$\dot{p}^+ = +\frac{\gamma}{2m}p^+ + (\frac{\gamma^2}{4m} - k)x^-, \tag{4.49}$$

$$\dot{p}^- = -\frac{\gamma}{2m}p^- + (\frac{\gamma^2}{4m} - k)x^+, \tag{4.50}$$

$$\dot{x}^+ = -\frac{\gamma}{2m}x^+ + \frac{1}{m}p^-, \tag{4.51}$$

$$\dot{x}^- = +\frac{\gamma}{2m}x^- + \frac{1}{m}p^+, \tag{4.52}$$

Now we have three cases depending on the relation between damping and harmonic coefficients

Case I: The under damping ; $k > \frac{\gamma^2}{4m}$

$${\Omega_u}^2 = \frac{k}{m} - \frac{\gamma^2}{4m^2}, \tag{4.53}$$

$$x^+(t) = e^{-\frac{\gamma}{2m}t}\{x^+(0)\cos\Omega_u t + \frac{p^-(0)}{m\Omega_u}\sin\Omega_u t\} \tag{4.54}$$

$$x^-(t) = e^{+\frac{\gamma}{2m}t}\{x^-(0)\cos\Omega_u t + \frac{p^+(0)}{m\Omega_u}\sin\Omega_u t\} \tag{4.55}$$

$$p^+(t) = e^{+\frac{\gamma}{2m}t}\{p^+(0)\cos\Omega_u t - m\Omega_u x^-(0)\sin\Omega_u t\} \tag{4.56}$$

$$p^-(t) = e^{-\frac{\gamma}{2m}t}\{p^-(0)\cos\Omega_u t - m\Omega_u x^+(0)\sin\Omega_u t\} \tag{4.57}$$

Case II: Aperiodic case (over-damping); $k < \frac{\gamma^2}{4m}$

$$\Omega_o{}^2 = \frac{\gamma^2}{4m^2} - \frac{k}{m}, \tag{4.58}$$

$$x^+(t) = e^{-\frac{\gamma}{2m}t}\{x^+(0)\cosh\Omega_o t + \frac{p^-(0)}{m\Omega_o}\sinh\Omega_o t\} \tag{4.59}$$

$$x^-(t) = e^{+\frac{\gamma}{2m}t}\{x^-(0)\cosh\Omega_o t + \frac{p^+(0)}{m\Omega_o}\sinh\Omega_o t\} \tag{4.60}$$

$$p^+(t) = e^{+\frac{\gamma}{2m}t}\{p^+(0)\cosh\Omega_o t + m\Omega_o x^-(0)\sinh\Omega_o t\} \tag{4.61}$$

$$p^-(t) = e^{-\frac{\gamma}{2m}t}\{p^-(0)\cosh\Omega_o t + m\Omega_o x^+(0)\sinh\Omega_o t\} \tag{4.62}$$

Case III: Critical case ; $k = \frac{\gamma^2}{4m}$

$$\Omega = 0,$$

$$x^+(t) = x^+(0)e^{-\frac{\gamma}{2m}t} + \frac{p^-(0)}{m}te^{-\frac{\gamma}{2m}t} \tag{4.63}$$

$$x^-(t) = x^-(0)e^{\frac{\gamma}{2m}t} + \frac{p^+(0)}{m}te^{\frac{\gamma}{2m}t} \tag{4.64}$$

$$p^+(t) = p^+(0)e^{\frac{\gamma}{2m}t} \tag{4.65}$$

$$p^-(t) = p^-(0)e^{-\frac{\gamma}{2m}t} \tag{4.66}$$

In this chapter we introduced classical damped harmonic oscillator in Bateman and Caldirola-Kanai formulations. We found Lagrangian and corresponding hamiltonian. Now at the next chapter we pass to quantum frame in order to understand dissipation in quantum theory.

CHAPTER 5

QUANTUM DAMPED HARMONIC OSCILLATOR

5.1. Hamiltonian Formulation and Canonical Quantization

In 1925, P. A. M. Dirac observed that various quantum mechanical relations can be obtained from the corresponding classical relations just by replacing classical Poisson brackets by commutators, as follows:

$$\{,\} \rightarrow \frac{1}{i\hbar}[,]$$

where the classical Poisson brackets for functions of $a(p,q)$ and $b(p,q)$ as are defined as

$$\{a,b\} = \frac{\partial a}{\partial q}\frac{\partial b}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial b}{\partial q} \tag{5.1}$$

Now we can pass from the classical formalism to the quantum formalism by replacing the classical dynamical variables p, q, a, b, etc., by the operators p, q, A, B, etc., in Hilbert space (see Appendix C) of states in such a way that Poisson bracket is replaced by the quantum commutator,

$$[A,B] = AB - BA. \tag{5.2}$$

Poisson brackets and quantum-mechanical commutators satisfy similar algebraic properties,

- $[A,A] = 0$
- $[A,B] = -[B,A]$
- $[A,c] = 0$
- $[A+B,C] = [A,C] + [B,C]$
- $[A,BC] = [A,B]C + B[A,C]$
- $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

where c is a number. The last relation is known as the Jacobi identity.

Also the equation of motion for a dynamical variable A reads

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H] + \frac{\partial A}{\partial t} \tag{5.3}$$

where H is the Hamiltonian operator corresponding to the classical Hamiltonian function H(p,q). (For details see Appendix F)

5.2. Quantization of the Bateman's Damped Harmonic Oscillator

In this section we are going to quantize the motion of Bateman's damped harmonic oscillator. We will follow the Feshbach and Tikochinsky approach. (Feshbach and Tikochinsky 1977)

Now we are going to write the Hamiltonian operator of the quantum damped harmonic oscillator. For this aim we shall change the classical Hamiltonian equation (4.9) to the Hamiltonian operator,

$$H = \frac{p_x p_y}{m} + \frac{\gamma}{2m}(yp_y - xp_x) + (k - \frac{\gamma^2}{4m})xy. \tag{5.4}$$

Here x, y, p_x, p_y are all operators in a linear space satisfying the Heisenberg commutation relation rules:

$$[x, y] = [p_x, p_y] = 0$$

$$[p_x, x] = [p_y, y] = -i\hbar$$

Then we can introduce the following annihilation and creation operators:

$$a = \frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right), \quad b = \frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right), \tag{5.5}$$

$$a^+ = \frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right), \quad b^+ = \frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_y}{\sqrt{m}} + i\sqrt{m}\Omega y\right) \tag{5.6}$$

where a^+, b^+ are the Hermitian conjugates of a, b and also

$$\Omega = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}},$$

is the common frequency of the two oscillators (Eq. (5.5) and Eq. (5.6)), assuming Ω is real $\frac{k}{m} > \frac{\gamma^2}{4m^2}$ (i.e. under damping). The commutation relations that corresponds to the set are

$$[a, a^+] = [b, b^+] = 1, \quad [a, b] = [a, b^+] = [a^+, b^+] = 0. \tag{5.7}$$

Now the Feshbach and Tikochinsky quantum Hamiltonian becomes

$$H = \hbar\Omega[a^+b + b^+a] - \frac{i\gamma\hbar}{4m}[(a^2 - a^{+2}) - (b^2 - b^{+2})]. \tag{5.8}$$

By using a further linear unitary transformation in complex operator space:

$$a = \frac{1}{\sqrt{2}}(A + B), \quad b = \frac{1}{\sqrt{2}}(A - B),$$

$$a^+ = \frac{1}{\sqrt{2}}(A^+ + B^+), \quad b^+ = \frac{1}{\sqrt{2}}(A^+ - B^+), \tag{5.9}$$

These new operators obey the same algebra with Eq. (5.7),

$$[A, A^+] = [B, B^+] = 1, \tag{5.10}$$

the other commutators are zero. The new variables A and B can be written in terms of the old ones as follows

$$A = \frac{a + b}{\sqrt{2}}, \quad , B = \frac{a - b}{\sqrt{2}} \tag{5.11}$$

Also the variables A and B are written in terms of x,y and p_x,p_y as,

$$A = \frac{1}{2\sqrt{\hbar\Omega}}\left[\frac{1}{\sqrt{m}}(p_x + p_y) - i\sqrt{m}\Omega(x + y)\right], \tag{5.12}$$

$$B = \frac{1}{2\sqrt{\hbar\Omega}}\left[\frac{1}{\sqrt{m}}(p_x - p_y) - i\sqrt{m}\Omega(x - y)\right]. \tag{5.13}$$

Then transformed H is written as

$$H = H_0 + H_1, \tag{5.14}$$

$$H_0 = \hbar\Omega[A^+A - B^+B], \tag{5.15}$$

$$H_1 = \frac{i\Gamma}{2}[A^+B^+ - AB] \tag{5.16}$$

where $\Gamma = \hbar\gamma/m$.We see that $[H_0, H_1] = 0$, this is why they have common eigen-states.

Here the states generated by B^+ represent the sink where the energy is dissipated by the quantum damped oscillator flows: the B-oscillator represents the reservoir or heat bath coupled to the A-oscillator. (Vitiello 2001).

Now we want to find the eigenvalues of H_0. The eigenvalues of A^+A and B^+B can be found by starting from the ground state

$$A \mid 0, 0 >= 0, \quad B \mid 0, 0 >= 0 \tag{5.17}$$

but

$$A^+ \mid 0,0>=\mid 1,0>, \quad B^+ \mid 0,0>=\mid 0,1> \tag{5.18}$$

applying n_A and n_B times the creation operators to the ground state

$$(A^+)^{n_A} \mid 0,0>=\sqrt{n_A!} \mid n_A,0>, \quad (B^+)^{n_B} \mid 0,0>=\sqrt{n_B!} \mid 0,n_B> \tag{5.19}$$

where $n_A, n_B = 0, 1, 2, ...$ Hence

$$\mid n_A,n_B>=\frac{1}{\sqrt{n_A!n_B!}}(A^+)^{n_A}(B^+)^{n_B} \mid 0,0> \tag{5.20}$$

If we apply A^+ to the state $\mid n_A,n_B>$, it becomes one increased as

$$A^+ \mid n_A,n_B>=\sqrt{n_A} \mid n_A+1,n_B>$$

and applying A to the state $\mid n_A,n_B>$, it becomes one decreased as

$$A \mid n_A,n_B>=\sqrt{n_A} \mid n_A-1,n_B> .$$

So the eigenvalues of the operator A^+A ,

$$A^+A \mid n_A,n_B>=n_A \mid n_A,n_B>, \tag{5.21}$$

and

$$B^+B \mid n_A,n_B>=n_B \mid n_A,n_B> . \tag{5.22}$$

Then we can write eigenvalues of H_0

$$H_0 \mid n_A,n_B>=\hbar\Omega(n_A-n_B) \mid n_A,n_B> \tag{5.23}$$

so that eigenvalues of H_0 are $\hbar\Omega(n_A - n_B)$ with eigen-states $\mid n_A,n_B>$ belonging to Hilbert space. Here H_0 represents the difference of two free oscillator Hamiltonian.

Our next step is to find eigenvalues of H_1 to reach the full Hamiltonian Eq.(5.14). To that end, we define new operators (Dekker 1981),

$$\phi_0=\frac{1}{2}(A^+A+B^+B), \tag{5.24}$$

$$\phi_x=\frac{1}{2}(A^+B^++AB), \tag{5.25}$$

$$\phi_y=\frac{1}{2}(A^+B^+-AB), \tag{5.26}$$

$$\phi_z=\frac{1}{2}(A^+A^++BB^+). \tag{5.27}$$

Clearly

$$H_0 = 2\hbar\Omega\phi_0, \quad H_1 = i\Gamma\phi_y. \tag{5.28}$$

Their commutators as spin-like operators produce $su(1,1)$ algebra,

$$[\phi_x, \phi_y] = i\phi_z, \tag{5.29}$$

$$[\phi_z, \phi_y] = i\phi_x, \tag{5.30}$$

$$[\phi_x, \phi_z] = i\phi_y. \tag{5.31}$$

Also ϕ_0 commutes with ϕ_x, ϕ_y, ϕ_z. So ϕ_0 is the Casimir operator for the algebra,satisfying relation on pseudo sphere,

$$\phi_z^2 - \phi_x^2 - \phi_y^2 = \phi_0^2 - \frac{1}{4}. \tag{5.32}$$

Let's introduce

$$j = \frac{1}{2}(n_A - n_B), \quad m = \frac{1}{2}(n_A + n_B) \tag{5.33}$$

and re-label the eigen-states $\mid n_A, n_B >$ as $\mid j, m >$. Using the Baker-Hausdorff relation we reach

$$e^{\mu\phi_x}\phi_y e^{-\mu\phi_x} = \phi_y \cos\mu + i\phi_z \sin\mu. \tag{5.34}$$

If we take $\mu = \pm\frac{\pi}{2}$ we obtain ,

$$\phi_y = \pm i e^{\pm\frac{\pi}{2}\phi_x}\phi_z e^{\pm\frac{\pi}{2}\phi_x}. \tag{5.35}$$

The eigenvalue equation can be written as

$$\phi_y \mid \psi_{j,m}^{(\pm)} >= \pm i\left(m + \frac{1}{2}\right) \mid \psi_{j,m}^{(\pm)} > \tag{5.36}$$

where,

$$\mid \psi_{j,m}^{(\pm)} >= e^{\pm\frac{\pi}{2}\phi_x} \mid j, m > \tag{5.37}$$

The eigenfunctions $\mid \psi_{j,m}^{(\pm)} >$ do not belong in the ordinary Hilbert space because they can not be normalized in the usual manner (Dekker 1977).

We see that there are two sets of eigen-states, one of them is with positive imaginary eigenvalues $i\left(m + \frac{1}{2}\right)$, the other is with negative imaginary eigenvalues $-i\left(m + \frac{1}{2}\right)$. The eigen-states with negative imaginary eigenvalues are the decaying states, with positive eigenvalues states are the growing states.

As a result the eigenvalues of H are

$$H \mid j,m >= \left[2\hbar\Omega j \pm i\Gamma\left(m+\frac{1}{2}\right)\right] \mid j,m >, \tag{5.38}$$

or

$$H \mid n_A,n_B >= \left[\hbar\Omega(n_A-n_B) \pm i\frac{\Gamma}{2}(n_A+n_B+1)\right] \mid n_A,n_B > . \tag{5.39}$$

Now for the next step we find wave functions corresponding to Hamiltonian (5.39). This is why we start from the Eq. (5.17) and multiply it from the left by $< x,y \mid$ to write the ground state wave function $\psi_0(x,y) =< x,y \mid 0,0 >$ in coordinate representation as follow

$$< x,y \mid A \mid 0,0 > \;=\; 0, \tag{5.40}$$

$$< x,y \mid B \mid 0,0 > \;=\; 0 \tag{5.41}$$

where A and B are given by equations (5.12) and (5.13). Then we get

$$\frac{1}{2\sqrt{\hbar\Omega}}\left[\frac{1}{\sqrt{m}}(p_x+p_y) - i\sqrt{m}\Omega(x+y)\right]\psi_0(x,y) \;=\; 0, \tag{5.42}$$

$$\frac{1}{2\sqrt{\hbar\Omega}}\left[\frac{1}{\sqrt{m}}(p_x-p_y) - i\sqrt{m}\Omega(x-y)\right]\psi_0(x,y) \;=\; 0. \tag{5.43}$$

Adding and subtracting equations (5.43) and (5.42) , and replacing $p_x = -i\hbar\frac{\partial}{\partial x}$ and $p_y = -i\hbar\frac{\partial}{\partial y}$, we get

$$(\alpha^{'}\frac{\partial}{\partial x} - \alpha x)\psi_0(x,y) = 0, \tag{5.44}$$

$$(\alpha^{'}\frac{\partial}{\partial y} - \alpha y)\psi_0(x,y) = 0. \tag{5.45}$$

Here $\alpha^{'} = \frac{-i\hbar}{2\sqrt{\hbar m\Omega}}$ and $\alpha = \frac{i}{2}\sqrt{\frac{m\Omega}{\hbar}}$. Solving the above equations we get the wave function

$$\psi_0(x,y) \;=\; Cexp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\beta^2}\right)\right] \tag{5.46}$$

where $\beta = \sqrt{\frac{\hbar}{m\Omega}}$ and C is some constant.

In the following, we multiply Eq.(5.20) from the left by $< x,y \mid$,

$$< x,y \mid n_A,n_B > \;=\; < x,y \mid \frac{1}{\sqrt{n_A!n_B!}}(A^+)^{n_A}(B^+)^{n_B} \mid 0,0 > . \tag{5.47}$$

We rewrite A^+ and B^+ (since they are adjoints of equations (5.12) and (5.13))substituting the values of p_x and p_y as mentioned before , we get the form,

$$A^+ = \alpha\left[\left(x+\beta^2\frac{\partial}{\partial x}\right)+\left(y+\beta^2\frac{\partial}{\partial y}\right)\right] \tag{5.48}$$

$$B^+ = \alpha\left[\left(x+\beta^2\frac{\partial}{\partial x}\right)-\left(y+\beta^2\frac{\partial}{\partial y}\right)\right] \tag{5.49}$$

where $\alpha=\frac{i}{2\beta}$ and $\beta^2=\frac{\hbar}{m\Omega}$ Then the wave function becomes

$$\begin{aligned}\psi_{n_An_B}(x,y) &= \left(\frac{m\Omega}{4\hbar}\right)^{n_A+n_B}\frac{(-1)^{n_A+n_B}}{\sqrt{n_A!n_B!}}\left[\left(x+\beta^2\frac{\partial}{\partial x}\right)+\left(y+\beta^2\frac{\partial}{\partial y}\right)\right]^{n_A}\\ &\quad .\left[\left(x+\beta^2\frac{\partial}{\partial x}\right)-\left(y+\beta^2\frac{\partial}{\partial y}\right)\right]^{n_B}<x,y\mid 0,0>\end{aligned} \tag{5.50}$$

where

$$<x,y\mid 0,0>=\psi_0(x,y)=Cexp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\beta^2}\right)\right]$$

As a result we have found the wave function for Bateman quantum damped oscillator.

5.3. Quantization of the Caldirola-Kanai's Damped Harmonic Oscillator

In this section we quantize the Caldirola-Kanai's damped harmonic oscillator . We define Hamilton operator in terms of the operators q and p and then use Schrödinger equation to solve the problem in coordinate representation.

We begin by transferring the classical Hamiltonian function given at Eq. (4.29) to quantum Hamiltonian. For this aim we write q and p in terms of operators

$$H=\frac{p^2}{2me^{\frac{\gamma t}{m}}}+\frac{1}{2}me^{\frac{\gamma t}{m}}\omega^2q^2, \tag{5.51}$$

where $p\rightarrow -ih\frac{\partial}{\partial q}$, $q\rightarrow q$.

Substituting Eq.(5.51) into time dependent Schrödinger equation we get

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi &= H\psi\\ &= \left(\frac{p^2}{2me^{\frac{\gamma t}{m}}}+\frac{1}{2}me^{\frac{\gamma t}{m}}\omega^2q^2\right)\psi\end{aligned} \tag{5.52}$$

Now we use coordinate transformation. We transform the coordinates (q,t) to new coordinates $(q^{'},t^{'})$ to exclude explicit time dependence in H:

$$(q,t)=(q(q^{'},t^{'}),t(q^{'},t^{'}))$$

where $q^{'} = e^{\frac{\gamma}{2m}t}q$ and $t^{'} = t$. Also we need to write the Jacobian of the given transformation of coordinates to see if the coordinates are linearly independent.

Definition 5.1 *Jacobian J is determinant $J = det\hat{J}$ of the Jacobian matrix $\hat{J}$ given by*

$$\hat{J} = \begin{pmatrix} \frac{\partial q}{\partial q^{'}} & \frac{\partial q}{\partial t^{'}} \\ \frac{\partial t}{\partial q^{'}} & \frac{\partial t}{\partial t^{'}} \end{pmatrix}. \tag{5.53}$$

Remark 5.1 *New coordinates are independent if $J \neq 0$.*

We find jacobian of the transformation as $J = e^{\frac{\gamma}{2m}t} \neq 0$ so coordinates are independent. The derivatives can be found as

$$\begin{aligned} \frac{\partial}{\partial q} &= \frac{\partial q^{'}}{\partial q}\frac{\partial}{\partial q^{'}} + \frac{\partial t^{'}}{\partial q}\frac{\partial}{\partial t^{'}} \\ &= e^{\frac{\gamma}{2m}t}\frac{\partial}{\partial q^{'}}, \end{aligned} \tag{5.54}$$

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial q^{'}}{\partial t}\frac{\partial}{\partial q^{'}} + \frac{\partial t^{'}}{\partial t}\frac{\partial}{\partial t^{'}} \\ &= \frac{\gamma}{2m}q^{'}\frac{\partial}{\partial q^{'}} + \frac{\partial}{\partial t^{'}}. \end{aligned} \tag{5.55}$$

Writing Eq.(5.52) in terms of the new variables we get

$$i\hbar\frac{\partial\psi}{\partial t^{'}} = \left[\frac{p^{'2}}{2m} + \frac{m\omega^2}{2}q^{'2} + \frac{\gamma}{2m}q^{'}p^{'}\right]\psi. \tag{5.56}$$

where $p^{'} = -i\hbar\frac{\partial}{\partial q^{'}}$.

The next step is to diagonalize Hamiltonian operator. Let's consider $p^{'}$ as

$$p^{'} = k - \frac{\gamma}{2}q^{'} \tag{5.57}$$

the Hamiltonian becomes

$$H(k, q^{'}) = \left[\frac{k^2}{2m} + \left(\frac{m\Omega^2}{2}\right)q^{'2}\right], \tag{5.58}$$

where $\Omega^2 = \omega^2 - \frac{\gamma^2}{4m^2}$ and $k = p^{'} + \frac{\gamma}{2}q^{'}$

Since

$$[q, p] = i\hbar \tag{5.59}$$

and

$$q = e^{-\frac{\gamma}{2m}t}q', \quad p = e^{\frac{\gamma}{2m}t}p', \tag{5.60}$$

by substituting (5.60) into (5.59) we get

$$\left[q', p'\right] = i\hbar, \quad \left[q', k\right] = i\hbar. \tag{5.61}$$

Now we write the Schrödinger equation for the Hamiltonian (5.58),

$$i\hbar\frac{\partial\psi}{\partial t'} = \left[\frac{k^2}{2m} + \left(\frac{m\Omega^2}{2}\right)q'^2\right]\psi \tag{5.62}$$

or equivalently;

$$i\hbar\frac{\partial\psi}{\partial t'} = \frac{-\hbar^2}{2m}\left[\frac{\partial}{\partial q'} + \frac{i\gamma}{2\hbar}q'\right]^2\psi + \left(\frac{m\Omega^2}{2}\right)q'^2\psi. \tag{5.63}$$

We use the following gauge transformation [1]

$$\left[\frac{\partial}{\partial q'} + \frac{i\gamma}{2\hbar}q'\right] = e^{-\frac{i\gamma}{4\hbar}q'}\frac{\partial}{\partial q'}e^{\frac{i\gamma}{4\hbar}q'} \tag{5.64}$$

and [2]

$$\left[\frac{\partial}{\partial q'} + \frac{i\gamma}{2\hbar}q'\right]^2\psi = e^{-\frac{i\gamma}{4\hbar}q'}\frac{\partial^2}{\partial q'^2}e^{\frac{i\gamma}{4\hbar}q'}\psi \tag{5.65}$$

Changing the wave function ψ to another wave function Ψ

$$\psi = e^{-\frac{i\gamma}{4\hbar}q'}\Psi \tag{5.66}$$

and substituting Ψ into the Schrödinger equation (5.63) we get

$$i\hbar\frac{\partial\Psi}{\partial t'} = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial q'^2} + \left(\frac{m\Omega^2}{2}\right)q'^2\Psi \tag{5.67}$$

[1]Theorem:(Baker- Hausdorf) If A and B are two fixed non-commuting operators and ξ is a parameter, then

$$e^{\xi A}Be^{-\xi A} = B + \xi[A, B] + \frac{\xi^2}{2!}[A, [A, B]] + \frac{\xi^3}{3!}[A, [A, [A, B]]] + ...$$

For our case $A = q'$,$B = \frac{\partial}{\partial q'}$ and after the fist two terms the rest is zero.

[2]Theorem: If A and B are two fixed non-commuting operators and if A^{-1} exists, then we have

$$AB^nA^{-1} = (ABA^{-1})^n,$$

where n is an integer. (For more details see reference $(Louisell1964)$)

or

$$i\hbar\frac{\partial\Psi}{\partial t'} = \underbrace{\left(\frac{p'^2}{2m}+\frac{m\Omega^2}{2}q'^2\right)}_{\hat{H}}\Psi, \tag{5.68}$$

The Schrödinger equation has form of harmonic oscillator with fixed mass and frequency.

Our next step is to solve Eq. (5.68) . First we rewrite Eq. (5.68) in the form

$$\frac{\partial\Psi}{\partial t'}+\frac{i\hat{H}}{\hbar}\Psi=0 \tag{5.69}$$

Next we multiply this equation from the left by the integrating factor U^{-1} (see appendix[D])

$$U^{-1}=\exp\left(\frac{it'\hat{H}}{\hbar}\right)$$

we obtain the equation

$$\frac{\partial}{\partial t'}\left[\exp\left(\frac{it'\hat{H}}{\hbar}\right)\Psi\right]=0 \tag{5.70}$$

Integrating over the time gives

$$\exp\left(\frac{it'\hat{H}}{\hbar}\right)\Psi(q',t')-\Psi(q',0)=0 \tag{5.71}$$

Multiplying this equation by U gives the result

$$\Psi(q',t')=\exp\left(-\frac{it'\hat{H}}{\hbar}\right)\Psi(q',0)=U\Psi(q',0) \tag{5.72}$$

Suppose that in solution (5.72) we chose initial state to be an eigen-state of $\hat{H}$. Call it φ_n, so that

$$\Psi_n(q',0)=\varphi_n(q') \tag{5.73}$$

$$\hat{H}\varphi_n=E_n\varphi_n$$

where $E_n=\hbar\Omega(n+\frac{1}{2})$, $(n=0,1,2,...)$ and since the known property [3]we find the wave function

$$\Psi_n(q',t)=\exp\left(-\frac{it'\hat{H}}{\hbar}\right)\varphi_n=\exp\left(-\frac{it'\hat{E}_n}{\hbar}\right)\varphi_n. \tag{5.74}$$

[3]Let $A\varphi_n=a_n\varphi_n$ then let the function $f(x)=\sum_{t=0}^{\infty}b_l x^l$, one can write $f(A)\varphi_n=f(a_n)\varphi_n$ (see (Liboff 1998))

Substituting Eq.(5.74) into Eq.(5.68) we reach the following result,

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial q^{'2}}+\frac{m\Omega^2}{2}q^{'2}\right]\varphi_n=E_n\varphi_n \tag{5.75}$$

This equation is the time-independent Schrödinger equation and the solution of this equation in terms of Hermite polynomials is given in Appendix $[D]$, which, in terms of $q^{'}$, leads to

$$\varphi_n(\sqrt{\frac{m\Omega}{\hbar}}q^{'})=A_nH_n\left(\sqrt{\frac{m\Omega}{\hbar}}q^{'}\right)e^{-\frac{m\Omega}{2\hbar}q^{'2}} \tag{5.76}$$

where A_n is the normalization constant. Substituting the above equation into Eq.(5.74) we get,

$$\Psi_n(q^{'},t)=A_n\exp\left(-\frac{it^{'}E_n}{\hbar}\right)H_n\left(\sqrt{\frac{m\Omega}{\hbar}}q^{'}\right)e^{-\frac{m\Omega}{2\hbar}q^{'2}}. \tag{5.77}$$

Then we substitute previous equation into (5.66) ,

$$\psi=\sum_n A_n\exp\left(-\frac{i\gamma}{4\hbar}q^{'}\right)\exp\left(-\frac{it^{'}E_n}{\hbar}\right)H_n\left(\sqrt{\frac{m\Omega}{\hbar}}q^{'}\right)\exp\left(-\frac{m\Omega}{2\hbar}q^{'2}\right) \tag{5.78}$$

in terms of the variables (q,p,t) we reach our final result,

$$\psi=\sum_n A_n\exp\left(-\frac{i\gamma}{4\hbar}e^{\frac{\gamma t}{2m}}q\right)\exp\left(-\frac{itE_n}{\hbar}\right)H_n\left(\sqrt{\frac{m\Omega}{\hbar}}e^{\frac{\gamma t}{2m}}q\right)\exp\left(-\frac{m\Omega}{2\hbar}e^{\frac{\gamma t}{m}}q^2\right) \tag{5.79}$$

where $\Omega^2=\omega^2-\frac{\gamma^2}{4m^2}$. As a result we have found the wave function for the time dependent Caldirola's-Kanais damped harmonic oscillator.

Part II

EULER EQUATION

CHAPTER 6

GENERALIZED ANALYTICAL MECHANICS WITH HIGHER DERIVATIVES

In the first part we discussed damped oscillator equation as the second order constant coefficient equation.

There exist several problems in which equations of motion are higher degree than two. This is why the Lagrangian should include more than first derivatives of the generalized coordinates.

In this chapter, to describe dissipative systems with higher derivative, we should extend our general approach formulated at chapter 3 to include higher derivatives.

Example: *Radiation Damping, (Mendes et al. 2006)*

We begin with the problem of the radiation damping. The equation of motion of one-dimensional radiation damping , without external force,is

$$m\ddot{x} - \gamma\dddot{x} = 0 \tag{6.1}$$

where γ and m are independent of time. Since the system (6.1) is dissipative, a Lagrangian description leading to a consistent canonical formalism is not available. To develop a canonical formalism we are required to consider (6.1) along with its time-reversed image

$$m\ddot{y} + \gamma\dddot{y} = 0 \tag{6.2}$$

so that the composite system is conservative. The system (6.1) and (6.2) can be derived from the following Lagrangian

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(\dot{x}\ddot{y} - \ddot{x}\dot{y}), \tag{6.3}$$

where x is the radiation damping coordinate and y corresponds to the time-reversed counterpart. So, the system made of the radiation damping and of its time-reversed image globally behaves as a closed system.

As it is clear from the example, the Lagrangian formalism for radiation damping includes second derivatives. Hence equations of motion includes the third derivative. So we will extend our formulation now.

To start generalization of analytical mechanics for higher derivatives we restrict our consideration with linear dissipative systems having higher order derivatives. This why Lagrangian function we consider as a general quadratic form in terms of $(q_i, \dot{q}_i, ..., q_i^{(N-1)})$,

$$L = L(q_i, \dot{q}_i, ..., q_i^{(N-1)}) \tag{6.4}$$

where $q^{(n-1)}$ represents $(n-1)th$ derivative of q. In the next section we derive generalized Lagrange and Hamiltonian formalisms for higher derivatives.

6.1. Generalized Lagrange and Hamiltonian Formalisms for Higher Derivatives

The Euler-Lagrange equation can be generalized for the higher derivatives as following

$$\sum_{m=0}^{l}(-1)^m \frac{d^m}{dt^m}\left(\frac{\partial L}{\partial q_i^m}\right) = 0, \tag{6.5}$$

where $q_i^m = d^m q_i/dt^m$, $i = 1, ..., n$, $m = 0, ..., l$. We observe that they will have order $2m$ where m is the number of derivatives figuring in L. To pass Hamiltonian formalism, Legendre transformation is needed. For the non-singular case, Euler-Lagrange equations can be transformed into equivalent Hamiltonian form. This theorem is called "*Ostrogradsky's theorem*" (Dubrovin et al. 1984)

If the Lagrangian is in the form of $L = L(u, u^{'}, ..., u^{(l)})$, one can introduce canonical variables q_i and p_i, $(i = 1, ..., l)$, defined by

$$q_1 = u, q_2 = u^{'}, ..., q_l = u^{(l-1)} \tag{6.6}$$

$$\begin{aligned}
p_1 &= \frac{\partial L}{\partial u^{'}} - \left(\frac{\partial L}{\partial u^{''}}\right)^{'} + ... + (-1)^{l-1}\left(\frac{\partial L}{\partial u^l}\right)^{l-1} \\
p_2 &= \frac{\partial L}{\partial u^{''}} - \left(\frac{\partial L}{\partial u^{'''}}\right)^{'} + ... + (-1)^{l-2}\left(\frac{\partial L}{\partial u^l}\right)^{l-2} \\
&\vdots \\
p_l &= \frac{\partial L}{\partial u^l},
\end{aligned}$$

and then set

$$H(p,q) = -L + u^{'}p_1 + u^{''}p_2 + ... + u^{l}p_l. \tag{6.7}$$

Theorem 6.1 *(Ostrogradsky). For non-singular Lagrangian* $L = L(u, u^{'}, ..., u^{l})$*, the Euler-Lagrange equation are equivalent to the Hamiltonian system*

$$q_{\alpha}^{'} = \frac{\partial H}{\partial p_{\alpha}}, \quad p_{\alpha}^{'} = -\frac{\partial H}{\partial q_{\alpha}}, \quad \alpha = 1, ..., l \tag{6.8}$$

where H *is given by* (6.7).

Now as an example we give Hamiltonian formalism of the radiation damping. In the radiation damping case the coordinates and moments are given as follow:

$$q_1 = x, \quad q_2 = \dot{x}, \quad \widetilde{q_1} = y, \quad \widetilde{q_2} = \dot{y}$$

$$p_1 = \frac{\partial L}{\partial \dot{x}} - \left(\frac{\partial L}{\partial \ddot{x}}\right)^{'} = m\dot{y} + \gamma\ddot{y}, \quad \widetilde{p_1} = \frac{\partial L}{\partial \dot{y}} - \left(\frac{\partial L}{\partial \ddot{y}}\right)^{'} = m\dot{x} - \gamma\ddot{x}$$

$$p_2 = \frac{\partial L}{\partial \ddot{x}} = -\frac{\gamma}{2}\dot{y}, \quad \widetilde{p_2} = \frac{\partial L}{\partial \ddot{y}} = \frac{\gamma}{2}\dot{x}$$

substituting into Legenre transformation we reach the next Hamiltonian formalism for the radiation damping,

$$H = \frac{p_1\widetilde{p_1}}{m}.$$

In the next section we consider an example of generalized Bateman type Lagrangian.

6.2. Generalized Bateman's Lagrangian

In this section we consider Bateman's type dual Lagrangian with higher derivatives for description of the constant coefficient equations of arbitrary order. We can write this Lagrangian in the following form ,

$$\begin{aligned} L &= a_{2N}\frac{d^N y}{dt^N}\frac{d^N x}{dt^N} + \frac{1}{2}a_{2N-1}\left\{\frac{d^{N-1}y}{dt^{N-1}}\frac{d^N x}{dt^N} - \frac{d^N y}{dt^N}\frac{d^{N-1}x}{dt^{N-1}}\right\} \\ &\quad -a_{2N-2}\frac{d^{N-1}y}{dt^{N-1}}\frac{d^{N-1}x}{dt^{N-1}} - \frac{1}{2}a_{2N-3}\left\{\frac{d^{N-2}y}{dt^{N-2}}\frac{d^{N-1}x}{dt^{N-1}} - \frac{d^{N-1}y}{dt^{N-1}}\frac{d^{N-2}x}{dt^{N-2}}\right\} \\ &\quad + \ldots + \frac{1}{2}a_1\left\{y\frac{dx}{dt} - \frac{dy}{dt}x\right\} + a_0 xy, \end{aligned} \tag{6.9}$$

here $a_{2N}, a_{2N-1}, ..., a_0$ terms are constants.

The Euler-Lagrange equations are:

$$\sum_{q=0}^{N}(-1)^q \frac{d^q}{dt^q}\left(\frac{\partial L}{\partial v_q^i}\right) = 0 \tag{6.10}$$

where $v_q^i = \frac{d^q x^i}{dt^q}$, $i = 1, 2$ and $x^1 = x$, $x^2 = y$.

Substituting Eq. (6.9) into Eq. (6.10) we get equations of motion in the form of the constant coefficient equation of order $2N$ and its time reversed image:

$$a_{2N}\frac{d^{2N}x}{dt^{2N}} + a_{2N-1}\frac{d^{2N-1}x}{dt^{2N-1}} + ... + a_1\frac{dx}{dt} + a_0 x = 0, \tag{6.11}$$

$$a_{2N}\frac{d^{2N}y}{dt^{2N}} - a_{2N-1}\frac{d^{2N-1}y}{dt^{2N-1}} + ... - a_1\frac{dy}{dt} + a_0 x = 0 \tag{6.12}$$

CHAPTER 7

THE EULER EQUATION

In this chapter firstl we introduce general Euler type differential equation of degree N and it's representation in terms of constant coefficient equation of degree N. After that the Lagrangian formalism of the Euler equation will be given. Also we discuss some related examples.

The Euler equation also known as the Cauchy-Euler differential equation is named after Euler $(1707-1783)$. It is a second order differential equation given by

$$b_2x^2\frac{d^2y}{dx^2}+b_1x\frac{dy}{dx}+b_0y=0 \tag{7.1}$$

where b_0, b_1, b_2 are constants. This equation can be transformed to the second order constant coefficient equation coinciding with the damped harmonic oscillator equation, considered at section 4.1. As was shown there, the variational formulation of the equation needs generalization of analytical mechanics by considering non-stationary pseudo-Riemannian metric in Lagrangian function.

Definition 7.1 *The homogeneous Euler Equation of degree N for function $y=y(x)$ has the generic form*

$$\sum_{k=0}^{N}b_kx^k\frac{d^ky}{dx^k}=0,\quad b_N\neq 0 \tag{7.2}$$

where b_k are constants and the power of x is always equal to the order of the derivative of y in each term.

Solution of this differential equation is based on transformation to the constant coefficient differential equation which can be easily solved.

Theorem 7.1 *The Euler differential equation (7.2) can be transformed to a linear homogeneous ordinary differential equation degree N with constant coefficients,*

$$\sum_{k=0}^{N}a_k\frac{d^ky}{dt^k}=0 \tag{7.3}$$

by using the substitution $x = e^t$.

Proof:

Substituting $x = e^t$, derivatives of y become;

$$\begin{aligned}
x\frac{dy}{dx} &= x\frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt} = D_t y, \\
x^2\frac{d^2y}{dx^2} &= x^2\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) = D_t\left(D_t - 1\right)y, \\
x^3\frac{d^3y}{dx^3} &= \left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) = D_t\left(D_t - 1\right)\left(D_t - 2\right)y, \\
&\vdots \\
x^k\frac{d^ky}{dx^k} &= \prod_{s=0}^{k-1}(D_t - s)y \qquad (7.4) \\
&\vdots \\
x^N\frac{d^Ny}{dx^N} &= D_t\left(D_t - 1\right)\left(D_t - 2\right)...\left(D_t - N + 1\right)y
\end{aligned}$$

where $D_t \equiv \frac{d}{dt}$.

Inserting there into equation (7.2) we reach the following differential equation with constant coefficients

$$\begin{aligned}
&b_N D_t\left(D_t - 1\right)\left(D_t - 2\right)...\left(D_t - N + 1\right)y + ...+ \\
&+b_2 D_t\left(D_t - 1\right)y + b_1 D_t y + b_0 y = 0. \qquad (7.5)
\end{aligned}$$

For solution of equation (7.5) see Appendix A.

Combining terms at the same order derivatives in equation (7.5), we can write the equation according to powers of D_t

$$a_N {D_t}^{(N)}y + a_{N-1}{D_t}^{(N-1)}y + ... + a_1{D_t}^{(1)}y + a_0 y = \sum_{k=0}^{N} a_k D_t^{(k)}y = 0,$$

where ${D_t}^{(k)} = \frac{d^k}{dt^k}$. ■

The relation between constants $a_1, ..., a_N$ and $b_1, ..., b_N$

$$\sum_{k=0}^{N} a_k\frac{d^ky}{dt^k} = \sum_{k=0}^{N} b_k x^k\frac{d^ky}{dx^k} \qquad (7.6)$$

is given at Appendix E.

7.1. Euler Equation in Physical and Mathematical Problems

We consider here three problems which appears in the Euler equations. First problem is the Dirichlet problem for circular disk. The second one is a quantum mechanical problem, particle in a sphere, that leads to the spherical Bessel equations and written in the radial form it produces the Euler differential equation. The third problem is the Euler equation for radiation damping.

7.1.1. Dirichlet Problem for Circular Disk

Suppose that we have a thin circular disk of radius a, $r < a$, (with constant thermal properties and no sources) with the temperature prescribed on the boundary as illustrated in Figure 7.1,

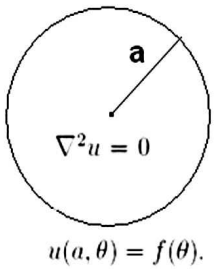

Figure 7.1. Drichlet Problem for Circular Disk

If the temperature on the boundary is independent of time then we determine the equilibrium temperature distribution. The temperature satisfies Laplace's equation $\nabla^2 u = 0$. The geometry of this problem suggests to use polar coordinates, so that $u = u(r, \theta)$. The Laplace equation in polar coordinate is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{7.7}$$

In particular, if on the circle $r = a$ the temperature distribution is prescribed by the function $f(\theta)$, then the Dirichlet boundary condition is

$$u(a, \theta) = f(\theta). \tag{7.8}$$

Mathematically, we need conditions at the end points of the coordinate system, $r = 0, a$ and $\theta = -\pi, \pi$. Here only $r = a$ corresponds to a physical boundary. Polar coordinates

are singular at $r = 0$; for physical reasons we will prescribe that temperature is bounded at the origin

$$\mid u(0,\theta) \mid < \infty. \tag{7.9}$$

The conditions are needed at $\theta = \pm\pi$. Here $\theta = \pi$ and $\theta = -\pi$ correspond to the same points. We say that the temperature is continuous there and the heat flow in the θ direction is continuous, which implies the following periodicity condition;

$$u(r,\theta) = u(r,\theta + 2\pi). \tag{7.10}$$

This satisfies homogeneous boundary condition. Now we use the method of separation of variables:

$$u(r,\theta) = \Phi(\theta)R(r). \tag{7.11}$$

From Laplace's equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Phi(\theta) + \frac{1}{r^2}R(r)\frac{d^2\Phi}{d\theta^2} = 0. \tag{7.12}$$

Dividing by $\frac{1}{r^2}\Phi(\theta)R(r)$ we get

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\theta^2} = \lambda. \tag{7.13}$$

Equation (7.13) yields two ordinary differential equations:

$$\frac{d^2\Phi}{d\theta^2} + \lambda\Phi = 0. \tag{7.14}$$

$$r^2R^{''} + rR^{'} - \lambda R = 0, \tag{7.15}$$

where primes denote derivatives with respect to r. Equation (7.15) is an example of Euler differential equation.

First we solve θ-dependent problem given at Eq. (7.14). We determine the general solution of the problem

$$\Phi(\theta) = A\cos\alpha\theta + B\sin\alpha\theta, \quad \lambda = \alpha^2 > 0 \tag{7.16}$$

$$\Phi(\theta) = A + B\theta, \quad \lambda = 0, \tag{7.17}$$

$$\Phi(\theta) = Ae^{\alpha\theta} + Be^{-\alpha\theta}, \quad \lambda = -\alpha^2 < 0 \tag{7.18}$$

Now requiring periodicity condition $\Phi(\theta) = \Phi(\theta + 2\pi)$, we get the general solution for our problem,

$$\lambda_0 = 0, \ \ \Phi_0(\theta) = A, \tag{7.19}$$

$$\lambda_n = n^2, \ \ \Phi_n(\theta) = A_n\cos n\theta + B_n\sin n\theta. \tag{7.20}$$

where $(n = \pm 1, \pm 2, ...)$.

Second we solve r-dependent equation given at Eq. (7.15). For $\lambda_0 = 0$ we get the solution

$$R_0(r) = C_0 + D_0 \ln r \tag{7.21}$$

Since for $r = 0$, $u(r, \theta)$ must be continuous then for this case $D_0 = 0$ and $R_0(r) = C_0$. For $\lambda_n = n^2$ we have Euler equation,

$$r^2 R_n'' + r R_n' - n^2 R_n = 0. \tag{7.22}$$

To solve this equation we substitute for $r = e^t$ and get following constant coefficient equation

$$\ddot{R}_n - n^2 R_n = 0, \tag{7.23}$$

dots denote derivative with respect to t variable. Then setting for $R = e^{\eta t}$ and substituting into the constant coefficient equation, we find the solution $\eta = \pm n$. We get $R = e^{\pm nt} = (e^t)^{\pm n} = r^{\pm n}$. We write the general solution of this equation

$$R_n(r) = C_n r^n + \frac{D_n}{r^n}. \tag{7.24}$$

For continuity at $r = 0$, we must have $D_n = 0$. So we get

$$R_n(r) = C_n r^n. \tag{7.25}$$

The product of equations (7.20) and (7.25) yields the following result

$$u(r, \theta) = \sum_{n=0}^{\infty} R_n(r)\Phi_n(\theta) \tag{7.26}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) r^n \tag{7.27}$$

Using the Dirichlet condition $u(a, \theta) = f(\theta)$ in the closed interval $[0, 2\pi]$ we write fourier series of $f(\theta)$

$$u(a, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) a^n \tag{7.28}$$

and find the constants

$$a_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta, \quad (n = 0, 1, 2, ...) \tag{7.29}$$

$$b_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta, \quad (n = 1, 2, 3, ...) \tag{7.30}$$

The above results determines the steady-state temperature distribution inside a circular disk.

7.1.2. Particle in a Sphere

We consider spherical Bessel Equation as an example of Euler Equation. In quantum mechanics the problem of quantum mechanical particle in a sphere of radius a is solved by using spherical Bessel functions. Quantum theory requires that the wave function Ψ, describing our particle, satisfy

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\Psi = E\Psi, \tag{7.31}$$

Here m is the mass of the particle, $V(r)$ is potential energy. The boundary conditions (i) $\Psi(r \leq a)$ remains finite, (ii) $\Psi(a) = 0$. This corresponds to a potential $V = 0$, $r \leq a$ and $V = \infty$, $r > a$. Let us determine the minimum value of the energy for which our problem has a solution. Eq. (7.31) is Helmholtz's equation, the radial part of which is the Euler equation,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{n(n+1)}{r^2}\right]R = 0, \tag{7.32}$$

with $k^2 = 2mE/\hbar^2$, for $n = 0$, we get

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + k^2r^2R = 0, \tag{7.33}$$

substituting $r = e^t$ we have

$$\frac{d^2R}{dt^2} + \frac{dR}{dt} + (ke^t)^2R = 0 \tag{7.34}$$

if we consider $(ke^t) = \omega(t)$ then we have damped harmonic oscillator with time dependent frequency. The solution is

$$R = Aj_0(kr) + Bn_0(kr). \tag{7.35}$$

where $j_0(kr) = \frac{\sin kr}{kr}$ and $n_0(kr) = -\frac{\cos kr}{kr}$ We chose the orbital angular momentum index $n = 0$, since any angular dependence with $n > 0$ would raise the energy. The spherical Neumann function is rejected because of its divergent behavior at the origin. To satisfy the second boundary condition, we require

$$ka = \frac{\sqrt{2mE}}{\hbar}a = \alpha,$$

where α is a root of j_0, that is $j_0(\alpha) = 0$. This is the effect of limiting the allowable energies to a certain discrete set or, in other words, application of condition (ii) quantizes the energy E. The smallest α is the first zero of j_0 ,

$$\alpha = \pi$$

and

$$E_{min} = \frac{\pi\hbar^2}{2ma^2}$$

which means that for any finite sphere the particle energy will have a positive minimum or zero point energy.

7.1.3. Euler Equation for Radiation Damping

The Euler equation of motion of the radiation damping

$$m\ddot{x} - \gamma\dddot{x} = 0 \tag{7.36}$$

can be reduced to the Euler equation by the following transformation $\tau = e^t$, then

$$\frac{d}{dt} = \tau\frac{d}{d\tau}. \tag{7.37}$$

Inserting the relation (7.37) into the Eq.(7.36), we find

$$m\left(\tau\frac{d}{d\tau}\right)^2 x - \gamma\left(\tau\frac{d}{d\tau}\right)^3 x = 0. \tag{7.38}$$

Reordering the Eq.(7.38) we get the Euler equation for radiation damping in the next form,

$$\gamma\tau^3\frac{d^3x}{d\tau^3} + (3m-\gamma)\tau^2\frac{d^2x}{d\tau^2} + (\gamma - m)\tau\frac{dx}{d\tau} = 0. \tag{7.39}$$

7.2. Lagrangian Formalism for The Euler Equation $N = 2$

In this section we find Lagrangian formalisms for the Euler equation for the generic case $N = 2$. The general solution of the problem of arbitrary degree N is out of the scope of this thesis. We will consider here only solution of the problem for the second order Euler equation. Let's write the Euler equation in the operator form (discussed at Appendix E).

$$D_E^2 y = \left[b_2 x^2\frac{d^2}{dx^2} + b_1 x\frac{d}{dx} + b_0\right] y = 0 \tag{7.40}$$

The Lagrangian formulation of Eq. (7.40) is related to self-adjoint property of differential operator D_E^2.

7.2.1. Self-Adjoint Differential Operators

Let $\mathcal{L}$ be a generic linear second order operator in the following form,

$$\mathcal{L} = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x) \tag{7.41}$$

where $p_0(x), p_1(x), p_2(x)$ are real functions of x. It results in the differential equation

$$\mathcal{L}u(x) = p_0(x)\frac{d^2}{dx^2}u(x) + p_1(x)\frac{d}{dx}u(x) + p_2(x)u(x) = 0. \tag{7.42}$$

Adjoint operator $\mathcal{L}^\dagger$ is determined by the next expression

$$< v \mid \mathcal{L}u >=< \mathcal{L}^\dagger v \mid u > \tag{7.43}$$

(where $< u \mid v >$ is inner product in vector space defined as $< u \mid v >= \int_a^b uvdx$)so that adjoint differential equation is

$$\mathcal{L}^\dagger v = 0 \tag{7.44}$$

or

$$(p_0 v)^{''} - (p_1 v)^{'} + p_2 v = 0 \tag{7.45}$$

Definition 7.2 *$\mathcal{L}$ is called self-adjoint if $\mathcal{L} = \mathcal{L}^\dagger$. Then equations for u and v are the same,*

$$p_0 u^{''} + p_1 u^{'} + p_2 u = 0,$$

$$p_0 v^{''} + p_1 v^{'} + p_2 v = 0.$$

According to this in the self-adjoint case we can consider the bilinear form

$$< u \mid \mathcal{L}u >=< \mathcal{L}u \mid u > \tag{7.46}$$

Proposition 7.1 *For the linear second order differential equation (7.42) to be self-adjoint it is necessary and sufficient to have*

$$p_0^{'} = p_1$$

Proof. Let adjoint operator $\mathcal{L}^\dagger$ (as same as Eq. (7.45))

$$\begin{aligned} \mathcal{L}^\dagger u(x) &= \frac{d^2}{dx^2}[p_0 u] - \frac{d}{dx}[p_1 u] + p_2 u \\ &= p_0\frac{d^2u}{dx^2} + (2p_0^{'} - p_1)\frac{du}{dx} + (p_0^{''} - p_1^{'} + p_2)u \end{aligned} \tag{7.47}$$

under the condition $\mathcal{L} = \mathcal{L}^{\dagger}$. In comparison of Eq. (7.42) and (7.47) the necessary and the sufficient condition for self-adjoint L is that

$$p_0^{'}(x) = \frac{dp_0(x)}{dx} = p_1(x).\blacksquare \tag{7.48}$$

Hence when this condition is satisfied

$$\mathcal{L}u(x) = \mathcal{L}^{\dagger}u(x) = \frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x). \tag{7.49}$$

and the operator $\mathcal{L}$ is said to be *self-adjoint*. Here for the self-adjoint case $p_0(x)$ is replaced by $p(x)$ and $p_2(x)$ by $q(x)$.

Proposition 7.2 *Every second order non-self adjoint linear differential equation where $p_0^{'}(x) \neq p_1(x)$ can be always transformed into required self adjoint form, by multiplying it by an integrating factor μ,*

$$((\mu p_0)u^{'})^{'} + (\mu p_2)u = 0$$

Proof Consider (7.42) with $p_0^{'} \neq p_1$. If we multiply $\mathcal{L}$ by an integrating factor, μ with

$$\mu = \frac{1}{p_0(x)} exp\left[\int^x \frac{p_1(t)}{p_0(t)}dt\right], \tag{7.50}$$

(see appendix B for the proof of μ)we obtain

$$\begin{aligned}\frac{1}{p_0(x)}\exp\left[\int^x \frac{p_1(t)}{p_0(t)}dt\right]\mathcal{L}u(x) &= \frac{d}{dx}\left\{\exp\left[\int^x \frac{p_1(t)}{p_0(t)}dt\right]\frac{du(x)}{dx}\right\} \\ &\quad + \frac{p_2(x)}{p_0(x)}\exp\left[\int^x \frac{p_1(t)}{p_0(t)}dt\right]u(x)\end{aligned} \tag{7.51}$$

we require $p_0 \neq 0$.As a result the Eq. (7.51) has been put into the self-adjoint form. ■

According to bilinear form discussed at chapter 3 we have definition

Definition 7.3 *Classical action is in bilinear form*

$$S = \frac{1}{2} < v \mid \mathcal{L} \mid u >= \frac{1}{2}\int_a^b v(x)\mathcal{L}u(x)dx$$

where $L = v(x)\mathcal{L}u(x)$ is the Lagrangian.

If $\mathcal{L}$ is self adjoint then action is

$$S = \frac{1}{2} < u \mid \mathcal{L} \mid u >= \frac{1}{2}\int_a^b u(x)\mathcal{L}u(x)dx$$

Using Hamilton's principle of least action to this action we get equations

$$\mathcal{L}u(x) = 0$$

$$\mathcal{L}v(x) = 0$$

and in self-adjoint case they coincide.

Now we will consider generic Euler equation (7.40) of degree two,

$$D_E^2 y = \left[b_2 x^2 \frac{d^2}{dx^2} + b_1 x \frac{d}{dx} + b_0\right] y = 0.$$

Comparing it with Eq. (7.42) we find

$$p_0 = b_2 x^2, \quad p_1 = b_1 x, \quad p_2 = b_0.$$

If the coefficients satisfy $b_1 = 2b_2$ then Eq. (7.48) is satisfied since $p_0^{'} = p_1$ and corresponding equation is self adjoint. But in generic case, if the operator is not self-adjoint, following above procedure, we can get self adjoint operator using integrating factor μ. We find the integrating factor for Eq.(7.40),

$$\mu = \frac{1}{b_2} x^{\frac{b_1}{b_2}-2}. \tag{7.52}$$

Multiplying Eq.(7.40) by (7.52) we get our action formalism as follows,

$$S = \frac{1}{2} < y \mid \mu D_E^2 \mid y > \tag{7.53}$$

where Lagrangian is equal to

$$L = \frac{1}{2} y x^{\frac{b_1}{b_2}} \frac{d^2 y}{dx^2} + \frac{1}{2} y \frac{b_1}{b_2} x^{\frac{b_1}{b_2}-1} \frac{dy}{dx} + \frac{1}{2} y \frac{b_0}{b_2} x^{\frac{b_1}{b_2}-2} y. \tag{7.54}$$

or in terms of only zero and first derivatives

$$L = -\frac{1}{2} x^{\frac{b_1}{b_2}} \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}\frac{b_0}{b_2} x^{\frac{b_1}{b_2}-2} y^2. \tag{7.55}$$

As easy to check by variation of action functional or by using Ostrogradsky formula for higher degree Lagrangian in the section (6.2), we can obtain Euler equation of generic form as

$$b_2 x^2 \frac{d^2}{dx^2} y + b_1 x \frac{d}{dx} y + b_0 y = 0$$

Unfortunately for the generic Euler equation of degree $N > 2$ it is impossible to have self-adjointness by one integrating factor.

In the next section we consider a special case for Lagrangian formalism of the Euler equation when $N = 2l$.

7.3. Lagrangian Formulation of Euler Equation for Special Case $N = 2l$

In this section we find Lagrangian formalism for Euler equation degree $N = 2l$ of the special form

$$x^{2l}\left(\frac{d^{2l}y}{dx^{2l}}\right) + 2l^2x^{2l-1}\left(\frac{d^{2l-1}y}{dx^{2l-1}}\right) + ... + 2x\frac{dy}{dx} + y = 0. \tag{7.56}$$

The Lagrangian, for this Euler differential equation is

$$L = \sum_{n=0}^{l} \frac{1}{2}(-1)^{n+1}x^{2n}\left(\frac{d^n y}{dx^n}\right)^2. \tag{7.57}$$

It is easy to check by substituting (7.57) into the Eq. (6.5).

In the next section we consider some examples of the Euler equation. These examples show the relation of Euler equation with damped harmonic oscillator in the Bateman and Caldirola-Kanai's formalisms.

7.3.1. Examples of the Euler Equation

1. **Damped Harmonic Oscillator Formulation of the Euler Equation:**

One can reach damped harmonic oscillator differential equation from the Lagrangian formalism of the Euler equations. Let's consider the Lagrangian function $L = L(x, y, y')$ of the form

$$L = \frac{1}{2}x^2y'^2 - \frac{y^2}{2}, \quad y' = \frac{dy}{dx}. \tag{7.58}$$

The above equation is derivable from Lagrangian (7.55). Using the Euler-Lagrange equation ;

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0,$$

we find the equation of the motion in the form of the Euler differential equation:

$$x^2y'' + 2xy' + y = 0. \tag{7.59}$$

It is in self-adjoint form since $p_0 = x^2$, $p_1 = 2x$ and $p_0^{'} = p_1$. Now let's consider the general form of (7.59). This is the case where it takes the following form

$$b_2 x^2 \frac{d^2y}{dx^2} + b_1 x \frac{dy}{dx} + b_0 y = 0. \tag{7.60}$$

here $b_2 = 1, b_1 = 2, b_0$=1.

Using the transformation $x = e^t$, the equation becomes a constant coefficient equation;

$$b_2 \frac{d^2y}{dt^2} + (b_1 - b_2) \frac{dy}{dt} + b_0 y = 0 \tag{7.61}$$

This is the equation of the damped harmonic oscillator;

$$\frac{d^2y}{dt^2} + \left(\frac{b_1 - b_2}{b_2}\right) \frac{dy}{dt} + \frac{b_0}{b_2} y = 0 \tag{7.62}$$

which we can write as

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + {\omega_0}^2 y = 0 \tag{7.63}$$

Here $\gamma = \left(\frac{b_1 - b_2}{b_2}\right)$ is the damping parameter and $\omega_0 = \sqrt{\frac{b_0}{b_2}}$ is the angular frequency.

As a result starting from the Euler Lagrangian we reached the damped harmonic oscillator equation.

2. Bateman Formalism for the Euler Equation:

In this example we relate the dual Bateman Lagrangian equation with the one for Euler differential equation. Starting from the Bateman Lagrangian

$$L = x_\tau y_\tau - \frac{\gamma}{2}(x_\tau y - x y_\tau) - kxy \tag{7.64}$$

where $x_\tau = \frac{dx}{d\tau}, y_\tau = \frac{dy}{d\tau}$. In this case the equations of motion are constant coefficient equation,

$$\begin{aligned} x_{\tau\tau} + \gamma x_\tau + kx &= 0, \\ y_{\tau\tau} - \gamma y_\tau + ky &= 0. \end{aligned}$$

We apply the transformation $t = e^\tau$ to pass the Euler differential equation. It is

$$t^2 \ddot{x} + (\gamma + 1) t \dot{x} + kx = 0 \tag{7.65}$$

where $\dot{x} = \frac{dx}{dt}$. The adjoint equation of the Eq. (7.65) is

$$t^2 \ddot{y} + (1 - \gamma) t \dot{y} + (k + 1 - \gamma) y = 0. \tag{7.66}$$

When we have equation of motion we can find the corresponding Lagrangian;

$$\delta \int L d(t) = \delta \int y(t^2\ddot{x} + (\gamma+1)t\dot{x} + kx)d(t) = 0. \tag{7.67}$$

Using integration by parts we can reformulate the Lagrangian

$$L = -t^2\dot{x}\dot{y} + t[(1+\gamma)\dot{x}y - 2x\dot{y}] + kxy \tag{7.68}$$

This Lagrangian is Bateman's dual Lagrangian and it's equations of motion are now in the Euler equation form.

3. Caldirola-Kanai Representation in the The Euler Form:

In chapter 4 we have applied Caldirola-Kanai Lagrangian formalism to derive damped harmonic oscillator equations. Now we are going to show that Caldirola-Kanai Lagrangian (4.27) can be represented in the Euler form. We start from Lagrangian (7.57) for the case $l = 1$. Multiplying some elements of (7.57) by necessary constants, m, w and change the variables of the Lagrangian as $L = L(x, \dot{x}, t)$. Then we get

$$L = m\left(\frac{t^2\dot{x}^2}{2} - w^2\frac{x^2}{2}\right). \tag{7.69}$$

where $\dot{x} = \frac{dx}{dt}$. The corresponding equation of motion is in the Euler differential equation form

$$t^2\ddot{x} + 2t\dot{x} + w^2x = 0$$

We re-parameterize the action (2.2) $t = t(\tau)$ it as follows

$$\begin{aligned} S &= \int_{t_1}^{t_2} L(\dot{x}, x)dt, \\ &= \int \underbrace{L\left(\frac{dx}{d\tau}\frac{d\tau}{dt}, x\right)\frac{dt}{d\tau}}_{New Lagrangian} d\tau \\ &= \int_{\tau_1}^{\tau_2} \mathbb{L}(x_\tau, x)d\tau \end{aligned}$$

where $\mathbb{L} = L\left(\frac{dx}{d\tau}\frac{d\tau}{dt}, x\right)\frac{1}{\dot{\tau}}$ is modified Lagrangian and $x_\tau = \frac{dx}{d\tau}$. Now we will apply it to the Lagrangian at Eq. (7.69) for $t = e^\tau$

$$\begin{aligned} \mathbb{L} &= m\left(\frac{t^2}{2}\left(\frac{dx}{d\tau}\frac{d\tau}{dt}\right)^2 - w^2\frac{x^2}{2}\right)\frac{1}{\dot{\tau}}, \\ &= e^\tau\frac{m}{2}\left(x_\tau^2 - w^2x^2\right). \end{aligned} \tag{7.70}$$

As a result we get Eq. (7.70), which is Caldirola-Kanai Lagrangian with the equation of motion,

$$x_{\tau\tau} + x_\tau + w^2 x = 0.$$

In the next chapter we consider stationary and non-stationary quantum mechanical problems connected with Euler equation.

CHAPTER 8

QUANTUM EULER EQUATION

In this chapter we will discuss Quantum Euler equations. First we consider Euler equation in a stationary Schrödinger problem. Second we discuss exact solutions of quantum Euler equations in non-stationary case.

8.1. Quantum Euler Equation in Stationary Case

In this section we consider the Euler equation. Then we associate Euler differential operator with the Hamiltonian, called the Euler Hamiltonian and the stationary Schrödinger equation.

8.1.1. Euler Equation in Heisenberg-Weyl Representation

General homogeneous Euler Differential equations are those in which every derivative operator $\frac{d}{dx}$ is combined by a factor of x. When we set for x and $\frac{d}{dx}$ the following relations

$$q = x, \quad p = -i\hbar\frac{d}{dx} \tag{8.1}$$

where $\hbar = 1$, this time x stands for coordinate and $-i\frac{d}{dx}$ stands for momentum,satisfying the commutation relations

$$[q, p] = i, \quad [q, q] = 0, \quad [p, p] = 0, \tag{8.2}$$

so they have both a quantum mechanical meaning. This why we can set the Heisenberg-Weyl algebra for the Euler equation. The definition of the algebra is the following,

Definition 8.1 The Heisenberg-Weyl algebra *is a real three-dimensional Lie algebra whose generators satisfy the commutation relations*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0. \tag{8.3}$$

Introducing generators in terms of operators q and p we get the Heisenberg-Weyl algebra representation of the Euler equations

$$e_1 = ip = \frac{d}{dx}, \quad e_2 = iq = ix, \quad e_3 = iI, \tag{8.4}$$

satisfying the commutation rules in (8.3).

According to this the Euler equation (7.2) can be written in terms of the operators q and p acting on the function $\Psi \equiv y(x)$:

$$\sum_{k=0}^{N} b_k q^k p^k \Psi = 0. \tag{8.5}$$

Defining the *Euler Hamiltonian*,

$$H = \sum_{k=1}^{N} b_k q^k p^k \tag{8.6}$$

and denoting the term for $k = 0$, b_0 as E, the Euler equation get the form of the Schrödinger equation,

$$H\Psi = E\Psi \tag{8.7}$$

where H is (8.6) and $E = b_0$ (8.7) ,then every solution of the Euler equation gives the solution of quantum mechanical problem with hamiltonian (8.6).

In the next section we consider Euler equation in Fock-Bragmann representation.

8.1.2. Euler Equation in Fock-Bargmann Representation

In this section we discuss complexified Euler equation. In this case the operators x, $\frac{d}{dx}$ are replaced by complex operators

$$x \to z = x + iy, \quad \frac{d}{dx} \to \frac{d}{dz} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}). \tag{8.8}$$

so that

$$\sum_{k=0}^{N} b_k z^k \frac{d^k}{dz^k} f(z) = 0 \tag{8.9}$$

This allow us to give interpretation of Euler differential equation as operator equation in *Fock-Bargmann* representation.

Definition 8.2 *There is a realization of Hilbert space (see appendix C for definition of Hilbert space), where any state vector is described by an entire analytical function, this realization is called Fock-Bargmann representation.*

According to this we can associate operators z, $\frac{d}{dz}$ with bosonic creation and annihilation operators

$$a = \frac{d}{dz}, \quad a^+ = z, \tag{8.10}$$

where they satisfy the commutation relation

$$[a, a] = [a^+, a^+] = 0, \quad [a, a^+] = 1. \tag{8.11}$$

Definition 8.3 *Euler differential operator can be defined in the Hamiltonian form in terms of creation and annihilation operators as follows*

$$H = \hbar w \sum_{k=1}^{N} b_k (a^+)^k (a)^k. \tag{8.12}$$

where for dimensional reasons we have introduced Planck's constant $\hbar$ and the characteristic frequency w.

We define vacuum states,

$$a \mid 0 >= 0$$

and n-particle state,

$$\mid n >= \frac{1}{\sqrt{n!}} (a^+)^n |0> .$$

Since $\{\mid n >\}$ is complete,

$$\mid \Psi >= \sum_{n=0}^{\infty} c_n \mid n >,$$

for $\forall\ \Psi$ are elements of Hilbert space.

Definition 8.4 *The number operator N is defined in terms of creation and annihilation operators $N = a^+ a$. When the number operator acts on a state n, it has the eigenvalue n*

$$N|n >= n|n> .$$

To get the Euler operator (8.12) in terms of the number operator N, we rewrite $(a^+)^k (a)^k$ as a polynomial of N

$$(a^+)^k (a)^k = P_k(N).$$

$P_k(N)$ can be found using the commutation relations (8.11) and stating $k = 0, 1, 2, ..., l$ as follows,

$$\begin{aligned}
P_0(N) &= 1 \\
P_1(N) &= a^+a = N, \\
P_2(N) &= a^+a^+aa = a^+\underbrace{a^+a}a = a^+([a^+,a] + aa^+)a \\
&= a^+(aa^+ - 1)a = a^+aa^+a - a^+a \qquad (8.13) \\
&= N(N-1) \\
&\cdot \\
&\cdot \\
&\cdot \\
P_l(N) &= (a^+)^l(a)^l = N(N-1)(N-2)...(N-(l-1))
\end{aligned}$$

Substituting the terms into the Eq. (8.12) we get Euler Hamiltonian operator in terms of number operators,

$$H(N) = \hbar\omega \sum_{k=1}^{l} b_k P_k(N), \qquad (8.14)$$

or in explicit form,

$$H(N) = \left[b_l N\left(N-1\right)\left(N-2\right)...\left(N-l+1\right) + ... + b_2 N\left(N-1\right) + b_1 N + b_0\right].$$

If we re-arrange the above equation in terms of powers of N we get

$$H(N) = c_l N^l + c_{l-1}N^{l-1} + ... + c_1 N + c_0.$$

where c_l are constants. Acting on the state $|n>$ we have

$$H(N)|n> = E(n)|n>,$$

with the discrete spectrum

$$E(n) = P_l(n) = \hbar\omega\left(c_l n^l + c_{l-1}n^{l-1} + ... + c_0\right). \qquad (8.15)$$

8.1.3. The Euler Hamiltonian in Terms of the Standard Harmonic Oscillator Hamiltonian

In this part we consider Euler hamiltonian operator (8.12) in terms of the standard harmonic oscillator creation and annihilation operators a^+, a,

$$a = \sqrt{\frac{mw}{2\hbar}}\left(q + \frac{ip}{mw}\right), \quad a^+ = \sqrt{\frac{mw}{2\hbar}}\left(q - \frac{ip}{mw}\right), \tag{8.16}$$

and standard harmonic oscillator hamiltonian operator H_0 is given by the form

$$H_0 = \frac{1}{2m}\left(p^2 + m^2w^2q^2\right). \tag{8.17}$$

where the q and p are the quantum mechanical coordinate and momentum operators respectively, m is mass, w is frequency.

They satisfy the algebra

$$[a, a^+] = \frac{i}{2\hbar}[p, q] - \frac{i}{2\hbar}[q, p] = 1 \tag{8.18}$$

where $[q, p] = i\hbar$.

Our aim is to write Eq. (8.14) in terms of the standard harmonic oscillator hamiltonian operator H_0. Hence we start from $k = 1$

$$a^+a = \frac{p^2}{2m\hbar w} + \frac{mw}{2\hbar}q^2 + \frac{i}{2\hbar}(qp - pq) = \frac{H_0}{\hbar w} - \frac{1}{2}, \tag{8.19}$$

then

$$H_0 = \hbar w(a^+a + \frac{1}{2}) = \hbar w(N + \frac{1}{2}), \quad k = 1. \tag{8.20}$$

to get higher terms $k = 2, 3, ..., l$ we use Eq. (8.13) for simplicity

$$\begin{aligned}
a^+a^+aa &= N(N-1) = \left(\frac{H_0}{\hbar w} - \frac{1}{2}\right)\left(\frac{H_0}{\hbar w} - \frac{3}{2}\right), \ k = 2, \\
a^+a^+a^+aaa &= N(N-1)(N-2) = \left(\frac{H_0}{\hbar w} - \frac{1}{2}\right)\left(\frac{H_0}{\hbar w} - \frac{3}{2}\right)\left(\frac{H_0}{\hbar w} - \frac{5}{2}\right), \ k = 3, \\
&\ . \\
&\ . \\
&\ . \\
(a^+)^l(a)^l &= N(N-1)(N-2)...(N-(l-1)), \\
&= = \left(\frac{H_0}{\hbar w} - \frac{1}{2}\right)\left(\frac{H_0}{\hbar w} - \frac{3}{2}\right)...\left(\frac{H_0}{\hbar w} - \frac{(l-1)}{2}\right), \ k = l.
\end{aligned}$$

In this way we get Eq. (8.14) in terms of H_0, that can be written as

$$H = P_N(H_0)$$

8.2. Quantum Euler Equation in Non-Stationary Case

In the present section we consider the time dependent Heisenberg equation with Hamilton operator given by the Euler Hamiltonian. These models are examples of non-linear quantum systems.

In the previous section we have defined Euler Hamiltonian to refer quantum mechanical Hamiltonian that can be written as functions of pq (Bender 1994). The examples of Euler hamiltonian can be given as $H = pqpq, H = p^3q^3 + q^3p^3, H = exp(pq + qp)$.

If we have a quantum mechanical Hamiltonian H, we can write down the Heisenberg operator equations of motion [For details see appendix E].

$$\dot{q} = \frac{\partial H}{\partial p}, \;\; \dot{p} = -\frac{\partial H}{\partial q}. \tag{8.21}$$

In terms of commutation relations we can formalize these as follows

$$\dot{q} = i[H, q], \quad \dot{p} = i[H, p] \tag{8.22}$$

These equations determine the time evolution of operators $q(t), p(t)$ in Heisenberg picture. Where $q(t)$ and $p(t)$ satisfy the commutation relation

$$[q(t), p(t)] = i. \tag{8.23}$$

In general it is difficult to find solutions to (8.21) for given hamiltonian H. But in the case of harmonic oscillator when Hamiltonian is quadratic

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 \tag{8.24}$$

it is simple, since the equations are linear. To solve this problem we use the given Hamiltonian (8.24) to establish the time evolution of the operators $q(t)$, $p(t)$. The equations of motion as follow

$$\dot{q} = \frac{\partial H}{\partial p} = p, \;\; \dot{p} = -\frac{\partial H}{\partial q} = -q. \tag{8.25}$$

can be solved with the initial conditions $q(0) = q(t_0)$, $p(0) = p(t_0)$ and to yield the solution;

$$q(t) = q(0)\cos(t) + p(0)\sin(t), \tag{8.26}$$

$$p(t) = p(0)\cos(t) - q(0)\sin(t). \tag{8.27}$$

Now we show how to solve exactly the large class of non-linear Heisenberg equations arising from Hamiltonian's of Euler type.

8.2.1. Examples for Non-linear Heisenberg Equations

Example 1 :

Let's consider the following system of operator equations:

$$\dot{q} = q(pq)^n, \quad \dot{p} = -p(qp)^n. \tag{8.28}$$

We can derive exact solution to the system of equations (8.28) satisfying the commutation relation (8.23). To find the solution we multiply $\dot{q}$ from the left by p and $\dot{p}$ from the right by q at (8.28) and add them as follows

$$\begin{aligned} p\dot{q} + \dot{p}q &= pq(pq)^n - p(qp)^n q \\ \frac{d}{dt}(pq) &= (pq)^{n+1} - \underbrace{pq}_{1}\underbrace{pq}_{2}p...q\underbrace{pq}_{n+1} \\ &= (pq)^{n+1} - (pq)^{n+1} = 0 \end{aligned} \tag{8.29}$$

Since left hand side of the Eq. (8.29) is total time derivative of pq and right hand side is equal to zero we can say $q(t_0)p(t_0) = q_0p_0$ and $p(t_0)q(t_0) = p_0q_0$ are all constant. Then the solution of the equation (8.28) is

$$q(t) = q_o e^{(p_o q_o)^n t} \tag{8.30}$$

$$p(t) = p_o e^{-(q_o p_o)^n t} \tag{8.31}$$

Example 2 :

For this example we consider a more complicated system of operator equation,

$$\dot{q} = q\sum_n \alpha_n (pq)^n \tag{8.32}$$

$$\dot{p} = -p\sum_n \alpha_n (qp)^n \tag{8.33}$$

where α_n are arbitrary numerical coefficients and the sum over n needs to be convergent. We derive the exact solution of the system satisfying the commutation relation (8.23),

$$\begin{aligned} q(t) &= q_0 exp(\sum_n \alpha_n (p_0q_0)^n t) \\ p(t) &= p_0 exp(-\sum_n \alpha_n (q_0p_0)^n t). \end{aligned} \tag{8.34}$$

The example 2 is the generalization of example 1. As a result if we are given the set of Heisenberg operator equations of motion it can be expressed in the form of (8.34) by use of the commutation relation (8.23).

Example 3 :

Now we use the same procedure to solve Heisenberg equations of motion for the given Euler Hamiltonian at the Eq. (8.6) ,

$$H_N = \sum_{n=0}^{N} b_n q^n p^n.$$

Using the Heisenberg operator equations of motion (8.22) we find by induction,
for q,

$$\begin{aligned}
n=0, \quad \dot{q}_0 &= i[H_0, q] = i[b_0, q] = 0, \\
n=1, \quad \dot{q}_1 &= i[H_1, q] = i[b_1 qp + b_0, q] = b_1 q, \\
n=2, \quad \dot{q}_2 &= i[H_2, q] = i[b_2 q^2 p^2 + b_1 qp + b_0, q] = 2b_2 q^2 p + b_1 q, \\
&\ldots \\
n=N, \quad \dot{q}_N &= i[H_N, q] = \sum_{n=0}^{N} \alpha_n n q^n p^{n-1}.
\end{aligned} \tag{8.35}$$

for p,

$$\begin{aligned}
n=0, \quad \dot{p}_0 &= i[H_0, p] = i[b_0, p] = 0, \\
n=1, \quad \dot{p}_1 &= i[H_1, p] = i[b_1 qp + b_0, p] = -b_1 p, \\
n=2, \quad \dot{p}_2 &= i[H_2, p] = i[b_2 q^2 p^2 + b_1 qp + b_0, p] = -2b_2 qp^2 - b_1 p, \\
&\ldots \\
n=N, \quad \dot{p}_N &= i[H_N, p] = -\sum_{n=0}^{N} \alpha_n n q^{n-1} p^n.
\end{aligned} \tag{8.36}$$

Multiplying Eq. (8.35) by p from the right, Eq. (8.36) by q from the left and taking the sum and writing it in terms of the derivative of the product we find the result as in previous examples, again $\frac{d}{dt}(qp) = 0$. This show that the product of $q(t)p(t) = p_0 q_0$ is constant. This is why equations (8.35) and (8.36) can be reduced to (8.32) and (8.33) as follows,

$$\dot{q} = q \sum_{n=0}^{N} \alpha_n n q^{n-1} p^{n-1} = q \sum_{n=0}^{N} \alpha_n n (qp)^{n-1} \tag{8.37}$$

$$\dot{p} = -p \sum_{n=0}^{N} \alpha_n n q^{n-1} p^{n-1} = -p \sum_{n=0}^{N} \alpha_n n (qp)^{n-1} \tag{8.38}$$

with the solutions,

$$\begin{aligned} q(t) &= q_0 exp\left(\sum_{n=0}^{N} \alpha_n n (p_0 q_0)^{n-1} t\right), \\ p(t) &= p_0 exp\left(-\sum_{n=0}^{N} \alpha_n n (q_0 p_0)^{n-1} t\right). \end{aligned} \tag{8.39}$$

Example 4 :

In this example we consider the homogeneous Euler Hamiltonian of degree 4,

$$H = apq^2p + bqp^2q + c(pqpq + qpqp) + d(q^2p^2 + p^2q^2), \tag{8.40}$$

where a, b, c, d are arbitrary coefficients. Using the Heisenberg equation (8.22) for this hamiltonian (8.40) we get

$$\begin{aligned} \dot{q} &= (a + c + 2d)(q^2p + pq^2) + 2(b + c)qpq \\ \dot{p} &= -(b + c + 2d)(qp^2 + p^2q) - 2(a + c)pqp \end{aligned}$$

Now we use the commutation relation (8.23) to re-order previous expression. It can be re-written as

$$\begin{aligned} \dot{q} &= \alpha qpq \\ \dot{p} &= -\alpha pqp \end{aligned}$$

where $\alpha = (2a + 2b + 4c + 4d)$. Then we get the solution depends only α

$$\begin{aligned} q(t) &= q_0 exp(\alpha_n p_0 q_0 t) \\ p(t) &= p_0 exp(-\alpha_n q_0 p_0 t). \end{aligned} \tag{8.41}$$

In this section we discussed Euler equation in Heisenberg picture, now we consider another representation of the Euler equation.

8.3. Quantum Euler Equation for $\mathbf{N = 2}$ in the Schrödinger Picture

In this section we write quantum Euler equation in Schrödinger picture (see Appendix F).

At the previous chapter we have found Lagrangian formalisms of the Euler equation for the cases $N = 2$, Eq.(7.55) and $N = 2l$, Eq. (7.57). When we set $l = 1$ in

Eq. (7.57) we see that the equations that represent the Lagrangians have exactly the same form. This form can be written as follows $(x \to t, y \to x)$,

$$L = \frac{1}{2}mt^2\dot{x} - \frac{1}{2}kx^2, \tag{8.42}$$

where m and k are all constants, t is time and x is coordinate. Corresponding equation of motion is ,

$$mt^2\ddot{x} + 2t\dot{x} + kx = 0. \tag{8.43}$$

Using the Legendre transformation $H = p\dot{x} - L$ where $p = \frac{\partial L}{\partial x}$ we find the Hamiltonian function for (8.42) in the next form,

$$H = \frac{1}{t^2}\frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{8.44}$$

Substituting Eq.(8.44) into the time dependent Schrödinger equation we get

$$\begin{aligned} H\psi(x,t) &= i\hbar\frac{\partial\psi}{\partial t} \\ (\frac{1}{t^2}\frac{p^2}{2m} + \frac{1}{2}kx^2)\psi(x,t) &= i\hbar\frac{\partial\psi}{\partial t}. \end{aligned}$$

Since the momentum operator $p = -i\hbar\frac{d}{dx}$ we get the following second order differential equation

$$-\frac{\hbar^2}{2mt^2}\psi_{xx} + \frac{k}{2}x^2\psi = i\hbar\psi_t \tag{8.45}$$

or

$$\psi_{xx} + mt^2kx^2\psi = i\hbar 2mt^2\psi_t. \tag{8.46}$$

To solve Eq.(8.46) we use coordinate transformation. We want to pass from the coordinates (x,t) to new coordinates $(x^{'},t^{'})$. This why we write

$$(x,t) = (x(x^{'},t^{'}),t(x^{'},t^{'}))$$

where $x^{'} = x\sqrt{t}$ and $t^{'} = t$. Now also we need to write the Jacobian of the given transformation of coordinates to understand if the transformation is independent or dependent. We find jacobian of the transformation as $J = \sqrt{t} \neq 0$ so the transformation is independent. Then the derivatives can be found as

$$\begin{aligned} \frac{\partial}{\partial x} &= \sqrt{t'}\frac{\partial}{\partial x'} \\ \frac{\partial^2}{\partial x^2} &= t^{'}\frac{\partial^2}{\partial x'^2} \\ \frac{\partial}{\partial t} &= \frac{x^{'}}{2t'}\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}. \end{aligned} \tag{8.47}$$

Substituting the derivative terms Eq.(8.47) into Eq.(8.46) we get

$$-\frac{\hbar^2}{2mt^{'}}\psi_{x^{'}x^{'}}+\frac{k}{2t^{'}}x^{'2}\psi=i\hbar\left(\frac{x^{'}}{2t^{'}}\psi_{x^{'}}+\psi_{t}^{'}\right). \tag{8.48}$$

Multiplying the previous equation with $t^{'}$ and then substituting $t^{'}=e^{\tau}$ we have

$$i\hbar\psi_{\tau}=-\frac{\hbar^2}{2m}\psi_{x^{'}x^{'}}-i\hbar\frac{x^{'}}{2}\psi_{x^{'}}+\frac{k}{2}x^{'2}\psi \tag{8.49}$$

or in terms of momentum operator $p^{'}$

$$i\hbar\psi_{\tau}=\left(\frac{p^{'2}}{2m}+\frac{x^{'}p^{'}}{2}+\frac{k}{2}x^{'2}\right)\psi \tag{8.50}$$

where $p^{'}=-i\hbar\frac{d}{dx^{'}}$

The following step is diagonalizing Eq.(8.50), by this reason we substitute for $p^{'}\rightarrow p^{'}+\frac{m}{2}x^{'}$ into the Schrödinger equation and we get it in a diagonalized form as follows

$$i\hbar\psi_{\tau}=-\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x^{'}}+\frac{im}{2\hbar}x^{'}\right]^2\psi+\left(\frac{k}{2}-\frac{m}{8}\right)x^{'2}\psi+\frac{i\hbar}{4}\psi \tag{8.51}$$

Now we apply the gauge transformation ,

$$\left[\frac{\partial}{\partial x^{'}}+\frac{im}{2\hbar}x^{'}\right]^2\psi=e^{-\frac{im}{4\hbar}x^{'}}\frac{\partial^2}{\partial x^{'2}}e^{\frac{im}{4\hbar}x^{'}}\psi \tag{8.52}$$

and we transform the wave function ψ to another wave function Ψ according to

$$\psi=e^{-\frac{im}{4\hbar}q^{'}}\Psi. \tag{8.53}$$

Substituting ψ into the Schrödinger equation (8.51) we reach time dependent Schrödinger equation in terms of Ψ,

$$i\hbar\Psi_{\tau}=-\frac{\hbar^2}{2m}\Psi_{x^{'}x^{'}}+\left(\frac{k}{2}-\frac{m}{8}\right)x^{'2}\Psi+\frac{i\hbar}{4}\Psi \tag{8.54}$$

As was shown at section (5.3) we write for Ψ,

$$\Psi_n(x^{'},\tau)=\exp\left(-\frac{i\tau E_n}{\hbar}\right)\varphi_n. \tag{8.55}$$

where $E_n=\left[\hbar\omega(n+\frac{1}{2})+\frac{i\hbar}{4}\right]$

Now we substitute Eq.(8.55) into the Eq.(8.54) we reach the following form,

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^{'2}}+\left(\frac{k}{2}-\frac{m}{8}\right)x^{'2}\right]\varphi_n=E_n\varphi_n \tag{8.56}$$

This is the time-independent Schrödinger equation which has a solution in terms of Hermite polynomials given in Appendix D which leads in terms of $x^{'}$,

$$\varphi_n\left(\sqrt{\frac{m\Omega}{\hbar}}x^{'}\right) = A_n H_n\left(\sqrt{\frac{m\Omega}{\hbar}}x^{'}\right) e^{-\frac{m\Omega}{2\hbar}x^{'2}} \tag{8.57}$$

where A_n is the normalization constant and $\Omega^2 = \left(\frac{k}{m} - \frac{1}{4}\right)$.

Substituting the above equation into Eq.(8.55) we get,

$$\Psi_n(x^{'}, t) = A_n \exp\left(-\frac{i\tau E_n}{\hbar}\right) H_n\left(\sqrt{\frac{m\Omega}{\hbar}}x^{'}\right) e^{-\frac{m\Omega}{2\hbar}x^{'2}}. \tag{8.58}$$

Then we substitute previous equation into (8.53) ,

$$\psi = A_n \exp\left(-\frac{im}{4\hbar}x^{'}\right) \exp\left(-\frac{i\tau E_n}{\hbar}\right) H_n\left(\sqrt{\frac{m\Omega}{\hbar}}x^{'}\right) \exp\left(-\frac{m\Omega}{2\hbar}x^{'2}\right) \tag{8.59}$$

in terms of the variables (x, t) we reach our final result,

$$\psi = A_n \exp\left(-\frac{im}{4\hbar}x\sqrt{t}\right) t^{\left(-\frac{iE_n}{\hbar}\right)} H_n\left(\sqrt{\frac{m\Omega}{\hbar}}x\sqrt{t}\right) \exp\left(-\frac{m\Omega}{2\hbar}x^2 t\right) \tag{8.60}$$

Finally we found the wave function for the Hamiltonian of the Euler equation.

CHAPTER 9

CONCLUSION

In the present thesis, first we have given the generalization of analytical mechanics to describe dissipation. Then starting from classical damped harmonic oscillator we have passed Calfirola-Kanai and the Bateman oscillators as the quantum analogs of a classical dissipative oscillator. We have observed that Calfirola-Kanai oscillator is a one-dimensional oscillator with an exponentially increasing mass in time. On the other hand, also we have observed that the Bateman oscillator is a two-dimensional system, one subsystem describing the damped oscillator and the other describing an amplified oscillator or a heat bath. We have showed that the total energy is conserved since the energy dissipated away from the damped oscillator is transferred to the other. We have quantized the two models and found the wave functions of them.

Then we have formulated generalized analytical mechanics for higher derivatives since for Euler equation the Lagrangian function should include more than first derivatives. This led us to construct Euler equation in general form. Reducing the Euler equation to constant coefficient equation, coinciding with damped harmonic oscillator equation, we gave the general solution of it. Then we have showed the Lagrangian of the Euler equation for the cases $N = 2$. For this formalism self-adjointness is required. After we found Bateman and Caldirola formulations for Euler equation and showed they can be reduced to damped harmonic oscillator form.

Moreover, we have discussed quantum Euler equations in stationary and non-stationary cases. We have defined Euler Hamiltonians where they can be written as functions of (pq). We have given some examples for non-linear Heisenberg equations in the form of Euler Hamiltonians, we have found exact solutions for them.

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APPENDIX A

SOLUTION OF THE CONSTANT COEFFICIENT EQUATION

The equation

$$\{a_k D_t (D_t - 1)(D_t - 2)...(D_t - k + 1) + ... + a_2 D_t (D_t - 1) + a_1 D_t + a_0\} y = 0$$

is a constant coefficient equation of degree k. By using another notations we get the following equation ;

$$b_k {y_t}^{(k)} + b_{k-1} {y_t}^{(k-1)} + ... + b_1 {y_t}^{(1)} + b_0 y = 0 \tag{A.1}$$

where ${y_t}^{(k)} = \frac{d^k y}{dt^k}$. To solve such an equation we can make a substitution $y = e^{\lambda t}$ to form the characteristic equation,

$$P(\lambda) = a_k \lambda^k + a_{k-1} \lambda^{k-1} + ... + a_1 \lambda^1 + a_0 = 0 \tag{A.2}$$

For the solution of the characteristic equation the following cases are possible;
1) All roots $\lambda_1, \lambda_2, ..., \lambda_k$ of the characteristic equation are real and distinct. Then the general solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + ... + c_k e^{\lambda_k t} \tag{A.3}$$

2) There are n equal roots $\lambda_1 = \lambda_2 = ... = \lambda_n$ $(n \leqslant k)$, and the other roots are real and distinct. Then the general solution is

$$y = e^{\lambda_1 t} \left(c_1 + c_2 t + ... + c_n t^{n-1}\right) + c_{n+1} e^{\lambda_{n+1} t} + c_{n+2} e^{\lambda_{n+2} t} + ... + c_n e^{\lambda_n t} \tag{A.4}$$

3) There are n equal complex conjugate roots $\lambda = \alpha \pm i\beta$ $(2n \leqslant k)$, and the other roots are real and distict

$$y = e^{\alpha t} \cos \beta t \left(A_1 + A_2 t + ... + A_n t^{n-1}\right) + e^{\alpha t} \sin \beta t \left(B_1 + B_2 t + ... + B_n t^{n-1}\right) +$$

$$+ C_{2n+1} e^{2n+1t} + C_{2n+2} e^{2n+2t} + ... + C_n e^{nt} \tag{A.5}$$

4) In the general case, when there are r different roots $\lambda_1, \lambda_2, ..., \lambda_r$ of multiplicities $n_1, n_2, ..., n_r$ respectively, the right hand side of the characteristic equation can be represented as

$$P(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} ... (\lambda - \lambda_r)^{n_r} \tag{A.6}$$

where $n_1 + n_2 + ... + n_r = k$.

And the general solution is

$$y = \sum_{p=1}^{r} e^{\lambda_p t} \left(C_{p,0} + C_{p,1} t + ... + C_{p,n_p-1} t^{n_p-1} \right) \tag{A.7}$$

$C_{p,l}$: arbitrary constant

APPENDIX B

THE INTEGRATING FACTOR AND SELF-ADJOINT DIFFERENTIAL EQUATION

Considering the differential equation

$$p_0 u^{''} + p_1 u^{'} + p_2 u = 0 \tag{B.1}$$

multiplying above equation with the integrating factor we get,

$$\mu p_0 u^{''} + \mu p_1 u^{'} + \mu p_2 u = 0$$

writing the first term in terms of total derivative we reach

$$(\mu p_0 u^{'})^{'} - (\mu p_0)^{'} u^{'} + \mu p_1 u^{'} + \mu p_2 u = 0.$$

To get self-adjoint form the such of second and third terms must be zero, which leads

$$(\mu p_0)^{'} u^{'} = \mu p_1 u^{'}, \tag{B.2}$$

multiplying left hand side of the equation $(B.2)$ with p_0 and integrating it,

$$(\mu p_0)^{'} = (\mu p_0)\frac{p_1}{p_0} \Longrightarrow \int \frac{d(\mu p_0)}{\mu p_0} = \int \frac{p_1}{p_0} dx$$

we find μ as same as Eq. (7.50)

$$\mu = \frac{1}{p_0(x)} exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt\right] . \blacksquare \tag{B.3}$$

APPENDIX C

QUANTUM FUNDAMENTALS

C.1. Postulates of Quantum Mechanics

Postulate C.1 *(**Observables and Operators**) To any self-consistently and well-defined observables in physics such as energy, momentum, number of particles, there corresponds an operator (call it A) such that measurement of A yields values (call these measured values a) which are eigenvalues of A. The values, a, are those values for which the equation is called **eigenvalue equation***

$$A\psi = a\psi$$

has a non-zero solution ψ. The function ψ is called the eigenfunction of A corresponding to the eigenvalue a.

Postulate C.2 *(**Measurement in Quantum Mechanics**) Measurement of the observable A that yields the value a leaves the system in the state ψ_a, where ψ_a is the eigenfunction of A that corresponds to eigenvalue a.*

Postulate C.3 *(**The State Function and Expectation Values**) The state of a system at any instant of time may be represented by a **state** or **wave function** ψ which is continuous and differentiable. All information regarding the state of the system is contained in the wave function. Let a system be in the state $\psi(x,t)$, the average of any physical observable C relevant to that system at time t is*

$$< C >= \int \psi^* C \psi dx.$$

*The average, $< C >$ is called the **expectation value** of C*

Postulate C.4 *(**Time Development of the State Function**) The state function $\psi(x,t)$ for a system (e.g. a single particle) develops in time according to the equation*

$$-i\hbar \frac{\partial \psi(x,t)}{\partial t} = H\psi(x,t).$$

This equation is called the time-dependent Schrödinger equation. The operator H is the Hamiltonian operator. It is given by

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x,t)$$

Here $V(x,t)$ is the potential that particle is placed in.

The Schrödinger equation is a linear, first order partial differential equation in time. Therefore, it completely determines the wave function $\psi(x,t)$ at any time t.

Stationary States: Assume that the system is isolated, in other words that the potential V does not depend explicitly on time. Consider the eigenfunctions of the Hamiltonian, which are defined by the eigenvalue equation

$$H\psi_\alpha(x) = E_\alpha\psi_\alpha(x)$$

where the E_α are the energy eigenvalues, which are real. These particular wave functions correspond to states of the system which have well-defined values E_α of the energy. Using these eigenfunctions, we obtain solutions of the Schrödinger equation, by choosing

$$\psi(x,t) = \psi_\alpha e^{-iE_\alpha t/\hbar}.$$

The time dependence of such states is periodic, with angular frequency $w = E_\alpha/\hbar$. These are called stationary states. Stationary state whose energy is well defined "does not evolve". Neither the probability law nor the expectation value of its position evolves with time. In order for a system to evolve with time, it must be a superposition of at least two stationary states with different energies.

C.2. Hilbert Space

Definition C.1 *Hilbert space is an infinite dimensional vector (function) space* `H` *with an inner product*
$< \psi, \phi >$ *such that the norm defined by*

$$|| \psi ||= \sqrt{< \psi, \psi >}$$

turns `H` *into a complete space.*

A Hilbert space `H` has the following properties:

1. The space is linear. A function space is linear under the following two conditions: (a) If a is a constant and a ψ is any element of the space. (b) If ψ and ϕ are any two elements of the space, then $\psi + \phi$ is also element of the space.

2. There is an *inner product* , $< \psi, \phi >$, for any two elements in the space. For functions defined in the interval $a \leq x \leq b$ we may take

$$< \psi, \phi >= \int_a^b \psi(x)\phi(x)dx.$$

3. Any element of H has a norm that is related with the inner product as follows

$$|| \psi ||= \sqrt{< \psi, \psi >}.$$

4. H is complete.

An example of a Hilbert space is given by the set of functions of L^2 space. This is the set of square-integrable functions defined on the whole x interval,

$$|| \psi ||^2= \int_\infty^\infty \psi(x)^*\psi(x)dx < \infty.$$

C.3. Linear Operators

A *linear operator* A is correspondence, or rule, by which a vector $A\psi$ is associated with any vector ψ, in such a way that
(i) if $\phi = \psi$, then $A\phi = A\psi$;
(ii) $A(c\psi) = c(A\psi)$;
(iii) $A(\psi + \phi) = A\psi + A\phi$.
If A and B are linear operators, and ψ is any vector,
(a) cA denotes the operator defined by $(cA)\psi = c(A\psi)$;
(b) $A + B$ denotes the operator defined by $(A + B)\psi = A\psi + B\psi$;
(c) AB denotes the operator defined by $(AB)\psi = A(B\psi)$

Commuting Operators: An important concept in quantum mechanics is the commutator $[A, B]$ of two operators defined by

$$[A, B] = AB - BA$$

If $[A, B] = 0$, then A and B are called commuting operators. If $[A, B] \neq 0$, then A and B are called non-commuting operators.

Theorem C.5 *Observable A and B commute if and only if there exists a basis of eigenvalues common to both.*

Adjoint Operators: The adjoint operator can be formally defined within the Dirac notation by demanding that if $< \phi \mid$ and $\mid \phi >$ are corresponding bras and kets, then

$$< \phi \mid A^+ \equiv < \omega \mid, \quad A \mid \phi > \equiv \mid \omega >$$

should also be corresponding bras and kets. From the fact that $< \omega \mid \psi >^* = < \psi \mid \omega >$, it follows that

$$< \phi \mid A^+ \mid \psi >^* = < \psi \mid A \mid \phi >$$

for all ϕ and ψ. Several useful properties of the adjoint operator are

$$\begin{aligned} (cA)^+ &= c^* A^+ \\ (A+B)^+ &= A^+ + B^+ \\ (AB)^+ &= B^+ A^+ \end{aligned}$$

where c is a complex number.

Hermitian Operators (Self-Adjoint Operators): An operator A that is equal to its adjoint A^+ is called self-adjoint. This means that it satisfies

$$< \phi \mid A \mid \psi > = < \psi \mid A \mid \phi >^* \tag{C.1}$$

An operator that only satisfies Eq. $(C.1)$ is called Hermitian, in analogy with a Hermitian matrix, for which $M_{ij} = M_{ji}^*$.

The following theorems are useful in identifying Hermitian operators on a vector space with complex scalars, (Liboff 1998).

Theorem C.6 *If A is a Hermitian operator then all of its eigenvalues are real.*

Theorem C.7 *Eigenvectors corresponding to distinct eigenvalues of a Hermitian operator are orthogonal.*

Theorem C.8 *If A and B are self-adjoint operators, each of which possesses a complete set of eigenvectors, and if $AB = BA$, then there exists a complete set of vectors which are eigenvectors of both A and B.*

APPENDIX D

STATIONARY SCHRÖDINGER EQUATION FOR THE STANDARD HARMONIC OSCILLATOR

The Hamiltonian of the stationary Schrödinger equation given by

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{D.1}$$

where k is a real positive constant. It can be shown that the angular frequency is $w = \sqrt{k/m}$, where m is the mass of the oscillator. Now we will solve the stationary Schrödinger equation and find the stationary eigen-states for this system, and then we will find the energy eigenvalues of the oscillator.

Eq. $(D.1)$ can be written as

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{mw^2}{2}x^2. \tag{D.2}$$

Thus eigenvalue equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{mw^2}{2}x^2\psi(x) = E\psi(x) \tag{D.3}$$

We define $\varepsilon = \frac{2E}{\hbar w}$ and we change the variable $\xi = \sqrt{\frac{mw}{\hbar}}x$, hence we have

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{d\xi}\frac{d\xi}{dx}\right) = \frac{d^2\psi}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 = \frac{m\omega}{\hbar}\frac{d^2\psi}{d\xi^2} \tag{D.4}$$

Therefore,

$$\frac{\hbar\omega}{2}\frac{d^2\psi(\xi)}{d\xi^2} + E\psi(\xi) - \frac{\hbar\omega}{2}\xi^2\psi(\xi) = 0 \tag{D.5}$$

or

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\varepsilon - \xi^2)\psi = 0 \tag{D.6}$$

For large ξ the dominant part of the previous differential equation is

$$\frac{d^2\psi(\xi)}{d\xi^2} - \xi^2\psi = 0 \tag{D.7}$$

The solution for this equation points to the asymptotic behavior of the wave function for large ξ :

$$\psi(\xi) \sim e^{-\xi^2/2} \tag{D.8}$$

So we can assume

$$\psi(\xi) = H(\xi)e^{-\xi^2/2} \tag{D.9}$$

Substituting $(D.9)$in Eq. $(D.7)$ yields

$$\begin{aligned} \frac{d^2\psi(\xi)}{d\xi^2} &= \frac{d}{d\xi}[H^{'}e^{-\xi^2/2} - \xi H(\xi)e^{-\xi^2/2}] \\ &= [H^{''}(\xi) - 2\xi H^{'}(\xi) + (\xi^2 - 1)H(\xi)]e^{-\xi^2/2} \end{aligned} \tag{D.10}$$

Thus we have

$$[H^{''}(\xi) - 2\xi H^{'}(\xi) + (\xi^2 - 1)H(\xi)]e^{-\xi^2/2} + (\varepsilon - \xi^2)H(\xi)e^{-\xi^2/2} = 0 \tag{D.11}$$

We obtain the Hermite polynomials differential equation.

$$H^{''} - 2\xi H^{'} + (\varepsilon - 1)H = 0 \tag{D.12}$$

The wave function's behavior around $\xi = 0$ $(x = 0)$ is accounted for by these polynomials. In order to solve this equation we substitute

$$H(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$$

so that Eq. $(D.12)$ becomes

$$\sum_{n=0}^{\infty}[a_{n+2}(n+1) - 2na_n + (\varepsilon - 1)a_n]\xi^n = 0 \tag{D.13}$$

Therefore all the coefficients of this series must vanish:

$$a_{n+2}(n+1) + (\varepsilon - 2n - 1)a_n = 0 \tag{D.14}$$

or

$$a_{n+2} = \frac{2n + 1 - \varepsilon}{(n+2)(n+1)}a_n \tag{D.15}$$

D.1. Hermite Polynomials

The Hermite polynomials $H_n(x)$ are defined by the relation

$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d^n}{dx^n}e^{-x^2}\right)$$

$n = 0, 1, 2, ...$ The $H_n(x)$ are the solutions to the differential equation

$$\frac{d^2H_n(x)}{dx^2} - 2x\frac{dH_n(x)}{dx} + 2nH_n(x) = 0$$

The orthogonality relation for $H_n(x)$ is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \sqrt{\pi} 2^n n! \delta_{mn}$$

Two important recurrence relations for $H_n(x)$ are

$$\frac{dH_n(x)}{dx} = 2nH_n(x), \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

The first few Hermite polynomials are given below:

$$\begin{aligned} H_0(x) &= 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12 \end{aligned}$$

APPENDIX E

EULER EQUATION

Here our aim is to find the relation between the constants of the constant coefficient equation and Euler's equation

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{N} b_k x^k \frac{d^k y}{dx^k} \tag{E.1}$$

we will follow two ways. The first way we use matrices and the second way without using a matrix formulation it gives direct equivalence.

$1-)$ For the first way before writing the matrix elements, let's find the relations between them. We begin with the following definitions.

Definition E.1 *Euler differential operator of degree N D_E^N is defined by the next formula*

$$D_E^N = \sum_{k=0}^{N} b_k x^k \frac{d^k}{dx^k} \tag{E.2}$$

where b_k are constants.

Definition E.2 *Constant coefficient differential operator of degree N D_C^N is defined by the formula*

$$D_C^N = \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} \tag{E.3}$$

where a_k are constants.

After giving definitions and using the equations in (7.4) we write the relation starting from $N = 1$,

For the case $N = 1$

$$\begin{aligned} D_E^1 &= b_0 + b_1 x \frac{d}{dx} = b_0 + b_1 \frac{d}{dt} \\ D_C^1 &= a_0 + a_1 \frac{d}{dt} \end{aligned}$$

Result for $N = 1$,

$$\begin{aligned} a_0 &= b_0 \\ a_1 &= b_1 \end{aligned} \tag{E.4}$$

For the case $N = 2$

$$\begin{aligned} D_E^2 &= b_0 + b_1 x \frac{d}{dx} + b_2 x^2 \frac{d^2}{dx^2} \\ &= b_0 + b_1 \frac{d}{dt} + b_2 \frac{d}{dt}(\frac{d}{dt} - 1) \\ &= b_0 + (b_1 - b_2)\frac{d}{dt} + b_2 \frac{d^2}{dt^2} \\ D_C^2 &= a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} \end{aligned}$$

Result for $N = 2$,

$$\begin{aligned} a_0 &= b_0 \\ a_1 &= b_1 - b_2 \\ a_2 &= b_2 \end{aligned} \tag{E.5}$$

For the case $N = 3$

$$\begin{aligned} D_E^3 &= b_0 + b_1 x \frac{d}{dx} + b_2 x^2 \frac{d^2}{dx^2} + b_3 x^3 \frac{d^3}{dx^3} \\ &= b_0 + b_1 \frac{d}{dt} + b_2 \frac{d}{dt}(\frac{d}{dt} - 1) + b_3 \frac{d}{dt}(\frac{d}{dt} - 1)(\frac{d}{dt} - 2) \\ &= b_0 + (b_1 - b_2 + 2b_3)\frac{d}{dt} + (b_2 - 3b_3)\frac{d^2}{dt^2} + b_3 \frac{d^3}{dt^3} \\ D_C^3 &= a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + a_3 \frac{d^3}{dt^3} \end{aligned}$$

Result for $N = 3$,

$$\begin{aligned} a_0 &= b_0 \\ a_1 &= b_1 - b_2 + 2b_3 \\ a_2 &= b_2 - 3b_3 \\ a_3 &= b_3 \end{aligned} \tag{E.6}$$

For the case $N = 4$

$$\begin{aligned} D_E^4 &= b_0 + b_1 x \frac{d}{dx} + b_2 x^2 \frac{d^2}{dx^2} + b_3 x^3 \frac{d^3}{dx^3} + b_4 x^4 \frac{d^4}{dx^4} \\ &= b_0 + b_1 \frac{d}{dt} + b_2 \frac{d}{dt}(\frac{d}{dt} - 1) + b_3 \frac{d}{dt}(\frac{d}{dt} - 1)(\frac{d}{dt} - 2) + b_4 \frac{d}{dt}(\frac{d}{dt} - 1)(\frac{d}{dt} - 2)(\frac{d}{dt} - 3) \\ &= b_0 + (b_1 - b_2 + 2b_3 - 6b_4)\frac{d}{dt} + (b_2 - 3b_3 + 11b_4)\frac{d^2}{dt^2} + (b_3 - 6b_4)\frac{d^3}{dt^3} + b_4 \frac{d^4}{dt^4} \\ D_C^4 &= a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + a_3 \frac{d^3}{dt^3} + a_4 \frac{d^4}{dt^4} \end{aligned}$$

Result for $N = 4$,

$$\begin{aligned} a_0 &= b_0 \\ a_1 &= b_1 - b_2 + 2b_3 - 6b_4 \\ a_2 &= b_2 - 3b_3 + 11b_4 \\ a_3 &= b_3 - 6b_4 \\ a_4 &= b_4 \end{aligned} \tag{E.7}$$

Now we can write the relations at $(E.4)$, $(E.5)$, $(E.6)$ and $(E.7)$ in terms of matrices. They are respectively as follows,

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \tag{E.8}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \tag{E.9}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{E.10}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & -6 \\ 0 & 0 & 1 & -3 & 11 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \tag{E.11}$$

These matrices are upper triangular matrices, where repeated indices are equal to 1. They can be rewritten for N dimension as

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} M_0^0 & M_0^1 & M_0^2 & . & . & . & M_0^{N-1} & M_0^N \\ M_1^0 & M_1^1 & . & . & . & . & . & . \\ M_2^0 & . & M_2^2 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ M_{N-1}^0 & . & . & . & . & . & . & M_{N-1}^N \\ M_N^0 & . & . & . & . & & M_N^{N-1} & M_N^N \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ . \\ b_{N-1} \\ b_N \end{pmatrix}, \quad \text{(E.12)}$$

or this formalism can be written as in terms of,

$$a_n = \sum_{k=0}^{N} M_n^k b_k \quad \text{(E.13)}$$

where $n = 0, 1, ..., N$, $M_n^k = 1$ for $n = k$ and $M_n^k = 0$ for $n > k$.

The next step is to find the relation between matrix elements, M_n^k. This relation is given by the following formula,

$$M_n^k = (-1)^{k+n} \left[(k-1)|M_n^{k-1}| + |M_{n-1}^{k-1}| \right] \quad \text{(E.14)}$$

Substituting $(E.14)$ into $(E.13)$ we get the relation between the constants in terms of matrices give by

$$a_n = \sum_{k=0}^{N} (-1)^{k+n} \left[(k-1)|M_n^{k-1}| + |M_{n-1}^{k-1}| \right] b_k \quad \text{(E.15)}$$

where $n = 0, 1, ..., N$.

Since we didn't give the proof of the above correspondence, it is just a hypothesis.

Now we pass the second way.

$2-)$ Our aim is to formalize the relation of constants of the Euler equation and constant coefficient equation in an direct way. Writing the Euler differential operator defined at the Eq. $(E.2)$ in terms of multiplication formula we get

$$\begin{aligned} D_E^N &= \sum_{k=1}^{N} b_k \prod_{s=0}^{k-1} (D_t - s), \\ &= \sum_{l=1}^{N} a_l D_t^l. \end{aligned}$$

If we act this operator on a state $e^{\lambda t}$ for $N = 1$ then we get

$$D_E^1 e^{\lambda t} = \tag{E.16}$$

$$\begin{aligned} D_E^N e^{\lambda t} &= \sum_{k=1}^{N} b_k \prod_{s=0}^{k-1} (\lambda - s) e^{\lambda t}, \\ &= \sum_{l=1}^{N} a_l \lambda^l e^{\lambda t}. \end{aligned}$$

differentiating the previous term we can write a_l in the form of,

$$a_l = \frac{1}{l!} \frac{d^l}{d\lambda^l} \left[\sum_{k=1}^{N} b_k \prod_{s=0}^{k-1} (\lambda - s) \right]_{\lambda=0}. \tag{E.17}$$

where $l = 1, 2, ..., N$. Therefore a_l can be calculated in an explicit way.

APPENDIX F

QUANTUM DYNAMICS

F.1. Dynamical Behavior of a Quantum System

To give time development of a quantum system, we postulate the existence of a hamiltonian H for the system require that the state vector for the system $\mid \psi(t) >$ change in accordance with the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t} \mid \psi(t) >= H \mid \psi(t) > \tag{F.1}$$

where H is to be treated as an observable of the system and must be therefore Hermitian.

Two cases may arise

(i) In the first case, the system is conservative and H is explicitly independent of time. In that case we may formally integrate $(F.1)$ and obtain

$$\mid \psi(t) >= U(t, t_0) \mid \psi(t_0) > \tag{F.2}$$

where

$$U(t, t_0) = exp[-\frac{iH(t - t_0)}{\hbar}] \tag{F.3}$$

and $\mid \psi(t_0) >$ is the state of the system at time t_0. This solution can be verified by differentiation and substitution back into $(F.1)$.

Since H is hermitian it follows from $(F.3)$ that

$$U^{+}(t, t_0) = exp[\frac{iH(t - t_0)}{\hbar}] = U^{-1}(t, t_0) \tag{F.4}$$

which shows that U is a unitary operator

Therefore it can be said that the state of the system at time t develops from the state at time t_0 by a unitary transformation.

(ii) In the second case, Hamilton explicitly contains time dependence, $H = H(t)$. This time unitary operator U is given by the next relation,

$$U(t, t_0) = exp\left[-i\int_{t_0}^{t} H(t^{'})dt^{'}/\hbar\right] \tag{F.5}$$

If we multiply $(F.1)$ on both sides dt and integrate from t_0 to t we get

$$\mid \psi(t) >= \mid \psi(t_0) > -\frac{i}{\hbar}\int_{t_0}^{t} H(t^{'}) \mid \psi(t^{'}) > dt^{'} \tag{F.6}$$

and repeating this process for $\mid \psi(t^{'}) >$ and then infinite number of time we get,

$$\begin{aligned} \mid \psi(t) > &= \left[-\frac{i}{\hbar}\int_{t_0}^{t} H(t^{'})dt^{'} + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} H(t^{'})dt^{'}\int_{t_0}^{t} H(t^{''})dt^{''} + ...\right] \mid \psi(t_0) > \\ &\equiv U(t,t_0) \mid \psi(t_0) > . \end{aligned}$$

F.1.1. The Schrödinger Picture

In Schrödinger picture the observables (p, q, H) are time independent. The state vector $\mid \psi_S(t) >$ (subscript S indicates that it is in Schrödinger picture)is function of time, this why time dependent. This description in which the basis vector are stationary and the dynamical state vector $\mid \psi_S(t) >$ moves is called Schrödinger picture of quantum mechanics.

F.1.2. The Heisenberg Picture

In Heisenberg picture the observables (p, q, H) are time dependent, the state vector $\mid \psi_H >$ is not depended on time (remains stationary).

The state vectors in the two pictures are related by the next equation

$$\mid \psi_S(t) >= U(t,t_0) \mid \psi_H(t_0) > \tag{F.7}$$

where subscript H indicates the Heisenberg picture. The vector $\mid \psi_H(t_0) >$ is stationary while $\mid \psi_S(t_0) >$ is moving.

Also an operator in Heisenberg picture is defined as

$$A_H(t) = U^{+}(t,t_0)A_S U(t,t_0) \tag{F.8}$$

If we differentiate both sides of $(F.8)$ with respect to t and and use U and its adjoint U^{+} in Schrödinger equation, respectively given as follows

$$i\hbar\frac{dU}{dt} = HU, \quad -i\hbar\frac{dU^t}{dt} = HU^{+},$$

also using $U^+U = UU^+ = 1$ we obtain

$$\begin{aligned} i\hbar\frac{dA_H}{dt} &= U^+A_SHU - U^+HA_SU + i\hbar U^+\frac{\partial A_H}{\partial t}U \\ &= U^+A_SUU^+HU - U^+HUU^+A_SU + i\hbar U^+\frac{\partial A_H}{\partial t}U \\ &\equiv [A_H, H_H] + i\hbar\frac{\partial A_H}{\partial t} \end{aligned} \tag{F.9}$$

where

$$H_H = U^+(t, t_0)H_SU(t, t_0).$$

The equation $(F.9)$ is called ***Heisenberg equation of motion*** for the observable A. If

$$\frac{dA_H}{dt} = 0,$$

then A_H is a constant of the motion.