SOLITONIC SOLUTIONS AND PARTICLE LOCALIZATION IN HIGHER DIMENSIONAL SPACES

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Solitonic Solutions and Particle Localization in Higher Dimensional Spaces

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ABSTRACT

First the basic extra dimensional theory (i.e. Kaluza-Klein theory) and the main aspects of solitons are reviewed. Then the current status of extra dimensional models and the application of solitons (especially kinks) in this context are studied. The extra dimensional models for fermion localization are considered with a particular emphasis. Finally a variation of these models which tries to incorporate the best aspects of various models for fermion localization is presented.

ÖZ

İlk aşamada ekstra boyut teorilerinin kaynağı olan Kaluza-Klein teorisi ve solitonların temel kavramları gözden geçirildi. Daha sonra ekstra boyut teorilerinin şu andaki durumu ve solitonların (özellikle de kinklerin) bu çerçeve içinde uygulamaları tartışıldı. Ekstra boyutlu fermiyon lokalizasyon modelleri özel bir önemle ele alındı. Son olarak değişik fermiyon lokalizasyon modellerinin önemli yönlerini birleştirmeye çalışan bir model sunuldu.

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Chapter 1

INTRODUCTION

The first rigorous model which increases the number of space-time dimensions from four to five was proposed by T. Kaluza [1] - [4] and O. Klein [5] in the early 1920's and this idea was based on a theory of unification of electromagnetism with gravitation. In this approach electromagnetism was considered as the force of gravity corresponding to the (fifth) extra dimension. This idea was extended to include all the other forces such as weak and strong interactions in early 60's and attracted increasing attention starting from early 1970's when it is recognized that string theories (the only unification scheme which can unify gravity with other forces) need more than four dimensions i.e., 10 in the case of superstring theory, 26 in the case of bosonic string theory [6, 7]. Starting from [8], the idea of extra dimensions received even more attention because it was shown that it is possible to consider extra dimensions of the order 100 μm without conflict with modern experimental data. In addition, the new studies on string theories showed that it is possible to construct string theories at these scales. This has lead to intense interest of phenomenologicists and theorists because this made it possible to detect extra dimensions in the next generation of experiments. Moreover it is shown that the use of extra dimensions can throw light on the problem of explaining fermion families and chirality.

In this study we review the current status of extra dimensional models and give particular emphasis on the explanation of right handedness and left handedness of fermion in the context of extra dimensional models by means of kink solutions.

In chapter 2 we begin with brief overview of Kaluza-Klein theory. After historical introduction to Kaluza-Klein theory we show that introducing a fifth dimension allows one to combine the general theory of relativity with the Maxwell's electromagnetism. In this context, gauge symmetry is explained as a geometric symmetry of space-time. We see that it is possible to extract also Brans-Dicke scalar field theory from Kaluza-Klein idea. Then, we indicate how these ideas can be used for quantization of electric charge and some estimation on

the scale of extra dimensions. Moreover, generalization of Kaluza-Klein theory to higher dimensions is briefly discussed.

In chapter 3, we review some of the standard facts on soliton theory. After giving a few aspects of the theory, we discuss the soliton solutions of some systems in nonlinear field theory in various dimensions.

Chapter 4 is devoted to the study of localization of fields in extra dimensions. It is explained how some solitons in nonlinear field theory discussed in chapter 3 motivates one to construct new extra dimensional models [9].

In chapter 5, we first briefly sketch the recent models on extra dimensions starting from ADD (Arkani-Hamed, Dimopoulos and Dvali) model. The main goal of these models is to explain some longstanding puzzles in high energy physics. We show that in 5.1, by proposing the standard model fields living on a brane embedded in a higher dimensional space-time while gravity allows the extra dimensions to be as large as 1μm and the hierarchy between the Planck scale and the weak scale is explained. It is shown that one may explain the origin of fermion masses by using overlap of fermion wave functions in extra dimensions [10].

In 5.2, alternative extra dimensional models called RS (Randall Sundrum Models), based on a curved space-time metric in contrast to the previous extra dimensional models, is discussed. In section 5.2.1 RS1 model [11] is treated and it is shown that it gives a solution to the hierarchy problem. In the second RS model [12], by using a noncompact extra dimension, it is shown that the massless graviton (zero mode) is localized on the brane whereas massive modes are nonlocalized with continuous spectrum. These massive modes are responsible for the corrections of Newton's law which does not contradict with experiments at distances $r \gg k^{-1}$ when the parameter k chosen to be $k \sim M_{pl}$.

In 5.3, we address the problem of fermion localization in RS models. First, we review domain wall solution obtained by [13] in the framework of RS model. Then, we consider a model [14] where the problem of fermion families and chirality is discussed and some progress has been made. We replace the complicated form of scalar potential by the usual ϕ^4 potential used in the study of [13] and repeat the similar procedure as done in [14] for this solution.

In this way, we get an improved model of fermion localization and chirality (i.e. the disparity between the right handed and left handed weak interactions).

Chapter 2

KALUZA-KLEIN THEORY

2.1 Five-dimensional Kaluza-Klein Theory (Unification of Gravity with Electromagnetism)

2.1.1 Historical Introduction to Kaluza-Klein Theory

When we achieve a true understanding of nature, even familiar ideas such as space and time, the number of dimensions in which we live will have to be derived. The reason why we seem to live in three space dimensions appeared surprisingly long time ago. The first scientists to publish on the question may have been Paul Ehrenfest. He realized in 1917 that the equations describing the motion of planets around the sun, and similar equations for electrons bound to nuclei in atoms, only had stable solutions if the dimension of space is three. However, the first rigorous idea of extra dimensions was probably stimulated by Minkowski, showing the time as an extra dimension.

In 1864 James Clerk Maxwell find the proper equations that govern the unified electromagnetic phenomena. Albert Einstein later realized in 1905 that Maxwell equations obey the principle of special relativity, that the laws of physics should be invariant to all observers who are in uniform relative motion. In special relativity, which treats time as a fourth dimension, time t and the space coordinates x, y, z are collectively denoted x^μ where the index μ runs over $0, 1, 2, 3$. The 0 refers to time and $1, 2, 3$ to the three space coordinates (x, y, z):

$$(x^0, x^1, x^2, x^3) = (ict, x, y, z)$$

Throughout this report, Greek indices $\alpha, \beta, ...$ run over $0, 1, 2, 3$, and capital Latin indices $A, B, ...$ run over $0, 1, 2, 3, 5$.

The successes of Maxwell's unified electromagnetic theory and Einstein's special relativity could be understood geometrically if time, along with space, were considered part of a four dimensional spacetime manifold via $x^0 = ict$. Maxwell's electromagnetic field also has four components collectively denoted

A_μ. The field depend on its position in spacetime and so is a function of the 4 spacetime coordinates x^μ.

In 1916, Einstein introduced the principle of general relativity, that the laws of physics should be same to all observers. In this theory, gravitational field can be represented by a second rank tensor, $g_{\mu\nu}$, which also has geometrical meaning - pseudo Riemannian metric, the quantity that describes the infinitesimal distance ds between two points in four dimensional spacetime.

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{2.1}$$

Note that the Minkowski geometry of flat spacetime must be replaced by the pseodo-Riemannian geometry of curved spacetime for which the metric tensor is itself a function of the spacetime coordinates.

Inspired by the close ties between Minkowski's four dimensional spacetime and Maxwell's unification of electricity and magnetism, Nordstöm [15] in 1914 and (independently) Theodor Kaluza [1] in 1921 were the first to try unifying gravity with electromagnetism in a theory of five dimensions. In other words,physics was to take place - for as-yet unknown reasons - on a four-dimensional hypersurface in a five-dimensional universe (Kaluza's cylinder condition). Kaluza was able to do by the ingenious device of postulating a fifth dimension with coordinate θ. The five coordinates are denoted collectively x^A where the index A runs over $0, 1, 2, 3, 5$.

$$(x^0, x^1, x^2, x^3, x^5) = (t, x, y, z, \theta)$$

He imagined a five dimensional pseudo Riemannian geometry with metric tensor $\hat{g}_{AB}(x)$ which describes the infinitesimal distance ds between two points in this five dimensional spacetime.

$$\mathrm{d}\hat{s}^2 = \hat{g}_{AB}(x)\,\mathrm{d}x^A \mathrm{d}x^B \tag{2.2}$$

Similar to Einstein's gravity theory, where distances depend locally on the riemannian metric identified with gravitational potential, he introduce extra dimension such that the generalized distance depends also on the electromagnetic potential . Although Kaluza's idea was attractive, it suffered from two obvious drawbacks. Firstly, there is no obvious reason the dependence of the fields on the extra coordinate θ. Secondly, if there is a fifth dimension why have not we seen it? The resolution of both these problems was supplied by Oskar Klein [1] in

[1]Not to be confused with Felix Klein, inventor of the Klein bottle.

1926 [5]. He show that Kaluza's cylinder condition would arise naturally if the fifth dimension has a circular topology so that the fifth coordinate θ is periodic, $0 \leq \theta \leq 2\pi$.

2.1.2 Effective 4-Dimensional Action

Five-dimensional Kaluza-Klein theory unifies electromagnetism with gravitation by using a theory of Einstein vacuum gravity in five dimensions. Thus the initial theory has five-dimensional general coordinate invariance. However, it is assumed that one of the spatial dimensions compactifies so as to have the geometry of a circle S^1 of very small radius. Then, there is a residual four-dimensional general coordinate invariance, and an Abelian gauge invariance associated with transformations of the coordinate of the compact manifold, S^1, as we will see in 2.1.3. Put another way, the original five-dimensional general coordinate invariance is spontaneously broken in the ground state. In this way, we arrive at an ordinary theory of gravity in four dimensions, together with a theory of an Abelian gauge field, with connections between the parameters of the two theories because they both derive the same initial five-dimensional Einstein gravity theory.

The ground-state metric after compactification is (We are not going to discuss the compactification mechanism here and for a brief review on compactification mechanisms, see [16])

$$\hat{g}^{(0)}_{AB} = \mathrm{diag}\{\eta_{\mu\nu}, -\tilde{g}_{55}\}.$$

Here

$$\eta_{\mu\nu} = (1, -1, -1, -1) \tag{2.3}$$

is the metric of Minkowski space, M_4, and

$$\tilde{g}_{55} = \tilde{R}^2 \tag{2.4}$$

is the metric of the compact manifold S^1, where $\tilde{R}$ is the radius of the compact dimension. The identification of the gauge field arises from an expansion of the metric about the ground state. Quite generally, the coordinates or gauge are chosen so as to write the 5-dimensional metric tensor in the form:

$$\hat{g}_{AB} = \begin{pmatrix} g_{\alpha\beta}(x,\theta) - B_\alpha(x,\theta)B_\beta(x,\theta)\Phi(x,\theta) & -B_\alpha(x,\theta)\Phi(x,\theta) \\ -B_\beta(x,\theta)\Phi(x,\theta) & -\Phi(x,\theta) \end{pmatrix} \tag{2.5}$$

To extract the graviton and the Abelian gauge field A_α it is necessary to replace $\Phi(x,\theta)$ by its ground-state value $\tilde{g}_{55}$, and to use the ansatz without θ dependence:

$$\hat{g}_{AB} = \begin{pmatrix} g_{\alpha\beta}(x) - \xi^2\, A_\alpha(x) A_\beta(x) \tilde{g}_{55} & \xi A_\alpha(x) \tilde{g}_{55} \\ \xi A_\beta(x) \tilde{g}_{55} & -\tilde{g}_{55} \end{pmatrix} \tag{2.6}$$

Here we have rescaled the electromagnetic potential A_α by a constant ξ in order to get the right multiplicative factors in the action later on.

It is well known that fundamental physical laws can be put in the form a principle of least action. For example, Einstein's equations are derivable from the following the Einstein-Hilbert action functional

$$I = \frac{-1}{16\pi G} \int \mathrm{d}^4x \sqrt{(-\det g)}\; R \tag{2.7}$$

where G is the 4-dimensional gravitational constant , g is the determinant of the 4-dimensional metric tensor and R is the 4-dimensional Ricci scalar. So 4-dimensional Einstein's equations in vacuum can be found by the variation of this action ($\delta I = 0$):

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0 \quad (\mu,\nu = 0,1,2,3)$$

where $G_{\mu\nu}$ is the Einstein tensor in 4-dimensions.

Original Kaluza-Klein idea can be put in the form of the variational principle, that is, 5-D vacuum action functional can be divided into two parts corresponding to 4-D Einstein vacuum gravity and Maxwell equations.[2] .

$$\bar{I} = \frac{-1}{16\pi \bar{G}} \int \mathrm{d}^5x \sqrt{(-\det \bar{g})}\; \bar{R} \tag{2.8}$$

where $\bar{G}$ is five dimensional gravitational constant , $\bar{R}$ - five dimensional scalar curvature defined as $\bar{R} = g^{AB} R_{AB}$, R_{AB} - five dimensional Ricci tensor.

Five dimensional Ricci tensor in terms of the 5D Christoffel symbols is given by,

$$R_{AB} = (\Gamma^C_{AB})_{,C} - (\Gamma^C_{AC})_{,B} + \Gamma^C_{AB}\Gamma^D_{CD} - \Gamma^C_{AD}\Gamma^D_{BC} \tag{2.9}$$

where

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD} \{ g_{DA,B} + g_{DB,A} - g_{AB,D} \}. \tag{2.10}$$

Here a comma denote the partial derivative

[2] Additional matter fields or supergravitational actions can also be considered.

$$(\Gamma^C_{AB})_{,C} = \frac{\partial \Gamma^C_{AB}}{\partial x_C}$$

$$g_{DA,B} = \frac{\partial g_{DA}}{\partial x_B}$$

Putting $A \to \alpha$, $B \to \beta$ in (2.9) gives us the 4D part of the quantity. Expanding some summed terms on the right hand side by letting $C \to \lambda\,, 5$ etc. and rearranging gives

$$\begin{aligned}\hat{R}_{\alpha\beta} = (\Gamma^\lambda_{\alpha\beta})_{,\lambda} + (\Gamma^5_{\alpha\beta})_{,5} - (\Gamma^\lambda_{\alpha\lambda})_{,\beta} - (\Gamma^5_{\alpha 5})_{,\beta} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} + \\ \Gamma^\lambda_{\alpha\beta}\Gamma^5_{\lambda 5} + \Gamma^5_{\alpha\beta}\Gamma^D_{5D} - \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\mu} - \Gamma^5_{\alpha\lambda}\Gamma^\lambda_{\beta 5} - \Gamma^D_{\alpha 5}\Gamma^5_{\beta D}\end{aligned} \tag{2.11}$$

Part of this is the conventional Ricci tensor that only depends on indices 0,1,2,3, so

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - (\Gamma^5_{\alpha 5})_{,\beta} + \Gamma^\lambda_{\alpha\beta}\Gamma^5_{\lambda 5} + \Gamma^5_{\alpha\beta}\Gamma^D_{5D} - \Gamma^5_{\alpha\lambda}\Gamma^\lambda_{\beta 5} - \Gamma^D_{\alpha 5}\Gamma^5_{\beta D} \tag{2.12}$$

To evaluate this we need the Christoffel symbols.

$$\begin{aligned}\hat{\Gamma}^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\beta\gamma} + \frac{\tilde{g}_{55}}{2}\xi^2(A_\beta F^\alpha_{\ \gamma} + A_\gamma F^\alpha_{\ \beta}) \\ \Gamma^\alpha_{55} &= 0 \\ \Gamma^5_{55} &= 0 \\ \Gamma^\alpha_{\beta 5} &= \frac{1}{2}\tilde{g}_{55}\xi F^\alpha_{\ \beta} \\ \Gamma^5_{5\beta} &= \frac{-1}{2}\tilde{g}_{55}\xi^2 A^\lambda F_{\lambda\beta} \\ \Gamma^5_{\alpha\beta} &= \frac{\xi}{2}(A_{\alpha;\beta} + A_{\beta;\alpha}) - \frac{\tilde{g}_{55}\xi^3}{2}[A^\lambda(A_\alpha F_{\lambda\beta} + A_\beta F_{\lambda\alpha})]\end{aligned} \tag{2.13}$$

where $A_{\alpha;\beta} = A_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta}A_\lambda$ and a semicolon denotes the ordinary 4D covariant derivative. With the help of (2.10) we have $\Gamma^D_{\alpha D} = \Gamma^\lambda_{\alpha\lambda}$ and $\Gamma^D_{4D} = 0$. Using these results, we can find the components of the 5D Ricci tensor:

$$\begin{aligned}\hat{R}_{\alpha\beta} &= R_{\alpha\beta} + \frac{\tilde{g}_{55}}{2}\xi^2 F_{\alpha\lambda}F_\beta^{\ \lambda} + \frac{\tilde{g}_{55}}{2}\xi^2(A_\alpha F^\gamma_{\ \beta;\gamma}) + \frac{\tilde{g}_{55}}{2}\xi^2(A_\beta F^\gamma_{\ \alpha;\gamma}) \\ &+ \frac{\tilde{g}^2_{55}}{4}\xi^3 A_\alpha A_\beta F^{\mu\nu}F_{\mu\nu} \\ R_{5\alpha} &= \frac{\tilde{g}_{55}}{2}\xi F^\lambda_{\ \alpha;\lambda} + \frac{\tilde{g}^2_{55}}{4}\xi^3 A_\alpha F^{\mu\nu}F_{\mu\nu} \\ R_{55} &= \frac{\tilde{g}^2_{55}}{4}\xi^2 F^{\mu\nu}F_{\mu\nu}\end{aligned} \tag{2.14}$$

And inverse of the metric tensor in (2.6) can be calculated as

$$\hat{g}^{AB} = \begin{pmatrix} g^{\alpha\beta} & -\xi A_\alpha \\ -\xi A_\beta & -\tilde{g}^{-1}_{55} + \xi^2 A^\mu A_\mu \end{pmatrix} \tag{2.15}$$

From definition, Ricci scalar is

$$\bar{R} = g^{AB} R_{AB}. \tag{2.16}$$

Substituting the components of the Ricci tensor into (2.16) we find

$$\bar{R} = R + \frac{\tilde{g}_{55}}{4} \xi^2 F^{\mu\nu} F_{\mu\nu} \tag{2.17}$$

As for Einstein-Hilbert action in 5D, it is easy to see that determinant of 5D metric tensor can be expressed by the 4D one by using some properties of determinants.

$$\det \bar{g} = -\tilde{g}_{55} \det g \tag{2.18}$$

Therefore, 5D action functional can be divided into two parts.

$$\bar{I} = \frac{-2\pi\tilde{R}}{16\pi\bar{G}} \int \mathrm{d}^4x \sqrt{-\det g}\, R - \frac{\xi^2 \tilde{g}_{55}}{16\pi\bar{G}} \int \mathrm{d}^4x \sqrt{-\det g}\, F^{\mu\nu} F_{\mu\nu} \tag{2.19}$$

The four dimensional gravitational constant G is identified as

$$G = \bar{G}/2\pi\tilde{R}$$

and to obtain standard normalization for the Gauge field we must then choose

$$\xi^2 = \frac{16\pi G}{\tilde{g}_{55}} = \frac{\kappa^2}{\tilde{R}^2} \tag{2.20}$$

where

$$\kappa^2 = 16\pi G \tag{2.21}$$

Then the effective four-dimensional action is

$$I = [\underbrace{\frac{-1}{16\pi G} \int \mathrm{d}^4x \sqrt{-\det g}\, R}_{4D\ Einstein's Action}] - [\underbrace{\frac{1}{4} \int \mathrm{d}^4x \sqrt{-\det g}\, F^{\mu\nu} F_{\mu\nu}}_{Maxwell's Action}] \tag{2.22}$$

2.1.3 Abelian Gauge Symmetry and Extra Dimensions

The key idea how extra dimensions lead to unification of electomagnetism with gravity lies in the fact that coordinate transformations associated with the coordinate θ of the compact manifold can be interpreted as gauge transformations, as we now show. Consider the following transformation

$$\theta \to \theta' = \theta + \xi\varepsilon(x) \tag{2.23}$$

For a general coordinate transformations, the metric transforms as a second rank tensor:

$$g_{AB} = g_{A'B'} \frac{\partial x^{A'}}{\partial x^A} \frac{\partial x^{B'}}{\partial x^B} \tag{2.24}$$

For the particular transformation (2.23), the off-diagonal elements of the metric become

$$g_{5\mu} = -B_\mu \tilde{g}_{55} = \xi A_\mu \tilde{g}_{55}$$
$$\xi A_\mu \tilde{g}_{55} = g'_{5\mu} \frac{\partial x^{\mu'}}{\partial x^\mu} + g'_{55} \frac{\partial x^{5'}}{\partial x^\mu}$$

$$\xi A_\mu \tilde{g}_{55} = \xi A'_\mu \tilde{g}_{55} - \tilde{g}_{55} \xi \, \partial_\mu \varepsilon(x)$$

Thus the transformation (2.23) of the coordinates of the compact manifold induces an Abelian gauge transformation on A_μ

$$A'_\mu = A_\mu + \partial_\mu \varepsilon \tag{2.25}$$

This means that the compact manifold is providing the internal symmetry space for the Abelian gauge group, and internal symmetry has now to be interpreted as just another space-time symmetry, but associated with the extra spatial dimension.

2.1.4 Conformal Rescaling and Brans-Dicke Theory

Starting from the general parametrisation of the metric (2.5), it is also possible to extract a massless scalar field. It means that if $\Phi \neq$ constant, then Kaluza's five dimensional theory contains besides electromagnetic effects a Brans-Dicke-type scalar-tensor field theory [17]. This becomes clear when one considers the case in which the electromagnetic potentials vanish, A_μ=0 (see [18, 19]). Without the cylinder condition, this would be no more than a choice of coordinates, and would not entail any loss of algebraic generality. (It would be exactly analogous to the common procedure in ordinary electrodynamics of choosing coordinates in which either the electric or magnetic field disappears.) With the cylinder condition, however, we are effectively working in a special set of coordinates, so that the theory is no longer invariant with respect to general five-dimensional coordinate transformations. The restriction A_μ=0 is, therefore, a physical and not a merely mathematical one, and restricts us to the graviton-scalar sector of the theory. This is acceptable in some contexts - in homogenous and isotropic situations, for example, where off-diagonal metric coefficients would pick out preferred directions; or in early-universe models which are dynamically dominated by scalar fields. In the absence of the A_μ-fields, then

$$\hat{g}^{AB} = \begin{pmatrix} g^{\alpha\beta} & 0 \\ 0 & -\Phi(x) \end{pmatrix} \tag{2.26}$$

Substituting this metric into the Einstein-Hilbert action for five-dimensions, we get the effective four-dimensional action

$$I = \frac{-2\pi}{16\pi\bar{G}} \int \mathrm{d}^4x \sqrt{-\det g}\ (\Phi)^{1/2} R. \tag{2.27}$$

The $(\Phi)^{1/2}$ multiplying the four-dimensional curvature scalar may be removed by a Weyl scaling,

$$\begin{aligned} g_{\mu\nu} &\to g_{\mu\nu}\ (\Phi)^{-1/3} \\ \Phi &\to (\Phi)^{2/3}. \end{aligned} \tag{2.28}$$

Here, five-dimensional metric tensor is rescaled by the same factor as the four-dimensional one. Then

$$I = \frac{-1}{16\pi G} \int \mathrm{d}^4x \sqrt{-\det g}\ \left[R - \frac{1}{6} (\Phi)^{-2} \partial^\mu \Phi \partial_\mu \Phi \right] \tag{2.29}$$

or with further change of variables(in terms of dilaton field)

$$\sigma = \frac{\kappa^{-1}}{\sqrt{3}} \ln(\Phi) \tag{2.30}$$

we have

$$I = \frac{-1}{16\pi G} \left(\int \mathrm{d}^4x\ \sqrt{-\det g}\ R + \frac{1}{2} \int \mathrm{d}^4x\ \sqrt{-\det g}\ \partial^\mu \sigma \partial_\mu \sigma \right) \tag{2.31}$$

This is a particular case of four-dimensional Brans-Dicke theory.

2.1.5 Geometrical Quantization of Electric Charge

Since fifth coordinate has a circular topology S^1, all fields can be Fourier expanded. (Any quantity $f(x,\theta)$, where $x = (x_0, x_1, x_2, x_3)$ becomes periodic; $f(x,\theta) = f(x,\theta + 2\pi)$)

$$g_{\alpha\beta}(x,\theta) = \sum_{n=-\infty}^{\infty} g_{\alpha\beta}^{(n)}(x)\ e^{in\theta} \tag{2.32}$$

$$A_\alpha(x,\theta) = \sum_{n=-\infty}^{\infty} A_\alpha^{(n)}(x)\ e^{in\theta} \tag{2.33}$$

$$\phi(x,\theta) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x)\ e^{in\theta} \tag{2.34}$$

where the superscript (n) refers to the n th Fourier mode. The expansion of the fields into Fourier modes suggests a possible mechanism to explain charge quantization.

The simplest kind of matter is a massless five-dimensional scalar field ϕ. Equations of motion corresponding the massless scalar ϕ (Klein-Gordon),

$$\left[\Box_x - \tilde{R}^{-2}\frac{\partial^2}{\partial\theta^2}\right]\phi(x,\theta) = 0 \tag{2.35}$$

where $\Box_x$ is four-dimensional d'Alembertian operator. Then, this gives the equations for the Fourier components:

$$\left[\Box_x + m_n^2\right]\phi^{(n)}(x) = 0 \tag{2.36}$$

where

$$m_n^2 = \frac{n^2}{\tilde{R}^2}. \tag{2.37}$$

So a massless state in the higher dimensional theory show up in the lower dimensional theory as a tower of equally spaced massive states. Here the fields $\phi^{(n)}(x)$ are the mass eigenstates in four dimensions, and the field $\phi^{(0)}(x)$ is the only massless one. The other fields $\phi^{(n)}(x)$ have masses of order $\tilde{R}^{-1}$, which we would expect to be comparable to the Planck mass. These modes carry a momentum in the extra dimension and if the radius of the extra dimension is small enough, then the momenta in extra dimension of the n=1 modes will be so large as to put them beyond the reach of the experiment. Hence only the n=0 modes, which are independent of extra coordinate, will be observable, as required in Kaluza's theory because the Kaluza-Klein ansatz amounts to simply dropping the θ dependency of $g_{\alpha\beta}$, A_α, and ϕ, giving the effective four-dimensional " low-energy " theory of the graviton $g^{(0)}_{\alpha\beta}$, photon $A^{(0)}_\alpha$ and a scalar ϕ^0. If we apply the coordinate transformation

$$\theta \to \theta' = \theta + \xi\varepsilon(x) \tag{2.38}$$

to the field $\phi(x,\theta)$ of (2.34), we have

$$\phi^{(n)}(x) \to \exp[in\xi\varepsilon(x)]\phi^{(n)}(x) \tag{2.39}$$

Since the Abelian gauge field transforms as in (2.25)(Compare with the minimal coupling rule $\partial_\alpha \to \partial_\alpha + ieA_\alpha$ of quantum electrodynamics), this means that $\phi^{(n)}(x)$ has charge

$$q = -n\xi = -n\frac{\kappa}{\tilde{R}} \tag{2.40}$$

where we have used the normalization condition (2.20). Thus charge is quantized in units of $\kappa/\tilde{R}$. The radius of the compact manifold may now be estimated from

$$\tilde{R}^2 = \frac{\kappa^2}{e^2} = \frac{4G}{(e^2/4\pi)} \tag{2.41}$$

Thus, identifying e with the quantum of electric charge

$$\tilde{R} \approx 10^{-33}\,\mathrm{cm}$$

As we have seen, a characteristic feature of Kaluza-Klein theories is the appearance of the infinite set of KK modes (called the KK tower of modes) and upon compactification of the additional dimensions, all fields which propagate in the bulk are Fourier expanded into a complete set of modes-(KK) tower of states, with mode numbers n labeling the KK excitations. Similar to a particle in a box, the momentum of the bulk field is quantized in the compactified dimensions, given by $p^2 = n^2/\tilde{R}^2$. From the 4d perspective of an observer, each allowed momentum in the compactified volume appears as a KK excitation of the bulk field with mass given in (2.37) . This builds a KK tower of states where each KK excitation carries identical spin and gauge quantum numbers. Kaluza-Klein states are a generic feature of models with compactified dimensions. The above assumes that all additional dimensions are of the same size and are flat. In more complicated compactifications, the Fourier expansion must be generalized, and the mass formula no longer takes on the above simple form. Correspondingly, a characteristic signature of the existence of extra dimensions would be detection of series of KK excitations with a spectrum of the form (2.37). So far no evidence of such excitations has been observed in high energy experiments. The bound on the size $\tilde{R}$, derived from the absence of signals of KK excitations of the particles of the Standard Model in the available experimental data, is

$$m \sim \frac{1}{R} \gtrsim 1\ \mathrm{TeV}.$$

2.2 (4+D) Dimensional Kaluza-Klein Theory (Unification of Gravity with Non-Abelian Gauge Fields)

2.2.1 Introduction to Higher Dimensional Kaluza-Klein Theory

Various modifications of Kaluza's five-dimensional scheme, including Klein's idea of compactifying the extra dimension were suggested by Einstein, Jordan, Bergmann, and a few others over the years, but it was not extended to more than five dimensions until theories of the strong and weak nuclear interactions were developed. The obvious question was whether these new forces could be unified with gravity and electromagnetism by the same method. The key to extending the Kaluza-Klein formalism to strong and weak nuclear interactions lies in

recognizing that electromagnetism has been effectively incorporated into general relativity by adding $U(1)$ local gauge invariance to the theory, in the form of local coordinate invariance with respect to θ, as shown in 2.1.3. To extend the same approach to more complicated symmetry groups, one goes to higher dimensions [20], [21], [22], [23], and [24]. In the five-dimensional case, an abelian gauge group arose from the coordinate transformation

$$\theta \to \theta' = \theta + \xi\varepsilon(x) \tag{2.42}$$

on the single coordinate θ of the compact manifold. In the (4+D)-dimensional case we must look for symmetries of the compact manifold which generalize (2.42). The appropriate transformations to study are the isometries of the manifold.

Isometry Group of a Manifold

Let us denote the coordinates of ordinary four-dimensional space by x^μ, and the coordinates of the compact manifold K by y^n. An isometry of K is a coordinate transformation $y \to y'$ which leaves the form of the metric $\tilde{g}_{mn}$ for K invariant:

$$y \to y' \ : \qquad \tilde{g}'_{mn}(y') = \tilde{g}_{mn}(y') \tag{2.43}$$

Isometries form a group, with generators t_a and structure constants C_{abc}, in the following way. The general infinitesimal isometry is

$$I + i\,\varepsilon^a\,t_a \ : \qquad y^{n'} = y^n + \varepsilon^a\,\xi^n_a(y) \tag{2.44}$$

where the infinitesimal parameters ε^a are independent of y, and Killing vectors ξ^n_a, which are associated with the independent infinitesimal isometries, obey the algebra:

$$\xi^m_b\,\partial_m\xi^n_c - \xi^m_c\,\partial_m\xi^n_b = -C_{abc}\,\xi^n_a \tag{2.45}$$

Correspondingly, by considering the commutator of two infinitesimal isometries, we can show that

$$[t_a, t_b] = i\,C_{abc}\,t_c. \tag{2.46}$$

For instance, the N-dimensional sphere S^N has isometry group $SO(N+1)$, and the 2N(real) - dimensional complex projective plane CP^N has isometry group $SU(N+1)$. The isometry group for the compact manifold S^1 of the five-dimensional theory is just the $SO(2)$ (or $U(1)$)group of transformations of (2.42). As we shall discuss later, it is possible to choose the compact manifold to obtain the isometry group $SU(3) \times SU(2) \times U(1)$, which is the (observed)gauge group of electroweak and strong interactions.

2.2.2 Effective 4-Dimensional Action

The ground state metric for the compactified (4+D)-dimensional theory may be written as

$$\hat{g}^{(0)}_{AB} = \mathrm{diag}\{\eta_{\mu\nu}, -\tilde{g}_{mn}(y)\} \tag{2.47}$$

where $\eta_{\mu\nu}$ is the metric of Minkowski space M_4 as in 2.1.2, and $\tilde{g}_{mn}(y)$ is the metric of the compact manifold. The non-Abelian gauge fields of the theory may be displayed by the expansion about the ground state

$$\hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} - \tilde{g}_{mn}(y)\,\xi^m_a(y)\,A^a_\mu\,\xi^n_a(y)\,A^a_\nu & \xi^n_a(y)\,A^a_\mu \\ \xi^m_a(y)\,A^a_\nu & -\tilde{g}_{mn}(y) \end{pmatrix} \tag{2.48}$$

The action for Einstein gravity in (4+D) dimensions is

$$\bar{I} = \frac{-1}{16\pi\bar{G}} \int \mathrm{d}^{4+D}x\,\sqrt{-\det\bar{g}}\,\bar{R} \tag{2.49}$$

where $\bar{R}$ is the (4+D)-dimensional curvature scalar, and $\bar{G}$ is the gravitational constant for (4+D)-dimensions. Substituting the ansatz (2.48) for $\hat{g}_{AB}$, and integrating over the compact degrees of freedom y gives an effective four-dimensional action

$$\begin{aligned} I &= -(\int \mathrm{d}^D y\,\sqrt{-\det\tilde{g}})\,(16\pi\bar{G})^{-1}\int \mathrm{d}^4x\,\sqrt{-\det g}\,R \qquad (2.50)\\ &\quad -(\int \mathrm{d}^D y\,\sqrt{-\det\tilde{g}}\,\xi^m_a(y)\xi^n_b(y)\,\tilde{g}_{mn}(y)) \\ &\quad \times(16\pi\bar{G})^{-1}\frac{1}{4}\int \mathrm{d}^4x\,\sqrt{-\det g}\,F^a_{\mu\nu}(F^{\mu\nu})^b \qquad (2.51) \end{aligned}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\nu - C_{abc}\,A^b_\mu A^c_\nu \tag{2.52}$$

and R denoting the four-dimensional curvature scalar. The four-dimensional gravitational constant G is thus identified by

$$\kappa^{-2} = (16\pi G)^{-1}\int \mathrm{d}^D y\,|\det\tilde{g}|^{1/2}$$

and standard normalization of the gauge fields requires the Killing vectors to be scaled so that

$$<\xi^m_a\xi^n_b\tilde{g}_{mn}>=\kappa^2\delta_{ab} \tag{2.53}$$

where we have introduced the notation (of Weinberg 1983)

$$<f(y)>=\frac{\int \mathrm{d}^D y|\det\tilde{g}|^{1/2}f(y)}{\int \mathrm{d}^D y|\det\tilde{g}|^{1/2}} \tag{2.54}$$

Then we have the standard action for Einstein gravity plus non-Abelian gauge fields in four-dimensions:

$$I = -(16\pi G)^{-1} \int \mathrm{d}^4x \,|\det g|^{1/2} - \frac{1}{4} \int \mathrm{d}^4x \,|\det g|^{1/2} F^a_{\mu\nu} (F^{\mu\nu})^a \tag{2.55}$$

2.2.3 Non-Abelian Gauge Symmetry and Extra Dimensions

Non-Abelian gauge transformations arise by considering the effect on the components $\hat{g}_{\mu\nu}$ of the metric of the infinitesimal isometry with x-dependent parameters:

$$y^n \rightarrow \; y^n + \xi^n_a(y)\, \varepsilon^a(x) \tag{2.56}$$

We then find

$$A^a_\mu \rightarrow \; A^{a'}_\mu = A^a_\mu + \partial_\mu \varepsilon^a(x) + C_{abc}\, \varepsilon^b A^c_\mu \tag{2.57}$$

which is just the usual Yang-Mills transformation if we display the gauge coupling constant g explicitly by writing

$$C_{abc} = g f_{abc} \tag{2.58}$$

and

$$t_a = g\, T_a \tag{2.59}$$

so that

$$[T_a, T_b] = i f_{abc} T_c \tag{2.60}$$

Thus, non-Abelian gauge transformations are generated by x-dependent infinitesimal isometries of the compact manifold K.

2.3 Modern Kaluza-Klein Theory

Many of the major developments in fundamental physics of the past century arose from identifying and overcoming contradictions between existing ideas. For example incompatibility of Maxwell equations and Galilean invariance led to Einstein to propose the special theory of relativity. Similarly, the inconsistency of special relativity with Newtonian gravity led him to develop the general theory of relativity. We are now facing another crisis of the same character. Namely, general relativity appears to be incompatible with quantum field theory. Any straightforward attempt to quantize general relativity leads to a nonrenormalizable theory. This means that the theory is inconsistent and needs to be modified at short distances (i.e. high energies). The way that string theory does this is

to give up one of the basic assumptions of quantum field theory, the assumption that elementary particles are mathematical points, and instead to develop a quantum field theory of one-dimensional extended objects, called strings. There are few consistent theory of this type, but superstring theory shows great promise as a unified quantum theory of all fundamental forces including gravity. For our point of view, the most important point is that string theory requires extra dimensions, in other words, string theory lives in extra dimensions. For example, bosonic string theory is described in 26 spacetime dimensions. One can construct string theories using the idea of compactification and this is the reason how Kaluza-Klein theories become important nowadays. This idea amounts to take 26 dimensional spacetime as the product of a (25-d) dimensional compact manifold $\mathcal{M}$ with euclidean signature and a (d+1) dimensional Minkowski space $R^{d,1}$ like Kaluza-Klein. Then in the limit when the size of the compact manifold is sufficiently small so that the present day experiments can not resolve this distance. The world will effectively appear to be (d+1) dimensional. Choosing d=3 will give us a (3+1) dimensional theory. Of course we can not choose arbitrary manifold $\mathcal{M}$ for this purpose; It must satisfy the equations of motion of the effective field theory that comes out of the string theory. There are many known examples of manifolds satisfying these restrictions e.g. tori of different dimensions, Calabi Yau manifolds, etc. The simplest class of compact manifolds are tori, i.e. product of circles. The effect of this compactification is to periodically identify some of the bosonic fields in the string world-sheet theory[3] - the fields which represent coordinates tangential to the compact circles. One effect of this is that the momentum carried by any string state along any of these circles is quantized in units of $1/R$, where R is the radius of the circle. But there is another important effect: we now have new states that correspond to strings wrapped around a compact circle. For such states, as we go once around the string, we also go once around the compact circle. These states are known as winding states or string solitons and play a crucial role in the anaysis of duality symmetries.

[3]As a string evolves in time it sweeps out a two dimensional surface in spacetime, which is called the world sheet of the string.

Chapter 3

SOLITONS IN NONLINEAR FIELD THEORY

3.1 Why Solitons ?

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped, a large solitary elevation, a rounded, smooth and well-defined heap of water....I followed it on horseback, and overtook it still rolling on a rate of eight or nine miles an hour, preserving its original figure...after a chase of one or two times. I lost it in the windings of the channel.*
>
> J. Scott Russel, 1834

The physics of 20th century, which was initiated by Maxwell's completion of the theory of electromagnetism, can, with some justification, be called the era of *linear* physics. With few exceptions, the methods of theoretical physics have been dominated by linear equations(Maxwell, Schrödinger), linear mathematical objects(vector spaces, in particular Hilbert spaces), and linear methods(Fourier transforms, perturbation theory, linear response theory).

Beginning with the Navier-Stokes equations, the importance of nonlinearity naturally was recognized in the Einstein's gravitational theory and the interactions of particles in solids, nuclei, and quantized fields. However, it was hardly possible to treat the effects of nonlinearity exactly, except as a perturbation to the basic solution of the linearized theory.

During the last decade, it has become more widely recognized in many areas of " field theory " that nonlinearity can result in qualitatively new phenomena which can not be constructed via perturbation theory starting from linearized equations. By " field theory " we mean all those areas of theoretical physics for which the description of physical phenomena leads one to consider field equations, or partial differential equations of the form

$$\phi_t \quad or \quad \phi_{tt} = F(\phi, \phi_x, ...) \tag{3.1}$$

for one- or many-component " fields " $\phi(t, x, y, ...)$ (or their quantum analogs). These include classical areas, such as hydro- and magnetohydrodynamics, some areas of meteorology, oceanography, and plasma physics, as well as elementary particle physics, solid state physics, and nonlinear optics. Nonlinear field equations arising in those areas admit new type of solitary wave solutions or solitons, playing crucial rule in all above mentioned fields. The reason why we are interested in solitons lies in the fact that it plays very important role in higher dimensional field theories, for example, in magnetic monopole solutions, in string theory, and localization of particles in higher dimensions.

3.2 Solitary Waves and Solitons

Solitary waves and solitons are defined as certain special solutions of nonlinear wave equations [25]. In order to fully appreciate special solutions, we recall properties of the simplest relativistic wave equations, namely

$$\begin{aligned} \Box\phi &= \partial^\mu \partial_\mu \phi \\ &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi(x,t) = 0 \end{aligned} \tag{3.2}$$

where ϕ is a real scalar field in (1+1)-dimensions, and, c is the velocity of light. As it is well known, solution to this equation is both linear and dispersionless. As a result it has two features of relevance to our discussion.

(i) Any real well-behaved function of the form $f(x \pm ct)$ is a solution of Eq.(3.2). In particular, if we choose a localized function f, we can construct a localized wave packet

$$f(x - ct) = \int \mathrm{d}k \, (a_1(k)\cos(kx - \omega t) + a_2(k)\sin(kx - \omega t)) \tag{3.3}$$

that will travel with uniform velocity $\pm c$ and without distortion of its shape. The fact that the wave packet $f(x - ct)$ travels undistorted with velocity c lies in the fact that all its plane-wave components have the same phase velocity $\omega/k = c$.

(ii) Since the wave equation is linear, superposition principle suggests us that if it is given two localized wave packet solutions $f_1(x - ct)$ and $f_2(x + ct)$, their sum $f_3(x, t) = f_1(x - ct) + f_2(x + ct)$ is also a solution. At large negative time $f_3(x, t)$ consists of the two packets widely separated and approaching each other essentially undistorted and after a finite time, they will collide. After collision they will asymptotically separate into the same two packets retaining their original shapes and velocities. For the system (3.2), this property holds for more than two packets as well.

These two features clearly hold for Eq.(3.2) since that particularly simple system is both linear and dispersionless. However, typical wave equations in many branches of physics are much more complicated: They can contain non-linear terms, dispersive terms, and several coupled wave fields with more than one space-dimensionality. Now one can ask if such equations admit solutions with the properties (i) and even (ii) despite of their complexity compared with (3.2)? Answer is affirmative as we will show.

Note that adding the simplest kinds of terms to (3.2) tends to destroy these nice features, even in (1+1) dimensions. Consider for example the Klein-Gordon equation in two dimensions,

$$(\Box + m^2c^2)\phi(x,t) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2c^2\right)\phi(x,t) = 0 \tag{3.4}$$

This equation is still linear and plane waves $\cos(kx \pm \omega t)$ and $\sin(kx \pm \omega t)$ still form a complete set of solutions. But now $\omega^2 = k^2c^2 + m^2c^4$ and so the equation is dispersive i.e., different wavelengths travel at different velocities $\omega(k)/k$. Any localized wave packet having, at $t = 0$, the form

$$\int \mathrm{d}k \;\; (a_1(k)\cos(kx) + a_2(k)\sin(kx)) \tag{3.5}$$

will spread as time goes on. Thus, feature (i) is lost, and so is the feature (ii). Similarly, consider adding a simple non-linear term to Eq.(3.2) as in

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(x,t) + \phi^3(x,t) = 0. \tag{3.6}$$

Not all solutions of this equation are known, but one can persuade oneself through numerical or approximate calculations that an arbitrary wave packet will spread.

It is however possible that for some equations where both dispersive and non-linear terms are present, their effects might balance each other in such a way that some special solutions essentially admits feature (i). This can happen in one, two or three space dimensions, and such solutions are called solitary waves. In a small subset of these cases when feature (ii) is also exhibited, those solutions are called solitons.

Our definition of solitons will be in terms of the energy density rather than the wave-fields themselves. This means that we are restricting ourselves to those field equations that have an associated energy density $\varepsilon(\mathbf{x}, t)$ which is some function of the fields $\phi_i(x, t)$. Its space integral is the conserved total energy functional $E[\phi_i]$. A large class of equations, including field equations in particle

physics, satisfy this property. For physical systems having an energy bounded from below we can without loss of generality set the minimal value reached by E as equal to zero. Given this framework, we shall use the adjective "localized" for those solutions to the field equation whose energy density $\varepsilon(\mathbf{x}, t)$ at any finite time t is localized in space, i.e. it is finite in some bounded region of space and falls to zero at spatial infinity sufficiently fast as to be integrable. Note that for those systems where $E[\phi_i] = 0$ if and only if $\phi_i(x, t) = 0$, a localized solutions as defined above also has the fields $\phi_i(x, t)$ themselves localized in space. For instance, Eq.(3.6) has an associated conserved energy given by

$$E[\phi] = \int_{-\infty}^{\infty} \mathrm{d}x \ \left[\frac{1}{2c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{4} \phi^4 \right] \tag{3.7}$$

which is minimized by $\phi(x, t) = 0$. Localized solutions of this system, if any, would asymptotically go to $\phi(x, t) = 0$ as $x \to \pm\infty$, at any given time t. The derivatives $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial t}$ must also vanish in this limit. By contrast, the equation

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \phi + \phi^3 = 0 \tag{3.8}$$

has an associated energy

$$E[\phi] = \int_{-\infty}^{\infty} \mathrm{d}x \ \left[\frac{1}{2c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{4} \left(\phi^2 - 1 \right)^2 \right]. \tag{3.9}$$

where $E[\phi]$ is minimized by $\phi(x, t) = \pm 1$ and this is one of the simplest example of "spontaneous symmetry breaking" in the quantized version of field systems. Now a localized solution must approach $\phi = \pm 1$ as $x \to \pm\infty$ at any instant.

Given localization in the sense of energy density, a solitary wave is defined as localized non-singular solution of non-linear field equation (or coupled equations, when several fields are involved) whose energy density, as well as being localized, has a space-time dependence of the form (travelling wave)

$$\varepsilon(\mathbf{x}, t) = \varepsilon(\mathbf{x} - \mathbf{u}t) \tag{3.10}$$

where $\mathbf{u}$ is some velocity vector. In other words, the energy density should move undistorted with constant velocity. Among systems that do have an associated energy density this definition permits a larger class of solutions than the one given by Scott[26] who require that the fields themselves have such a travelling wave space-time dependence.

Note that Eq.(3.10) defines solitary waves in one or more space dimensions. Moreover, any static (time-independent) localized solution is automatically a solitary wave, with the velocity $\mathbf{u} = 0$. Many of the solitary waves which we will

discuss will be obtained as static solutions. However, for systems with relativistic (or Galilean) invariance, the moving single solutions is trivially obtained by boosting the static solution, i.e. transforming to a moving coordinate frame.

We now turn to solitons: These are solitary waves with an added requirement given below, which is a somewhat generalized and precisely stated version of feature (ii). Consider some (possibly coupled) non-linear equation(s). Let them have a solitary wave solution whose energy density is some localized function $\varepsilon_0(\mathbf{x}-\mathbf{u}t)$. Consider any other solution of this system which in the far past consists of N such solitary waves, with arbitrary initial velocities and positions. Then, the energy density $\varepsilon(\mathbf{x},t)$ of this solution will have the following form

$$\varepsilon(\mathbf{x},t) \to \sum_{i=1}^{N} \varepsilon_0(\mathbf{x}-\mathbf{a}_i-\mathbf{u}_i t), \qquad as \quad t \to -\infty \tag{3.11}$$

Given this configuration at $t=-\infty$, it will then evolve in time as governed by the non-linear evolution equations. Suppose this evolution is such that

$$\varepsilon(\mathbf{x},t) \to \sum_{i=1}^{N} \varepsilon_0(\mathbf{x}-\mathbf{a}_i-\mathbf{u}_i t+\delta_i) \qquad as \ t \to +\infty \tag{3.12}$$

where δ_i are some constant vectors. Then such a solitary wave is called a soliton. In other words, solitons are those solitary waves whose energy density profiles are asymptotically (as $t \to \infty$) restored to their original shapes and velocities. The vectors δ_i represent the possibility that the solitons may suffer a bodily displacement compared with their pre-collision trajectories. This displacement should be the sole residual effect of collisions if they are to be solitons. Obviously this is a remarkable property for solutions of a non-linear field equation to have.

While all solitons are solitary waves, the converse is clearly not true. In order to find a solitary-wave solution to a given non-linear equation, we only need to look for one localized solution satisfying Eq.(3.10). This is often hard enough to do, but several equations have yielded solitary waves by now. In contrast, to ensure that a solution is a soliton we must find not merely that solution, but infinitely many time-dependent solutions consisting of arbitrary numbers of solitons, and check that Eq.(3.11) and Eq.(3.12) are satisfied. Thus, it is very hard to tell, given a non-linear wave equation, whether it even permits soliton solutions, letting alone evaluate them explicitly. Of course a large body of powerful techniques has been developed for solving soliton-bearing equations and studying their properties. These include the inverse scattering method, Bäcklund transformations, the use of conserved quantities, Hirota bilinear method etc. While these

techniques offer elegant ways of solving and understanding such systems, they are not as yet very helpful towards identifying new equations carrying solitons, or in deciding whether a given equation has this property. Not surprisingly then, very few soliton-bearing equations have been found.

3.3 Some Solitary Waves in 2 Dimensions : Kinks

We shall now present some examples of solitary waves, beginning with the simplest one. As mentioned earlier, any localized static (time-independent) solution is a solitary wave. We shall therefore concentrate in this section on static solutions in the simplest context where they occur, namely scalar fields in two (one space+one time) dimensions. Consider first a single scalar field $\phi(x,t)$ whose dynamics is governed by the Lorentz-invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - \mathrm{U}(\phi) \tag{3.13}$$

where henceforth a dot or a prime represents differentiation with respect to time or the space variable x, respectively, and the velocity of light c is set equal to one. The potential $U(\phi)$ is any positive semi-definite function of ϕ, reaching a minimum value of zero for some value or values of ϕ. When the variational action principle

$$\delta\left[\int \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\ \mathcal{L}(\mathrm{x},\mathrm{t})\right] = 0 \tag{3.14}$$

is applied to this Lagrangian, one obtains the wave equation

$$\Box\phi = \ddot{\phi} - \phi'' = -\frac{\partial U}{\partial \phi} \tag{3.15}$$

whose non-linear terms depend on the choice of $U(\phi)$. The equation conserves the total energy functional E given by

$$E[\phi] = \int_{-\infty}^{\infty} \mathrm{d}x\ \left[\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + U(\phi)\right] \tag{3.16}$$

Let the absolute minima of $U(\phi)$ occur at M points $M \geq 1$, which are also its zeros. That is, let

$$U(\phi) = 0 \qquad for \qquad \phi = g^{(i)}\ ;\ \ i = 1, ..., M \tag{3.17}$$

Then the energy functional is also minimized when the field $\phi(x,t)$ is constant in space-time and takes any one of these values. That is,

$$E[\phi] = 0$$

if and only if

$$\phi(x,t) = g^{(i)} \ ; i = 1, ..., M \tag{3.18}$$

Now, we are interested in static solutions, for which Eq.(3.15) reduces to

$$\phi''(x) = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial U}{\partial \phi} \tag{3.19}$$

Further, a solitary wave must have finite energy and the localized energy density. In view of Eq.(3.17), its field must approach one of the values $g^{(i)}$, as $x \to \pm\infty$. If the $U(\phi)$ has a unique minimum at $\phi = g$, then our static solution must reach $\phi(x) \to g$ as $x \to \pm\infty$. If there are several degenerate minima, then ϕ must tend to any one of the $g^{(i)}$ as $x \to -\infty$, and either the same or any other of the $g^{(i)}$ as $x \to \infty$.

Subject to these boundary conditions, we solve Eq.(3.19) for $\phi(x)$. Since this is an ordinary second-order differential equation, it can easily be solved by quadrature for any $U(\phi)$. Before doing that, it will be useful to notice that Eq.(3.19) has a mechanical analogue. Such mechanical analogues to static solutions have been pointed out by several people (see for example Coleman [27]). If we think of the variable x as time and ϕ as the coordinate of a unit-mass point particle, then Eq.(3.19) is just Newton's second law for this particle's motion in a potential given by $[-U(\phi)]$. The solution ϕ represents the motion of this analogue particle. The total conserved energy of this motion is given by

$$W = \frac{1}{2}(\mathrm{d}\phi/\mathrm{d}x)^2 - U(\phi) \tag{3.20}$$

The boundary conditions discussed earlier demand that as $x \to \pm\infty$, $U(\phi) \to 0$ and $(\partial\phi/\partial x) \to 0$, hence $W = 0$. The energy W of the analogue particle is not to be confused with the energy E, given in Eq.(3.16) of the original field system. For a static solution $\phi(x)$, E is given by

$$E = \int_{-\infty}^{\infty} \left[\frac{1}{2}(\frac{\mathrm{d}\phi}{\mathrm{d}x})^2 + U(\phi)\right] \ \mathrm{d}x \tag{3.21}$$

and clearly represents the total action functional of the analogue particle's motion. Our static solution therefore corresponds to some finite action, zero-energy trajectory of the particle. Finally, upon multiplying Eq.(3.19) by ϕ' and integrating once, we have

$$\int \phi'\phi'' \ dx = \int \frac{\mathrm{d}U}{\mathrm{d}\phi}\phi' \ \mathrm{d}x \tag{3.22}$$

or

$$\frac{1}{2}(\phi')^2 = U(\phi) \tag{3.23}$$

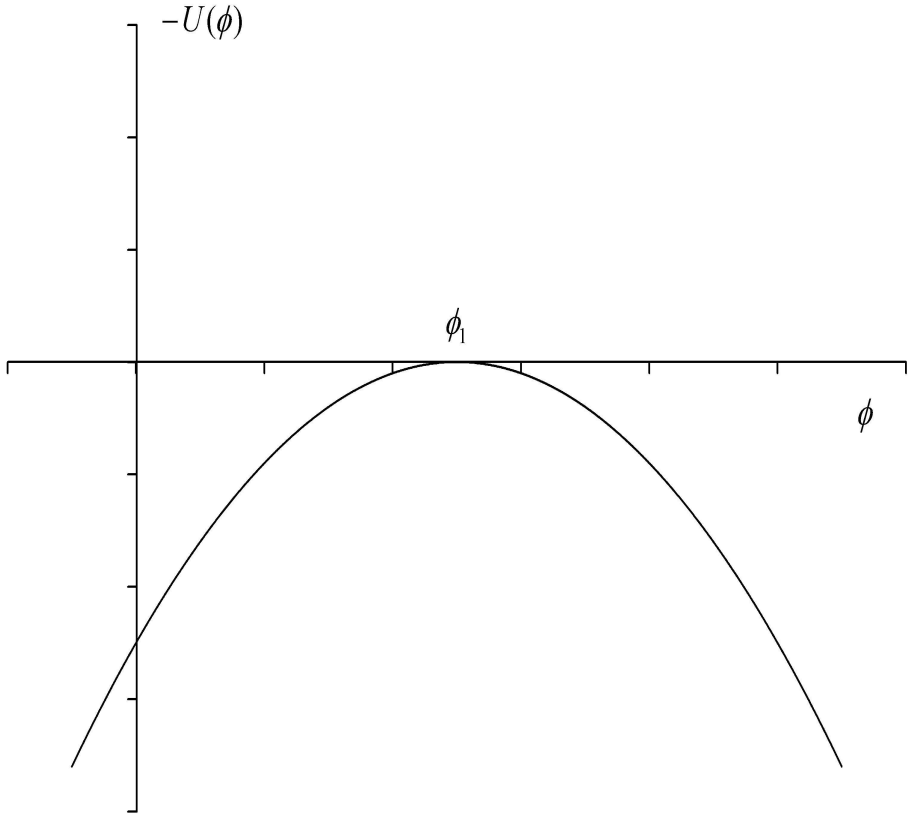

Figure 3.1: The potential $(-U(\phi))$ of the analogue particle when $U(\phi)$ has a unique minimum at ϕ_1.

Since both ϕ' and $U(\phi)$ vanish at $x \to -\infty$, the integration constant is zero. Eq. (3.23) is just a virial theorem for the analogue-particle.

Armed with this mechanical analogy, we consider potential first potential $U(\phi)$ which has a unique minimum, at $\phi = \phi_1$, where $U(\phi_1) = 0$. The analogue particle sees a potential $[-U(\phi)]$ as in Figure(3.1), with a maximum at $\phi = \phi_1$ and a negative value for all other ϕ.

Once the particle takes off from $\phi = \phi_1$ in either direction, it will not return. Its kinetic energy will never be zero again since its zero total energy W will always be larger than its potential energy $[-U(\phi)]$. Consequently, the particle can never stop and turn back towards ϕ_1. In terms of the static field solution $\phi(x)$, this means that once we fix the boundary condition as $\phi = \phi_1$ and $\mathrm{d}\phi/\mathrm{d}x = 0$ at $x = -\infty$, the same condition at $x = +\infty$ will not be satisfied by a non-trivial non-singular solution. Therefore, without explicitly solving Eq.(3.19) and independent of the details of $U(\phi)$, we see that if $U(\phi)$ has a unique absolute minimum, no static solitary wave exists. Of course, the trivial solution $\phi(x) = \phi_1$

for all x, is permitted.

Next, let $U(\phi)$ have two or more degenerate minima, where it vanishes. For example, let $U(\phi)$ has three minima at ϕ_1 , ϕ_2 and ϕ_3. The boundary conditions now state that the particle must leave any one of these points at $x = -\infty$ and end up at $x = \infty$ at any of them. This is now possible. The particle can take off from the top of the hill ϕ_1 at $x = -\infty$, and roll up to the top of the hill ϕ_2 asymptotically as $x = \infty$. Or, it can begin at ϕ_2 and end up at ϕ_3. Or it can undergo the reverse of these two motions. These are the elementary four non-trivial possibilities for this example. Thus, the mechanical analogy helps us conclude that (i) when $U(\phi)$ has a unique absolute minimum, there can be no static solitary wave, and (ii) when $U(\phi)$ has n discrete degenerate minima, we can have $2(n-1)$ types of solutions, connecting any two neighboring minima, as x varies from $-\infty$ to ∞.

Apart from these general consideration, one can also explicitly solve Eq.(3.19) by quadrature. We have found from Eq.(3.23),

$$\mathrm{d}\phi/\mathrm{d}x = \pm[2U(\phi)]^{1/2}. \tag{3.24}$$

Upon integration

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{\mathrm{d}\bar{\phi}}{[2U(\bar{\phi})]^{1/2}} \tag{3.25}$$

where the integration constant x_0 is any arbitrary point in space where the field has value $\phi(x_0)$. Our earlier discussion tells us that as $x \to \pm\infty$, $\phi(x)$ must approach any two neighboring minima of $U(\phi)$ and $\phi(x)$ lies between these two minima. Consequently, $U(\phi)$ will vanish only as $x \to \pm\infty$, and be positive for finite x. The integrand in Eq.(3.25) will therefore be non-singular except at the end points if $x \to \infty$ or $x_0 \to -\infty$. The solution $\phi(x)$ can be obtained in principle explicitly, given an x_0, and a $\phi(x_0)$, by integrating Eq.(3.25) and inverting it. In practice, it may be possible to do this analytically only for some $U(\phi)$. Note that varying x_0, keeping $\phi(x_0)$ fixed, merely shifts the same solution in x-space. This is just a reflection of the translational invariance of the Eq.(3.15).

As an illustration of this method, let us consider the 'kink' solution of the ϕ^4 theory (Dashen et al.[28], Goldstone and Jackiw [29], Polyakov [30]). The Lagrangian density has the form of Eq.(3.13) with

$$U(\phi) = \frac{1}{4}\lambda(\phi^2 - m^2/\lambda)^2 \tag{3.26}$$

where λ and m^2 are positive constants. The equation of motion,

$$\ddot{\phi} - \phi'' = m^2\phi - \lambda\phi^3 \tag{3.27}$$

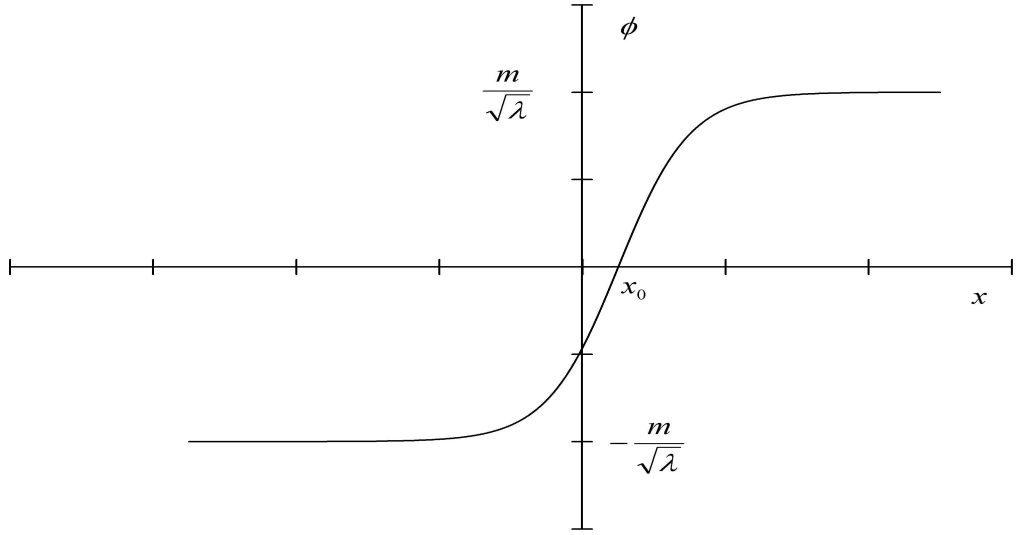

Figure 3.2: A schematic plot of the static kink solution

is essentially the same as Eq.(3.8) except for constants. Here $U(\phi)$ vanishes at two degenerate minima $\phi = \pm m/\sqrt{\lambda}$. Consequently localized solutions must tend to $\pm m/\sqrt{\lambda}$ as $x \to \pm\infty$. In particular, static solutions can be of two types, as per earlier arguments. They can begin from $\phi = -m/\sqrt{\lambda}$ at $x = -\infty$ and end up with $\phi = +m/\sqrt{\lambda}$ at $x = \infty$, or vice versa. Specifically, the static equation

$$\phi'' = \frac{\mathrm{d}U}{\mathrm{d}\phi} = \lambda\phi^3 - m^2\phi \tag{3.28}$$

can be solved using Eq.(3.25) to give

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{\mathrm{d}\bar{\phi}}{\sqrt{\lambda/2}(\bar{\phi}^2 - m^2/\lambda)} \tag{3.29}$$

Upon choosing $\phi(x_0) = 0$, integrating over $\bar{\phi}$ and inverting, we have

$$\phi(x) = \pm(m/\sqrt{\lambda}) \tanh[(m/\sqrt{2})(x - x_0)]. \tag{3.30}$$

The solution with the plus sign plotted in Figure 3.2 is called the 'kink' and the one with the minus sign the 'antikink'. The effect of translational invariance is explicitly seen, since a change in x_0 merely shifts the solution in space. The other symmetries of the Lagrangian, under $x \leftrightarrow -x$ and separately under $\phi \leftrightarrow -\phi$ are reflected in the relations which take on a particularly simple form when x_0 is chosen equal to zero:

$$\phi_{kink}(x) = -\phi_{antikink}(x) = \phi_{antikink}(-x). \tag{3.31}$$

The energy density of the kink solution,

$$\varepsilon(x) = \frac{1}{2}(\phi')^2 + U(\phi) = 2U(\phi) \tag{3.32}$$

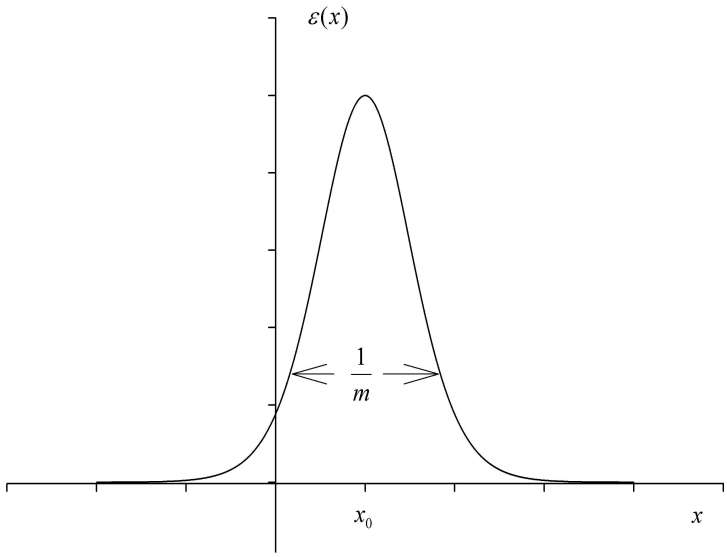

Figure 3.3: The energy density of the kink. It is localized, with a width characterized by $\frac{1}{m}$.

using Eq.(3.24)

$$\varepsilon(x) = (m^4/2\lambda)\,\mathrm{sech}^4[m(x-x_0)/\sqrt{2}] \tag{3.33}$$

is plotted in Figure 3.3 and is clearly localized near x_0. The total kink energy, sometimes called the classical kink mass, M_{cl} is given by

$$M_{cl} = \int_{-\infty}^{\infty} \mathrm{d}x\; \varepsilon(x) = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \tag{3.34}$$

and is finite. The kink in this model is therefore a legitimate solitary wave. So is the antikink. It resembles a 'lump' of matter in the sense that it is static, self-supporting localized packet of energy. The resemblence to an extended particle goes further: because the system is Lorentz invariant, given the static solution of Eq.(3.30), one can Lorentz-transform it to obtain a moving kink solution. Remembering that ϕ is a scalar field, we need only to transform the coordinate variables in Eq.(3.30). This gives

$$\phi_u(x,t) = \frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}\left(\frac{(x-x_0)-ut}{\sqrt{1-u^2}}\right)\right] \tag{3.35}$$

where $1 < u < -1$ is the velocity. This is a solution of the field Eq.(3.27) as can be verified by substitution. Whereas the spatial width of the static kink, in the sense of its energy density (3.33), is characterized by $1/m$, the corresponding width of the moving kink in Eq.(3.35) is $\sqrt{1-u^2}/m$, as would happen from Lorentz contraction for a lump of matter. Further, the energy of the time-dependent solution (3.35) as per (3.16) is

$$E[\phi_u] \;=\; \int_{-\infty}^{\infty} \mathrm{d}x\;\left[\frac{1}{2}(\phi_u)^2 + \frac{1}{2}(\phi_u')^2 + U(\phi_u)\right]$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{m^4}{2\lambda(1-u^2)} \operatorname{sech}^4\left(\frac{m}{\sqrt{2}} \frac{x-x_0-ut}{\sqrt{1-u^2}}\right) \\
&= \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} \frac{1}{\sqrt{1-u^2}} = \frac{M_{cl}}{\sqrt{1-u^2}}
\end{aligned} \tag{3.36}
$$

where M_{cl} is the static kink energy in (3.34). The relationship of Eq.(3.36) to Eq.(3.34) is again the same as the Einstein mass-energy equation for a particle. Therefore it should not be surprising that in the quantum version of this theory, the kink solution leads to a particle-state. Another important feature of $\phi_{kink}(x)$ is that it is not analytical in the nonlinear coupling constant λ near zero. Thus it can not be obtained by mere perturbation expansion starting from the linear equation. Since ϕ_{kink} is non-perturbative, so are many consequences which flow from it in the quantized theory.

3.4 Topological Indices

In the last section we discussed a single scalar field in one space dimension with more than one ground state admits time independent solutions of finite energy (kink or antikink solutions). Besides these features of solutions , stability of them is another interesting feature. Now we will show that these solutions are stable under small perturbations [27]. The equation of motion for our system was given by (3.15)

$$\Box\phi + \frac{\partial U}{\partial \phi} = 0$$

Let us consider a solution of the form

$$\phi(x,t) = f(x) + \delta(x,t), \tag{3.37}$$

where $f(x)$ is our time-independent solution and δ is the small perturbation. Inserting this expression in the equation of motion and only retaining terms of first order in the perturbation, we find

$$\Box\delta + \frac{\partial^2 U(f)}{\partial \phi^2}\delta = 0. \tag{3.38}$$

This equation is invariant under time translations, so we can express a general small perturbation as a superposition of modes. That is to say, the general solution is of the form

$$\delta(x,t) = \Re e \sum_n a_n e^{i\omega_n t}\psi_n(x), \tag{3.39}$$

where a's are arbitrary complex coefficients, and the ψ's and ω's obey the equation

$$-\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2} + U''(f)\psi_n = \omega_n^2\psi_n. \tag{3.40}$$

Note that this is a one-dimensional Schrödinger equation, with potential $U''(f)$. Our solution is stable under small perturbations if and only if none of the energy eigenvalues of this Schrödinger equation are negative. We will now show that this is always the case. Spatial translation invariance tells us that if $f(x)$ is a solution of the equation of motion, so is $f(x+a)$. Thus we already know an energy eigenfunction of our Schrödinger equation,

$$\psi_0 = \mathrm{d}f/\mathrm{d}x. \tag{3.41}$$

We also know the associated eigenvalue; it is zero. Since f is always a monotone function of x; therefore ψ_0 has no nodes. It is a well-known theorem that for a one dimensional Schrödinger equation with arbitrary potential the eigenfunction with no nodes is the eigenfunction of the lowest energy[31].

This stability indicates the presence of a conserved current and charge and as we will see that solutions of the equations of motion can be grouped according to their topological conserved quantity which is called topological charge. Now let us recall our discussion of a single scalar field $\phi(x,t)$ in two dimensions. Let the potential $U(\phi)$ in Eq.(3.13) have a discrete number of degenerate absolute minima, where it vanishes. Now we are interested in non-singular finite energy solutions, of which solitary waves and solitons are special cases. Therefore the field, whether static or time-dependent, must at any instant t, to a minimum of $U(\phi)$ at every point on spatial infinity, in order that the energy E in Eq.(3.16) be finite. In one space-dimension, spatial infinity consists of two points, $x = \pm\infty$. Consider $x = \infty$, for instance. Let, at some given instant t_0,

$$\lim_{x\to\infty} \phi(x,t_0) \equiv \phi(\infty,t_0) = \phi_1 \tag{3.42}$$

where ϕ_1 has to be one of the minima of $U(\phi)$. Then, as time develops (either forward or backward, starting from t_0), the field $\phi(x,t)$ will change continuously with t at every x as governed by the differential equation Eq.(3.15). In particular, $\phi(\infty,t)$ will be some continuous function of t. On the other hand, since the energy of that solution is conserved and remains finite, $\phi(\infty,t)$ must always be one of the minima of $U(\phi)$, which are a discrete set. It can not jump from ϕ_1 to another of the discrete minima if it is to vary continuously with t. Therefore $\phi(\infty,t)$ must remain stationary at ϕ_1. The same arguments apply to $x = -\infty$, where $\phi(-\infty,t) = \phi_2$, must also be time-independent and a minimum of $U(\phi)$, but not necessarily the same as ϕ_1 in the case of degenerate minima.

We can therefore divide the space of all finite-energy non-singular solutions into sectors, characterized by two indices, namely, the time-independent values of $\phi(x = \infty)$ and $\phi(-\infty)$. These sectors are topologically unconnected, in the sense that fields from one sector can not be distorted continuously into another without violating the requirement of finite energy. In particular, since time evolution is an example of continuous distortion, a field configuration from any one sector stays within that sector as time evolves. Of course, when $U(\phi)$ has a unique minimum, there is only one permissible value for both $\phi(\infty)$ and $\phi(-\infty)$ and therefore only one sector of solutions exists.

As illustration consider the system Eq.(3.26) in the preceding section. The potential has two degenerate minima, at $\phi = (\pm m/\sqrt{\lambda})$. Consequently, all finite-energy non-singular solutions of this system, whether static or time-dependent, fall into four topological sectors. These are characterized by the pairs of indices $(-m/\sqrt{\lambda}, m/\sqrt{\lambda})$, $(m/\sqrt{\lambda}, -m/\sqrt{\lambda})$, $(-m/\sqrt{\lambda}, -m/\sqrt{\lambda})$ and $(m/\sqrt{\lambda}, m/\sqrt{\lambda})$ respectively, which represent the values of $(\phi(x = -\infty), \phi(x = \infty))$. Thus, the kink, the antikink, and the trivial constant solutions $\phi(x) = \mp(m/\sqrt{\lambda})$, are members of the four sectors respectively. When a kink from the far left and an antikink from the far right approach one another, the field configuration belongs to the $(-m/\sqrt{\lambda}, -m/\sqrt{\lambda})$ sector. Even though we may not be able to calculate easily what happens after they collide, we can be sure that the resulting field configuration will always stay in the $(-m/\sqrt{\lambda}, -m/\sqrt{\lambda})$ sector.

Topological charge can be defined as

$$Q = (\sqrt{\lambda}/m)[\phi(x = \infty) - \phi(-\infty)] \tag{3.43}$$

with an associated conserved current,

$$j^\mu = (\sqrt{\lambda}/m)\varepsilon^{\mu\nu}\partial_\nu\phi \tag{3.44}$$

where $\mu, \nu = 0, 1$ and $\varepsilon^{\mu\nu}$ is the antisymmetric tensor. Clearly

$$\partial_\mu j^\mu = 0 \qquad \textit{and} \qquad Q = \int_{-\infty}^{\infty} \mathrm{d}x\ j_0. \tag{3.45}$$

Here divergencelessness of the current follows from independently of the equations of motion and Q is just the difference between the two indices $(\sqrt{\lambda}/m)\phi(\infty)$ and $(\sqrt{\lambda}/m)\phi(-\infty)$. The adjective topological is sometimes bestowed on solitary waves which have $Q \neq 0$. Waves with $Q = 0$ are non-topological. Thus the kink and the antikink solutions (3.30) are topological solutions, while trivial solutions $\phi(x) = \pm(m/\sqrt{\lambda})$ are non-topological.

Note that these topological indices are different from the more familiar conserved quantities like energy, momentum, charge etc. The latter, as is very well known in classical and quantum field theory, can be traced to the existence of continuous symmetries of the Lagrangian, such as under time translation, space translation, internal groups and so on. By contrast, the topological indices are boundary conditions, conserved because of finiteness of energy. Indeed, in many cases, these indices are closely related to a certain kind of breaking of some symmetry. That is, suppose the Lagrangian and $U(\phi)$ are invariant under some symmetry transformation acting on $\phi(x)$. If $U(\phi)$ had a unique minimum at some $\phi = \phi_0$, then ϕ_0 itself must remain invariant under that transformation. But in order to get non-trivial topological sectors, we need to have two or more degenerate minima. In that case while the full set of minima is invariant under the transformation, each individual minimum need not be so. For instance, the system (3.26) which permits four topological sectors, has a $U(\phi)$ invariant under $\phi \to -\phi$. But its two minima, $\phi = -m/\sqrt{\lambda}$ and $\phi = m/\sqrt{\lambda}$ are not separately invariant. Rather, they are transformed into one another. This fact has great importance in the quantum theory as well as the statistical mechanics of the field system and is called spontaneous symmetry breaking. So we observe the relation of non-trivial topological sectors to the existence of several degenerate minima of the potential, which in turn is connected to spontaneous symmetry breaking.

3.5 Solitons of the Sine-Gordon System

Historically, sine-Gordon equation first appeared in differential geometry. It enters geometry in the following way:

On a two-dimensional Riemannian manifold, there exist some special coordinates in some neighborhood of any point such that

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{d}v^2 + 2\,\mathrm{d}u\,\mathrm{d}v\cos\theta(u,v). \tag{3.46}$$

In terms of such coordinates, a simple computation shows that the statement that the manifold has constant negative curvature is equivalent to

$$\partial^2\theta/\partial u\,\partial v = \alpha\sin\theta, \tag{3.47}$$

where α is a constant related to the magnitude of the curvature. This is the sine-Gordon equation, in light-cone coordinates.

Sine-Gordon system can be also considered as the equations of motion

governed by the following Lagrangian density,

$$\mathcal{L}(\mathrm{x},\mathrm{t}) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + (m^4/\lambda)\{\cos[(\sqrt{\lambda}/m)\phi]\}. \tag{3.48}$$

This system has been used in the study of a wide range of phenomena, including propagation of crystal dislocations, of splay waves in membranes, of magnetic flux in Josephson lines, Bloch wall motion in magnetic cyristals, and two-dimensional models of elementary particles. This last application has been especially interesting due to the fact that its quantized form (with ϕ regarded as a boson field) is equivalent to the massive Thirring model, which is a model for interacting fermions in one space dimension.

If this Lagrangian density expand in powers of the coupling constant λ,

$$\mathcal{L}(\mathrm{x},\mathrm{t}) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{\lambda\phi^4}{4!} - \frac{\lambda^2\phi^6}{m^2 6!} + \ldots \tag{3.49}$$

As $\lambda \to 0$, this is just the free Klein-Gordon system, and the $O(\lambda)$ term is the familiar quartic coupling. The field equation from Eq.(3.43) is the sine-Gordon equation

$$\Box\phi + (m^3/\sqrt{\lambda})\sin[(\sqrt{\lambda}/m)\phi] = 0 \tag{3.50}$$

To eliminate some of the unwanted constants, let us make the following substitution

$$\bar{x} = mx, \qquad \bar{t} = mt \qquad and \qquad \bar{\phi} = (\sqrt{\lambda}/m)\phi. \tag{3.51}$$

Under this change of variables, the Lagrangian density becomes

$$\bar{\mathcal{L}}(\bar{\mathrm{x}},\bar{\mathrm{t}}) = (m^4/\lambda)\left[\frac{1}{2}(\bar{\partial}_\mu\bar{\phi})(\bar{\partial}^\mu\bar{\phi}) + (\cos(\bar{\phi}) - 1)\right]. \tag{3.52}$$

Then the equation of motion simply reads

$$\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} - \frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \sin[\bar{\phi}(\bar{x},\bar{t})] = 0 \tag{3.53}$$

and the conserved energy is

$$E = \frac{m^3}{\lambda}\int \mathrm{d}\bar{x}\ \left[\frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial\bar{t}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial\bar{t}}\right)^2 + \left(1 - \cos\bar{\phi}\right)\right]. \tag{3.54}$$

The Lagrangian and the field equation admits the following discrete symmetries

$$\bar{\phi}(\bar{x},\bar{t}) \to -\bar{\phi}(\bar{x},\bar{t}) \tag{3.55}$$

and

$$\bar{\phi}(\bar{x},\bar{t}) \to \bar{\phi}(\bar{x},\bar{t}) + 2N\pi\ ;\ N = \ldots -2,-1,0,1,2,\ldots \tag{3.56}$$

Consistent with these symmetries, the energy E vanishes at the absolute minima of

$$U(\bar{\phi}) = 1 - \cos\bar{\phi} \tag{3.57}$$

which are,

$$\bar{\phi}(\bar{x}, \bar{t}) = 2N\pi. \tag{3.58}$$

As we know from earlier discussion, all finite energy configurations, whether static or time-dependent, can be divided into an infinite number of topological sectors, each characterized by a conserved pair of integer indices (N_1, N_2) corresponding to the asymptotic values $2N_1\pi$ and $2N_2\pi$ that the field must approach as $\bar{x}$ tends to $-\infty$ and ∞ respectively. If on physical grounds we decide that only $\bar{\phi}$ modulo 2π is meaningful, as will happen in applications where $\bar{\phi}$ is an angle variable, then only the topological charge

$$Q = N_1 - N_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\bar{x}\, \frac{\partial\bar{\phi}}{\partial\bar{x}} \tag{3.59}$$

matters. By $\bar{\phi}$ modulo 2π, we mean that at any one space-time point $\bar{\phi}(\bar{x}, \bar{t})$ can be picked modulo 2π. At other points, it is fixed by continuity requirements.

Let us begin with static localized solutions. Our general considerations tells us that for a single scalar in one space-dimension, static solutions must connect only neighbouring minima of $U(\bar{\phi})$. That is, they must carry $Q = \pm 1$. Explicit solutions are easily obtained using Eq.(3.25):

$$\bar{x} - \bar{x}_o = \pm\int_{\bar{\phi}(\bar{x}_0)}^{\bar{\phi}(\bar{x})} \frac{\mathrm{d}\phi}{\sqrt{2U(\phi)}} = \pm\int_{\bar{\phi}(\bar{x}_0)}^{\bar{\phi}(\bar{x})} \frac{\mathrm{d}\phi}{2\sin(\phi/2)}. \tag{3.60}$$

This is easily integrated to give

$$\bar{\phi}(x) = 4\arctan[\exp(\bar{x} - \bar{x}_0)] = \phi_{sol}(\bar{x} - \bar{x}_0) \tag{3.61}$$

or

$$\bar{\phi}(x) = -4\arctan[\exp(\bar{x} - \bar{x}_0)] = \bar{\phi}_{antisol}(\bar{x} - \bar{x}_0) = -\bar{\phi}_{sol} \tag{3.62}$$

The solution with the plus sign (3.61) goes from $\bar{\phi} = 0$ to $\bar{\phi} = 2\pi$(fig 5a), or equivalently from 2π to 4π, 4π to 6π etc. It corresponds to $Q = 1$, and is often called the soliton of the system. The other solution (3.62) has $Q = -1$ and is called the antisoliton. Each has energy $M_s = 8m^3/\lambda$ as calculated by inserting Eq.(3.61) and Eq.(3.62) into Eq.(3.54). Moving soliton solutions can, as before, be obtained on Lorentz-transforming of Eq.(3.61), i.e. on replacing $\bar{x} - \bar{x}_0$ by $[(\bar{x} - \bar{x}_0 - u\bar{t})/\sqrt{1-u^2}]$. The solution Eq.(3.61) is roughly similar to the kink although the function, in detail, is different.

3.6 Vortex Lines

In addition to these two dimensional (1+1) models, we have more complicated solutions in higher dimensions for example vortex lines and monopole solutions as we will see.

Now consider a complex scalar field in 2-dimensional space. The 'boundary' of this space is the circle S^1 at infinity. The Lagrangian and Hamiltonian are

$$\mathcal{L}=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2-\frac{1}{2}|\nabla\phi|^2-V(\phi), \tag{3.63}$$

$$\mathcal{H}=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}|\nabla\phi|^2+V(\phi). \tag{3.64}$$

Now let us consider a static configurations with, for example,

$$V(\phi)=(a^2-\phi^*\phi)^2 \tag{3.65}$$

so that $V=0$ on the boundary. Then the field on the boundary takes the value

$$\phi=ae^{in\theta}\quad(r\to\infty) \tag{3.66}$$

where r and θ are polar coordinates in the plane, a is a constant, and, to make ϕ single-valued, n is an integer. From Eq.(3.66), we have

$$\nabla\phi=\frac{1}{r}(inae^{in\theta})\hat{\theta}. \tag{3.67}$$

Then as $r\to\infty$

$$\mathcal{H}=\frac{1}{2}|\nabla\phi|^2=\frac{n^2a^2}{2r^2} \tag{3.68}$$

and the energy (mass) of the static configuration is

$$E\approx\int^{\infty}\mathcal{H}\ \ \mathrm{r}\ \mathrm{d}r\ \mathrm{d}\theta=\pi n^2a^2\int^{\infty}\frac{1}{r}\ \mathrm{d}r. \tag{3.69}$$

This is logarithmically divergent, so the kink, as it stands, cannot be generalized to two dimensions - nor to more than two, for it turns out that in all these cases the energy is divergent. To avoid these difficulties we add a gauge field and replace standard derivative with the covariant one.

$$D_\mu\phi=\partial_\mu\phi+ieA_\mu\phi. \tag{3.70}$$

At the boundaries for Eq.(3.66) the gauge field of the form

$$\mathbf{A}=-\frac{1}{e}\nabla(\mathrm{n}\theta)\quad(r\to\infty), \tag{3.71}$$

i.e.

$$A_r \to 0, \qquad A_\theta \to -\frac{n}{er} \qquad (r \to \infty) \tag{3.72}$$

we find that at $r = \infty$

$$D_\theta \phi = \frac{1}{r}\left(\frac{\partial \phi}{\partial \theta}\right) + ieA_\theta \phi = 0, \qquad D_r \phi = 0 \tag{3.73}$$

so $D_\mu \phi \to 0$ on the boundary at infinity. The Lagrangian is now modified by gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu \phi|^2 - V(\phi). \tag{3.74}$$

Since Eq.(3.72) is a pure gauge,

$$A_\mu \to \partial_\mu \chi \qquad (r \to \infty), \tag{3.75}$$

then $F_{\mu\nu} \to 0$. For a static configuration $\mathcal{H} = -\mathcal{L}$, and with $V(\phi)$ given by Eq.(3.65) we have $\mathcal{H} \to 0$ as $r \to \infty$, making possible a field configuration of finite energy. We shall now see that the effect of adding the gauge field is to give the soliton magnetic flux. Consider the integral $\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ round the circle S^1 at infinity. By Stokes' theorem, this is $\int \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \Phi$, the flux enclosed, hence

$$\Phi = \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \oint A_\theta \, r \, \mathrm{d}\theta = -\frac{2\pi n}{e}, \tag{3.76}$$

and the flux is quantized. So we have, after all, constructed a 2-dimensional field configuration of a charged scalar field and a gauge field (the electromagnetic field!). It carries magnetic flux, and since $D_\mu \phi \to 0$ and $F_{\mu\nu} \to 0$ on the boundary at infinity, it appears to have finite energy. It is clear that by adding a third dimension (the z-axis) on which the fields have no dependence, this configuration becomes a vortex line. Apart from the presence of the scalar field, it is the same as the solenoid under the Bohm-Aharanov effect; and just as that effect is attributable to the topology of the gauge group $U(1)$, so here also the same topology ensures stability of the vortex.

The Lagrangian (3.74) with $V(\phi)$ given by Eq.(3.65) is that of the Higgs model, that is, scalar electrodynamics with spontaneous symmetry breaking. Actually this Lagrangian is the relativistic version of Landau-Ginzburg free energy, which describes superconductivity. It is known that on the occasions when magnetic flux does penetrate superconductors (that is, in type II superconductors), it creates quantized flux lines, called Abrikosov vortices. Thus the above solutions for the field ϕ are describing the BCS condensate state in superconductors. Now, let us discuss how the idea of vortex lines work for in (3+1) dimensions. For example the Higgs Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |(\partial_\mu + ieA_\mu)\phi|^2 - m^2\phi^*\phi - \lambda(\phi^*\phi)^2. \tag{3.77}$$

Spontaneous symmetry breaking has to appear when $m^2 < 0$, and the vacuum is then given by

$$|\phi|_{vac} = a = \left(\frac{-m^2}{2\lambda}\right)^{1/2}. \tag{3.78}$$

The equations of motion obtained from Eq.(3.77) are

$$D^\mu(D_\mu\phi) = -m^2\phi - 2\lambda\phi|\phi|^2, \tag{3.79}$$

$$ie(\phi\,\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + 2e^2A_\mu|\phi|^2 = \partial^\nu F_{\mu\nu}. \tag{3.80}$$

These equations allow the solutions (3.66) and (3.72) at infinity. Since by construction $D_\mu\phi = 0$ as $r \to \infty$, the left-hand side of Eq.(3.79) vanishes; and so does the right-hand side when ϕ takes on its vacuum value of Eq.(3.78). Since A_μ is a pure gauge (see Eq.(3.75)) $F_{\mu\nu} = 0$ as $r \to \infty$, so the right-hand side of Eq.(3.80) vanishes. In view of Eq.(3.66) and Eq.(3.72) the left-hand side vanishes identically when $\mu = r$, and when $\mu = \theta$. Hence our particular choices for A_μ and ϕ are allowed by the equations of motion.

As r becomes finite, and particularly as $r \to 0$, of course, the values of A_μ and ϕ change. Let us now treat the problem as one in three dimensions, with cylindrical symmetry about the z-axis. Then, since there is magnetic flux, the magnetic field component B_z must be non-zero, which means that A cannot be a pure gauge everywhere. Also, continuity requires that $\phi \to 0$ as $r \to 0$; since this is not the vacuum value, the 2-dimensional soliton will have an energy, and the vortex will have a corresponding mass per unit length. The forms of A and ϕ can be found from the equations of motion. Taking B with a z-component only, and A with a θ component only, we have

$$B = B_z = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\,A(r)\right], \qquad A(r) = A_\theta = A. \tag{3.81}$$

In addition, ϕ is of the form

$$\phi = \chi(r)\mathrm{e}^{in\theta} \tag{3.82}$$

with

$$\chi(r) \xrightarrow[r\,\to\,0]{} 0 \qquad \chi(r) \xrightarrow[r\,\to\,\infty]{} a. \tag{3.83}$$

In the static case, the equation of motion (3.79) then becomes

$$(\partial_i + ieA_i)^2\phi - (m^2 + 2\lambda|\phi|^2)\phi = 0 \tag{3.84}$$

which gives

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\chi}{\mathrm{d}r}\right) - \left[\left(\frac{n}{r} - eA\right)^2 + m^2 + 2\lambda\chi^2\right]\chi = 0 \tag{3.85}$$

On the other hand, the θ component of Eq.(3.80) gives

$$-\frac{ie}{r}(2in)\chi^2 + 2e^2 A\chi^2 = -\partial_i F_{\theta i}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rA)\right) - 2e\left(\frac{n}{r} + eA\right)\chi^2 = 0. \tag{3.86}$$

One should now solve the coupled non-linear equations of motion (3.85) and (3.86). No exact analytical solution, however, has yet been found. In the approximation where $\chi \simeq a$ is a constant (i.e. for $r \to \infty$), Nielsen and Olesen [32] found

$$A = -\frac{n}{er} - \frac{c}{e}K_1(|e|ar) \xrightarrow[r \to \infty]{} -\frac{n}{er} - \frac{c}{e}\left(\frac{\pi}{2|e|ar}\right)^{1/2} \mathrm{e}^{-|e|ar} + \ldots$$

with magnetic field

$$B_z = c\chi K_0(|e|ar) \to \frac{c}{e}\left(\frac{\pi a}{2|e|r}\right)^{1/2} \mathrm{e}^{-|e|ar} + \ldots \tag{3.87}$$

where c is a constant of integration and K_1 and K_0 are modified Bessel functions. For nonhomogenous deviations from the vacuum state

$$\chi(r) = a + \rho(r);$$

then one has

$$\rho(r) \simeq \mathrm{e}^{-\sqrt{-m^2}r} \tag{3.88}$$

($-m^2 > 0$). Why these solutions are stable? As with the kink case, the reason is topological. The Lagrangian is invariant under a symmetry group -in this case $U(1)$, the electromagnetic gauge group. The group space of $U(1)$ is a circle S^1, since an element of $U(1)$ may be written $\exp(i\theta) = \exp(i(\theta + 2\pi))$, so the space of all values of θ is a line with $\theta = 0$ identified with $\theta = 2\pi$, and the line becomes a circle S^1. The field ϕ in Eq.(3.66) is a representation basis of $U(1)$, but it is also the boundary value of the field in a 2-dimensional space. This boundary is clearly S^1(the circle $r \to \infty$, $\theta = (0 \to 2\pi)$). Hence ϕ defines a mapping of the boundary S^1 in physical space onto the group space S^1:

$$\phi : \ S^1 \to S^1, \tag{3.89}$$

the mapping being specified by the integer n. Now a solution characterized by fixed value of n is stable since it cannot be continuously deformed into a solution with different value of n(a rubber band which fits twice round a circle cannot be

continuously deformed into one which goes once round the circle). This is to say that the *first homotopy group* of S^1, the group space of $U(1)$, is not trivial:

$$\pi_1(S^1) = \mathbb{Z}. \tag{3.90}$$

$\mathbb{Z}$ is the additive group of integers (see [33] and [34]). The status of a topological argument like this is that it provides a very general condition which must be fulfilled in order that solitons exist in a particular model. If, as in the model above, the topological argument indicates that soliton solutions are possible in principle then one goes to the equations of motion to find them. Topology therefore provides possibilities of soliton existence and stability arguments.

Another type of nontrivial topological solitons are magnetic monopoles in gauge theory. Historically, magnetic monopoles was introduced to make Maxwell equations symmetric between electricity and magnetism (electromagnetic duality). Now let us review the properties of magnetic monopole in electrodynamics.

3.7 The Dirac Monopole

The Dirac monopole is based upon a straightforward generalization of the electric monopole [35]. By analogy, the electric field $\mathbf{E}$ of a point electric charge can be generalized to the magnetic field $\mathbf{B}$ of a point magnetic monopole:

$$\mathbf{E} = \mathrm{e}\frac{\mathbf{r}}{\mathrm{r}^3} \longrightarrow \mathbf{B} = \mathrm{g}\frac{\mathbf{r}}{\mathrm{r}^3} \tag{3.91}$$

(we are using Gaussian units). Then Maxwell's equations are generalized to include a nontrivial divergence of the magnetic field:

$$\nabla \cdot \mathbf{E} = 4\pi \mathrm{e}\ \delta^3(r) \longrightarrow \nabla \cdot \mathbf{B} = 4\pi \mathrm{g}\ \delta^3(r) \tag{3.92}$$

Since $\mathbf{B}$ is radial, the total flux through a sphere surrounding the origin is

$$\Phi = 4\pi r^2 B = 4\pi g. \tag{3.93}$$

Consider a particle with electric charge e in the field of this monopole. The wave function for a free particle is

$$\psi = |\psi| \exp\left[\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - \mathrm{Et})\right]. \tag{3.94}$$

In the presence of an electromagnetic field, $\mathbf{p} \longrightarrow \mathbf{p} - (\mathrm{e/c})\mathbf{A}$, so

$$\psi \longrightarrow \psi \exp\left(-\frac{ie}{\hbar c}\,\mathbf{A}\cdot\mathbf{r}\right)$$

or the phase α changes by

$$\alpha \longrightarrow \alpha - \frac{e}{\hbar c}\,\mathbf{A}\cdot\mathbf{r}$$

Consider a closed path at fixed r, θ, with ϕ ranging from 0 to 2π. The total change in phase is

$$\begin{aligned}
\Delta\alpha &= \frac{e}{\hbar c}\oint \mathbf{A}\cdot \mathrm{d}\mathbf{l} \\
&= \frac{e}{\hbar c}\int \nabla\times A\cdot \mathrm{d}\mathbf{S} \qquad\qquad (\text{Stokes}'\,\text{Theorem}) \\
&= \frac{e}{\hbar c}\int \mathbf{B}\cdot \mathrm{d}\mathbf{S} \\
&= \frac{e}{\hbar c}(\text{Flux through cap}) = \frac{e}{\hbar c}\Phi(r,\theta) \qquad (3.95)
\end{aligned}$$

$\Phi(r,\theta)$ is the flux through the cap defined by a particular r and θ. As θ is varied the flux through the cap varies. As $\theta \to 0$ the loop shrinks to a point and the flux passing through the cap approaches zero:

$$\Phi(r,0) = 0.$$

As the loop is lowered over the sphere the cap encloses more and more flux until, eventually, at $\theta \to \pi$ we should have, from Eq.(3.93),

$$\Phi(r,\pi) = 4\pi g. \qquad (3.96)$$

However, as $\theta \to \pi$ the loop has again shrunk to a point so the requirement that $\Phi(r,\pi)$ is finite entails, from Eq.(3.95), that A is singular at $\theta = \pi$. This argument holds for all spheres of all possible radii, so it follows that $\mathbf{A}$ is singular along the entire negative z axis. This is known as the Dirac string. It is clear that by a suitable choice of coordinates the string may be chosen to be along any direction, and, in fact, need not be straight, but must be continuous.

The singularity in $\mathbf{A}$ gives rise to the so-called Dirac veto - that the wave function vanish along the negative z-axis. Its phase is therefore indeterminate there and referring Eq.(3.95) there is no necessity that as $\theta \to \pi$, $\Delta\alpha \to 0$. However, we must have $\Delta\alpha = 2\pi n$ in order for ψ to be single-valued. From Eq.(3.95) and Eq.(3.96) we then have

$$\begin{aligned}
2\pi n &= \frac{e}{\hbar c}4\pi g, \\
eg &= \frac{1}{2}n\hbar c. \qquad (3.97)
\end{aligned}$$

This is the Dirac quantization condition. It implies that if there exist a magnetic charge anywhere in the universe all electric charges will be quantized:

$$e = n\frac{\hbar c}{2g}. \tag{3.98}$$

This is a possible explanation for the observed quantization of electric charge (also electric charge quantization can be realized in Kaluza-Klein theory as we discussed in 2.1.5.) In units $\hbar = c = 1$, Eq.(3.97) becomes

$$eg = \frac{1}{2}n. \tag{3.99}$$

We would like to make a final remark about Dirac magnetic monopoles. One can find fault with the previous presentation because of the existence of the singular Dirac string. Although the Dirac string can be moved in any direction and also has no physical consequences, one suspects that there is another formulation of the monopole in which the Dirac string is absent. This new representation of the magnetic monopole uses the theory of fibre bundles. It has the advantage that the representation is completely nonsingular and also is formulated in a well-established mathematical formalism.

Let $\mathbf{A}$ be the vector potential for the previous monopole, in which the Dirac string goes through the south pole. However, there is, of course, another vector potential $\tilde{\mathbf{A}}$ in which the Dirac string runs through the north pole. Our strategy is to split the sphere surrounding the magnetic monopole into two pieces along the equator. For the northern hemisphere, we take the field configuration $\mathbf{A}$ and simply throw away the Dirac string running through the south pole. In the southern hemisphere we take the field configuration $\tilde{\mathbf{A}}$ (and throw away the Dirac string that runs through the north pole).

Thus, $\mathbf{A}$ defines the monopole field in the northern hemisphere, while $\tilde{\mathbf{A}}$ describes the field in the southern hemisphere. Neither $\mathbf{A}$ nor $\tilde{\mathbf{A}}$ are singular.However, there is a price we have to pay for this sophisticated construction; that is, we have to piece together these two distinct patches in order to cover the sphere. We will 'glue' the two vector potentials along the equator. The final gluing process between these two different field configurations is accomplished by making a gauge transformation between them along the equator; that is;

$$\mathbf{A} = \tilde{\mathbf{A}} + \nabla\Omega \tag{3.100}$$

Since a gauge transformation cannot affect the physics, we now have a description that covers the entire sphere. To see how this gluing is actually accomplished,

let us write down the explicit representation of the vector fields. For $\mathbf{A}$, we have:

$$\begin{aligned} A_x &= -g\frac{y}{r(r+z)} \\ A_y &= g\frac{x}{r(r+z)} \\ A_z &= 0 \end{aligned} \tag{3.101}$$

Actually, a more convenient description of the monopole field is given in terms of spherical coordinates. Let θ be the polar angle, which is 0 at the north pole and π along the south pole. Let ϕ be the azimuthal angle, which ranges from 0 to 2π. Then the field configuration is given by:

$$A_r = 0$$

$$A_\theta = 0 \tag{3.102}$$

$$A_\phi = \pm g\frac{1 \mp \cos\theta}{r\sin\theta} \tag{3.103}$$

Notice that we have two solutions, given by the sign of $\pm$. The $-$ solution corresponds to $\mathbf{A}$, while the $+$ corresponds to $\tilde{\mathbf{A}}$. We can now 'glue' the two configurations together along the equator by a gauge transformation:

$$\tilde{A}_\phi = A_\phi - \frac{2g}{r\sin\theta} = A_\phi - (i/e)S\nabla_\phi S^{-1} \tag{3.104}$$

where

$$S = \mathrm{e}^{2ige\phi} \tag{3.105}$$

There is also another way to get rid of non physical singularities by introducing a scalar potential for magnetic field and a vector potential for electric field [95].

3.8 't Hooft-Polyakov Monopole

The previous discussion of magnetic monopoles is in fact not compelling, because ordinary electrodynamics does not require that monopoles should exist. Electrodynamics without monopoles is perfectly consistent theory. However, non-abelian gauge theory coupled to scalar fields possess monopole solutions given by independently 't Hooft [36] and Polyakov [30] and this fact is related to spontaneous symmetry breaking. They are not artificially superimposed on the theory as was the case with the Dirac formulation of monopoles. Instead, in this case the magnetic monopoles arises naturally from the equations of motion. Assume a theory with gauge group $O(3)$ and the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{\lambda}{4!}(\phi^a\phi^a)^2. \tag{3.106}$$

One can show that there exists a solution with the asymptotic behavior ($r \to \infty$):

$$\begin{aligned} A_i^a &\longrightarrow -\epsilon_{iab}\frac{r^b}{er^2} \\ A_0^a &\longrightarrow 0 \\ \phi^a &\longrightarrow (-6m^2/\lambda)\frac{r^a}{r} \end{aligned} \tag{3.107}$$

such that ϕ^a is covariantly constant at infinity (i.e.,$D_\mu\phi^a = 0$). This is the 't Hooft-Polyakov monopole. To compare this $O(3)$ monopole, with the usual Dirac monopole, we define a new Maxwell tensor $F_{\mu\nu}$ that will reduce to the usual one when ϕ^a becomes fixed in isospin space:

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e|\phi|^3}\,\epsilon_{abc}\,\phi^a(\partial_\mu\phi^b)(\partial_\nu\phi^c) \\ A_\mu &\equiv \frac{1}{|\phi|}\phi^a A_\mu^a \end{aligned} \tag{3.108}$$

With this definition, we can now calculate the magnetic and electric charge of the monopole. We find that $A_\mu = 0$ ($r \neq 0$) and that;

$$F_{0i} = 0, \qquad F_{ij} = -\frac{1}{er^3}\,\epsilon_{ijk}\,r^k, \qquad B_k = \frac{r^k}{er^3} \tag{3.109}$$

With this value of the magnetic field, then, we can show that the total flux through a sphere surrounding the monopole is given by $4\pi/e$. But the total flux of a monopole is $4\pi g$, so the monopole magnetic charge then obeys the constraint:

$$eg = 1 \tag{3.110}$$

which is twice the Dirac case. To reveal the topological nature of these monopole solutions, we remark that the sole contribution to $F_{\mu\nu}$ comes from the Higgs sector, since $A_\mu = 0$. The magnetic current is given by $K^\mu = \partial_\nu\tilde{F}^{\mu\nu}$ and can be written entirely in terms of Higgs field $\hat{\phi}^a = \phi^a/|\phi|$. A direct calculation shows that the conserved magnetic current is,

$$K^\mu = -\frac{1}{2e}\,\epsilon^{\mu\nu\rho\sigma}\,\epsilon_{abc}\,\partial_\nu\hat{\phi}^a\,\partial_\rho\hat{\phi}^b\,\partial_\sigma\hat{\phi}^c \tag{3.111}$$

Since $\partial_\mu K^\mu = 0$, the corresponding magnetic charge can be written as

$$\begin{aligned} M &= \frac{1}{4\pi}\int K^0\,\mathrm{d}^3x \\ &= -\frac{1}{8e\pi}\int \epsilon^{ijk}\,\epsilon_{abc}\,\partial_i\hat{\phi}^a\,\partial_j\hat{\phi}^b\,\partial_k\hat{\phi}^c\,\mathrm{d}^3x \qquad (3.112) \\ &= -\frac{1}{8e\pi}\oint_{S^2} \epsilon^{ijk}\,\epsilon_{abc}\,\hat{\phi}^a\,\partial_j\hat{\phi}^b\,\partial_k\hat{\phi}^c\,\mathrm{d}S_i \qquad (3.113) \end{aligned}$$

where we have integrated by parts, so that this volume integral becomes a two-dimensional surface integral taken over S_2 at infinity, which is the boundary of the static field $\hat{\phi}$. Comparing this with the definition of the winding number, we find the magnetic charge M is proportional to the winding number that maps the sphere S^2 onto S^2. But we know topologically that

$$\pi_2(S^2) = \mathbb{Z} \tag{3.114}$$

so we are left with $M = n/e$, where n is the winding number. Finally, the previous results may be generalized to more complicated phenomenologically acceptable groups. The key element of this monopole solution was the existence of a function $\hat{\phi}$ that smoothly mapped S^2 onto S^2. If we have a group G that is broken down to the subgroup H, then monopole solutions will exist if there are nontrivial mappings of S^2 onto G/H; that is:

$$\pi_2(G/H) = \mathbb{Z} \tag{3.115}$$

where G/H is called the coset space. Any gauge theory with this group property may have monopole solutions. For example, this can be satisfied if H has $U(1)$ factors. For example, the GUT (Grand Unified Theory) based on $SU(5)$ can be shown to have monopole solutions because it has a nontrivial homotopy group. In addition, these monopoles have finite energy and mass given, roughly $137M_W$, where M_W is a vector meson mass, so the monopole can be extremely heavy.

3.9 The Kaluza-Klein Monopole

K-K theory unifying gravity with electromagnetism suggests that monopoles could appear as a pure gravitational configuration on 5-dimensions and it was found by Sorkin [37], Gross and Perry [38]. The main point of their construction is that 4 dimensional gravitational instantons [1] solve the static equations of 4+1 dimensional gravity. The solution is a generalization of the self-dual Euclidean Taub-NUT solution [39], and is described by the following metric

$$ds^2 = -dt^2 + V(dx^5 + 4m(1-\cos\theta)d\phi)^2 + \frac{1}{V}(dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2), \tag{3.116}$$

where $\frac{1}{V} = 1 + \frac{4m}{r}$ and (r, θ, ϕ) are polar coordinates. From this solution the Taub-NUT instanton can be found by setting $dt = 0$. There is singularity point at $r = 0$, which is a so called NUT singularity and it is absent if x^5 is periodic

[1] instantons are classical localized finite-action solutions to the Euclidean equations of motion which obey special properties.

with period $16\pi m$ [40]. Since we are interested in constructing solutions of the five-dimensional field equations that approach the vacuum solution: $g_{AB} = \eta_{AB}$ (or $V = 1$, $A_\mu = 0$, $g_{\mu\nu} = \eta_{\mu\nu}$) at spatial infinity, we must identify $16\pi m$ with $2\pi R$. Accordingly,

$$m = \frac{1}{8}R = \frac{\sqrt{\pi G}}{2e}. \tag{3.117}$$

The gauge field A_μ is clearly that of a monopole from 5D Kaluza-Klein metric $ds^2 = V(dx^5 + A_\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu$, which is exactly Eq.(2.6),

$$A_\phi = 4m(1 - \cos\theta), \tag{3.118}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{4m\mathbf{r}}{r^3}. \tag{3.119}$$

and it is easily seen it has a Dirac string singularity running from $r = 0$ to ∞. This singularity is not a physical one if and only if the period of x^5 is equal to that of $16\pi m$. This is the geometrical analogue of Dirac's argument. The metric is regular on the half axis $\theta = 0$, but has a singularity at $\theta = \pi$ since the $(1 - \cos\theta)$ term in the metric means that a small loop around this axis does not shrink to a zero length at $\theta = \pi$. By a change of coordinates $x'^5 = x^5 + 8m\phi$ the metric becomes regular at $\theta = \pi$ but not at $\theta = 0$. In order to get rid of singularities one can then use $(x^5, t, r, \theta, \phi)$ to cover the northern hemisphere, and $(x'^5, t, r, \theta, \phi)$ to cover the southern hemisphere. Since x^5 and ϕ are periodic with periods $2\pi R$ and 2π respectively, $2\pi R$ must be identified with $16\pi m$.

The magnetic charge of our monopole is thus fixed by the radius of the Kaluza-Klein circle. Scaling the magnetic field so as to have the proper normalization, $B \to (16\pi G)^{-1/2}B$, we find that the magnetic charge g is

$$g = \frac{4m}{\sqrt{16\pi G}} = \frac{R}{2\sqrt{16\pi G}} = \frac{1}{2e}. \tag{3.120}$$

Thus, our monopole has one unit of Dirac charge. The above geometrical interpretation of the Dirac monopole indicates on existence of intimate relation between quantum theory and Kaluza-Klein 5-dimensional gravity [41].

Chapter 4

LOCALIZATION OF FIELDS ON DOMAIN WALLS AND BRANES

The possibility that our space has more than three spatial dimensions has been attracting continuing interest for many years, starting from Kaluza-Klein idea. Strong motivation for considering space as multidimensional one comes from theories which incorporate gravity in a reliable manner - the string theory and the so-called M-theory. In parallel to developments in the fundamental theory, studies along more phenomenological lines have recently give rise to new insights on whether and how extra dimensions may manifest themselves, and whether and how they may help to solve long-standing puzzles in particle physics, for example, hierarchy problem, cosmological constant problem, etc.

An important issue in multi-dimensional theories is the mechanism by which extra dimensions are hidden, so that the space-time is effectively four-dimensional insofar as known physics is concerned. Until recently, the main emphasis was put on theories of Kaluza-Klein type, where extra dimensions are compact and essentially homogenous. In that picture, the compactness of extra dimensions provides the space-time to be effectively four-dimensional at distances exceeding the compactification scale (size of extra dimensions). Hence the size of the extra dimensions must be microscopic ; a 'common wisdom' was that this size was roughly of the order of the Planck scale as we discussed in 2.1.5. With the Planck length $\ell_{P\ell} \sim 10^{-33}$ cm and the corresponding energy scale $M_{P\ell} \sim 10^{19}$ GeV, probing extra dimensions directly appeared to be hopeless. However, one of the problem with the Klein's idea on small extra space is that it can not explain why the extra space has to be markedly different in topology and in size from other space coordinates. One of alternative approaches attempting to address this problem is that the particles are trapped inside of the four-dimensional hypersurface by a potential well [9],[42] or by topological reasons [43].

So, the emphasis has recently shifted towards "brane world" picture which assumes that ordinary matter is trapped to a three-dimensional submanifold -

brane- embedded in multi-dimensional space. This idea provides a way basically different from the standad compactification. In the brane world scenario, extra dimensions may be large, and even infinite; as we will see that they may then have experimentally observable effects. Certainly, the potential detectability of large and infinite extra dimensions is one of the main reason why they are interesting. Another reason is that lower dimensional manifolds, p-branes, are inherent in string/M-theory. Some kinds of p-branes are capable of carrying matter fields; for example, D-branes have gauge fields residing on them (for a review, see [44]. Hence, the general idea of brane world appears naturally in M-theory context, and, indeed, realistic brane-world models based on M-theory have been proposed [45]. Even though the phenomenological models to be discussed in this thesis may have nothing to do with M-theory p-branes, one hopes that some of their properties will have counterparts in the fundamental theory. We note in this regard that the term "brane" has quite different meaning in different context; we shall use this term for any three-dimensional submanifold to which ordinary matter is trapped, irrespectively of the trapping mechanism.

4.1 Localization of Scalar Fields via Domain Wall

Localization of scalar fields on domain wall or branes was considered as simple field-theoretical models by V.Rubakov and M.Shaposhnikov [9]. One possible mechanism of breaking the translational invariance in $(3+N)+1$ - dimensional space-time is associated with the compactification of extra dimensions; in this way one arrives at Kaluza-Klein type theories. Within this approach the space-time manifold is assumed to be $M^{(3,1)} \times R^1$, where $M^{(3,1)}$ is the usual Minkowski space and R^1 is some compact manifold. The main purpose here is to discuss another mechanism of the translation invariance breaking for the space $M^{(3+N,1)}$ with $(3+N)$ spatial dimensions, and ordinary particles confined inside a potential well, which is sufficiently narrow along N directions and flat along three others. The origin of this well can be purely dynamical; the simplest possibility is that the well is formed due to a nonlinearity of classical equations of motion. To illustrate the idea, consider the toy quantum field model with the Lagrangian describing one scalar field φ in the (4+1)-dimensional space-time $M^{(4,1)}$ with the metric $g_{AB} = \mathrm{diag}(1,-1,-1,-1,-1)$.

$$\mathcal{L} = \frac{1}{2}\partial_A\varphi\,\partial^A\varphi - V(\varphi), \qquad A = 0,1,...,4 \tag{4.1}$$

where

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \frac{m^2}{\lambda}\right)^2 \tag{4.2}$$

which is exactly the same as Eq.(3.26). From Eq.(4.1), equation of motion can be found as,

$$\partial_A \partial^A \varphi - m^2\varphi + \lambda\varphi^3 = 0. \tag{4.3}$$

Similar to discussion in section 3.3, the classical equations of motion admit a kink solution $\varphi^{\rm cl}(x^4)$, which is independent of three spatial coordinates $(x^1, x^2, x^3) = \mathbf{x}$ and the time x^0. The form of this solution coincides with $(1+1)$ dimensional kink,

$$\varphi^{\rm cl}(x^4) = (m/\sqrt{\lambda})\tanh(mx^4/\sqrt{2}). \tag{4.4}$$

of the Eq.(4.3).

This classical field provides a potential well discussed above, if it is narrow in the fourth direction. It can be realized that m is sufficiently large since the energy density of this configuration is localized in the vicinity of the hyperplane $x^4 = 0$ within a region of thickness $\sim 1/m$ as shown in Figure 3.3.

In the WKB approximation, the spectrum of quantum fluctuations in the presence of the domain wall can be easily solved by the linearized equation of motion for the field $\varphi' = \varphi - \varphi^{\rm cl}$,

$$-\partial_A \partial^A \varphi' + m^2\varphi' - 3\lambda(\varphi^{\rm cl})^2\varphi' = 0. \tag{4.5}$$

There exist three types of perturbations:

(i) the first one is

$$\varphi'(x^0, \mathbf{x}, \mathrm{x}^4) = [\mathrm{d}\varphi^{\rm cl}/\mathrm{d}x^4]\exp(-i\mathbf{k}\cdot x + \mathrm{iE}x^0), \tag{4.6}$$

$$E^2 = \mathbf{k}^2, \tag{4.7}$$

and the corresponding particles are confined inside the wall; (ii) the second one is

$$\varphi'(x^0, \mathbf{x}, \mathrm{x}^4) = u(x^4)\exp(-i\mathbf{k}\cdot x + \mathrm{iE}x^0), \tag{4.8}$$

$$E^2 = \mathbf{k}^2 + \frac{3}{2}m^2, \tag{4.9}$$

where $u(x^4)$ is a normalizable solution [28] to the following Schrödinger equation (see the explicit form of the solution [46])

$$[-\partial_4^2 - m^2 + 3\lambda(\varphi^{\rm cl})^2]u = \frac{3}{2}m^2 u \tag{4.10}$$

and these perturbations are also confined. (iii)there exist perturbations which are not confined inside the domain wall; At large x^4 these are

$$\varphi'(x^0, \mathbf{x}, \mathrm{x}^4) = \exp(-i\mathbf{k}\cdot\mathbf{x}) - \mathrm{i}k^4 x^4 + iEx^0, \tag{4.11}$$

$$E^2 = \mathbf{k}^2 + (\mathrm{k}^4)^2 + 2m^2. \tag{4.12}$$

Therefore the spectrum of quantum fluctuations around the kink solution includes a zero mode which corresponds to translational symmetry of the theory in (i), one massive mode in (ii) and a continuous states in (iii). For low enough energies only the discrete modes are excited, and effectively the theory describes fields moving inside the potential well along the plane $x^4 = 0$. This model provides an example of dynamical localization of fields on the hyperplane which plays the role of our three-dimensional space embedded into the four-dimensional space. This hyperplane is referred to as a wall or 3-brane. If the energy is high enough the modes of the continuous spectrum are excited, this leads to a manifestation of particles escaping into the fifth dimension.

4.2 Localization of Spinor Fields via Domain Wall

In a similar way fermions coupled to the scalar fields can be localized on the wall [9]. The models under discussion can naturally account for massless fermions living in (3+1) dimensions. To see this, let us introduce fermions into the above model with the lagrangian

$$\mathcal{L}_\Psi = \mathrm{i}\bar{\Psi}\Gamma^A\partial_A\Psi + h\bar{\Psi}\Psi\varphi. \tag{4.13}$$

Here Ψ is a four component spinor, h is Yukawa coupling constant, and the minimal representation of spinors in (4+1)-dimensions can be chosen to be four dimensional. The (4+1)-dimensional Cllifford algebra can then be constructed from the usual four dimensional one by adding the γ_5 matrix to close the algebra. So (4+1)-dimensional gamma matrices,

$$\Gamma^\mu = \gamma^\mu, \qquad \mu = 0, 1, ..., 3, \qquad \Gamma^4 = i\gamma^5,$$

γ^μ,γ^5 being the standard Dirac matrices. Dirac equation corresponding to fermions in the domain wall background field is

$$i\Gamma^A\partial_A\Psi + h\varphi^{\mathrm{cl}}\Psi = 0. \tag{4.14}$$

A key point is that there exists a zero mode solution to Eq.(4.14). For this mode one has $i\gamma^\mu\partial_\mu\Psi^{(0)} = 0$, and the Dirac equation (4.14) becomes

$$\gamma^5\partial_4\Psi^{(0)} = h\varphi^{\mathrm{cl}}\Psi^{(0)}. \tag{4.15}$$

Then solution to Eq.(4.15) is

$$\Psi^{(0)}(x^0, \mathbf{x}, \mathrm{x}^4) = \exp\left[-h\int_0^{x^4} \varphi^{\mathrm{cl}}(x'^4)\, \mathrm{d}x'^4\right] \times \psi(x^0, \mathbf{x}) \tag{4.16}$$

where $\psi(x^0, \mathbf{x})$ is a left-handed massless (3+1)-dimensional spinor ($\gamma^5\psi = \psi$), which is the usual solution of the four-dimensional Weyl equation. The wave function (4.16) is also confined inside the domain wall, and the corresponding particles are just massless fermions in the (3+1)-dimensional world. The zero mode (4.16) is localized near $x^4 = 0$, i.e.,at the domain wall, and at large $|x^4|$ it decays exponentially, because domain wall solution

$$\varphi^{\mathrm{cl}} \to m/\sqrt{\lambda},\ x^4 \to \infty$$

and

$$\varphi^{\mathrm{cl}} \to -m/\sqrt{\lambda},\ x^4 \to -\infty.$$

that is,

$$\Psi^{(0)} \propto \exp(-h|x^4|),\ h > 0.$$

Of course, there exist excitations not confined inside the wall, but their energy exceeds $mh/\sqrt{\lambda}$ and they can be created only in high energy collisions. Massless four-dimensional fermions localized on the domain wall, zero modes, are meant to mimick our matter. They propagate with the speed of light along the domain wall, but do not move along x^4. At low energies, their interactions can produce only zero modes again, so physics is effectively four-dimensional. Zero modes interacting at high energies, however, will produce continuum modes, the extra dimension will open up, and particles will be able to leave the brane, escape to $|x^4| = \infty$ (if the size of the extra dimension is infinite) and literally disappear from our world. For our four-dimensional observer (composed of particles trapped to the brane), these high energy processes will look like $e^+\ e^- \to$ nothing or $e^+\ e^- \to \gamma +$ nothing. It is worth noting that the existence of massless (3+1)-dimensional fermions is closely related to the existence of a (1+1)-dimensional kink. From Eq.(4.15) one can also realized that right-handed spinor does not lead to localization since its solution exponentially growing. Another point here is that if background field were antikink, then right-handed massless fermions would play the localization role.

The above construction is straightforwardly generalized to more than one extra dimensions. This is done by considering, instead of the domain wall, topo-

logical defects of higher dimensions: the Abrikosov-Nielsen-Olesen vortex in six-dimensional space-time, 't Hooft-Polyakov monopole in seven dimensional space-time, etc. Explicit expressions for fermion zero modes in various backgrounds are given in [47],[48],[49]. The number of fermion zero modes may be greater than one, so from one family of multi-dimensional fermions one can obtain several four-dimensional families. This possibility of explaining the origin of three Standard Model generations has been considered in [50] and [51]

Chapter 5

PHENOMENOLOGICALLY VIABLE EXTRA DIMENSIONAL MODELS AND FERMION LOCALIZATION

In chapter 2 we explained basics of the traditional Kaluza-Klein approach, which will be essential to us and also outlined some recent ideas, which lie in the basis of the new type Kaluza-Klein models. In contrast to the main goal of the traditional Kaluza-Klein theories unifying of various types of interactions within a gravity interaction in the multidimensional space-time, the aim of the new models with extra dimensions is to solve some long-standing problems, for example, hierarchy problem and cosmological constant problem, etc. Hierarchy problem can be stated as follows: As it is well known that the electroweak scale is defined to be the energy scale in the Standard Model description of elementary particle physics at which the electromagnetic interaction unifies with the weak interaction. The Planck energy scale is theoretically calculated to lie at $M_{P\ell} = \sqrt{\frac{\hbar c}{G}} = 10^{19}\ GeV$ or at 10^{-35} m, while electroweak scale is roughly $10^{3}\ GeV$, or 10^{-19}m. At the Planck scale, a theory of Quantum Gravity should be revealed, and it is hoped that the gravitational interaction unifies with the remaining three interactions described by the Standard Model. The hierarchy of sixteen orders of magnitude between these two scales (namely $M_{EW}/M_{P\ell} \sim 10^{-16}$) is called the hierarchy problem. This problem, however, can not be studied in the context of the chapter 4 because gravity is excluded in that section. In the next two sections we consider two different approaches to solve hierarchy problem by including gravitation.

5.1 Large Extra Dimensions

5.1.1 General Aspects of Large Extra Dimensions

One of the approach, [8], [52] hereafter called ADD model (Arkani-Hamed, Dimopoulos, Dvali), is to neglect the brane tension and consider compact extra dimensions. In this way Kaluza-Klein picture is reintroduced. The size of extra

dimension R is not so small i.e., not in the scale of standard model. The distances at which non-gravitational interactions cease to be four-dimensional are determined by the dynamics on the brane, and are much smaller than R. Only gravity becomes multi-dimensional at scales just below R. The four-dimensional law of gravitational attraction has been established experimentally down to distances of about 0.2 mm [53], so the size of extra dimension is allowed to be large as 0.1 mm.

This possibility opens up a new way to address the hierarchy problem. In multi-dimensional theories, the four dimensional Planck scale is not a fundamental parameter. Rather, the mass scale of multi-dimensional gravity, which we denote simply by M, is fundamental, as it is this latter scale that enters the full multi-dimensional gravitational action,

$$S = -\frac{1}{16\pi G_{(D)}} \int \mathrm{d}^D X \, \sqrt{g^{(D)}} \, R^{(D)} \tag{5.1}$$

where

$$G_{(D)} = \frac{1}{M^{D-2}} \equiv \frac{1}{M^{d+2}} \tag{5.2}$$

is the fundamental D-dimensional Newton's constant, $d = D - 4$ is the number of extra dimensions, and $\mathrm{d}^D X = \mathrm{d}^4 x \, \mathrm{d}^d z$, z being extra coordinates.

In ADD picture, the long-distance four-dimensional gravity is mediated by the graviton zero mode whose wave function is homogenous over extra dimensions. Hence the four-dimensional effective action describing long-distance gravity is obtained from Eq.(5.1) by taking the metric to be independent of extra coordinates z. The integration over z is then trivial, and the effective four-dimensional gravitational action is

$$S_{eff} = \frac{V_d}{16\pi G_{(D)}} \int \mathrm{d}^4 x \, \sqrt{g^{(4)}} \, R^{(4)} \tag{5.3}$$

where $V_d \sim R^d$ is the volume of extra dimensions. We see that the four-dimensional Planck mass is, up to a numerical factor of order one, equal to

$$M_{P\ell} = M(MR)^{\frac{d}{2}} \tag{5.4}$$

If the size of extra dimensions is large compared to the fundamental length M^{-1}, the Planck mass is much larger than the fundamental gravity scale M. One way push this line of reasoning to extreme and suppose that the fundamental gravity scale is of the same order as the electroweak scale, $M \sim 1$ TeV. Then the hierarchy between $M_{P\ell}$ and M_{EW} is entirely due to the large size of extra dimensions. The hierarchy problem becomes now the problem of explaining why

R is large. This is certainly an interesting reformulation. Assuming that $M \sim$ 1 TeV, one calculates from Eq.(5.4) the value of R,

$$R \sim M^{-1} \left(\frac{M_{P\ell}}{M} \right)^{\frac{2}{d}} \sim 10^{\frac{32}{d}} \cdot 10^{-17}\text{cm} \tag{5.5}$$

For one extra dimension one obtains unacceptably large value of R. An interesting case is $d = 2$ when roughly $R \sim 1$ mm. This observation [52], [54], [55] stimulated recent activity in experimental search for deviations from Newton's gravity law at sub-milimeter distances. Experimental data show that the mass scale $M \sim 1$ TeV is excluded for $d = 2$ by astrophysics and cosmology. A more realistic value $M \sim 30$ TeV implies $R \sim 1 - 10$ μm. This motivates search for deviations from Newton's law in a micro-meter range, which is difficult but not impossible and that is the main reason why extra dimensions become important. For $d > 2$, Eq.(5.5) results in smaller values of R. For example, for $d = 3$ and $M \sim 1$ TeV one obtains $R \sim 10^{-6}$cm. Search for violation of Newton's law at these scales appears hopeless. For $d = 6$ (full dimensionality of space-time $D = 10$, as suggested by superstring theory), one has $R \sim 10^{-12}$cm, which is still much larger than the electroweak scale, $(1 \text{ TeV})^{-1} \sim 10^{-17}$cm. We note, however, that the compactification scales of different extra dimensions are not guaranteed to be of the same order; if some of these are much smaller than the others, the situation with deviations from Newton's gravity in spaces with $d > 2$ may be similar to that of $d = 2$. In other words, deviations from Newton's gravity law may occur in micro-meter range even for $d > 2$. Now let us see what kind of modification of Newton's gravity is needed when compact extra dimensions are introduced. We assume that the space is of the structure $M_4 \times T^n$, where M_4 is the 3+1 dimensional Minkowski space, and T^n is an n dimensional torus. The analysis we will carry out here is similar to the discussion of the massless scalar in section 2.1.5, where the role of the scalar is taken over by the Newtonian potential. Let us denote the coordinates with a vector $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ corresponds to the M^4 and $\mathbf{y}$ to the T^n. For simplicity we assume that the torus is described by a quadratic lattice and uniform length of a cycle is $2\pi R$, i.e.,

$$\mathbf{y} \equiv \mathbf{y} + 2\pi R \tag{5.6}$$

The $n + 4$ dimensional Newton potential V_{n+4} of a point particle with mass μ located at the origin is given by

$$\triangle_{n+3} V_{n+4} = (n + 1)\Omega_{n+2} G_{n+4} \mu \delta^{n+3}(\mathbf{x}, \mathbf{y}), \tag{5.7}$$

where $\triangle_{n+3}$ is the three dimensional flat Laplacian and Ω_{n+2} is the volume of a unit $n+2$ sphere. Any solution (5.7) should be periodic under (5.6). This can be

ensured by expanding the potential in terms of eigenfunctions $\psi_{\mathbf{k}}(\mathbf{y})$ of a Laplace operator. The eigenvalue equation is

$$\triangle_n \psi_{\mathbf{k}}(\mathbf{y}) = -m_k^2 \psi_{\mathbf{k}}(\mathbf{y}). \tag{5.8}$$

Thus an orthonormal set of eigenfunctions is

$$\psi_{\mathbf{k}}(\mathbf{y}) = \frac{1}{(2\pi R)^{n/2}} \mathrm{e}^{\frac{\mathbf{k}\cdot\mathbf{y}}{R}} \tag{5.9}$$

where $\mathbf{k}$ is an n-dimensional vector with integer entries. We expand the higher dimensional Newton potential into a series of the eigenfunctions with $r = |\mathbf{x}|$ dependent coefficients.

$$V_{n+4} = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(r) \psi_{\mathbf{k}}(\mathbf{y}). \tag{5.10}$$

Substituting this ansatz into Eq.(5.7), Fourier coefficients can be found

$$\phi_{\mathbf{k}}(r) = -\frac{\Omega_n G_{n+4}\, \mu\, \psi_{\mathbf{k}}^*(0)}{2} \frac{1}{r}\, \mathrm{e}^{-\frac{|\mathbf{k}|}{R}}. \tag{5.11}$$

Now, we consider the case that all particles with which we can test the gravitational potential are localized at $\mathbf{y} = 0$. (This is the natural from the brane picture since we can test gravity only with matter which is confined to live on the brane.) We are interested in the Newton potential at $\mathbf{y} = 0$. This leads to

$$V_4 \equiv V_{n+4} = -\frac{G_{4\mu}}{r} \sum_{\mathbf{k}} \mathrm{e}^{-r\frac{|\mathbf{k}|}{R}}, \tag{5.12}$$

where the four dimensional and the higher dimensional Newton constant are related via

$$G_4 = \frac{\Omega_n\, G_{n+4}}{2(2\pi R)^n}. \tag{5.13}$$

For $\mathbf{k} = 0$ we obtain the usual four-dimensional Newton potential. The other terms are additive Yukawa potentials. They arise due to the change of massive Kaluza Klein gravitons. Experimentalists usually parametrize deviations from Newton's law with the following expression [56]

$$V_4(r) = -\frac{G_4 \mu}{r} \left(1 + \alpha \mathrm{e}^{\frac{r}{\lambda}}\right). \tag{5.14}$$

However, it is important fact that no deviation from Newton's law up to the order of micrometers has been observed so far.

5.1.2 Fermion Localization in Large Extra Dimension

Suggestions that extra dimensions may not be compact [8, 9, 12, 57, 58, 59], or large [52, 60] can provide new insights for a solution of gauge hierarchy problem, of cosmological constant problem, and give new possibilities for model building. One of the interesting questions, related to these ideas, is localization of different fields on a brane. It has been shown that the graviton [12] and the massless scalar field [61] have normalizable zero modes on branes of different types, that the abelian vector fields are not localized in the Randall-Sundrum (RS) model in five dimensions but can be localized in some higher-dimensional generalizations of it [62]. In contrast, in [61] it was shown that fermions do not have normalizable zero modes in five dimensions, while in [62] the same result was derived for a compactification on a string [63] in six dimensions. It is known, though, that fermion interaction with a scalar domain wall in five dimensions can lead to localization of chiral fermions [9, 10].

5.1.3 Arkani-Hamed-Shmaltz Model

One Chiral Fermion in 5 Dimensions

The action a five dimensional fermion Ψ coupled to the background scalar Φ is

$$S = \int \mathrm{d}^4\mathbf{x}\,\mathrm{d}x_5 \bar{\Psi}\left[i\hat{\partial}_4 + i\gamma^5 + \Phi(x_5)\right]\Psi. \tag{5.15}$$

where $\hat{\partial} = \gamma^\mu\partial_\mu$ and the coordinates of our 3+1 dimensions are represented by $\mathbf{x}$ whereas the fifth coordinate is x_5; five dimensional fields are denoted with capital letters whereas four-dimensional fields will be lower case. This Dirac operator is separable, and it is convenient to expand the Ψ fields in a product base

$$\Psi(\mathbf{x}, x_5) = \sum_n < x_5|Ln > \; P_L\psi_n(\mathbf{x}) + \sum_n < x_5|Rn > \; P_R\psi_n(\mathbf{x}) \tag{5.16}$$

$$\bar{\Psi}(\mathbf{x}, x_5) = \sum_n \bar{\psi}_n(\mathbf{x})P_R < Ln|x_5 > + \sum_n \bar{\psi}_n(\mathbf{x})P_L < Rn|x_5 > \tag{5.17}$$

where the ψ_n are arbitrary four-dimensional Dirac spinors and $P_{L,R} = (1 \pm i\gamma_5)/2$ are chiral projection operators. We use a bra-ket notation for the eigenfunctions which diagonalize the x_5-dependent part of the Dirac operator; the kets $|Ln >$ and $|Rn >$ are solutions of

$$aa^\dagger|Ln >= (-\partial_5^2 + \Phi^2 + \dot{\Phi})|Ln >= \mu_n^2|Ln > \tag{5.18}$$

$$a^\dagger a|Rn >= (-\partial_5^2 + \Phi^2 - \dot{\Phi})|Rn >= \mu_n^2|Rn > \tag{5.19}$$

respectively. Here $\dot{\Phi} \equiv \partial_5 \Phi$, and $a^\dagger$ and a are "creation" and "annihilation" operators defined as

$$a = \partial_5 + \Phi(x_5) \tag{5.20}$$

$$a^\dagger = -\partial_5 + \Phi(x_5). \tag{5.21}$$

The $|Ln>$ and $|Rn>$ each form an orthonormal set and for non-zero μ_n^2 are related through $|Rn>= (1/\mu_n)a|Ln>$ as can be verified easily from Eq.(5.18) and Eq.(5.19). The eigenfunctions with vanishing eigenvalues need not be paired however. It is no accident that we use simple harmonic oscillator (SHO) notation. For the special choice $\Phi(x_5) = 2\mu^2 x_5$ the operators a and $a^\dagger$ become the usual SHO creation and annihilation operators up to a normalization factor $\sqrt{2}\mu$, and the operator $a^\dagger a$ becomes the number operator N. The eigenkets are then related to the usual SHO kets by $|Ln>= |n>$ and $|Rn>= |n-1>$. Expanding in $|Ln>$ and $|Rn>$ the action for a 5-d Dirac fermion (5.15) can be re-written in terms of a 4-d action for an infinite number of fermions

$$S = \int \mathrm{d}^4\mathbf{x} \left[\bar{\psi}_L i\hat{\partial}_4 P_L \psi_L + \bar{\psi}_R i\hat{\partial}_4 P_R \psi_R + \sum_{n=1}^{\infty} \bar{\psi}_n (i\hat{\partial}_4 + \mu_n)\psi_n \right]. \tag{5.22}$$

The first two terms correspond to 4-d two-component chiral fermions, they arise from the zero modes of (5.18) and (5.19). The third term describes an infinite tower of Dirac fermions corresponding to the modes with non-zero μ_n in the expansion. The zero mode wave functions are easily found by integrating $a^\dagger|Ln>= 0$ and $a|Rn>= 0$. The solutions

$$< x_5|L,0 > \sim \exp\left[-\int_0^{x_5} \Phi(s)\mathrm{d}s \right] \tag{5.23}$$

and

$$< x_5|R,0 > \sim \exp\left[\int_0^{x_5} \Phi(s)\mathrm{d}s \right] \tag{5.24}$$

are exponentials with support near the zeros Φ. In the infinite system that we are considering these modes cannot both be normalizable. It is easy to see that $|b,0>$ is normalizable if $\Phi(-\infty) < 0$ and $\Phi(+\infty) > 0$. Also if $\Phi(-\infty) > 0$ and $\Phi(+\infty) < 0$ then the mode $|f,0>$ is normalizable. In the other cases there is no normalizable zero mode. For definiteness let us now specialize to the SHO. Then

$$< x_5|L,0 >= \frac{\mu^{1/2}}{(\pi/2)^{1/4}} \exp\left[-\mu^2 x_5^2\right] \tag{5.25}$$

and $< x_5|R,0 >$ is not normalizable. Thus the spectrum of four dimensional fields contains one left-handed chiral fermion in addition to an infinite tower of massive Dirac fermions. The shape of the wave function of the chiral fermion is

Gaussian, centered at $x_5 = 0$. Note that coupling Ψ to $-\Phi$ would have rendered $< x_5|R, 0 >$ normalizable and we would have instead localized a massless right handed chiral fermion. For clarity, let us write the wave function of the massless chiral fermion in the chiral basis

$$\Psi(\mathbf{x}, x_5) = \begin{pmatrix} < x_5|L, 0 > \psi(\mathbf{x}) \\ 0 \end{pmatrix}. \tag{5.26}$$

Many Chiral Fermions

Eq.(5.15) can be generalized to the case of several fermion fields. For simplification, we can assume that all 5-d fermions couple to the same scalar Φ

$$S = \int \mathrm{d}^5 x \sum_{i,j} \bar{\Psi}_i \left[i\hat{\partial}_5 + \lambda\Phi(x_5) - m \right]_{ij} \Psi_j. \tag{5.27}$$

Here we allowed for general Yukawa couplings λ_{ij} and also included masses m_{ij} for the fermion fields. Mass terms for the five-dimensional fields are allowed by all the symmetries and should therefore be present in the Lagrangian. In the case that we will eventually be interested in - the standard model - the fermions carry gauge charges. This force the couplings λ_{ij} and m_{ij} to be block diagonal, with mixing only between fields with identical gauge quantum numbers. For simplicity λ_{ij} is taken as δ_{ij}, then m_{ij} can be diagonalized with eigenvalues m_i. We can find the massless four-dimensional fields with the analogy to the single fermion case. Each 5-d fermion Ψ_i leads to a single 4-d left chiral fermion. Similar to Eq.(5.25) the wave functions in the 5th coordinate are Gaussian, but in this case their localization are centered around the zeros of $\Phi - m_i$. In the SHO approximation this is at $x_5^i = m_i/2\mu^2$. The five dimensional action action describes a set of non-interacting four-dimensional chiral fermions localized at different 4-d "slices" in the 5th dimension at energies below μ. So we now exhibit the field content of the 5-d theory which can reproduce the chiral spectrum of the 4-d SM as localized zero modes. If λ's are chosen as positive, then left handed chiral Weyl spinors are localized and this implies that it is possible to construct the SM by means of only left handed spinors, the right handed fields are represented by their charge conjugates $\bar{\psi}^c$. Then the SM arises simply by choosing 5-d Dirac spinors (Q, U^c, D^c, L, E^c) transforming like the left handed SM Weyl fermions (q, u^c, d^c, l, e^c).

Yukawa Couplings

By this mechanism it is possible to generate hierarchial Yukawa couplings in four dimensions. The action of five-dimensional fermion fields for only one generation and the lepton sector only,

$$S = \int \mathrm{d}^5 x\, \bar{L} \left[i\hat{\partial}_5 + \Phi(x_5) \right] L + \bar{E}^c \left[i\hat{\partial}_5 + \Phi(x_5) - m \right] E^c + \kappa H L^T C_5 E^c. \quad (5.28)$$

where $C_5 = \gamma^0\gamma^2\gamma^5$. As discussed in the previous section, we find a left handed massless fermions l from L localized at $x_5 = 0$ and e^c from E^c localized at $x_5 = r \equiv m/(2\mu^2)$. For simplicity, the Higgs field is assumed to be delocalized inside the wall. We now determine what effective four-dimensional interactions between the light fields results from the Yukawa coupling in Eq.(5.28). So L and E^c will be expanded as in Eq.(5.16) and Eq.(5.17), then replace the Higgs field H by its lowest Kaluza-Klein mode which has an x_5-independent wave function. We get the Yukawa coupling

$$S_{Yuk} = \int \mathrm{d}^4\mathbf{x}\, \kappa h(\mathbf{x})\, l(\mathbf{x})\, e^c(\mathbf{x}) \int dx_5\, \phi_l(x_5)\phi_{e^c}(x_5). \quad (5.29)$$

Here $\phi_l(x_5)$ and $\phi_{e^c}(x_5)$ are the zero-mode wave functions for the lepton doublet and singlet respectively. ϕ_l is a Gaussian centered at $x_5 = 0$ whereas ϕ_{e^c} is centered at $x_5 = r$. The overlap of Gaussians is itself a Gaussian and we find

$$\int \mathrm{d}x_5\, \phi_l(x_5)\, \phi_{e^c}(x_5) = \frac{\sqrt{2}\mu}{\sqrt{\pi}} \int \mathrm{d}x_5\, \mathrm{e}^{-\mu^2 x_5^2}\, \mathrm{e}^{-\mu^2(x_5-r)^2} = \mathrm{e}^{-\mu^2 r^2/2}. \quad (5.30)$$

Any coupling between the two chiral fermions is necessarily exponentially suppressed because the two fields are separated in space. The coupling is then proportional to the exponentially small overlap of the wave functions. In this model it has not been imposed any chiral symmetries in the fundamental symmetry by $\mathcal{O}(1)$. Even with chiral symmetry maximally broken in the fundamental theory, an approximate chiral symmetry in the low energy, 4-d effective theory, has been obtained.

5.2 Randall-Sundrum Models

The discussion in previous section assumes that the extra dimensions are flat or at least weakly curved, but another possibility is to take the extra dimension be strongly curved or warped by a large cosmological constant. This models are based on solutions for five-dimensional background metric obtained

by L. Randall and R. Sundrum in Refs. [11, 12]. The form of the metric is

$$\mathrm{d}s^2 = a^2(z)\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2 \tag{5.31}$$

where z is the extra dimension and a is some function depending on it. Until now the energy density of the brane itself has been ignored, i.e., the gravitational field that the brane produces. Here we shall see that a gravitating brane induces an interesting geometry in multi-dimensional space, and that a number of novel properties emerge.

5.2.1 RS-1 Model

Let us consider two 3-branes [11]. One of the 3-branes are located at $\phi = 0$ with positive tension, the other with negative tension located at $\phi = \pi$, where $0 \leq \phi \leq \pi$. Allowing the negative tension brane to vibrate freely give rise to physical excitations of arbitrarily large negative energy [64]. To overcome this problem, the branes are placed at fixed points of an orbifold, that is, the fifth dimension is compactified on an orbifold $S^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is defined by the transformation $\phi \to -\phi$. An Orbifold is defined as the quotient space $\Gamma \equiv \mathcal{M}/\mathrm{G}$, where $\mathcal{M}$ is some manifold and G is a discrete group acting on $\mathcal{M}$. As we see that there are fixed points in $\mathcal{M}$, which do not transform under the action of G (For detailed discussion see [65]). The action in this model is of the form

$$S = S_{\mathrm{bulk}} + S_{vis} + S_{hid}, \tag{5.32}$$

where S_{vis} and S_{hid} denote the actions on the branes. For the bulk action we take the five-dimensional gravity with a bulk cosmological constant,

$$S_{\mathrm{bulk}} = \int \mathrm{d}^4x \int_{-\pi}^{\pi} \mathrm{d}\phi\, \sqrt{-G}(2M^3\, R - \Lambda), \tag{5.33}$$

where M denotes the five-dimensional Planck mass and G_{MN} is the five-dimensional metric. The branes are located in ϕ and the brane coordinates are identified with the remaining 5-D coordinates x^μ. Then the induced metrics on the branes are

$$g^{vis}_{\mu\nu} = G_{\mu\nu}|_{\phi=\pi} \qquad g^{hid}_{\mu\nu} = G_{\mu\nu}|_{\phi=0} \tag{5.34}$$

It is assumed that the fields being localized on the branes are in the trivial vacuum and take into account only nonzero vacuum energies on the branes. Calling those vacuum energies V_{vis} and V_{hid}, the brane actions read

$$S_{vis} + S_{hid} = -\int \mathrm{d}^4x\, (V_{vis}\sqrt{-g^{vis}} + V_{hid}\sqrt{-g^{hid}}) \tag{5.35}$$

Instead of working out the solutions to the system on an interval $S^1/\mathbb{Z}_2$, it is technically easier to construct a solution in a non compact space, such that the solution is periodic in

$$\phi \equiv \phi + 2\pi, \tag{5.36}$$

and even under

$$\phi \to -\phi. \tag{5.37}$$

Then the equations of motion read

$$\begin{aligned}\sqrt{-G}(R_{MN} - \frac{1}{2}G_{MN}R) &= -\frac{1}{4M^3}[\Lambda\sqrt{-G}G_{MN} \\ &+ V_{vis}\sqrt{-g^{vis}}g^{vis}_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N}\delta(\phi-\pi) \\ &+ V_{hid}\sqrt{-g^{hid}}g^{hid}_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N}\delta(\phi)].\end{aligned} \tag{5.38}$$

The delta functions appearing in Eq.(5.38) are defined on a real line. The most general metric ansatz possessing a four-dimensional Poincarè transformation as isometry is

$$\mathrm{d}s^2 = \mathrm{e}^{-\sigma(\phi)}\eta_{\mu\nu}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + r_c^2\,\mathrm{d}\phi^2 \tag{5.39}$$

where r_c is the radius of the extra dimension.

We could rescale ϕ such that the r_c dependence drops out, but that would change the periodicity condition [66]. Substituting this ansatz into the equation of motion yields

$$\frac{6\sigma'^2}{r_c^2} = -\frac{\Lambda}{4M^3}, \tag{5.40}$$

$$\frac{3\sigma''}{r_c^2} = \frac{V_{hid}}{4M^3 r_c}\delta(\phi) + \frac{V_{vis}}{4M^3 r_c}\delta(\phi-\pi) \tag{5.41}$$

The solution to Eq.(5.40) by direct integration

$$\sigma = r_c|\phi|\sqrt{\frac{-\Lambda}{24M^3}} + c \tag{5.42}$$

where c is the integration constant. Without loss of generality c can be chosen zero because it just amounts to an overall constant rescaling of the x^{μ}. So

$$\sigma = r_c|\phi|\sqrt{\frac{-\Lambda}{24M^3}} \tag{5.43}$$

The modulus function is defined as usual in the interval $-\pi < \phi < \pi$,

$$|\phi| = \begin{cases} -\phi & ,-\pi < \phi < 0 \\ \phi & ,0 < \phi < \pi \end{cases} \tag{5.44}$$

This ensures that the solution is even under $\phi \to -\phi$. We define the modulus function on the real line by the periodic continuation of Eq.(5.44). Away from the points at $\phi = 0$ and integer multiples of π, the second derivative of σ vanishes and Eq.(5.41) is fulfilled in those regions. In order to take into account the delta function sources in Eq.(5.41), this equation can be integrated over an infinitesimal neighborhood around the location of the brane sources. Integration near zero,

$$\int_{-\varepsilon}^{\varepsilon} \sigma'' \mathrm{d}\phi = \frac{r_c V_{hid}}{12M^3} \tag{5.45}$$

it is found

$$2r_c\sqrt{\frac{-\Lambda}{24M^3}} = \frac{r_c V_{hid}}{12M^3} \tag{5.46}$$

From this equation

$$V_{hid} = 24M^3 k \tag{5.47}$$

where

$$k^2 = \frac{-\Lambda}{24M^3} \tag{5.48}$$

By the same way integration near π leads to

$$V_{vis} = -24M^3 k \tag{5.49}$$

So solution is obtained if V_{vis}, V_{vis}, Λ are related in terms of a single scale k,

$$V_{hid} = -V_{vis} = 24M^3 k, \qquad k = \frac{-\Lambda}{24M^3} \tag{5.50}$$

This gives rise to the constraints on the parameters of the model. These constraints can be thought of as fine-tuning conditions for a vanishing effective cosmological constant in four-dimensions and this is equivalent to the usual cosmological constant problem. Solution is then

$$\mathrm{d}s^2 = \mathrm{e}^{-2kr_c|\phi|}\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + r_c^2\,\mathrm{d}\phi^2. \tag{5.51}$$

where k^2 is defined in Eq.(5.50), and k is taken to be positive (for a negative k, ϕ can be redefined as $\pi - \phi$).

We observe that by taking into account the back reaction of the branes onto the geometry, we obtain a metric which depends on the position in the compact direction. For the particular model we consider this dependence is exponential. That opens up an interesting alternative explanation for the large hierarchy between the Planck scale and the electroweak scale. We take all the input scales (M, Λ, r_c) to be the order of the Planck scale. First we should check

whether this provides the correct four dimensional Planck mass. To this end, we expand a general four-dimensional metric around the solution (5.51)

$$\mathrm{d}s^2 = \mathrm{e}^{-2kr_c|\phi|}(\eta_{\mu\nu} + \tilde{h}_{\mu\nu}(x,\phi))\mathrm{d}x^\mu \mathrm{d}x^\nu + r_c^2\,\mathrm{d}\phi^2 \tag{5.52}$$

Here, $\tilde{h}_{\mu\nu}$ represents tensor fluctuations around Minkowski space and is the physical graviton of the four-dimensional effective theory (and is the massless mode in the Kaluza-Klein decomposition of $G_{\mu\nu}$). In principle we should also allow the four-four component of the metric r_c^2 to fluctuate. Since r_c is an integration constant, such fluctuations will be seen as massless scalars in the effective four-dimensional theory. This is common problem known as moduli stabilization problem. We will assume here that some unknown mechanism gives a mass to the fluctuations of G_{55} and take it to be frozen at the classical value r_c^2. One can use the gauge

$$g_{55} = -1, \qquad g_{5\mu} = 0 \tag{5.53}$$

As the next step the field $h_{\mu\nu}$ is decomposed over an appropriate system of orthonormalized functions:

$$h_{\mu\nu}(x,\phi) = \sum_{n=0}^{\infty} h_{\mu\nu}^{(n)} \chi_n(\phi) \tag{5.54}$$

where

$$\chi_0(\phi) = 2\sqrt{kr_c}\,\mathrm{e}^{-2kr_c|\phi|}, \tag{5.55}$$

$$\chi_n(\phi) = N_n\left[C_1 Y_2\left(\frac{m_n}{k}\mathrm{e}^{kr_c\phi}\right) + C_2 J_2\left(\frac{m_n}{k}\mathrm{e}^{kr_c\phi}\right)\right],\ (n \neq 0). \tag{5.56}$$

Here J_2 and Y_2 are the Bessel functions, N_n are the normalization factors. The boundary conditions on the branes, that are due to the δ-function terms, fix the constants $C_1 = Y_1(m_n/k)$ and $C_2 = -J_1(m_n/k)$ and lead to the eigenvalue equation

$$J_1(\beta_n \mathrm{e}^{-kr_c\pi})Y_1(\beta_n) - Y_1(\beta_n \mathrm{e}^{-kr_c\pi})J_1(\beta_n) = 0. \tag{5.57}$$

The numbers β_n are related to m_n by $m_n = \beta_n k\ \mathrm{e}^{-kr_c\pi}$. For small $n \geq 1$ this equation reduces to the approximate one: $J_1(\beta_n) = 0$, and β_n's are equal to $\beta_n = 3.83, 7.02, 10.17, 13.32, ...$ for $n = 1, 2, 3, 4, ...$ The zero mode field $h_{\mu\nu}^{(0)}(x)$ describes the massless graviton. Within the five-dimensional picture it appears as a state localized on hidden brane(the one of which vacuum energy is denoted by V_{hid}).

Substituting Eq.(5.52) into action we can calculate the zero mode sector of the effective theory

$$S_{eff} \supset \int \mathrm{d}^4x \int_{-\pi}^{\pi} \mathrm{d}\phi\ 2M^3 r_c\,\mathrm{e}^{-2kr_c|\phi|}\sqrt{-g}\,R^{(4)} \tag{5.58}$$

where $R^{(4)}$ denotes the four-dimensional Ricci scalar made out of $g_{\mu\nu}(x)$ ($g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$), in contrast to the five-dimensional Ricci scalar, R, made out of $G_{MN}(x,\phi)$. Because the low-energy fluctuations do not change the ϕ dependence (the effective fields depend on x alone), we can explicitly perform the ϕ integral to obtain a purely four-dimensional action. From this the four-dimensional Planck mass $M_{P\ell}$ is given by

$$M_{P\ell}^2 = M^3 r_c \int_{-\pi}^{\pi} \mathrm{d}\phi \, \mathrm{e}^{-2kr_c|\phi|} = \frac{M^3}{k}[1 - \mathrm{e}^{-2kr_c\pi}]. \tag{5.59}$$

This equation tells us that choosing five-dimensional scales of the order of the Planck scale gives the correct order of magnitude for the four-dimensional Planck scale and it can be inferred that $M_{P\ell}$ depends on weakly on r_c in the large kr_c limit. Even though the exponential has very little effect in determining the Planck scale, it plays a crucial role in the determination of the visible sector masses, as we will see now.

In order to determine the matter field Lagrangian we need to know the coupling of the 3-brane fields to the low-energy gravitational fields, in particular, the metric $g_{\mu\nu}(x)$. From Eq.(5.34) it is easy to see that $g_{\mu\nu}^{hid} = g_{\mu\nu}$. However, this is not the case for the visible sector fields; $g_{\mu\nu}^{vis} = \mathrm{e}^{-2kr_c\pi} g_{\mu\nu}$. By properly normalizing the fields we can determine the physical masses. Consider for example a fundamental Higgs field being located at the visible brane,

$$S_{vis} \supset \int \mathrm{d}^4x \, \sqrt{-g_{vis}} \, \left[g_{vis}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - \upsilon_0)^2 \right], \tag{5.60}$$

which contains one mass parameter υ_0. Substituting Eq.(5.34) into this action yields

$$S_{vis} \supset \int \mathrm{d}^4x \, \sqrt{-g}\, \mathrm{e}^{-4kr_c\pi} \, \left[g^{\mu\nu} \mathrm{e}^{2kr_c\pi} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - \upsilon_0)^2 \right], \tag{5.61}$$

where the overall exponential factor originates from the determinant. Rescaling the Higgs field H such that the kinetic term in Eq.(5.61) takes its canonical form $H \to \mathrm{e}^{kr_c\pi} H$, we obtain

$$S_{vis} \supset \int \mathrm{d}^4x \, \sqrt{-g} \, \left[g^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - \mathrm{e}^{-2kr_c\pi}\upsilon_0)^2 \right]. \tag{5.62}$$

This means that a symmetry-breaking scale which is written as υ_0 into the model effectively is multiplied by a factor of $\mathrm{e}^{-kr_c\pi}$.

$$\upsilon_0 \to \upsilon_{eff} = \mathrm{e}^{-kr_c\pi}\upsilon_0. \tag{5.63}$$

This result is completely general: any mass parameter m_0 on the visible 3-brane in the fundamental higher-dimensional theory will correspond to a physical mass

$$m_0 \to m_{eff} = \mathrm{e}^{-kr_c\pi} m_0 \tag{5.64}$$

when an effective description in which kinetic terms are canonically normalized. If the quantity kr_c is chosen to be the order of 10, the exponential in Eq.(5.64) takes the Planck sized input masses to effective masses of the order of a TeV. Hence, in the above model it has been obtained that the TeV scale from the Planck scale without introducing large numbers, provided we live on the visible brane. Some problems with the above model are that the fine-tuning between the weak scale and the Planck scale is replaced by the fine-tuning between k and the brane separation r_c. This is related to the problem of treating the scalar field that describes the relative motion between the branes. For consistency reasons, this so-called radion has to be a massive field with the correct expectation value in order to maintain stability of the solution. For more detail about stability see Appendix A.

5.2.2 RS-2 Model

In this section we are going to consider a variant of the model presented in section 5.2.1, where the second brane is removed. To solve hierarchy problem, it was necessary that the observers live on the visible brane. However, we now give up the goal of solving the hierarchy problem. The construction of the single brane solution is very simple. The extra dimension is not compact anymore and therefore we use the coordinate y instead of ϕ. We do not impose the periodicity condition but still require a $\mathbb{Z}_2$ symmetry under

$$y \to -y \tag{5.65}$$

Further, we remove S_{vis} from the action (5.32). Since the extra dimension is not compact, we can rescale y in order to remove the r_c dependence of ansatz (5.51). Without loss of generality we take $r_c = 1$. Thus, in the single brane case, the solution for the metric is

$$\mathrm{d}s^2 = \mathrm{e}^{-2k|y|}\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2. \tag{5.66}$$

Let us consider small gravitational fluctuations around the background (5.66)

$$G_{MN} = \mathrm{e}^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu} \tag{5.67}$$

Now we need to determine whether the spectrum of the general fluctuations is consistent with four-dimensional experimental gravity. In order to get a Kaluza-Klein reduction down to four-dimensions, we need to do a separation of variables: $h(\mathbf{x}, y) = \psi(y)\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}$, where $\mathbf{p}^2 = m^2$ and m^2 permits a solution to the linearized equation of motion for tensor fluctuations following from Einstein's equations expanded about Eq.(5.66):

$$\left[\frac{-m^2}{2}\mathrm{e}^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2\right]\psi(y) = 0, \tag{5.68}$$

where boundary conditions which is Eq.(5.65) give rise to consider only even functions of y, describing the infinite half-line. Here the indices μ, ν are omitted without loss of generality, because we are free to choose the gauge, where $\partial^\mu h_{\mu\nu} = h^\mu_\mu = 0$.

It is more convenient to do a change of variables in such a way that the terms with m^2 will be unit. As we will see that then the form of the resulting equation become an analog non relativistic quantum mechanical problem, i.e. Schrödinger equation: So if the new coordinate z

$$z \equiv \frac{\mathrm{sgn}(y)}{k}\left(\mathrm{e}^{k|y|} - 1\right). \tag{5.69}$$

With

$$\hat{\psi}(z) = \psi(y)\,\mathrm{e}^{\frac{k|y|}{2}} \tag{5.70}$$

then Eq.(5.68) takes the following form

$$\left[-\frac{1}{2}\partial_z^2 + U(z)\right]\hat{\psi(z)} = m^2\hat{\psi(z)}, \tag{5.71}$$

Analogous to our previous discussion we plan to expand the solution h into a series eigenfunctions, that is, we are looking for solutions of the Eq.(5.71), which is exactly the same as Shrödinger equation with potential U, where

$$U(z) = \frac{15k^2}{8(k|z|+1)^2} - \frac{3k}{2}\delta(z). \tag{5.72}$$

The general behaviour of solution can be understood by the general shape of this analog non relativistic potential. Roughly speaking, this problem is equivalent to one dimensional motion of a particle in the presence of this potential. Near $z = 0$ the dominant term is coming from the Dirac delta-potential. So this leads to the localization of the particle near $z = 0$, i.e. in our case the particle will be graviton. However, to be more accurate let us consider the problem in more detail:

It can be easily note that the solution has both discrete and continuous spectrum. As it is seen from Eq.(5.72), potential consists of two parts; the one which is Dirac delta-function is responsible for a single normalizable state mode [67]. The other part corresponds to continuum modes. Let us discuss first zero mode, i.e. the solution to Eq.(5.71) with $m^2 = 0$. The zero mode is found to be

$$\hat{\psi}_0(z) \equiv \hat{\psi}(0, z) = \frac{N_0}{(k|z| + 1)^{3/2}}, \tag{5.73}$$

Now, we take $m > 0$. For $z > 0$ the general solution to the above equation can be written as a superposition of Bessel functions

$$\hat{\psi}(m, z) = \sqrt{|z| + \frac{1}{k}} \left[c_1 J_2 \left(m \left(|z| + \frac{1}{k} \right) \right) + c_2 Y_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \right], \tag{5.74}$$

where J_ν denotes the Bessel functions of the first kind whereas Y_ν stands for the Bessel functions of the second kind and $c_{1,2}$ are constants to be fixed below. Because the solution (5.74) is written as a function of $|z|$, the second derivative with respect to z in Eq.(5.71) will yield a term containing a $\delta(z)$ (and other terms). One can fix the ratio c_1/c_2 by matching the factor in front of this delta function with the factor in front of the delta function in Eq.(5.74). We will do this in an approximate way. The most severe corrections to Newton's law are to be expected from gravitons with small m (because they carry interactions over longer distances). In matching the coefficients of the delta functions, only a neighborhood around $z = 0$ matters. Therefore, we replace the Bessel functions by their asymptotics for small arguments, which are

$$J_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \sim \frac{m^2 \left(|z| + \frac{1}{k} \right)^2}{8}, \tag{5.75}$$

$$Y_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \sim -\frac{4}{\pi m^2 \left(|z| + \frac{1}{k} \right)^2} - \frac{1}{\pi}. \tag{5.76}$$

Plugging the asymptotic approximation into Eq.(5.74) and then into Eq.(5.71) one finds that the overall coefficient in front of the delta function vanishes if

$$\frac{c_1}{c_2} = \frac{4k^2}{\pi m^2}. \tag{5.77}$$

Hence, our general solution (5.74) reads

$$\hat{\psi}(m, z) = N_m \sqrt{|z| + \frac{1}{k}} \left[Y_2 \left(m \left(|z| + \frac{1}{k} \right) \right) + \frac{4k^2}{\pi m^2} J_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \right] \tag{5.78}$$

where we replaced $c_2 = N_m$ because this remaining integration constant will turn out to depend on the eigenvalue m.

Recall that the extra dimension y (or z) is not compact. Thus the eigenvalue m is continuous. Therefore, we normalize

$$\int \mathrm{d}z\, \hat{\psi}\,(m,z)\, \hat{\psi}\,(m',z) = \delta\,(m-m')\,, \tag{5.79}$$

for $m, m' > 0$. For $m \geq 0$ we impose the normalization condition

$$\int \mathrm{d}z\, \hat{\psi}_0\,(z)\, \hat{\psi}\,(m,z) = \delta_{m,0}, \tag{5.80}$$

such that the completeness relation reads

$$\hat{\psi}_0\,(z)\, \hat{\psi}_0\,(z') + \int_0^\infty \mathrm{d}m\, \hat{\psi}\,(m,z)\, \hat{\psi}\,(m,z') = \delta\,(z-z')\,. \tag{5.81}$$

The orthonormalization condition (5.79) fixes N_m. It turns out that the computation simplifies essentially in the approximation where the arguments of the Bessel functions are large, since the corresponding asymptotics yields plane waves. Explicitly, for large mz the Bessel functions are approximated by

$$\sqrt{z} J_2\,(mz) \;\sim\; \sqrt{\frac{2}{\pi m}} \cos\left(mz - \frac{5\pi}{4}\right), \tag{5.82}$$

$$\sqrt{z} Y_2\,(mz) \;\sim\; \sqrt{\frac{2}{\pi m}} \sin\left(mz - \frac{5\pi}{4}\right). \tag{5.83}$$

Because we are mainly concerned about large distance modifications of Newton's law we focus on the contribution of the "light" modes ($\frac{m^2}{k^2} \ll 1$). (Recall that k is of the order of the Planck mass.) Then Eq.(5.79) yields for the normalization constant (for $m > 0$)

$$N_m = \frac{\pi m^{\frac{5}{2}}}{(4k^2)}. \tag{5.84}$$

The condition (5.80) is satisfied for $m > 0$ to a good approximation. Evaluating Eq.(5.80) for $m = 0$ fixes

$$N_0 = \sqrt{k}. \tag{5.85}$$

Now, we expand $\hat{h}\,(x,z)$ into eigenfunctions $\hat{\psi}_0\,(z)$ and $\hat{\psi}\,(m,z)$ with x dependent coefficients $\varphi_m\,(x)$

$$\hat{h}\,(x,z) = \varphi_0\,(x)\, \hat{\psi}_0\,(z) + \int_0^\infty \mathrm{d}m\, \varphi_m\,(x)\, \hat{\psi}\,(m,z)\,. \tag{5.86}$$

In the presence of a point particle with mass μ at the origin, the non relativistic limit of linearized equation for h is modified as

$$\left[\Delta_3 - e^{-2k|y|}\left(\partial_y^2 + 4k\delta\,(y) - 4k^2\right)\right] h\,(x,y) = G\mu\delta^3(x)\delta(y), \tag{5.87}$$

By plugging the ansatz (5.86) into the wave equation for $\hat{h}$, we find that for $m \geq 0$ and $r = |x|$

$$\varphi_m(x) = -\frac{G\mu}{r} e^{-mr} a_m, \tag{5.88}$$

with the constants a_m taken such that

$$a_0 \hat{\psi}_0(z) + \int \mathrm{d}m \, a_m \hat{\psi}(m, z) = \delta(z). \tag{5.89}$$

Comparison with Eq.(5.81) yields

$$a_0 = \hat{\psi}_0(0) \quad , \quad a_m = \hat{\psi}(m, 0). \tag{5.90}$$

In the current setup we are interested in corrections to Newton's law as an observer on the brane at the origin would measure them. Defining the four dimensional Newton constant G_4 as

$$G_4 = Gk, \tag{5.91}$$

we find from Eq.(5.86)

$$\hat{h}(x, 0) = h(x, 0) = -\frac{G_4 \mu}{r} \left(1 + \int_0^\infty \mathrm{d}m \, \frac{m}{k^2} e^{-mr}\right), \tag{5.92}$$

where once again we took into account only modes with $m/k \ll 1$ such that we could use the asymptotics (5.75) and (5.76) in order to evaluate $\hat{\psi}(m, 0)$. Finally, performing the integral in (5.92) leads to

$$h(x, 0) = -G_4 \frac{\mu}{r} \left(1 + \frac{1}{r^2 k^2}\right). \tag{5.93}$$

Therefore, Fluctuations around the solution include a state with zero mass, which describes the massless graviton, and massive states. The massless graviton is localized on the brane, hence no contradiction with the Newton's law appears at distances $r \gg k^{-1}$ with the parameter k chosen to be $k \sim M_{P\ell}$. Non-zero KK states are non-localized and form the continuous spectrum starting from $m = 0$ (no mass gap). The RS2 model gives an elegant example of localized gravity with non-compact extra dimension.

For k being of the order of the Planck mass (5.93) is in very good agreement with the experimental values. This may look a bit surprising. Even though the extra dimension is not compact, we obtain a four dimensional Newton potential for observers who live on the brane at $y = 0$. This non trivial result finds its explanation in the exponentially warped geometry. It is this geometry which is responsible for the fact that the amplitude of the zero mode has its maximum at the brane and vanishes rapidly for finite z. On the other hand, the massive

modes reach their maximal amplitudes asymptotically far away from the brane. Therefore, they have very little influence on the gravitational interactions on the brane, although the masses of the extra gravitons can be arbitrarily small.

The corrections to Newton's law has power law behaviour at large r, in contrast to theories with compact dimensions where the corrections are suppressed exponentially at large distances. However, this correction is negligible at distances exceeding the anti-de Sitter radius k^{-1}. It has been explicitly shown in [68] and [69] that the tensor structure of the gravitational interactions at large distances indeed correspond to (the weak field limit) the four-dimensional general relativity. Note that the radion is absent in RS2 set up. We have already mentioned that in RS2 set up with one brane, extra dimension does not help to solve the hierarchy problem. It was pointed out, however, that modest extension of this set up leads to exponential hierarchy even if extra dimension is infinite (The Lykken-Randall model [70]). This model is a combination of the RS1 and RS2 models. Brane 1 is located at $y = 0$ and its tension determines the same background solution for the metric as in the RS2 model. Brane 2 is regarded as a probe brane, i.e. the tension $V_2 \ll V_1$, so that it does not affect the solution. The second brane is located at $z = z_c$, and the value of r_c is adjusted in such a way that

$$M_{P\ell}\, \mathrm{e}^{-kr_c\pi} \sim M_{P\ell} \cdot 10^{-15} \sim 1 \text{ TeV}. \tag{5.94}$$

This ensures that the hierarchy problem is solved on the second brane. Therefore, it is considered to be our brane, i.e. the brane where the SM is localized. Randall-Sundrum set up is analogous to the Horova-Witten scenario [71],[72], [73] which arises in M-theory. For a discussion of how the scenario above may arise from string theory compactifications is considered in [74], and supergravity solutions which also exhibit exponential hierarchies are worked out in [75]

Randall and Sundrum's theory may explain the existence of dark matter - which is invisible and makes up 90 percent of our universe. Dark matter emits or absorbs no light and is evident only through its gravity. It could simply come from another universe from which we can sense gravitons. In addition, Randall and Sundrum's theory can explain why dark matter is usually found in the halos around galaxies. According to the theory, large masses on different branes are attracted to each other through hyperspace with mutual gravitational pulls. Thus, a galaxy on our universe may be mirrored by a galaxy from another universe, with only the gravity from its edges apparent. The static Randall-Sundrum solution has been also extended to time dependent solutions and their cosmological properties have been extensively studied in [76] - [89].

5.3 Some Models of Fermion Localization in The Context of RS Models

5.3.1 Domain Wall Solutions in RS Models

In this thesis, by using the Randall-Sundrum type metric with scalar field which Ichinose [90, 91] used, we will show that the fermions interacting with this field are localized. The scalar potential $V(\phi)$ and metric tensor g_{AB} under consideration are,

$$V(\phi) = \lambda(\phi^2 - \upsilon^2)^2, \tag{5.95}$$

$$\mathrm{d}s^2 = \mathrm{e}^{-2\sigma(y)}\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2. \tag{5.96}$$

The action in this model is given by:

$$S = \int \mathrm{d}^5x\,\sqrt{-g}\left[\Lambda - \frac{1}{2}M^3R - \frac{1}{2}g^{AB}\partial_A\phi\partial_B\phi - V(\phi)\right]. \tag{5.97}$$

Equations of motion corresponding to this action are

$$6M^3(\frac{\mathrm{d}\sigma}{\mathrm{d}y})^2 = \frac{1}{2}(\frac{\mathrm{d}\phi}{\mathrm{d}y})^2 - \lambda(\phi^2 - \upsilon^2)^2 + \Lambda \tag{5.98}$$

$$3M^3\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} = (\frac{\mathrm{d}\phi}{\mathrm{d}y})^2. \tag{5.99}$$

Even though the equations of motion are nonlinear, they have been solved by usual perturbative technique. Now let us briefly discuss the solution to the equations of motion given in [13]. With the convenient dimensionless rescaled variables

$$\varphi = \upsilon^{-1}\phi, \tag{5.100}$$

$$z = \sqrt{\frac{3\Lambda M^3}{2\upsilon^4}}y. \tag{5.101}$$

Although σ is dimensionless, it is useful to define new dimensionless variables s and ζ given by

$$s = \frac{3M^3}{\upsilon^2}\sigma, \tag{5.102}$$

$$\zeta = \frac{\mathrm{d}s}{\mathrm{d}z}. \tag{5.103}$$

The equations of motion in terms of these variables read

$$\zeta' = (\varphi')^2, \tag{5.104}$$

$$\zeta^2 = 1 + \frac{3M^3}{4\upsilon^2}\varphi'^2 - \frac{\lambda\upsilon^4}{\Lambda}(\varphi^2 - 1)^2, \tag{5.105}$$

where the symbol $'$ denotes the differentiation with respect to z. These are first order differential equations whose general behaviour can be visualized in a phase space constructed by (φ, ζ). Flows of phase points are determined by

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\varphi} = \frac{\zeta'}{\varphi'} = \varphi' = \pm A\sqrt{(1-\varphi^2)^2 - \epsilon(1-\zeta^2)}, \tag{5.106}$$

where the dimensionless parameters A and ϵ are given by

$$A = \sqrt{\frac{4\lambda v^6}{3\Lambda M^3}}, \tag{5.107}$$

$$\epsilon = \frac{\Lambda}{\lambda v^4}. \tag{5.108}$$

It is sufficient to consider only the branch corresponding to the positive sign in Eq.(5.106). Since $\mathrm{d}\zeta/\mathrm{d}\varphi$ is a real function, it is clear that the allowed region of the phase space is restricted by the condition $C_\epsilon(\varphi, \zeta) \geq 0$, where

$$C_\epsilon(\varphi, \zeta) = (1-\varphi^2)^2 - \epsilon(1-\zeta^2). \tag{5.109}$$

The shape of the boundary of the forbidden region, which we name it the island, depends on ϵ. If the islands for different values of ϵ are drawn, it is seen that they are symmetric under the separate reflections of φ and ζ. When we assume that the domain wall is located at $z = 0$, it becomes clear that the origin of the phase space, $\varphi = \zeta = 0$, should be in the allowed region of the phase space. It is possible only when $0 < \epsilon \leq 1$.

The coefficient A given in Eq.(5.106) determines the initial flow direction at the origin $\varphi = \zeta = 0$. The same equation shows that for a given shape of island, which is determined by ϵ, flows in the phase space either terminate at the island or diverge indefinitely depending on A. There is unique $A(\epsilon)$ by which the flow line starting at the origin terminates at $\varphi = \zeta = 1$. When A is less than this critical value $A(\epsilon)$, the flow line reaches the island, and stays there forever. But if it is slightly larger than $A(\epsilon)$, the flow bypasses the island, and runs indefinitely. In this case it becomes unstable. The numerical result of the computation of this behaviour is given in [13]. To understand stabilities one should find $A(\epsilon)$ corresponding to the critical flow. Even though the flow equation is highly nonlinear, one may use the usual perturbation technique to solve it. We assume that φ and ζ both reach the critical values, $\varphi \to 1$ and $\zeta \to 1$, as $z \to \infty$. We have seen, for the stable solution, that $0 < \epsilon \leq 1$. It allows us to expand the flow equation (5.106) in terms of ϵ,

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\varphi} = A[1 - \varphi^2 - \frac{\epsilon}{2}\frac{1-\zeta^2}{1-\varphi^2} + \frac{\epsilon^2}{8}\frac{(1-\zeta^2)^2}{(1-\varphi^2)^3} + \ldots]. \tag{5.110}$$

One can solve this equation perturbatively under the conditions

$$\zeta(\varphi = 0) = 0, \qquad \zeta(\varphi = 1) = 1. \tag{5.111}$$

Firstly, we find $A(\epsilon)$ corresponding to this critical flow, and then solve (15). The curve $A(\epsilon)$ in the parameter space (ϵ, A) divides it up into the stable and unstable regions. The general formula for $A(\epsilon)$ is hidden in

$$\int_0^1 \frac{\mathrm{d}\zeta}{\mathrm{d}\varphi}\mathrm{d}\varphi = 1. \tag{5.112}$$

Using the ϵ independent part of Eq.(5.110), we have

$$1 = A(\epsilon)\int_0^1 (1-\varphi^2)\mathrm{d}\varphi = \frac{2}{3}A(\epsilon). \tag{5.113}$$

That is, to the order of $\mathcal{O}(\epsilon^0)$, $A(\epsilon) = \frac{3}{2}$. Then by Eq.(5.110), one has

$$\zeta = \frac{3}{2}(\varphi - \frac{1}{3}\varphi^3). \tag{5.114}$$

Substituting this in Eq.(5.110) again, one gets the following equation,

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\varphi} = A(\epsilon)[1-\varphi^2 - \frac{\epsilon}{2}\frac{(1-\frac{9}{4}(\varphi-\frac{\varphi^3}{3})^2)}{(1-\varphi^2)}]. \tag{5.115}$$

From this we find that $A(\epsilon)$ and $\zeta(\varphi)$, to the order $\mathcal{O}(\epsilon)$, are

$$\begin{aligned} A(\epsilon) &= \frac{\frac{3}{2}}{1-\frac{19}{40\epsilon}}, & (5.116)\\ \zeta(\varphi) &= \zeta_0(\varphi) + \epsilon\zeta_1(\varphi). & (5.117) \end{aligned}$$

Here ζ_0 is the same as Eq.(5.115), and

$$\zeta_1(\varphi) = -\frac{3}{80}(\varphi - 2\varphi^3 + \varphi^5). \tag{5.118}$$

To solve φ as a function of z, we combine Eq.(5.106) and Eq.(5.110),

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \frac{\mathrm{d}\zeta}{\mathrm{d}\varphi} = A(\epsilon)[1-\varphi^2 - \frac{\epsilon}{2}\frac{(1-\frac{9}{4}(\varphi-\frac{\varphi^3}{3})^2)}{(1-\varphi^2)}]. \tag{5.119}$$

This can be integrated to give the following

$$z = \frac{\epsilon}{12}\varphi + \frac{10-\epsilon}{30}\log\frac{1+\varphi}{1-\varphi}, \tag{5.120}$$

It is valid up to the order $\mathcal{O}(\epsilon)$. Solving φ in terms of z, one gets

$$\phi = \varphi\upsilon = \upsilon\left(\tanh\frac{3z}{2} + \frac{3}{2}\epsilon(\frac{z}{10} - \frac{1}{12}\tanh\frac{3z}{2})(1-\tanh^2\frac{3z}{2}) + \mathcal{O}(\epsilon^2)\right). \tag{5.121}$$

which has interesting z linear term. By neglecting higher order terms in ϵ, σ is given in the form

$$\begin{aligned} \sigma &= \frac{1}{15}(10-\epsilon)\ln[\cosh\frac{3z}{2}] + \frac{\epsilon}{60}\mathrm{sech}^2\frac{3z}{2} \\ &+ \frac{3\epsilon}{80}\mathrm{sech}^4\frac{3z}{2} - \frac{\epsilon}{10}z\tanh\frac{3z}{2} + \frac{\epsilon}{20}z\,\mathrm{sech}^2\frac{3z}{2}\tanh\frac{3z}{2} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{5.122}$$

5.3.2 Erdem's Model for Fermion Localization

In [14], it is shown that the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}g^{BC}\partial_B\phi_1\partial_C\phi_1 + \frac{1}{2}g^{BC}\partial_B\phi_2\partial_C\phi_2 \\ &+ \frac{3\mu^2}{2\lambda}\mathrm{e}^{2\sqrt{\lambda}/\mu\phi_1} + \frac{3\mu^2}{2\lambda}\mathrm{e}^{-2\sqrt{\lambda}/\mu\phi_2} \\ &+ V_1(\sigma)\left(|\phi_1| - \phi_3\right) + V_2(\sigma)\left(|\phi_2| - \phi_3\right) \end{aligned} \tag{5.123}$$

with

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B = \mathrm{e}^{2A}\eta_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu - (3ay^2+b)^2\mathrm{e}^{2B}\mathrm{d}y^2, \tag{5.124}$$

where $A = -\tanh\eta(y), \qquad B = -2\ln\cosh\eta(y) - \tanh\eta(y),$

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \qquad \mu = 0,1,2,3.$$

leads to domain wall and anti-domain wall solutions given by

$$\phi_1 = \phi_3 = \phi_{cl} = \frac{\mu}{\sqrt{\lambda}}\tanh\eta, \qquad \phi_2 = \phi_{Acl} = -\frac{\mu}{\sqrt{\lambda}}\tanh\eta \tag{5.125}$$

provided that $V_1(\sigma) = -V_2(\sigma)$ and

$$V_1(\sigma) = \frac{\mu}{\sqrt{\lambda}}\left(-5\,\mathrm{e}^{2\tanh\eta} + \frac{\eta''}{(\eta')^2(1-\tanh^2\eta)} - \frac{2\tanh\eta}{(1-\tanh^2\eta)}\right), \tag{5.126}$$

One considers fermions interacting with the domain wall and the anti-domain wall pair as follows:

$$\begin{aligned}\mathcal{L} &= i\bar{\Psi}\Gamma^\mu D_\mu\Psi + i\bar{\Psi}\Gamma^4\partial_4\Psi + g_1\bar{\Psi}\phi_1\Psi + g_2\bar{\Psi}\phi_2\Psi \\ &= i\bar{\Psi}\gamma^\mu\mathrm{e}^{\tanh\eta}D_\mu\Psi + i\bar{\Psi}(-i\gamma_5)\frac{1}{(3ay^2+b)}(1-\tanh^2\eta)^{-1}\mathrm{e}^{\tanh\eta}\frac{\partial\Psi}{\partial y} \\ &+ g_1\bar{\Psi}\phi_1\Psi + g_2\bar{\Psi}\phi_2\Psi, \end{aligned} \tag{5.127}$$

where

$$\Gamma^\mu = \mathrm{e}^{\tanh\eta}\gamma^\mu \tag{5.128}$$

$$\Gamma^4 = -i\gamma_5\frac{\mathrm{e}^{\tanh\eta}}{(3ay^2+b)}(1-\tanh^2\eta)^{-1} \tag{5.129}$$

$$D_\mu = \partial_\mu. \tag{5.130}$$

Hence in the presence of a background consisting of a domain wall-anti-domain wall pair, the five dimensional Dirac equation is

$$i\mathrm{e}^{\tanh\eta}\gamma^\mu D_\mu\Psi + \mathrm{e}^{\tanh\eta}\frac{1}{3ay^2+b}(1-\tanh^2\eta)\gamma_5\frac{\partial\Psi}{\partial y} + g\phi_{cl}\Psi = 0 \tag{5.131}$$

where $g = g_1 - g_2$ and $\phi_{Acl} = -\phi_{cl}$. It is considered that the solutions which propagate in the usual four dimensions as free fields whose form is

$$\Psi = \mathrm{e}^{-i[(\mathrm{e}^{2A})(p_0 x_0 - \mathbf{p}\cdot\mathbf{x})]}\chi(y). \tag{5.132}$$

At $\eta = 0$ for the free field solutions becomes

$$\gamma^\mu p_\mu \Psi + m\gamma_5 \Psi = 0 \tag{5.133}$$

where

$$m = \frac{1}{\chi}\left(\frac{\partial\chi}{\partial\eta}\right)|_{\eta=0} = \begin{pmatrix} m_L & 0 \\ 0 & m_R \end{pmatrix} \tag{5.134}$$

Then after replacing Eq.(5.132) and Eq.(5.133) in Eq.(5.131) one gets

$$-m\mathrm{e}^{-\tanh\eta}\gamma_5\Psi + \frac{1}{3ay^2+b}(1-\tanh^2\eta)^{-1}\mathrm{e}^{\tanh\eta}\gamma_5\frac{\partial\Psi}{\partial y} + \beta\tanh\eta\Psi = 0, \tag{5.135}$$

where $\beta = g\frac{\mu}{\sqrt{\lambda}}$. Eq.(5.135) may be written in terms of $\Psi_L = \frac{1}{2}(1-\gamma_5)\Psi$, and $\Psi_R = \frac{1}{2}(1+\gamma_5)\Psi$ as

$$\begin{aligned} & \frac{\partial\Psi_L}{\partial y} - \frac{\partial\Psi_R}{\partial y} + \eta'(1-\tanh^2\eta) \\ \times\ & \left[\left(\beta\tanh\eta\mathrm{e}^{-\tanh\eta} - m_L\mathrm{e}^{-2\tanh\eta}\right)\Psi_L\right. \\ +\ & \left.\left(\beta\tanh\eta\mathrm{e}^{-\tanh\eta} + m_R\mathrm{e}^{-2\tanh\eta}\right)\Psi_R\right] = 0 \end{aligned} \tag{5.136}$$

The solutions of Eq.(5.136) are

$$\begin{aligned} \Psi_R &= \exp\left[-\frac{1}{2}m_R\mathrm{e}^{-2\tanh\eta} - \beta(1+\tanh\eta)\mathrm{e}^{-\tanh\eta}\right]\psi_R, \\ \Psi_L &= \exp\left[-\frac{1}{2}m_L\mathrm{e}^{-2\tanh\eta} + \beta(1+\tanh\eta)\mathrm{e}^{-\tanh\eta}\right]\psi_L. \end{aligned} \tag{5.137}$$

where ψ is the solution of $(i\gamma^\mu\partial_\mu + m\gamma_5)\psi = 0$.

Different profiles of Ψ_R and Ψ_L in the extra dimension gives an explanation of the source of chirality. One gets phenomenologically interesting values for some of the values of the parameters. If one assumes that the photon is localized in a narrow range of η where the magnitude of the Ψ_L and Ψ_R are almost the same while the gauge bosons of the weak interactions can penetrate into the bulk more deeply (where the average magnitude of Ψ_R is suppressed with respect to that of Ψ_L) then one may explain why the electromagnetic interactions are vector-like, while weak interactions are chiral. For example for $m_L = 3$, $m_R = -0.3$ and $\beta = 3$ the average magnitude of the y dependent part of Ψ_R is about the same as Ψ_L at $-0.4 < \eta < -0.3$, while most of the values of η the y dependent part

in Ψ_L is much greater than that of Ψ_R. One may assume that the photon is localized about $\eta \simeq -0.35$(for instance, in the interval $-0.37 < \eta < -0.33$), while the weak bosons propagate in the region where $-0.3 < \eta < 0.4$. The average density of Ψ_L in the region $-0.3 < \eta < 0.4$ is much higher than that of Ψ_R. This may explain why the right handed weak currents are highly suppressed with respect to the left handed ones. The fact that the neutral weak currents have a right handed component while the charged weak currents are purely left handed could be understood if the wave function of the W bosons (compared to the wave function of the Z boson)is assumed to be localized in a smaller region, where the average value of Ψ_L is much greater than the average value of Ψ_R when compared to the broader region where the Z boson is localized. For example if we take $m_L = 3$, $m_R = -0.3$, $\beta = 3$ and assume that Z is localized in the region $-0.3 < \eta < 0.4$, while the W bosons are localized in $0 < \eta < 0.3$, then the average value of Ψ_L interacting with Z bosons is about 15 times that of the average value of Ψ_R while the average value of Ψ_L interacting with W bosons is about 50 times that of the Ψ_R. So in this way Z bosons have an appreciable amount of vector interactions, while W bosons are effectively purely left handed.

In fact, each of the curves describing the η dependence of Ψ_L and Ψ_R corresponds to three curves, the same in form but translated in the y-direction because to each value of $\eta = ay^3 + by + c$ there exist three values of y in general. One can assume that each of these equation $\eta(y) = z_{L(R)}$ (where $z_{L(R)}$ denotes the values of η overlapping with our brane) has three distinct real roots for each value of $z_{L(R)}$. As long as we chose the width of the brane in the fifth dimension sufficiently small one can find such pieces of curves provided the equation $\eta = z_{L(R)}$ has three distinct real roots for one value of $z_{L(R)}$ because the variation of $z_{L(R)}$ corresponds to the variation of the location of the curves $\eta = ay^3 + by + c$ in the η- direction. This does not change the property that there are three distinct real roots provided the variation is small enough (i.e. the width of the brane is small enough). These three curves could be interpreted as three generations of fermions. Eq.(5.133) suggest that the masses of all the generations of fermions are the same. In order to break this degeneracy one may either explicitly break degeneracy in an ad hoc way or one introduce a direct y dependence into the metric. The second way is more promising. However, in that case to find a appropriate Lagrangian which satisfies the Einstein equations becomes a rather non-trivial matter. So at this step degeneracy for the masses of the fermion generations is assumed (although this is not realistic). Despite the fact that the $\phi_{1(2)} - \Psi$ interactions do not discriminate between different fermions in the same

family, gravity does. So in principle there are universality breaking effects because of the gravitational interactions for different fermions. Nevertheless, we assume that these effects are so small (i.e. the four dimensional brane is so narrow in the fifth dimension) that they cannot be detected at present. The relevant part of $\Psi_{L(R)}$ at low energies is the portion of their curve in the η-direction which overlaps with the portion in the fifth coordinate in which our four dimensional world is located.

5.3.3 An Improved Model for Fermion Localization Through The Domain Wall Solutions

We follow the same strategy that we considered for the potential (5.126)to explain the fermion localization and chirality by means of more simple potential form given in Eq.(5.95) [92]. For this purpose we consider the following fermion-scalar interaction Lagrangian:

$$\begin{aligned}\mathcal{L} &= i\bar{\Psi}\Gamma^{\mu}D_{\mu}\Psi + i\bar{\Psi}\Gamma^{4}\partial_{4}\Psi + g\bar{\Psi}\gamma_{5}\phi\Psi \\ &= i\bar{\Psi}\gamma^{\mu}\mathrm{e}^{-\sigma}D_{\mu}\Psi + i\bar{\Psi}(-i\gamma_{5})\frac{\partial\Psi}{\partial y} + g\bar{\Psi}\gamma_{5}\phi\Psi, \end{aligned} \tag{5.138}$$

with the RS-like metric

$$\mathrm{d}s^{2} = \mathrm{e}^{-2\sigma(y)}\eta_{\mu\nu}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} - \mathrm{d}y^{2}. \tag{5.139}$$

Five dimensional gamma matrices Γ^{A} are defined with the help of the vielbein $E^{A}{}_{B}$ and flat space gamma matrices γ^{A}

$$\Gamma^{A} = E_{B}{}^{A}\gamma^{B} \tag{5.140}$$

where vielbeins are defined by

$$\begin{aligned} g_{AB} &= \eta_{AB}E^{A}{}_{C}E^{B}{}_{D} \\ \eta^{AB} &= g^{CD}E^{A}{}_{C}E^{B}{}_{D} \end{aligned} \tag{5.141}$$

i.e. they are, in some sense, square root of the metric. The coordinate vielbein for the metric Eq.(5.139)

$$\begin{aligned} E_{\mu}{}^{\nu} &= \mathrm{e}^{\sigma}\delta_{\mu}{}^{\nu} \\ E_{5}{}^{\mu} &= 0 \\ E_{\mu}{}^{5} &= 0 \\ E_{5}{}^{5} &= 1 \end{aligned} \tag{5.142}$$

The covariant derivative is defined as

$$D_A = \partial_A + \frac{1}{2}\omega^{BC}{}_A\,\sigma_{BC} \tag{5.143}$$

where ω^{BC} is spin connection and $\sigma_{BC} = \frac{1}{4}[\Gamma_B, \Gamma_C]$ are generators of five dimensional Lorentz group. Spin connection can be found from zero torsion condition,

$$T^A = \mathrm{d}E^A + \omega^A{}_B \wedge E^B = 0 \tag{5.144}$$

Here

$$\begin{aligned} E^A &= E^A{}_B\,\mathrm{d}x^B \qquad (1-form) \\ \omega^A{}_B &= \omega^A{}_{BC}\,\mathrm{d}x^C \qquad (1-form) \end{aligned} \tag{5.145}$$

Substituting these into Eq.(5.144) we get the components of spin connection,

$$\omega^{5\alpha}{}_\beta = -\frac{\sigma'\,\mathrm{e}^{-\sigma}}{(1+\mathrm{e}^{-\sigma(y)})}\delta^\alpha{}_\beta \tag{5.146}$$

and all other terms vanishes. Then Lorentz covariant derivative with spin connection

$$D_\mu \equiv \partial_\mu - \frac{\sigma'(y)\mathrm{e}^{-\sigma(y)}}{2(1+\mathrm{e}^{-\sigma(y)})}\gamma_\mu\gamma^4 \tag{5.147}$$

We are interested in the solutions near zero and we assume that σ' at zero is zero provided that $\mathrm{e}^{-\sigma}$ and Ψ does not blow up near zero. So under these restrictions (5.147) becomes just ordinary partial derivative and equations of motion derived from the Lagrangian (5.138),

$$i\gamma^\mu \mathrm{e}^\sigma \partial_\mu \Psi + \gamma_5 \frac{\partial \Psi}{\partial y} + g\,\gamma_5\,\phi\,\Psi = 0 \tag{5.148}$$

Let us consider the solutions propagating in the usual four dimensions as free fields in the form,

$$\Psi = \mathrm{e}^{-i[(\mathrm{e}^{-2\sigma})(p_0 x_0 - \mathbf{p}\cdot\mathbf{x})]}\chi(y). \tag{5.149}$$

We impose

$$\gamma^\mu p_\mu \Psi + m\,\Psi = 0. \tag{5.150}$$

This simply means that we take the lower and upper components of 4-spinor Ψ be dotted and undotted representations (i.e. left handed and right handed) of $SL(2,C)$, [33, 93]. Then after putting Eq.(5.149) and Eq.(5.150) in Eq.(5.148) we get

$$-m\,\mathrm{e}^{-\sigma}\Psi + \gamma_5 \frac{\partial \Psi}{\partial y} + g\,\gamma_5\,\phi\,\Psi = 0 \tag{5.151}$$

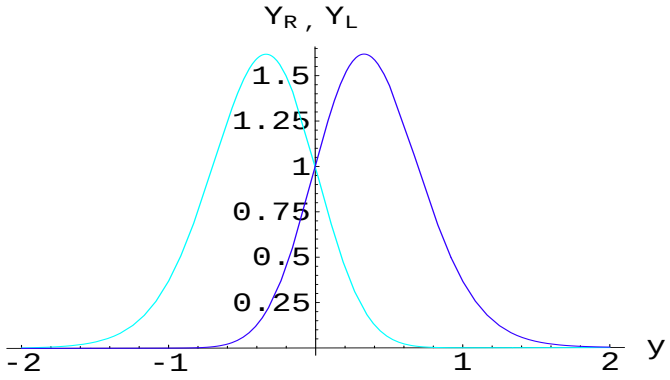

Figure 5.1: Localization of fields Ψ_R and Ψ_L.

By the help of projection operators $\frac{1}{2}(1-\gamma_5)$ and $\frac{1}{2}(1+\gamma_5)$, Eq.(5.151) may be written as

$$\frac{\partial \Psi_R}{\partial y} = -m_R e^{-\sigma}\Psi_R - g\,\phi\,\Psi_R \tag{5.152}$$

$$\frac{\partial \Psi_L}{\partial y} = m_L e^{-\sigma}\Psi_L - g\,\phi\,\Psi_L \tag{5.153}$$

solutions are given

$$\Psi_R = \exp(-m_R \int e^{-\sigma}\,dy - g\int \phi\,dy)\,\psi_R \tag{5.154}$$

$$\Psi_L = \exp(m_L \int e^{-\sigma}\,dy - g\int \phi\,dy)\,\psi_L \tag{5.155}$$

where $\psi_{(R)L}$ is the solution of Eq.(5.150). By using the Eq.(5.121) and Eq.(5.122), we get

$$\int \phi\,dy = \frac{1}{15}(10-\epsilon)\ln(\cosh(\frac{3\xi\,y}{2})) + \frac{\epsilon}{24}\mathrm{sech}^2(\frac{3\xi\,y}{2}) + \frac{\epsilon}{10}\xi\,y\tanh(\frac{3\xi\,y}{2}) \tag{5.156}$$

where $\xi = \sqrt{\frac{3\Lambda M^3}{2v^4}}$ is scaling factor defined in section 5.3.1 and

$$\begin{aligned}\int e^{-\sigma}\,dy = \int dy \exp(-(&\frac{1}{15}(10-\epsilon)\ln[\cosh\frac{3\xi y}{2}] + \frac{\epsilon}{60}\mathrm{sech}^2\frac{3\xi y}{2} + \frac{3\epsilon}{80}\mathrm{sech}^4\frac{3\xi y}{2} \\ &- \frac{\epsilon}{10}\xi y\tanh\frac{3\xi y}{2} + \frac{\epsilon}{20}\xi y\,\mathrm{sech}^2\frac{3\xi y}{2}\,\tanh\frac{3\xi y}{2}))\end{aligned} \tag{5.157}$$

We can see that the behavior of the functions near $z = 0$, Ψ_L and Ψ_R are found to be as $e^{ay+by^3-cy^2}$ and $e^{-ay-by^3-cy^2}$, respectively. Here a, b, c are positive constants. So with various values of parameters Ψ_L and Ψ_R are localized at different positions in extra dimensions (as shown in Figure 5.1 for $a = 1$, $b = 1$, $c = 5$, and $\xi = 1$) provided that m_L is a positive constant and m_R is a negative one. Although we have determined the general form of Ψ_R and Ψ_L for small

values of y, it is true for all values of y since dominating term in Eq.(5.154) and Eq.(5.155) are $\exp(-g\int \phi\, dy)$. For large values of y the shape of the graphs of Ψ_R and Ψ_L are distorted with respect to the graphs we obtained although the localization properties of Ψ_R and Ψ_L are preserved.

CONCLUSIONS

We have seen that the idea of extra dimensions is very attractive since one may understand almost all properties of elementary particles and fundamental forces through the use of extra dimensions. Although the idea of explaining the fundamental forces through the geometry of extra dimensions is old, the new studies suggest that some other properties of elementary particles may be explained by localizing different particles at different points in extra dimensions. Accounting for different masses and chirality of fermions is especially interesting since it can not be explained in other means. We have considered different recent studies in this perspective. We have combined different interesting aspects of some of these models to get an improved model of fermion localization and fermion chirality. The extra dimensions in the original Kaluza-Klein theory was at the Planck scale, so almost impossible to detect in a foreseenable future while there are modern extra dimensional models at scales as large as 100-200 μm. Moreover the extra dimensional models are becoming more realistic and phenomenologically more viable. However, there is still a long way to go. No extra dimension or indirect signature of it is detected so far. Although we have now a good framework to apply the theoretical results into phenomenological models, one still does not a single comprehensive extra dimensional model accounting for all properties of elementary particles simultaneously in a phenomenologically and experimentally perfect set up. There is intense current research in this direction. So we hope that extra dimensional approach will be successful in near future, at least up to a good portion of its ambitious program. The next generation of accelerators will also help to constrain the vast amount of alternatives for extra dimensional models. We think that the problem of explaining fermion properties through the use of extra dimensions is especially appealing.

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APPENDIX A

RADION STABILIZATION

In the previous section, we have already mentioned the internal metric component G_{55} gives rise to a massless field in effective description. This means that its vacuum expectation value r_c is very sensitive against any perturbation and rather unstable. For the discussion of the hierarchy problem it is important that the distance of the branes r_c is of the order of the Planck length. Therefore it is desirable to stabilize this distance, i.e. to give a mass to G_{55} in the effective description. In the present section we briefly discuss how a stabilization might be achieved via an additional scalar living in the bulk [94]. We will neglect the back reaction of the scalar field on the geometry. This means that we just consider a scalar field in the RS1 background constructed in the previous section. The action consists of out of three parts

$$S = S_{bulk} + S_{hid} + S_{vis} \tag{A.1}$$

where S_{bulk} defines the five-dimensional dynamics of the field and S_{hid} and S_{vis} its coupling to the respective brane. We choose

$$S_{bulk} = \frac{1}{2}\int d^4x \int_{-\pi}^{\pi} d\phi \, \sqrt{-G}(G^{MN}\partial_M\Phi \, \partial_N\Phi - m^2\Phi^2), \tag{A.2}$$

where Φ is the scalar field and G_{MN} is given in Eq.(5.51). The coupling to the branes is taken to be

$$S_{hid} = -\int d^4x \, \sqrt{-g^{hid}}\lambda_{hid}(\Phi^2 - \upsilon_{hid}^2)^2, \tag{A.3}$$

$$S_{vis} = -\int d^4x \, \sqrt{-g^{vis}}\lambda_{vis}(\Phi^2 - \upsilon_{vis}^2)^2, \tag{A.4}$$

where υ_i and λ_i are dimensionfull parameters whose values will be discussed below. With the ansatz that Φ does not depend on the x^μ for $\mu = 0, 1, 2, 3$ the equation of motion for the scalar is

$$
\begin{aligned}
0 &= -\frac{1}{r_c^2}\partial_\phi(\mathrm{e}^{-4kr_c|\phi|}\partial_\phi\Phi) + m^2\mathrm{e}^{-4kr_c|\phi|}\Phi 4\mathrm{e}^{-4kr_c|\phi|}\lambda_{hid}\Phi(\Phi^2 - \upsilon_{hid}^2)\frac{\delta(\phi)}{r_c} \\
&+ 4\mathrm{e}^{-4kr_c|\phi|}\lambda_{vis}\Phi(\Phi^2 - \upsilon_{vis}^2)\frac{\delta(\phi-\pi)}{r_c}, \qquad \text{(A.5)}
\end{aligned}
$$

Away from the boundaries at $\phi = 0, \pi$, this equation has the general solution

$$
\Phi(\phi) = \mathrm{e}^{2kr_c|\phi|}\left[A\mathrm{e}^{kr_c|\phi|\nu} + B\mathrm{e}^{-kr_c|\phi|\nu}\right], \qquad \text{(A.6)}
$$

with $\nu = \sqrt{4 + m^2/k^2}$ and the integration constant will be fixed below. Putting this solution into the scalar field action and integrating over ϕ yields an effective four-dimensional potential for r_c which has the form

$$
\begin{aligned}
V_\Phi(r_c) &= k(\nu+2)A^2(\mathrm{e}^{2\nu kr_c\pi} - 1) + k(\nu-2)B^2(1 - \mathrm{e}^{-2\nu kr_c\pi}) \\
&+ \lambda_{vis}\mathrm{e}^{-4\nu kr_c\pi}(\Phi(\pi)^2 - \upsilon_{vis}^2)^2 + \lambda_{hid}(\Phi(0)^2 - \upsilon_{hid}^2)^2. \qquad \text{(A.7)}
\end{aligned}
$$

The unknown coefficients A and B are determined by imposing appropriate boundary conditions on the 3-branes. We obtain these boundary conditions by inserting Eq.(A.6) into the equations of motion and matching the delta functions:

$$
\begin{aligned}
0 &= k[(2+\nu)A + (2-\nu)B] - 2\lambda_{hid}\Phi(0)[\Phi(0)^2 - \upsilon_{hid}^2], \qquad \text{(A.8)} \\
0 &= k\mathrm{e}^{2kr_c\pi}[(2+\nu)\mathrm{e}^{\nu kr_c\pi}A + (2-\nu)\mathrm{e}^{-\nu kr_c\pi}B] \\
&+ 2\lambda_{vis}\Phi(\pi)[\Phi(\pi) - \upsilon_{vis}^2]. \qquad \text{(A.9)}
\end{aligned}
$$

Instead of writing down and solving those equations explicitly we consider the simplified case that λ_{hid} and λ_{vis} are large enough for the approximation

$$
\Phi(0) = \upsilon_{hid}, \qquad \Phi(\pi) = \upsilon_{vis} \qquad \text{(A.10)}
$$

to be sufficiently accurate. In this approximation we get

$$
\begin{aligned}
A &= \upsilon_{vis}\mathrm{e}^{-(2+\nu)kr_c\pi} - \upsilon_{hid}\mathrm{e}^{-2\nu kr_c\pi}, \qquad \text{(A.11)} \\
B &= \upsilon_{hid}\left(1 + \mathrm{e}^{-2\nu kr_c\pi}\right) - \upsilon_{vis}\mathrm{e}^{-(2+\nu)kr_c\pi}, \qquad \text{(A.12)}
\end{aligned}
$$

where subleading powers of $\exp(-kr_c\pi)$ have been neglected. Now suppose that $m/k \ll 1$ so that $\nu = 2 + \epsilon$, with $\epsilon \simeq m^2/4k^2$ a small quantity. In the large kr_c limit, the potential becomes

$$
\begin{aligned}
V_\Phi(r_c) &= k\epsilon\upsilon_{hid}^2 + 4k\mathrm{e}^{-4kr_c\pi}\left(\upsilon_{vis} - \upsilon_{hid}\mathrm{e}^{-\epsilon kr_c\pi}\right)^2\left(1 + \frac{\epsilon}{4}\right) \\
&- k\epsilon\upsilon_{hid}\mathrm{e}^{-(4+\epsilon)kr_c\pi}\left(2\upsilon_{vis} - \upsilon_{hid}\mathrm{e}^{-\epsilon kr_c\pi}\right) \qquad \text{(A.13)}
\end{aligned}
$$

where terms of order ϵ^2 are neglected (but $\epsilon k r_c$ is not treated small). Ignoring terms proportional to ϵ, this potential has a minimum at

$$kr_c = \left(\frac{4}{\pi}\right) \frac{k^2}{m^2} \ln\left[\frac{\upsilon_{hid}}{\upsilon_{vis}}\right]. \tag{A.14}$$

With $\ln\left(\frac{\upsilon_{hid}}{\upsilon_{vis}}\right)$ of order unity, we only need m^2/k^2 of order 1/10 to get $kr_c \sim 10$. Clearly, no extreme fine tuning of parameters is required to get the right magnitude for kr_c. For instance, taking $\upsilon_{hid}/\upsilon_{vis} = 1.5$ and $m/k = 0.2$ yields $kr_c \simeq 12$. Even though this mechanism is the commonly established method for solving the problem of moduli stabilization, it is one of the most prominent lines of thought in the context of Randall Sundrum model.