Existence of Hamiltonians for some singular interactions on manifolds

Çağlar Doğan,[1,a)] Fatih Erman,[2,b)] and O. Teoman Turgut[3,4,c)]
[1]*Department of Physics, İstanbul University, Vezneciler 34134, İstanbul, Turkey*
[2]*Department of Mathematics, İzmir Institute of Technology, Urla 35430, İzmir, Turkey*
[3]*Department of Physics, Boğaziçi University, Bebek 34342, İstanbul, Turkey*
[4]*Feza Gürsey Institute, Kuleli Mahallesi, Şekip Ayhan Özışık Caddesi, No. 44, Kandilli 34684, İstanbul, Turkey*

(Received 13 November 2011; accepted 4 April 2012; published online 25 April 2012)

The existence of the Hamiltonians of the renormalized point interactions in two and three dimensional Riemannian manifolds and that of a relativistic extension of this model in two dimensions are proven. Although it is much more difficult, the proof of existence of the Hamiltonian for the renormalized resolvent for the non-relativistic Lee model can still be given. To accomplish these results directly from the resolvent formula, we employ some basic tools from the semigroup theory. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4705291]

I. INTRODUCTION

Typical field theory problems require a concept known as renormalization, which is a way of rendering infinite quantities to finite values to get physically meaningful results. This is a very hard problem, and it would be interesting to find some simple models in which the ideas can be tested in depth and a mathematically sound description can be presented as much as possible. This will illuminate the underlying mathematical and physical ideas in more complicated models.

There are indeed some simple models introduced in the past. One of them is the Dirac-delta potentials in quantum mechanics which was first studied rigorously by Berezin and Fadeev[1] and later discussed extensively by Albeverio *et al.*[2,3] These works show that Dirac-delta potential can be understood from the self-adjoint extension point of view, hence all could be made mathematically sound. Many body version of this problem on $\mathbb{R}^2$ is known as the formal non-relativistic limit of the $\lambda\phi^4$ scalar field theory in $(2+1)$ dimensions. All these are extensively discussed first in the unpublished thesis of Hoppe.[4] A similar model is the non-relativistic Lee model, which exhibits an additive divergence. We are not aware of a mathematically rigorous discussion of this model. Physically the relativistic simplified version of the Lee model is more important and there is quite of a bit of work from a nonperturbative point of view to understand the physics behind it (see the references in Ref. 5). The approach we follow is introduced in Ref. 11 by Rajeev, and recently we have introduced the generalizations of these models on to manifolds.[6–10] The rigorous understanding of the existence of the Hamiltonian left aside in our previous works. We would like to address this issue in the present work. There is a general approach which is exposed in the excellent book by Albeverio and Kurasov,[3] and it should be applicable in the Dirac-delta functions case for the manifolds. The self-adjoint extension point of view becomes complicated when we discuss field theories, it is usually hard to give a meaning to operator valued distributions and their extension theory is even more delicate. The other alternative which uses resolvent convergence of cut-off Hamiltonians is problematic when we use other regularization schemes, e.g., the powerful dimensional regularization. However, we

a)Electronic mail: doganc@istanbul.edu.tr.
b)Electronic mail: fatih.erman@gmail.com.
c)Electronic mail: turgutte@boun.edu.tr.

will use an alternative approach. In the problems that we deal with, the resulting operator is not given but instead the resolvent is renormalized.

To answer the existence we use the following theorem taken from semigroup theory. Let Δ be a subset of the complex plane. A family $J(E)$, $E \in \Delta$ of bounded linear operators on the Hilbert space $\mathcal{H}$ under consideration, which satisfies the resolvent identity

$$J(E_1) - J(E_2) = (E_1 - E_2) J(E_1) J(E_2) \tag{1}$$

for E_1 , $E_2 \in \Delta$ is called a pseudo resolvent on Δ.[12]

The following corollary (Corollary 9.5 in Ref. 12) gives the condition for which there exists a densely defined closed linear operator A such that $J(E)$ is the resolvent family of A: Let Δ be an unbounded subset of $\mathbb{C}$ and $J(E)$ be a pseudo resolvent on Δ. If there is a sequence $E_k \in \Delta$ such that $|E_k| \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} -E_k J(E_k) x = x, \tag{2}$$

for all $x \in \mathcal{H}$, then $J(E)$ is the resolvent of a unique densely defined closed operator A. As we will see in our case, the family satisfies $J(E)^\dagger = J(E^*)$ so it is a holomorphic family of type (A) in the sense of Kato.[13] Hence, it defines a self-adjoint operator.

Let us mention the possibility of using the results from Ref. 3 in the case of Dirac-delta functions. In the approach of Ref. 3, we consider an operator A with a dense domain, and consider the singular perturbation by an element ϕ in some dual space, formally,

$$A^- = A + \lambda \langle \phi, . \rangle \phi, \tag{3}$$

here the bracket refers to dual pairing. The interesting case is when we have $\phi \in \mathcal{H}_{-2}(A)$, where

$$\phi \in \mathcal{H}_{-2}(A) \ \text{ if } \ || \frac{1}{1+|A|} \phi || < \infty \tag{4}$$

and $||.||$ refers to the usual norm in the Hilbert space. Then the theorem in Ref. 3 states that the operator A^- provides a self-adjoint extension with a new domain. In our case,

$$|| \frac{1}{1+(-\nabla_g^2)} \delta_g(a, .) || = \int_0^\infty \mathrm{d}s\, s\, e^{-s}\, K_s(a, a; g) < \infty, \tag{5}$$

thus we have the same type of singular perturbation—so called form unbounded one. It is interesting to see how the conditions on Ricci curvature we found will arise in this approach.

In our presentation, by following the Rajeev's idea developed for the renormalization of several models (point interactions in quantum mechanics, Lee model, etc.), we will rigorously prove the existence of Hamiltonians for each model after the renormalization procedure in compact manifolds, whereas the noncompact case will be heuristically given by using the spectral theorem for the Laplace-Beltrami operator defined on it.

Our approach here is not to investigate the spectral properties of the models that we are dealing with. They have been already discussed in our previous works.[6–10]

The paper is organized as follows. In Sec. II, we give the proof of existence of Hamiltonian for the point interactions in two and three dimensional Riemannian manifolds. The analysis for two and three dimensional cases are different, so we show the calculations separately. In Sec. III, we give the same proof for the relativistic point interactions on two dimensional Riemannian manifolds. Finally, Sec. IV presents a short review of non-relativistic Lee model from our point of view and give the lower bound of the ground state energy of the system in three dimensional Riemannian manifolds. Then, the the proof of the existence of Hamiltonian for this model is given but it requires a little more work.

II. POINT INTERACTIONS IN TWO AND THREE DIMENSIONAL RIEMANNIAN MANIFOLDS

We adopt the natural units $\hbar = 1$ in the non-relativistic models discussed in this paper for simplicity. In Ref. 8, after the renormalization we have found the resolvent kernel corresponding to the Hamiltonian for the N point interactions (Dirac-delta interactions) in two and three dimensional Riemannian manifolds as

$$R(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} R_0(x, a_i|E)\Phi_{ij}^{-1}(E)R_0(a_j, y|E), \tag{6}$$

where

$$\Phi_{ij}(E) = \begin{cases} \int_0^\infty \mathrm{d}t \; K_t(a_i, a_i; g) \; \left(e^{-t\mu_i^2} - e^{tE}\right) & \text{if } i = j \\ -\int_0^\infty \mathrm{d}t \; K_t(a_i, a_j; g) \; e^{tE} & \text{if } i \neq j. \end{cases} \tag{7}$$

Here $\Re(E) < 0$ and $K_t(x, y; g)$ is the heat kernel on the Riemannian manifold, which is defined as the fundamental solution to the heat equation

$$\frac{1}{2m}\nabla_g^2 K_t(x, y; g) = \frac{\partial K_t(x, y; g)}{\partial t}. \tag{8}$$

In order to show that the resolvent kernel given in Eq. (6) corresponds to a unique densely defined closed operator H, we need to first prove that it satisfies the resolvent identity, i.e.,

$$R(x, y|E_1) - R(x, y|E_2) = (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}_g^D z \; R(x, z|E_1)R(z, y|E_2). \tag{9}$$

A detailed proof as well as all properties of the heat kernel that we use in this paper, are given in our previous work[8] and the relevant literature is also given there. Here we will just give the main idea of the proof for the completeness of this paper. If we substitute Eq. (6) into Eq. (9), we obtain

$$\begin{aligned} R_0(x, y|E_1) - R_0(x, y|E_2) &+ \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, y|E_1) \\ &- \sum_{i,j=1}^{N} R_0(x, a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2) \\ = (E_1 - E_2) \int_{\mathcal{M}} \mathrm{d}_g^D z \Bigg[& R_0(x, z|E_1)R_0(z, y|E_2) \\ &+ \sum_{i,j=1}^{N} R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2) \\ &+ \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, z|E_1)R_0(z, y|E_2) \\ + \sum_{i,j=1}^{N} \sum_{r,l=1}^{N} & R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, z|E_1) \\ &\times R_0(z, a_r|E_2)\Phi_{rl}^{-1}(E_2)R_0(a_l, y|E_2) \Bigg]. \end{aligned} \tag{10}$$

The term $R_0(x, y|E_1) - R_0(x, y|E_2)$ equals to the first term in the right-hand side of the equation above since the free resolvent kernel $R_0(x, y|E)$ must satisfy the resolvent identity (9). If we add and subtract the terms

$$\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, y|E_2), \tag{11}$$

$$\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2), \tag{12}$$

to the remaining terms in the equation above and rearrange, one can complete the proof for the resolvent identity (9) by showing that the difference of the principal matrix $\Phi_{ij}(E_2) - \Phi_{ij}(E_1)$ equals to the difference of free resolvent kernels $R_0(a_i, a_j|E_1) - R_0(a_i, a_j|E_2)$. It is easy to show this by using the formula expressing the free resolvent kernel as a Laplace transformation of the heat kernel and semigroup property of the heat kernel following a change of variable for the time variable in the heat kernel.[8] Equation (2) requires the following condition to complete the second part of the proof,

$$||E_k R(E_k) f + f|| \to 0, \tag{13}$$

as $k \to \infty$, where f belongs to the Hilbert space $\mathcal{H} = L^2(\mathcal{M})$ and the norm is the usual $L^2(\mathcal{M})$ norm. Let us choose the sequence $E_k = -k|E_0|$, where E_0 is below the lower bound E_* on the ground state energy which has been found in Ref. 8. Then, we must show that

$$|||E_k| R(E_k) f - f|| \to 0, \tag{14}$$

as $k \to \infty$. Using Eq. (6) and separating the free part, we get

$$\begin{aligned} |||E_k| R(E_k) f - f|| &\leq |||E_k| R_0(E_k) f - f|| \\ &\quad + |E_k| ||R_0(E_k) \Phi^{-1}(E_k) R_0(E_k) f||. \end{aligned} \tag{15}$$

It is well known that the first part of the sum converges to zero as $k \to \infty$, that is, the free resolvent corresponds to a densely defined closed operator (Laplacian). Moreover, the Laplacian on geodesically complete Riemannian manifolds is essentially self-adjoint in $L^2(\mathcal{M})$.[14,15] Therefore, we are going to investigate only the second term in two and three dimensions separately. Two dimensional analysis has been already worked out in Ref. 8 and we will just review it here and then give the detailed proof for the three dimensional case. Since the norm of an operator is smaller than its Hilbert-Schmidt norm: $||A|| \leq \mathrm{Tr}^{1/2}(A^\dagger A)$ with $A = R_0(E_k)\Phi^{-1}(E_k)R_0(E_k)$, we have

$$\begin{aligned} &|E_k| ||R_0(E_k) \Phi^{-1}(E_k) R_0(E_k) f|| \\ &\leq |E_k| \Bigg[\sum_{i,j,r,l=1}^{N} \int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x|E_k) R_0(x, a_l|E_k) \\ &\quad \times \int_{\mathcal{M}} \mathrm{d}_g^D y \; R_0(a_j, y|E_k) R_0(y, a_r|E_k) |\Phi_{ij}^{-1}(E_k)| \; |\Phi_{rl}^{-1}(E_k)| \Bigg]^{1/2}. \end{aligned} \tag{16}$$

Let us first consider the diagonal case $l = i$ and $r = j$ for the terms inside the bracket above.

$$\begin{aligned} &|E_k| \Bigg[\sum_{i,j=1}^{N} \int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x|E_k) R_0(x, a_i|E_k) \\ &\quad \times \int_{\mathcal{M}} \mathrm{d}_g^D y \; R_0(a_j, y|E_k) R_0(y, a_j|E_k) |\Phi_{ij}^{-1}(E_k)| \; |\Phi_{ji}^{-1}(E_k)| \Bigg]^{1/2} \\ &\leq |E_k| \Bigg[N^2 \max_{1 \leq i \leq N} \; \alpha_i(E_k) \; \max_{1 \leq j \leq N} \; \alpha_j(E_k) \; \max_{1 \leq i,j \leq N} \; |\Phi_{ij}^{-1}(E_k)|^2 \Bigg]^{1/2}, \end{aligned} \tag{17}$$

where we have defined $\alpha_i(E_k) = \int_{\mathcal{M}} \mathrm{d}_g^D y \; R_0(a_i, y|E_k) R_0(y, a_i|E_k)$ for simplicity. It is easy to see that

$$\begin{aligned} &\int_{\mathcal{M}} \mathrm{d}_g^D x \; R_0(a_i, x|E_k) R_0(x, a_l|E_k) \\ &\quad = \int_0^\infty \int_0^\infty \mathrm{d}t_1 \, \mathrm{d}t_2 \; K_{t_1+t_2}(a_i, a_l; g) e^{-(t_1+t_2)|E_k|} \\ &\quad = \int_0^\infty \mathrm{d}t \; t \; K_t(a_i, a_l; g) e^{-t|E_k|} \end{aligned} \tag{18}$$

by using the fact that the free resolvent kernel is just the Laplace transform of the heat kernel. The upper bound of the heat kernel was given in Refs. 16 and 17 and summarized in Ref. 8 for compact (with bounded Ricci) and Cartan-Hadamard manifolds.[9] We shall use the notation for the dimensionless constants coming from the bounds of the heat kernel as C with subscripts for simplicity since the exact form of these constants do not play any role here. The upper bound of the heat kernel for compact (with bounded Ricci) and Cartan-Hadamard manifolds is given in the following form

$$K_t(x, y; g) \leq \begin{cases} \left[\dfrac{C_1}{V(\mathcal{M})} + \dfrac{C_2}{(t/2m)^{D/2}}\right] \exp\left(-\dfrac{md^2(x, y)}{C_3 t}\right) & \text{for compact manifolds} \\ \dfrac{C_4}{(t/2m)^{D/2}} \exp\left(-\dfrac{md^2(x, y)}{C_5 t}\right) & \text{for Cartan-Hadamard manifolds,} \end{cases} \tag{19}$$

where $V(\mathcal{M})$ is the volume of the manifold and $d(x, y)$ is the geodesic distance between the point x and y. Then, on-diagonal upper bound of Eq. (18) for compact manifolds (with bounded Ricci) becomes

$$\max_{1\leq i\leq N} \alpha_i(E_k) \leq \frac{C_1}{V(\mathcal{M})|E_k|^2} + C_6 (2m)^{\frac{D}{2}} \; |E_k|^{\frac{D}{2}-2}, \tag{20}$$

where $C_6 = C_2\Gamma(2 - D/2)$. For Cartan-Hadamard manifolds, we get

$$\max_{1\leq i\leq N} \alpha_i(E_k) \leq C_4 \; (2m)^{\frac{D}{2}} \; |E_k|^{\frac{D}{2}-2}. \tag{21}$$

We have also

$$\begin{aligned} \max_{1\leq i,j\leq N} |\Phi^{-1}_{ij}|^2 &\leq \max_{1\leq i\leq N} \sum_{j=1}^{N} |\Phi^{-1}_{ij}|^2 = \max_{1\leq i\leq N} (\Phi^{-1}(E_k)\Phi^{-1}(E_k))_{ii} \leq \rho(\Phi^{-2}(E_k)) \\ &\leq ||\Phi^{-2}(E_k)|| \leq ||\Phi^{-1}(E_k)||^2, \end{aligned} \tag{22}$$

where we have used $\Phi^\dagger(E_k) = \Phi(E_k)$ for $E_k \in \mathbb{R}$ and ρ is the spectral radius.

In order to find the upper bound for the norm of the inverse principal matrix, we first decompose the principal matrix into two positive matrices

$$\Phi = D - K, \tag{23}$$

where D and K stand for the on-diagonal and the off-diagonal part of the principal matrix, respectively. Then, it is easy to see $\Phi = D(1 - D^{-1}K)$. The principal matrix is invertible if and only if $(1 - D^{-1}K)$, and $(1 - D^{-1}K)$ has an inverse if the matrix norm satisfies $||D^{-1}K|| < 1$. Then, we can write the inverse of Φ as a geometric series

$$\begin{aligned} \Phi^{-1} &= (1 - D^{-1}K)^{-1} D^{-1} \\ &= \left(1 + (D^{-1}K) + (D^{-1}K)^2 + \ldots\right) D^{-1}, \end{aligned} \tag{24}$$

and the norm has the following upper bound

$$\begin{aligned} ||\Phi^{-1}|| &= ||(1 - D^{-1}K)^{-1} D^{-1}|| \leq ||(1 - D^{-1}K)^{-1}|| \; ||D^{-1}|| \\ &\leq \frac{1}{1 - ||D^{-1}K||} \; ||D^{-1}||. \end{aligned} \tag{25}$$

Since we are not concerned with the sharp bounds on the norm of Φ^{-1} for this problem, we can choose $|E_k|$ sufficiently large such that $||D^{-1}K|| < 1/2$ without loss of generality and get

$$||\Phi^{-1}(E_k)|| \leq 2||D^{-1}(E_k)||. \tag{26}$$

Whenever $D^{-1} = \text{diag}(\Phi^{-1}_{11}, \Phi^{-1}_{22}, \ldots, \Phi^{-1}_{NN})$, then

$$||D^{-1}|| = \max_{1\leq i\leq N} |\Phi^{-1}_{ii}|. \tag{27}$$

The lower bound of the diagonal principal matrix for compact and Cartan-Hadamard manifolds, which was given in Ref. 8 leads to the upper bound of the inverse of the diagonal part of the principal matrix. Hence, we find

$$\max_{1\leq i\leq N} |\Phi_{ii}^{-1}(E_k)| \leq \begin{cases} C_7(2m)^{-1}\ln^{-1}\left(|E_k|/\mu^2\right) & \text{if } D=2 \\ C_8(2m)^{-3/2}\left[|E_k|^{1/2}-\mu\right]^{-1} & \text{if } D=3, \end{cases} \tag{28}$$

for compact manifolds and

$$\max_{1\leq i\leq N} |\Phi_{ii}^{-1}(E_k)| \leq \begin{cases} C_9(2m)^{-1}\ln^{-1}\left(\dfrac{|E_k|+\xi}{\mu^2+\xi}\right) & \text{if } D=2 \\ C_{10}(2m)^{-3/2}\left[(|E_k|+\xi)^{1/2}-\left(\mu^2+\xi\right)^{1/2}\right]^{-1} & \text{if } D=3, \end{cases} \tag{29}$$

for Cartan-Hadamard manifolds. Here ξ is a positive constant and defined in Ref. 8.

If we substitute the results of (20), (21) and (28), (29) into (17) for $D=2$, and take the limit $k \to \infty$, the result goes to zero. Since the norm is always positive, we prove

$$|| |E_k| R(E_k) f - f|| \to 0, \tag{30}$$

as $k \to \infty$. For the off-diagonal terms, we do not have to make a separate detailed analysis since one can easily show that these terms are essentially exponentially suppressed by the factor $e^{-\sqrt{2m|E_k|}d(a_i,a_j)}$ due to the upper bounds of the modified Bessel functions which are given in Ref. 8. Therefore, all off-diagonal terms exponentially vanish when we take the limit $k \to \infty$, which is enough for our purposes.

Summary for the two dimensional case: *We have first proven that the resolvent for this system after the renormalization satisfies the resolvent identity. Then, by using the fact that the norm of an operator is smaller than its Hilbert-Schmidt norm, we have completed the second part of the proof that* $|| |E_k| R(E_k) f - f|| \to 0$ *as* $k \to \infty$. *This tells us that the Hamiltonian after the renormalization procedure is densely defined self-adjoint operator due to the theorem given explicitly in the introduction part of our paper.*

We now move on to the three dimensional case for the second part of our proof, that is, we show that $|| |E_k| R(E_k) f - f|| \to 0$ as $k \to \infty$ in the three dimensional case, as well. However, this proof requires more delicate analysis. We now return to the second term in (15), and show that

$$|E_k| \Bigg[\int_{\mathcal{M}} d_g^3 x \sum_{i,j,r,l=1}^{N} R_0(x,a_i|E_k)\Phi_{ij}^{-1}(E_k) \int_{\mathcal{M}} d_g^3 z\; R_0(a_j,z|E_k) f^*(z) \\ \times R_0(x,a_r|E_k)\Phi_{rl}^{-1}(E_k)\int_{\mathcal{M}} d_g^3 y\; R_0(a_l,y|E_k) f(y) \Bigg]^{1/2} \tag{31}$$

goes to zero as $k \to \infty$ for any $f \in L^2(\mathcal{M})$. From our previous argument, we know that the inverse of the principal matrix Φ satisfies for three dimensional compact and Cartan-Hadamard manifolds,

$$\max_{1\leq i,j\leq N} |\Phi_{ij}^{-1}(E_k)| \leq \frac{C_{11}(2m)^{-3/2}}{|E_k|^{1/2}}, \tag{32}$$

where we define all the constant terms coming from the bounds of the heat kernel as C_{11} and ignore the term in the denominator for large values of $|E_k|$ for simplicity. Moreover, we can combine the two resolvents with the common variable x in Eq. (31). As a result, we can express this combination as in Eq. (18) and the diagonal upper bounds of it for $D=3$ have been given in Eqs. (20) and (21) for compact and Cartan-Hadamard manifolds, respectively. Once again, we skip the detailed calculations for the off-diagonal terms ($i \neq r$) in the above sum since they are exponentially suppressed by the factor $e^{-\sqrt{2m|E_k|}d(a_i,a_r)}$. We always concentrate on the least convergent part in the terms. Once we have achieved our goal for these terms, we are done.

Therefore, it is sufficient to deal with only the diagonal term ($r = i$) in Eq. (31). It is easy to show that it is smaller than the following term:

$$N^{3/2}|E_k|\left[\max_{1\leq i\leq N}\ \alpha_i(E_k)\right]^{1/2}\left[\max_{1\leq i,j\leq N}\ |\Phi^{-1}_{ij}(E_k)|\max_{1\leq i,l\leq N}\ |\Phi^{-1}_{il}(E_k)|\right]^{1/2}$$
$$\times\left[\max_{1\leq j\leq N}\left(\int_{\mathcal{M}}\mathrm{d}^3_g z\ R_0(a_j,z|E_k)|f(z)|\right)\max_{1\leq l\leq N}\left(\int_{\mathcal{M}}\mathrm{d}^3_g y\ R_0(y,a_l|E_k)|f(y)|\right)\right]^{1/2}. \quad (33)$$

Using Eqs. (20), (21), and (32) in the above equation, we get the upper bound of (33) for three dimensional compact manifolds

$$N^{3/2}|E_k|\left[\frac{C_1}{V(\mathcal{M})|E_k|^2}+\frac{C_6(2m)^{\frac{3}{2}}}{|E_k|^{1/2}}\right]^{1/2}\left[\frac{C_{11}(2m)^{-3/2}}{|E_k|^{1/2}}\right]$$
$$\times\left[\max_{1\leq j\leq N}\left(\int_{\mathcal{M}}\mathrm{d}^3_g z\ R_0(a_j,z|E_k)|f(z)|\right)\max_{1\leq l\leq N}\left(\int_{\mathcal{M}}\mathrm{d}^3_g y\ R_0(y,a_l|E_k)|f(y)|\right)\right]^{1/2} \quad (34)$$

and for three dimensional Cartan-Hadamard manifolds

$$N^{3/2}|E_k|\left[\frac{C_4(2m)^{\frac{3}{2}}}{|E_k|^{1/2}}\right]^{1/2}\left[\frac{C_{11}(2m)^{-3/2}}{|E_k|^{1/2}}\right]$$
$$\times\left[\max_{1\leq j\leq N}\left(\int_{\mathcal{M}}\mathrm{d}^3_g z\ R_0(a_j,z|E_k)|f(z)|\right)\max_{1\leq l\leq N}\left(\int_{\mathcal{M}}\mathrm{d}^3_g y\ R_0(y,a_l|E_k)|f(y)|\right)\right]^{1/2}. \quad (35)$$

All these imply that the term

$$\int_{\mathcal{M}}\mathrm{d}^3_g y\ R_0(a_j,y|E_k)|f(y)| \quad (36)$$

must decay at least faster than $|E_k|^{-1/4}$. We now recall that the free resolvent kernel is just the Laplace transform of the heat kernel

$$R_0(a_j,y|E_k)=\int_0^\infty \mathrm{d}t\ e^{-t|E_k|}K_t(a_j,y;g), \quad (37)$$

so that we can find an upper bound for it by using Eq. (19) and evaluating the integrals over t,

$$R_0(a_j,y|E_k)\leq\frac{mC_{12}}{d(a_j,y)}\exp\left[-2\left(\frac{md^2(a_j,y)|E_k|}{C_3}\right)^{1/2}\right]$$
$$+\frac{C_{13}d(a_j,y)\sqrt{m}}{V(\mathcal{M})\sqrt{|E_k|}}\left[1+\left(\frac{C_3}{md^2(a_j,y)|E_k|}\right)^{1/2}\right]\exp\left[-2\left(\frac{md^2(a_j,y)|E_k|}{C_3}\right)^{1/2}\right], \quad (38)$$

for three dimensional compact manifolds and

$$R_0(a_j,y|E_k)\leq\frac{mC_{14}}{d(a_j,y)}\exp\left[-2\left(\frac{md^2(a_j,y)|E_k|}{C_5}\right)^{1/2}\right], \quad (39)$$

for three dimensional Cartan-Hadamard manifolds. Here we have used the upper bound of the modified Bessel function $K_1(x)$ given in Ref. 8.

For simplicity, let us first consider the generic term which is common for both compact and Cartan-Hadamard manifolds and keep the inverse volume term aside for the moment. Then, we find for the generic term

$$\int_{\mathcal{M}} \mathrm{d}_g^3 y \; R_0(a_j, y|E_k)|f(y)| \leq m C_{12} \int_{\mathcal{M}} \mathrm{d}_g^3 y \; \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] \frac{|f(y)|}{d(a_j, y)}. \tag{40}$$

We now divide the integration region into two pieces:

$$\int_{B_\delta(a_j)} \mathrm{d}_g^3 y \; \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] \frac{|f(y)|}{d(a_j, y)} + \int_{\mathcal{M}\setminus B_\delta(a_j)} \mathrm{d}_g^3 y \; \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] \frac{|f(y)|}{d(a_j, y)}, \tag{41}$$

where $B_\delta(a_j)$ is the geodesic ball of radius δ centered at a_j. It is easily seen that

$$\begin{aligned}
&\int_{\mathcal{M}\setminus B_\delta(a_j)} \mathrm{d}_g^3 y \; \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] \frac{|f(y)|}{d(a_j, y)} \\
&\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^2|E_k|}{C_3}\right)^{1/2}\right] \int_{\mathcal{M}\setminus B_\delta(a_j)} \mathrm{d}_g^3 y \; \exp\left[-\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] |f(y)| \\
&\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^2|E_k|}{C_3}\right)^{1/2}\right] \int_{\mathcal{M}} \mathrm{d}_g^3 y \; \exp\left[-\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] |f(y)| \\
&\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^2|E_k|}{C_3}\right)^{1/2}\right] \left[\int_{\mathcal{M}} \mathrm{d}_g^3 y \; \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right]\right]^{1/2} ||f||,
\end{aligned} \tag{42}$$

where we have used the fact that $d(a_j, y) \geq \delta$ for all j and $y \in \mathcal{M} \setminus B_\delta(a_j)$ in the second line. We then find an upper bound in terms of the norm of the function $f(x)$ by using Cauchy-Schwarz inequality in the last line.

For compact manifolds, it is a simple matter to find the upper bound to the above integral

$$\begin{aligned}
&\int_{\mathcal{M}\setminus B_\delta(a_j)} \mathrm{d}_g^3 y \; \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right] \frac{|f(y)|}{d(a_j, y)} \\
&\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^2|E_k|}{C_2}\right)^{1/2}\right] \left[V(\mathcal{M}) \sup_{y \in \mathcal{M}} \left(\exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right]\right)\right]^{1/2} ||f|| \\
&\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^2|E_k|}{C_2}\right)^{1/2}\right] V^{1/2}(\mathcal{M}) ||f||,
\end{aligned} \tag{43}$$

due to the fact that the volume of a compact manifold is finite. For non-compact manifolds, it is useful to consider the above integral in the Riemann normal coordinates near one of the centers a_i. We further assume that the radius of the ball δ is less than the injectivity radius $\mathrm{inj}(a_i)$. Let us recall that in Gaussian spherical coordinates, the volume integral of a function h on a D dimensional Riemannian manifold $\mathcal{M}$ can be written as

$$\int_{\mathcal{M}} \mathrm{d}_g^D x \; h(x) = \int_{\mathbb{S}^{D-1}} \mathrm{d}\Omega \int_0^{\rho_\Omega} \mathrm{d}r \; r^{D-1} J(r, \theta) h(r, \theta) \,. \tag{44}$$

Here $\theta = (\theta_1, \ldots \theta_{D-1})$ denotes the direction in the tangent space around a point that we choose, and ρ_Ω refers to distance to the cut locus of the point in the direction θ. Hence, we get,

$$\int_{\mathcal{M}} \mathrm{d}_g^3 y \ \exp\left[-2\left(\frac{m d^2(a_j, y)|E_k|}{C_3}\right)^{1/2}\right. \\ = \int_{\mathbb{S}^2} \mathrm{d}\Omega \int_0^{\rho_\Omega} \mathrm{d}r \ r^2 J(r,\theta) \exp\left[-2\left(\frac{m r^2 |E_k|}{C_3}\right)^{1/2}\right]. \tag{45}$$

To proceed further we assume that $\mathcal{M}$ has Ricci tensor bounded from below by K_1, i.e., Ric(., .) $> K_1\, g(.,.)$ everywhere and the sectional curvature is bounded from above by K_2 on $\overline{B_\delta(a_i)}$. The upper bound on the sectional curvature is automatically satisfied, since there are finite number of Dirac-delta centers and we take the metric to be $\mathcal{C}^\infty(\mathcal{M})$. Then Bishop-Gunther volume comparison theorems state that the Jacobian factor of the Gaussian spherical coordinates in D dimensions satisfies,[18,19]

$$\frac{\mathrm{sn}_{K_2}^{D-1}(r)}{r^{D-1}} \leq J(r,\theta) \leq \frac{\mathrm{sn}_{K_1}^{D-1}(r)}{r^{D-1}} \ , \tag{46}$$

where

$$\mathrm{sn}_K(r) = \begin{cases} \frac{\sin(\sqrt{K} r)}{\sqrt{K}} & \text{if } K > 0 \\ r & \text{if } K = 0 \\ \frac{\sinh(\sqrt{-K} r)}{\sqrt{-K}} & \text{if } K < 0. \end{cases} \tag{47}$$

Then the upper bound of Eq. (45) becomes

$$\begin{aligned} &\int_{\mathbb{S}^2} \mathrm{d}\Omega \int_0^\infty \mathrm{d}r \frac{\sinh^2(\sqrt{-K_1} r)}{(-K_1)} \exp\left[-2\left(\frac{m r^2 |E_k|}{C_3}\right)^{1/2}\right] \\ &= \frac{1}{|K_1|^{3/2}} \int_{\mathbb{S}^2} \mathrm{d}\Omega \int_0^\infty \mathrm{d}r' \ \sinh^2(r') \exp\left[-2\left(\frac{m r'^2 |E_k|}{C_3 |K_1|}\right)^{1/2}\right] \\ &\leq \frac{\pi}{|K_1|^{3/2}} \int_0^\infty \mathrm{d}r' \ \exp\left[-2r'\left(\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2} - 1\right)\right] \\ &= \frac{\pi}{2|K_1|^{3/2}\left[\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2} - 1\right]} \end{aligned} \tag{48}$$

as long as $\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2} \geq 1$. Since we are interested in the limit $k \to \infty$, it is satisfied for sufficiently large values of $|E_k|$. Therefore, Eq. (42) is smaller than

$$\frac{\left(\frac{\pi}{2}\right)^{1/2}}{\left(\delta |K_1|^{3/4}\right)} \exp\left[-\left(\frac{m\delta^2 |E_k|}{C_3}\right)^{1/2}\right]\left[\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2} - 1\right]^{-1/2} ||f||. \tag{49}$$

If we choose $\delta = (m\sqrt{R})^{-1/3}|E_k|^{-1/3}$, where R is an appropriate scale coming from the Ricci tensor around a point, where Ricci tensor is non-zero. The prefactor multiplying the exponent goes to infinity whereas the exponent decays rapidly. In fact, it decays fast enough to make the expression as a whole go to zero as $k \to \infty$.

Let us go back to the first integral in Eq. (41) and write it in the Gaussian spherical coordinates,

$$\int_{\mathbb{S}^2} \mathrm{d}\Omega \int_0^\delta \mathrm{d}r \ r^2 J(r,\theta) \exp\left[-2\left(\frac{m r^2 |E_k|}{C_3}\right)^{1/2}\right] \frac{|f(r,\theta)|}{r} \ . \tag{50}$$

Let us now make the observation that there are constants A_+, A_-, which depend only on δ and K_i's such that

$$A_-(K_i, K_j) \leq \frac{\mathrm{sn}_{K_i}(r)}{\mathrm{sn}_{K_j}(r)} \leq A_+(K_i, K_j), \tag{51}$$

for $r \in [0, \delta]$. For this part of the integral, we use the following characterization of essential supremum: let us define

$$\Lambda(\epsilon) = \mu(\{r \in [0, \delta]|\, |r^{3/2} F(r)| > \epsilon\}), \tag{52}$$

where μ is the standard Lebesgue measure. Then we have

$$\operatorname*{Essup}_{r\in[0,\delta]} |r^{3/2} F(r)| = \inf_{\epsilon}\{\epsilon | \Lambda(\epsilon) = 0\}. \tag{53}$$

Let us use now $F(r) = \int_{\mathbb{S}^2} \mathrm{d}\Omega |f(r, \theta)|$, and using Bishop-Gunther bound for the first part as

$$\int_0^\delta \mathrm{d}r\; r^{3/2} \int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)| \frac{\exp\left[-2\left(\frac{mr^2|E_k|}{C_3}\right)^{1/2}\right]}{r^{1/2}} \frac{\mathrm{sn}^2_{K_1}(r)}{r^2}, \tag{54}$$

which is smaller than

$$\begin{aligned} A_+^2(K_1, 0) \int_0^\delta \mathrm{d}r\; r^{3/2} \int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)| \frac{\exp\left[-2\left(\frac{mr^2|E_k|}{C_3}\right)^{1/2}\right]}{r^{1/2}} \\ \leq A_+^2(K_1, 0)\Big(\operatorname*{Essup}_{r\in[0,\delta]} |r^{3/2} F(r)|\Big)\left(\int_0^\delta \mathrm{d}r\; \frac{\exp\left[-2\left(\frac{mr^2|E_k|}{C_3}\right)^{1/2}\right]}{r^{1/2}}\right) \\ \leq \Big(\operatorname*{Essup}_{r\in[0,\delta]} |r^{3/2} F(r)|\Big) \frac{A_+^2(K_1, 0)}{2(m/C_3)^{1/4} |E_k|^{1/4}}. \end{aligned} \tag{55}$$

If we take the limit $\delta = (m\sqrt{R})^{-1/3} |E_k|^{-1/3} \to 0$, we claim that the essential-supremum goes to zero. To see this, we observe by Markov inequality[20] that

$$\begin{aligned} \Lambda(\epsilon) &\leq \frac{1}{\epsilon} \int_0^\delta \mathrm{d}r\; |r^{3/2} F(r)| \\ &\leq \frac{1}{\epsilon}\left[\int_0^\delta \mathrm{d}r\; r\right]^{1/2} \left[\int_0^\delta \mathrm{d}r\; r^2 \left(\int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)|\right)^2\right]^{1/2} \\ &\leq \frac{1}{\epsilon}\frac{\delta}{\sqrt{2}}\left[\int_0^\delta \mathrm{d}r\; \frac{r^2}{\mathrm{sn}^2_{K_2}(r)} \mathrm{sn}^2_{K_2}(r) \int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)|^2 \int_{\mathbb{S}^2} \mathrm{d}\Omega\right]^{1/2} \\ &\leq \frac{1}{\epsilon}\frac{\delta}{\sqrt{2}} (4\pi)^{1/2} A_+(0, K_2) \left[\int_0^\delta \mathrm{d}r\; \mathrm{sn}^2_{K_2}(r) \int_{\mathbb{S}^2} \mathrm{d}\Omega\; |f(r,\theta)|^2\right]^{1/2} \\ &\leq \frac{1}{\epsilon}\frac{\delta}{\sqrt{2}} (4\pi)^{1/2} A_+(0, K_2) \left[\int_0^\delta \mathrm{d}r\; r^2 \int_{\mathbb{S}^2} \mathrm{d}\Omega\; J(r,\theta)|f(r,\theta)|^2\right]^{1/2} \\ &\leq \frac{1}{\epsilon}\frac{\delta}{\sqrt{2}} (4\pi)^{1/2} A_+(0, K_2) ||f||. \end{aligned} \tag{56}$$

For any $\epsilon > 0$, our choice of δ implies that we can make $\Lambda(\epsilon) = 0$. Thus, the infimum goes to zero in the limit as $k \to \infty$. As a result, we see that Eq. (31) is smaller than

$$C_{15}\left[\frac{A_+^2(K_1,0)}{2(1/C_3)^{1/4}}\left(\operatorname*{Essup}_{r\in[0,\delta]}|r^{3/2}F(r)|\right)\right.$$
$$\left.+|E_k|^{1/4}m^{1/4}\frac{\exp\left[-\left(\frac{m\delta^2|E_k|}{C_3}\right)^{1/2}\right]}{\left(\delta|K_1|^{3/4}\right)}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2}-1\right]^{-1/2}||f||\right], \tag{57}$$

and it goes to zero as $k \to \infty$. The repeated application of the same analysis leads us to the same conclusion for the other terms coming from the inverse volume term which has been omitted for simplicity. Indeed, all these terms decay with $|E_k|$ faster than the result that we have obtained above. This completes the proof of the existence of the Hamiltonian for point interactions in three dimensional Riemannian manifolds.

Before embarking on the relativistic version of this problem, we will briefly discuss about the resonances which may possibly appear since the same problem in flat spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ for even one delta center exhibits resonances as discussed in Ref. 2. According to the definition of the resonances defined in Ref. 2, they are identified by the poles of the resolvent in the second sheet of the complex energy plane. We are going to investigate the resonances for the one delta center problem in two and three dimensional non-flat spaces, namely $\mathbb{H}^2$ and $\mathbb{H}^3$. Let us first study the problem in $\mathbb{H}^3$.

(i) $\mathbb{H}^3$: The diagonal heat kernel can be explicitly calculated and given by[16]

$$K_t(a,a;\mathbb{H}^3) = \frac{e^{-\frac{t}{2mR^2}}}{(4\pi t/2m)^{3/2}}, \tag{58}$$

where R is the characteristic length scale which could be taken as the inverse square root of the absolute value of the sectional curvature (in mathematics literature, R is chosen as 1). The zeros of the principal matrix $\phi(E)$ determines the bound state spectrum or the resonance spectrum of the problem. The principal matrix (7) of one-center problem becomes a function since $N = 1$. Its explicit form can be easily obtained by substituting (58) into (7), so we get

$$\Phi(E) = \frac{(2m)^{3/2}}{4\pi}\left(\sqrt{\frac{1}{2mR^2}-E}-\sqrt{\frac{1}{2mR^2}+\mu^2}\right), \tag{59}$$

where $\Re(E) < 0$ in order to ensure the convergence of "time" integral in (7). In this case, the bound state energy is readily found as $E = -\mu^2$ by assuming the fact that energy is on the real line and negative. In Ref. 8, our main problem was to consider the bound states and we followed the minimal renormalization prescription. In general, one can always add an arbitrary constant term to the renormalized term. Keeping this in mind, we will choose the bare coupling constant as

$$\frac{1}{\lambda(\epsilon)} = \int_\epsilon^\infty \mathrm{d}t\; K_t(a,a;\mathbb{H}^3)e^{-t\mu^2} + \frac{1}{\lambda_R(\mu)}, \tag{60}$$

where $\lambda_R(\mu)$ is the renormalized coupling constant defined as in Ref. 8. In this prescription, the principal function becomes

$$\Phi(E) = \frac{(2m)^{3/2}}{4\pi}\left(\sqrt{\frac{1}{2mR^2}-E}-\sqrt{\frac{1}{2mR^2}+\mu^2}+\frac{1}{\lambda_R(\mu)}\right), \tag{61}$$

for $\Re(E) < 0$. However, we can also consider here the case where the principal function can be analytically continued into the complex energy plane. As it is clear from its explicit expression (61), it has a branch cut from $1/R^2$ to the positive infinity. Therefore, the poles of the resolvent which correspond to the solution $\Phi(E) = 0$ in the second sheet of the complex energy plane appear only if

$$\frac{1}{\lambda_R(\mu)} - \sqrt{\frac{1}{2mR^2}+\mu^2} \geq 0. \tag{62}$$

Otherwise, there is no solution. Hence, the solution for the resonance energy can easily be found under the above condition as

$$E_{\text{res}} = \frac{1}{2mR^2} - \left(\frac{1}{\lambda_R(\mu)} - \sqrt{\frac{1}{2mR^2} + \mu^2}\right)^2. \tag{63}$$

(ii) $\mathbb{H}^2$: Since the heat kernel for $\mathbb{H}^2$ is well known,[16] we have

$$K_t(a, a; \mathbb{H}^2) = \frac{\sqrt{2}}{(4\pi/2mR^2)^{3/2}} \frac{e^{-\frac{t}{8mR^2}}}{R^2} \int_0^\infty \mathrm{d}r \, \frac{r \, e^{-\frac{2mR^2 r^2}{4t}}}{\sqrt{\cosh r - 1}}, \tag{64}$$

where R is the characteristic length scale. Substituting this into the principal function, we find[6]

$$\Phi(E) = \frac{2m\sqrt{2}}{4\pi}\left[\psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} - 2mR^2 E}\right) - \psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + 2mR^2\mu^2}\right)\right] + \frac{1}{\lambda_R(\mu)}, \tag{65}$$

where ψ is the digamma function. An integral representation of the digamma function[21] is given by

$$\psi(z) = \int_0^\infty \mathrm{d}t \, \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\right), \tag{66}$$

where $\Re(z) > 0$ and it is a meromorphic function. However, due to the square root, the above function has a branch cut from the point $\frac{1}{8mR^2}$ to infinity. Similar to the above analysis, the poles of the resolvent appears in the second sheet of the energy plane only if

$$\frac{1}{\lambda_R(\mu)} - \psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + 2mR^2\mu^2}\right) \geq 0. \tag{67}$$

Then, the resonance energy E_r can be found from the solution of the equation

$$\psi\left(\frac{1}{2} - \sqrt{\frac{1}{4} - 2mR^2 E_r}\right) = -\gamma^2, \tag{68}$$

where $\gamma^2 = \frac{1}{\lambda_R(\mu)} - \psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + 2mR^2\mu^2}\right)$.

III. RELATIVISTIC POINT INTERACTIONS ON TWO DIMENSIONAL RIEMANNIAN MANIFOLDS

The resolvent for this system have been found in Ref. 9 and it is given by

$$R(E) = R_0(E) + R_0(E) b^\dagger \Phi^{-1}(E) b R_0(E), \tag{69}$$

where

$$b^\dagger = \sum_{i=1}^N \phi^{(-)}(a_i)\chi_i \tag{70}$$

and

$$\begin{aligned}\Phi(E) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^N \int_0^\infty \mathrm{d}s \, e^{-s^2/4} \int_0^\infty \mathrm{d}u \, \left(e^{s\mu_i\sqrt{u}} - e^{sE\sqrt{u}}\right) e^{-um^2} K_u(a_i, a_i; g)\chi_i^\dagger \chi_i \\ - \frac{1}{\sqrt{\pi}} \sum_{\substack{i,j \\ (i \neq j)}} \int_0^\infty \mathrm{d}s \, e^{-s^2/4} \int_0^\infty \mathrm{d}u \, e^{sE\sqrt{u}} e^{-um^2} K_u(a_i, a_j; g)\chi_i^\dagger \chi_j.\end{aligned} \tag{71}$$

Here $\phi^{(-)}(x)$ is the positive frequency part of the bosonic field and a_i stands for the position of one of the N Dirac-delta function potential centers and μ_i is the bound state energy for the single delta center at a_i. The operator χ_i, called angel operator, was first introduced for this purpose by Rajeev in Ref. 11 and it obeys orthofermionic algebra. $K_t(x, y; g)$ is the heat kernel on the Riemannian manifold, which is defined as the fundamental solution to the heat equation

$$\nabla_g^2 K_t(x, y; g) = \frac{\partial K_t(x, y; g)}{\partial t}. \tag{72}$$

We would like to first check whether $R(E)$ satisfies the resolvent identity (1). If we put the form of the resolvent in second quantized form,

$$R(E) = R_0(E) + R_0(E) b^\dagger \Phi^{-1}(E) b R_0(E), \tag{73}$$

into the above resolvent identity (1) and we simplify by purely algebraic operations to arrive at the following identity:

$$\Phi_{ij}(E_1) - \Phi_{ij}(E_2) + b_i(R_0(E_1) - R_0(E_2)) b_j^\dagger = 0, \tag{74}$$

where we stripped off the angels and wrote everything in terms of explicit matrix indices, and thus $\Phi(E) = \Phi_{ij}(E)\chi_i^\dagger \chi_j$ and also $b_i = \phi^{(+)}(a_i)$ and similarly for $b_j^\dagger$. Let us now verify the above identity, we note that

$$\Phi_{ij}(E_1) - \Phi_{ij}(E_2) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{d}s\, e^{-s^2/4} \int_0^\infty \mathrm{d}u\, [e^{sE_2\sqrt{u}} - e^{sE_1\sqrt{u}}] e^{-um^2} K_u(a_i, a_j; g). \tag{75}$$

Let us work out the other term, acting on *no particle Fock space*, this is the same calculation we have done for the renormalized term. For simplicity, we present the calculation in a formal eigenfunction expansion of the Laplace operator (which is rigorously valid for only compact manifolds)

$$\begin{aligned}
b_i(R_0(E_1) - R_0(E_2)) b_j^\dagger &= \sum_\sigma f_\sigma^*(a_i) \frac{a_\sigma}{\sqrt{\omega_\sigma}} \Big[\frac{1}{H_0 - E_1} - \frac{1}{H_0 - E_2}\Big] \sum_\lambda f_\lambda(a_j) \frac{a_\lambda^\dagger}{\sqrt{\omega_\lambda}} \\
&= \sum_\sigma f_\sigma^*(a_i) \frac{a_\sigma}{\sqrt{\omega_\sigma}} \sum_\lambda f_\lambda(a_j) \frac{a_\lambda^\dagger}{\sqrt{\omega_\lambda}} \Big[\frac{1}{H_0 + \omega_\lambda - E_1} - \frac{1}{H_0 + \omega_\lambda - E_2}\Big] \\
&= \sum_\sigma \sum_\lambda f_\sigma^*(a_i) f_\lambda(a_j) \Big[\frac{a_\lambda^\dagger}{\sqrt{\omega_\lambda}} \frac{a_\sigma}{\sqrt{\omega_\sigma}} + \frac{\delta_{\sigma\lambda}}{\sqrt{\omega_\lambda}\sqrt{\omega_\sigma}}\Big] \Big[\frac{1}{\omega_\lambda - E_1} - \frac{1}{\omega_\lambda - E_2}\Big] \\
&= \sum_\lambda f_\lambda^*(a_i) f_\lambda(a_j) \frac{1}{\omega_\lambda} \Big[\frac{1}{\omega_\lambda - E_1} - \frac{1}{\omega_\lambda - E_2}\Big] \\
&= \int_0^\infty \mathrm{d}s\, s \sum_\lambda f_\lambda^*(a_i) f_\lambda(a_j) \int_0^1 \mathrm{d}\zeta\; e^{-s\omega_\lambda} \Big[e^{s\zeta E_1} - e^{s\zeta E_2}\Big] \\
&= \int_0^\infty \mathrm{d}s\, s \int_0^1 \mathrm{d}\zeta\; \frac{s}{2\sqrt{\pi}} \int_0^\infty \frac{\mathrm{d}u}{u^{3/2}} e^{-s^2/4u - m^2 u} K_u(a_i, a_j; g) \Big[e^{s\zeta E_1} - e^{s\zeta E_2}\Big] \\
&= \int_0^\infty \mathrm{d}s\, s \int_0^1 \mathrm{d}\zeta\; \frac{s}{2\sqrt{\pi}} \int_0^\infty \mathrm{d}u\, e^{-s^2/4 - m^2 u} K_u(a_i, a_j; g) \Big[e^{s\sqrt{u}\zeta E_1} - e^{s\sqrt{u}\zeta E_2}\Big] \\
&= \int_0^\infty \mathrm{d}s\, \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{d}u\, e^{-s^2/4 - m^2 u} K_u(a_i, a_j; g) \Big[e^{s\sqrt{u}E_1} - e^{s\sqrt{u}E_2}\Big].
\end{aligned} \tag{76}$$

After calculating the ζ integral, we performed an integration by parts over the variable s. Hence we have found the required result (74).

Similar to the previous problem, let us choose the sequence $E_k = -k|E_0| = -|E_k|$, where E_0 is sufficiently below the lower bound E_* on the ground state energy which has been found in Ref. 9 and negative. We now want to show (30). Substituting the resolvent equation, written in the second quantized language, in this expression we obtain the following:

$$\lim_{k\to\infty} ||E_k[(H_0 - E_k)^{-1} + (H_0 - E_k)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j)(H_0 - E_k)^{-1}] f + f|| = 0. \tag{77}$$

The free resolvent, that is the first term in the above equation,

$$\lim_{k\to\infty} ||E_k R_0(E_k) f + f|| = \lim_{k\to\infty} ||E_k (H_0 - E_k)^{-1} f + f|| = 0 \tag{78}$$

already satisfies the resolvent equation hence we should look only into the second part. To see this, note that by the triangle inequality

$$\begin{aligned} \lim_{k\to\infty} ||E_k R(E_k) f + f|| &\leq \lim_{k\to\infty} \Big(||E_k (H_0 - E_k)^{-1} f + f|| \\ &+ ||E_k [(H_0 - E_k)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j)(H_0 - E_k)^{-1}] f|| \Big) \\ &\leq \lim_{k\to\infty} ||E_k [(H_0 - E_k)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j)(H_0 - E_k)^{-1}] f||. \end{aligned} \tag{79}$$

We choose a one-particle wave function of the form given below. Even though, this is the most general one-particle wave function one can write down, it does not include multi-particle wave functions. However, due to the mutually non-interacting nature of the particles involved, the total Hamiltonian appearing in the resolvent will be a sum of n identical, individual Hamiltonians in the case of a n-particle state and therefore will decay faster than in the one-particle case.

$$|\psi\rangle = \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \psi(x) \phi^{(-)}(x) |0\rangle = \sum_\sigma \hat{\psi}(\sigma) \frac{a_\sigma^\dagger}{\sqrt{\omega_\sigma}} |0\rangle. \tag{80}$$

A direct computation now reveals that

$$\langle \psi | \psi \rangle = \sum_\sigma \frac{|\hat{\psi}(\sigma)|^2}{\omega_\sigma}. \tag{81}$$

We verify that the limit

$$\lim_{k\to\infty} |E_k| \, || \sum_{i=1}^N \sum_{j=1}^N (H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j) (H_0 + |E_k|)^{-1} |\psi\rangle || \tag{82}$$

$$\leq \lim_{k\to\infty} |E_k| \, || \sum_{i=1}^N \sum_{j=1}^N |\Phi_{ij}^{-1}(E_k)| \; ||(H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \phi^{(+)}(a_j) (H_0 + |E_k|)^{-1} |\psi\rangle || \; ||, \tag{83}$$

converges to zero. An explicit computation reveals that

$$\phi^{(+)}(a_j)(H_0 + |E_k|)^{-1} |\psi\rangle = \sum_\sigma \frac{f_\sigma(a_j) \hat{\psi}(\sigma)}{(\omega_\sigma + |E_k|)\omega_\sigma} |0\rangle. \tag{84}$$

The action of $(H_0 + |E_k|)^{-1} \phi^{(-)}(a_i)$ onto this expression leads to

$$(H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \sum_\sigma \frac{f_\sigma(a_j) \hat{\psi}(\sigma)}{(\omega_\sigma + |E_k|)\omega_\sigma} |0\rangle = \sum_\sigma \frac{f_\sigma(a_j) \hat{\psi}(\sigma)}{(\omega_\sigma + |E_k|)\omega_\sigma} \sum_{\sigma'} \frac{f_{\sigma'}(a_i)}{(\omega_{\sigma'} + |E_k|)} \frac{a_{\sigma'}^\dagger}{\sqrt{\omega_{\sigma'}}} |0\rangle. \tag{85}$$

We have now a term like $|\Phi_{ij}^{-1}(E_k)| \; ||F(a_i, a_j|E_k)||$, which satisfies

$$|| \sum_{i,j=1}^N |\Phi_{ij}^{-1}(E_k)| \; ||F(a_i, a_j|E_k)|| \; || \leq \Big[\mathrm{Tr} \, |\Phi^{-1}(E_k)|^2 \Big]^{1/2} \Big[\mathrm{Tr} \, ||F(E_k)||^2 \Big]^{1/2}. \tag{86}$$

This can be used in the above norm, and we get after one more use of the Cauchy-Schwarz inequality,

$$\begin{aligned} &\lim_{k\to\infty} |E_k| ||(H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j)(H_0 + |E_k|)^{-1} |\psi\rangle || \\ &\leq N |E_k| \max_{1 \leq i,j \leq N} |\Phi_{ij}^{-1}(E_k)| \Bigg[\sum_{i=1}^N \sum_\sigma \frac{|f_\sigma(a_i)|^2}{(\omega_\sigma + |E_k|)^2 \omega_\sigma} \Bigg] \Big[\sum_\tau \frac{|\hat{\psi}(\tau)|^2}{\omega_\tau} \Big]^{1/2}, \end{aligned} \tag{87}$$

where we have used

$$\sum_{\sigma} \frac{|f_\sigma(a_j)\hat{\psi}(\sigma)|}{(\omega_\sigma + |E_k|)\omega_\sigma} \leq \Big[\sum_{\sigma} \frac{|f_\sigma(a_j)|^2}{(\omega_\sigma + |E_k|)^2\omega_\sigma} \Big]^{1/2} \Big[\sum_{\tau} \frac{|\hat{\psi}(\tau)|^2}{\omega_\tau} \Big]^{1/2}. \tag{88}$$

We recall that by choosing k sufficiently large we can make the off-diagonal elements as small as we like, while the diagonal elements increase. Therefore, without repeating the arguments of the previous section for sufficiently large values of $|E_k|$, we can show that

$$\max_{1 \leq i,j \leq N} |\Phi_{ij}^{-1}(E_k)| \leq ||\Phi_{ij}^{-1}(E_k)|| \leq 2 \max_{1 \leq i \leq N} |\Phi_{ii}^{-1}(E_k)| \tag{89}$$

where

$$\max_{1 \leq i \leq N} |\Phi_{ii}^{-1}(E_k)| \leq \frac{C_{16}}{\ln(|E_k|/(m - \mu_i^{min}))}, \tag{90}$$

and the constant C_{16} depends on the class of manifolds under consideration. In the above equation m should be superseded on Cartan-Hadamard manifolds by its counterpart m_{CH}, as defined in Ref. 9. Listed below are the values of this constant for compact Riemannian manifolds with positive Ricci curvature for flat space and for Cartan-Hadamard manifolds:

$$C_{16} = \begin{cases} 2\pi & \text{for flat and compact manifolds} \\ \frac{2\pi}{c(\delta)} & \text{for Cartan-Hadamard manifolds} \end{cases}. \tag{91}$$

Thus, essentially we are faced with the sum/integral,

$$I = \sum_{\sigma} \frac{|f_\sigma(a)|^2}{\omega_\sigma(\omega_\sigma + |E_k|)^2}. \tag{92}$$

We will work this out: First we recall that

$$\frac{1}{\omega_\sigma(\omega_\sigma + |E_k|)^2} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 \mathrm{d}\zeta \, \frac{\zeta}{[\omega_\sigma + \zeta|E_k|]^3}. \tag{93}$$

Let us now use the exponential form for the integrand,

$$\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 \mathrm{d}\zeta \, \frac{\zeta}{[\omega_\sigma + \zeta|E_k|]^3} = \frac{\Gamma(3)}{2\Gamma(2)\Gamma(1)} \int_0^1 \mathrm{d}\zeta \, \zeta \int_0^\infty \mathrm{d}s \, s^2 e^{-s\omega_\sigma} e^{-\zeta s|E_k|} \tag{94}$$

and use subordination for ω_σ,

$$e^{-s\omega_\sigma} = \frac{s}{2\sqrt{\pi}} \int_0^\infty \frac{\mathrm{d}u}{u^{3/2}} e^{-s^2/4u - m^2 u} e^{-\lambda(\sigma)u}, \tag{95}$$

where $\lambda(\sigma)$ is the eigenvalue of the Laplacian defined in Ref. 9. If we combine the last exponential with $|f_\sigma(a)|^2$ terms, we get the heat kernel at the same points, $K_u(a, a; g)$ and collecting them, we find

$$\begin{aligned} I &= \frac{\Gamma(3)}{4\sqrt{\pi}\Gamma(2)\Gamma(1)} \int_0^\infty \frac{\mathrm{d}u}{u^{3/2}} \int_0^1 \mathrm{d}\zeta \, \zeta \int_0^\infty \mathrm{d}s \, s^3 \, e^{-s^2/4u - m^2 u} K_u(a, a; g) e^{-\zeta s|E_k|} \\ &= \frac{\Gamma(3)}{4\sqrt{\pi}\Gamma(2)\Gamma(1)} \int_0^\infty \frac{\mathrm{d}u}{u^{3/2}} \int_0^\infty \mathrm{d}s \, s^3 \, e^{-s^2/4u - m^2 u} K_u(a, a; g) \int_0^1 \mathrm{d}\zeta \, \zeta e^{-\zeta s|E_k|} \\ &= \frac{\Gamma(3)}{4\sqrt{\pi}\Gamma(2)\Gamma(1)} \int_0^\infty \frac{\mathrm{d}u}{u^{3/2}} \int_0^\infty \mathrm{d}s \, s^3 \, e^{-s^2/4u - m^2 u} K_u(a, a; g) \frac{1}{s^2|E_k|^2} \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] \\ &\leq \frac{\Gamma(3)}{4\sqrt{\pi}|E_k|^2\Gamma(2)\Gamma(1)} \int_0^\infty \frac{\mathrm{d}u}{u^{3/2}} \int_0^\infty \mathrm{d}s \, s \, e^{-s^2/4u - m^2 u} \Big[\frac{C_{17}}{A(\mathcal{M})} + \frac{C_{18}}{u} \Big] \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] \end{aligned}$$

$$= \frac{1}{|E_k|^2} \int_0^\infty \mathrm{d}u \Big[\frac{C_{19}}{A(\mathcal{M}) u^{3/2}} + \frac{C_{20}}{u^{5/2}} \Big] \int_0^\infty \mathrm{d}s \, s \, e^{-s^2/4u - m^2 u} \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big]$$

$$= \frac{1}{|E_k|^2} \int_0^\infty \mathrm{d}s \, s \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] \int_0^\infty \mathrm{d}v \Big[\frac{C_{19} m}{A(\mathcal{M}) v^{3/2}} + \frac{C_{20} m^3}{v^{5/2}} \Big] e^{-(ms)^2/4v - v}, \tag{96}$$

where $A(\mathcal{M})$ is the area of the manifold. Here the most divergent contribution comes from the last term in the above expression, so we first analyze this term. By inspecting the following integral representation of the modified Bessel function $K_{3/2}(v)$:[22]

$$K_{3/2}(ms) = \frac{1}{2} \Big(\frac{ms}{2} \Big)^{3/2} \int_0^\infty \frac{\mathrm{d}v}{v^{3/2+1}} e^{-(ms)^2/4v - v}, \tag{97}$$

we obtain the following:

$$I_2 = \frac{m^{3/2} C_{21}}{|E_k|^2} \int_0^\infty \frac{\mathrm{d}s}{\sqrt{s}} K_{3/2}(ms) \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big]. \tag{98}$$

We now use another integral representation of the modified Bessel function $K_{3/2}(x)$,[22]

$$K_{3/2}(ms) = \frac{\Gamma(2) 2^{3/2} s^{3/2}}{\sqrt{\pi} m^{3/2}} \int_0^\infty \mathrm{d}t \, \frac{\cos(mt)}{(t^2 + s^2)^2}. \tag{99}$$

As a result we see that

$$\begin{aligned} I_2 &= \frac{C_{22}}{|E_k|^2} \int_0^\infty \mathrm{d}s \, s \int_0^\infty \mathrm{d}r \; s \frac{\cos(msr)}{(r^2 s^2 + s^2)^2} \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] \\ &\leq \frac{C_{22}}{|E_k|^2} \int_0^\infty \frac{\mathrm{d}s}{s^2} \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] \int_0^\infty \frac{\mathrm{d}r}{(r^2+1)^2} \\ &\leq \frac{C_{23}}{|E_k|^2} \int_0^\infty \frac{\mathrm{d}s}{s^2} \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] \\ &\leq \frac{C_{23}}{|E_k|} \int_0^\infty \frac{\mathrm{d}s}{s^2} \Big[1 - e^{-s} - s e^{-s} \Big]. \end{aligned} \tag{100}$$

We now note that the integral

$$\int_0^\infty \frac{\mathrm{d}s}{s^2} \Big[1 - e^{-s} - s e^{-s} \Big] \tag{101}$$

is actually convergent. The first term instead becomes

$$\begin{aligned} I_1 &= \frac{2\sqrt{\pi} C_{24}}{|E_k|^2} \int_0^\infty \mathrm{d}s \, \Big[1 - e^{-s|E_k|} - s|E_k| e^{-s|E_k|} \Big] e^{-ms} \\ &\leq \frac{C_{25}}{|E_k|^3} \int_0^\infty \mathrm{d}s \Big[1 - e^{-s} - s e^{-s} \Big] e^{-ms/|E_k|} \\ &\leq \frac{C_{26}}{A(\mathcal{M}) |E_k|^3}. \end{aligned} \tag{102}$$

Hence

$$\sum_\sigma \frac{|f_\sigma(a)|^2}{\omega_\sigma (\omega_\sigma + |E_k|)^2} \leq \frac{C_{27}}{|E_k|} + \frac{C_{26}}{A(\mathcal{M}) |E_k|^3} \tag{103}$$

is shown. As a result we see that

$$\lim_{k\to\infty} ||[1 - |E_k| R(E_k)] |\psi\rangle|| \leq \lim_{k\to\infty} |E_k| \Big[\frac{C_{28}}{|E_k| \ln(|E_k|)} + \frac{C_{29}}{A(\mathcal{M}) |E_k|^3 \ln(|E_k|)} \Big] \to 0, \tag{104}$$

which proves that our formula defines a densely defined closed operator.

IV. NON-RELATIVISTIC LEE MODEL IN TWO AND THREE DIMENSIONAL RIEMANNIAN MANIFOLDS

A. Introduction

Before starting directly to our proof for this model, let us summarize what the non-relativistic Lee model is about and give the renormalization of this model form our point of view.[7,10]

The Lee model[23] was first introduced as a simple renormalizable model which describes the interaction between a relativistic neutral bosonic field (e.g., "pions"), and two neutral fermionic fields (e.g., "nucleons"). Nucleons exist in two different intrinsic states. The particle corresponding to the Bose field is called the θ and the particles corresponding to the intrinsic states of the nucleon are called the V and the N particles. The fermionic field corresponding to the V and N particles are assumed to be spinless for simplicity. Only allowable process is given by

$$V \leftrightharpoons N + \theta \tag{105}$$

and the following process is not allowed:

$$N \leftrightharpoons V + \theta\,, \tag{106}$$

which makes the model rather simple. Although this model is not realistic, the important features of nucleon-pion system can be understood in a relatively simple way and one can get rid of the infinities without applying perturbation theory techniques. Moreover, the complete non-relativistic version of this model that describes one heavy particle sitting at some fixed point interacting with a field of non-relativistic bosons is as important as its relativistic counterpart. It is much simpler than its relativistic version because only an additive renormalization of the mass difference of the fermions is necessary. It has been studied in a textbook by Henley and Thirring for small number of bosons from the point of view of scattering matrix[24] and there are various other approaches to the model.[25–31] It is possible to look at the same problem from the point of view of the resolvent of the Hamiltonian in a Fock space formalism with arbitrary number of bosons (in fact there is a conserved quantity which allows us to restrict the problem to the direct sum of n and $n+1$ boson sectors). This is achieved in a very interesting unpublished paper by Rajeev,[11] in which a new non-perturbative formulation of renormalization for some models with contact interactions has been proposed. Our approach to Lee model is based on that work. Let us now shortly review how we non-perturbatively renormalize this model in this point of view, which was given in Ref. 7 and recently developed for the two dimensional case[10] and all the details about the construction of the model and the information about the bound state spectrum can be found in these references.

We start with the regularized Hamiltonian of the non-relativistic Lee model on $D = 2, 3$ dimensional Riemannian manifold $(\mathcal{M}, g)$ with a cut-off ϵ. Adopting the natural units ($\hbar = c = 1$), one can write down the regularized Hamiltonian,

$$H^{\epsilon} = H_0 + H_{I,\epsilon}, \tag{107}$$

where

$$H_0 = \int_{\mathcal{M}} \mathrm{d}_g^D x\; \phi_g^{\dagger}(x) \left(-\frac{1}{2m}\nabla_g^2 + m\right) \phi_g(x), \tag{108}$$

$$H_{I,\epsilon} = \mu(\epsilon)\left(\frac{1-\sigma_3}{2}\right) + \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x\; K_{\epsilon}(x, a; g)\left(\phi_g(x)\,\sigma_- + \phi_g^{\dagger}(x)\,\sigma_+\right). \tag{109}$$

Here $\phi_g^{\dagger}(x)$, $\phi_g(x)$ is the bosonic creation-annihilation operators defined on the manifold with the metric structure g_{ij} and λ is the coupling constant, and x, y refers to points on the manifold $\mathcal{M}$. Also, $K_{\epsilon}(x, a; g)$ is the heat kernel on a Riemannian manifold with metric structure g and it converges to the Dirac delta function $\delta_g(x, a)$ around the point a on $\mathcal{M}$ as we take the limit $\epsilon \to 0^+$. Similar to the flat case, $\mu(\epsilon)$ is defined as a bare mass difference between the V particle (neutron) and the N

particle (proton). One can easily see that there exists a conserved quantity for this model,

$$Q = -\left(\frac{1+\sigma_3}{2}\right) + \int_{\mathcal{M}} \mathrm{d}_g^D x \; \phi_g^\dagger(x)\phi_g(x).$$

Therefore, we can express the regularized Hamiltonian as a 2 × 2 block split according to $\mathbb{C}^2$,

$$H^\epsilon - E = \begin{pmatrix} H_0 - E & \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \, K_\epsilon(x,a;g)\, \phi_g^\dagger(x) \\ \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \, K_\epsilon(x,a;g)\, \phi_g(x) & H_0 - E + \mu(\epsilon) \end{pmatrix}. \tag{110}$$

Then, one can similarly construct the regularized resolvent of this Hamiltonian

$$R^\epsilon(E) = \frac{1}{H^\epsilon - E} = \begin{pmatrix} \alpha_\epsilon & \beta_\epsilon^\dagger \\ \beta_\epsilon & \delta_\epsilon \end{pmatrix} = \begin{pmatrix} a_\epsilon & b_\epsilon^\dagger \\ b_\epsilon & d_\epsilon \end{pmatrix}^{-1}, \tag{111}$$

where

$$\begin{aligned} \alpha_\epsilon &= \frac{1}{H_0 - E} + \frac{1}{H_0 - E}\, b_\epsilon^\dagger \; \Phi_\epsilon^{-1}(E)\, b_\epsilon \; \frac{1}{H_0 - E} \\ \beta_\epsilon &= -\Phi_\epsilon^{-1}(E)\, b_\epsilon \; \frac{1}{H_0 - E} \\ \delta_\epsilon &= \Phi_\epsilon^{-1}(E) \\ b_\epsilon &= \lambda \int_{\mathcal{M}} \mathrm{d}_g^D x \, K_\epsilon(x,a;g)\, \phi_g(x). \end{aligned} \tag{112}$$

The operator $\Phi_\epsilon(E)$, called principal operator, is given as

$$\Phi_\epsilon(E) = H_0 - E + \mu(\epsilon) - \lambda^2 \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \, K_\epsilon(x,a;g) K_\epsilon(y,a;g)\, \phi_g(x) \frac{1}{H_0 - E} \phi_g^\dagger(y). \tag{113}$$

We then make a normal ordering of this operator

$$\begin{aligned} \Phi_\epsilon(E) = H_0 - E - \lambda^2 \int_{\epsilon/2}^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \, K_s(x,a;g) K_s(y,a;g) \\ \times \phi_g^\dagger(x) e^{-(s-\epsilon/2)(H_0+2m-E)} \phi_g(y) + \mu + \lambda^2 \int_\epsilon^\infty \mathrm{d}s \, K_s(a,a;g) \big[e^{-s(m-\mu)} - e^{-(s-\epsilon)(H_0+m-E)} \big], \end{aligned} \tag{114}$$

and then choose $\mu(\epsilon)$ as

$$\mu(\epsilon) = \mu + \lambda^2 \int_\epsilon^\infty \mathrm{d}s \, K_s(a,a;g)\, e^{-s(m-\mu)}, \tag{115}$$

where μ is defined as the physical energy of the composite state which consists of a boson and the attractive heavy neutron at the center a. Then, we take the limit $\epsilon \to 0^+$ and find

$$\begin{aligned} \Phi(E) = H_0 - E + \mu + \lambda^2 \int_0^\infty \mathrm{d}s \, K_s(a,a;g) \left[e^{-s(m-\mu)} - e^{-s(H_0+m-E)} \right] \\ - \lambda^2 \int_0^\infty \mathrm{d}s \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \, K_s(x,a;g) K_s(y,a;g)\, \phi_g^\dagger(x) e^{-s(H_0+2m-E)} \phi_g(y). \end{aligned} \tag{116}$$

This is the finite form of the principal operator so that we have a well-defined explicit formula for the resolvent of the Hamiltonian in terms of the inverse of the principal operator $\Phi^{-1}(E)$. The bound states arise from the roots of the equation,

$$\Phi(E)|\Psi\rangle = 0, \tag{117}$$

corresponding to the poles in the resolvent. Hence, the principal operator determines the bound state spectrum.

B. The lower bound on the ground state energy

In this section, we will prove that there exists a lower bound on the ground state energy of the system for $D = 2, 3$ dimensional cases. The three dimensional and two dimensional cases were given separately in Refs. 7 and 10. But we will summarize them for completeness.

We will first restrict the energy E to the real axis. In order to give the proof that the energy E is bounded from below, we split the principal operator as

$$\Phi(E) = K(E) - U(E), \tag{118}$$

such that

$$K(E) = H_0 - E + \mu, \tag{119}$$

and

$$U(E) = U_1(E) + U_2(E) = -\lambda^2 \int_0^\infty \mathrm{d}t \; K_t(a, a; g) \left[e^{-t(m-\mu)} - e^{-t(H_0+m-E)}\right]$$

$$+ \lambda^2 \int_0^\infty \mathrm{d}t \int_{\mathcal{M}^2} \mathrm{d}_g^D x \, \mathrm{d}_g^D y \; K_t(x, a; g) \, K_t(y, a; g) \, \phi_g^\dagger(x) \, e^{-t(H_0+2m-E)} \, \phi_g(y). \tag{120}$$

It follows immediately that $K(E) \geq nm - E + \mu$, so it is a positive definite operator from our assumption $E < nm + \mu$. Due to the positivity of the heat kernel and since the difference of the two exponentials is a positive operator, the first integral term $U_1(E)$ is a negative operator. We thus remark that

$$U(E) \leq U_2(E). \tag{121}$$

This clearly forces

$$\Phi(E) \geq K(E) - U_2(E), \tag{122}$$

or rewriting it as

$$\Phi(E) \geq K(E)^{1/2} \, \left(1 - \tilde{U}_2(E)\right) \, K(E)^{1/2}, \tag{123}$$

where $\tilde{U}_2(E) = K(E)^{-1/2} \, U_2(E) \, K(E)^{-1/2}$ and $K(E)$, $U_2(E)$ are positive operators (so is $\tilde{U}_2(E)$). It must be emphasized that the unique square root of the positive self-adjoint operators $K(E)$ are well defined for all real values of E below μ. We will now show that by choosing E sufficiently small it is always possible to make the operator $\Phi(E)$ strictly positive, hence it becomes invertible, and has no zeros beyond this particular value of E (in Sec. IV C, the self-adjointness will be further clarified). Therefore, if we impose

$$||\tilde{U}_2(E)|| < 1, \tag{124}$$

then the principal operator $\Phi(E)$ becomes strictly positive. For Cartan-Hadamard manifolds, we have obtained in Ref. 7

$$||\tilde{U}_2(E)|| \leq n \, C_{30} m^{D/2} \, \frac{\lambda^2 \, \Gamma(2)}{\Gamma(1/2)^2} (nm + \mu - E)^{\frac{D}{2}-2} \Gamma(2 - \frac{D}{2}) \left[\frac{\sqrt{\pi}\Gamma(1 - \frac{D}{4})}{\Gamma(\frac{3}{2} - \frac{D}{4})} \right]^2. \tag{125}$$

Then the strict positivity of the principal operator (124) implies a lower bound for the ground state energy

$$E_{gr} \geq nm + \mu - \left(nC_{31} \lambda^2 m^{D/2} \right)^{\frac{1}{2-\frac{D}{2}}}, \tag{126}$$

where

$$C_{31} = C_{30} \, \frac{\pi \, \Gamma(2)\Gamma(1 - \frac{D}{4})^2 \Gamma(2 - \frac{D}{2})}{\Gamma(\frac{1}{2})^2 \Gamma(\frac{3}{2} - \frac{D}{4})^2}. \tag{127}$$

For the compact manifolds with Ricci curvature bounded from below by $-K \geq 0$, we have similarly obtained

$$||\tilde{U}_2(E)|| \leq n \, \frac{\lambda^2 \, \Gamma(2)}{\Gamma(1/2)^2} \Bigg[\frac{4}{V(\mathcal{M})\mu^{\frac{D}{2}}} + \frac{4A'^{1/2} m^{D/4} \pi^{1/2} \Gamma(2-\frac{D}{4})\Gamma(1-\frac{D}{4})}{\mu^{\frac{D}{4}} V(\mathcal{M})^{1/2} \Gamma(\frac{3}{2}-\frac{D}{4})} + \frac{A' m^{D/2} \pi \, \Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{4})^2}{\Gamma(\frac{3}{2}-\frac{D}{4})^2} \Bigg] \frac{1}{(nm+\mu-E)^{2-\frac{D}{2}}}, \tag{128}$$

so the lower bound of the ground state energy was found

$$E_{gr} \geq nm + \mu - \left(n\lambda^2 C_{32}\right)^{\frac{1}{2-\frac{D}{2}}} , \tag{129}$$

where

$$C_{32} = \frac{\Gamma(2)}{\Gamma(1/2)^2} \Bigg[\frac{4}{V(\mathcal{M})\mu^{\frac{D}{2}}} + \frac{4A'^{1/2} m^{D/4} \pi^{1/2} \Gamma(2-\frac{D}{4})\Gamma(1-\frac{D}{4})}{\mu^{\frac{D}{4}} V(\mathcal{M})^{1/2} \Gamma(\frac{3}{2}-\frac{D}{4})} + \frac{A' m^{D/2} \pi \, \Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{4})^2}{\Gamma(\frac{3}{2}-\frac{D}{4})^2} \Bigg]. \tag{130}$$

Therefore, the lower bounds on the ground state energies for different classes of manifolds (126) and (129) are of almost the same form up to a constant factor, so the form of the lower bound has a general character.

C. Existence of the Hamiltonian for the Lee model in two and three dimensional Riemannian manifolds

The explicit formula for the resolvent of the Hamiltonian in terms of the inverse of the principal operator $\Phi^{-1}(E)$ is given in Refs. 7 and 10 by

$$R(E) = \frac{1}{H-E} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \tag{131}$$

where

$$\begin{aligned}
\alpha &= \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \, b^\dagger \, \Phi^{-1}(E) \, b \, \frac{1}{H_0 - E} \\
\beta &= -\Phi^{-1}(E) \, b \, \frac{1}{H_0 - E} \\
\gamma &= -\frac{1}{H_0 - E} \, b^\dagger \, \Phi^{-1}(E) \\
\delta &= \Phi^{-1}(E) \\
b &= \lambda \phi_g(a).
\end{aligned} \tag{132}$$

Let us check that the resolvent identity $R(E_1) - R(E_2) = (E_1 - E_2)R(E_1)R(E_2)$ is satisfied, that is, we must have

$$\begin{pmatrix} \alpha(E_1) - \alpha(E_2) & \gamma(E_1) - \gamma(E_2) \\ \beta(E_1) - \beta(E_2) & \delta(E_1) - \delta(E_2) \end{pmatrix} = (E_1 - E_2) \begin{pmatrix} \alpha(E_1)\alpha(E_2) + \gamma(E_1)\beta(E_2) & \alpha(E_1)\gamma(E_2) + \gamma(E_1)\delta(E_2) \\ \beta(E_1)\alpha(E_2) + \delta(E_1)\beta(E_2) & \beta(E_1)\gamma(E_2) + \delta(E_1)\delta(E_2) \end{pmatrix}. \tag{133}$$

We first consider the first diagonal element of the above matrix. Using the fact that free resolvent satisfies the resolvent identity, we get

$$R_0(E_1)b^\dagger\Phi^{-1}(E_1)\left[\Phi(E_1)-\Phi(E_2)+b(R_0(E_1)-R_0(E_2))b^\dagger+E_1-E_2\right]\Phi^{-1}(E_2)bR_0(E_2)=0. \tag{134}$$

Let us look at the term in the square bracket more closely. By using the explicit expression of the principal operator (116), this term becomes

$$\begin{aligned}&\lambda^2\int_0^\infty \mathrm{d}t\; K_t(a,a;g)\left[e^{-t(H_0+m-E_2)}-e^{-t(H_0+m-E_1)}\right]\\&+\lambda^2\int_0^\infty \mathrm{d}t\int_{\mathcal{M}^2}\mathrm{d}_g^D x\,\mathrm{d}_g^D y\; K_t(x,a;g)K_t(y,a;g)\,\phi_g^\dagger(x)\left[e^{-t(H_0+2m-E_2)}-e^{-t(H_0+2m-E_1)}\right]\phi_g(y)\\&+\lambda^2\phi_g(a)\left[(H_0-E_1)^{-1}-(H_0-E_2)^{-1}\right]\phi_g^\dagger(a).\end{aligned} \tag{135}$$

One can shift the operator $\phi_g^\dagger(x)$ to the left

$$\frac{1}{H_0-E}\phi_g^\dagger(x)=\int_{\mathcal{M}}\mathrm{d}_g^D x'\;\phi_g^\dagger(x')\int_0^\infty \mathrm{d}t\; e^{-t(H_0+m-E)}\,K_t(x,x';g), \tag{136}$$

and shift the operator $\phi_g(x)$ to the right

$$\phi_g(x)\frac{1}{H_0-E}=\int_{\mathcal{M}}\mathrm{d}_g^D x'\;\int_0^\infty \mathrm{d}t\; e^{-t(H_0+m-E)}\,K_t(x,x';g)\phi_g(x'), \tag{137}$$

which we have also used in Ref. 7 for the renormalization. The last term in Eq. (135) can be normal ordered as

$$\begin{aligned}&\lambda^2\int_0^\infty \mathrm{d}t\; K_t(a,a;g)\left[e^{-t(H_0+m-E_1)}-e^{-t(H_0+m-E_2)}\right]\\&+\lambda^2\int_0^\infty \mathrm{d}t\int_{\mathcal{M}^2}\mathrm{d}_g^D x\,\mathrm{d}_g^D y\; K_t(x,a;g)K_t(y,a;g)\,\phi_g^\dagger(x)\left[e^{-t(H_0+2m-E_1)}-e^{-t(H_0+2m-E_2)}\right]\phi_g(y).\end{aligned} \tag{138}$$

Then we prove that

$$\Phi(E_1)-\Phi(E_2)+b(R_0(E_1)-R_0(E_2))b^\dagger+E_1-E_2=0. \tag{139}$$

The other term in the matrix equality (133)

$$\gamma(E_1)-\gamma(E_2)=(E_1-E_2)\left[\alpha(E_1)\gamma(E_2)-\gamma(E_1)\delta(E_2)\right] \tag{140}$$

can be written as

$$-R_0(E_1)b^\dagger\Phi^{-1}(E_1)\left[\Phi(E_1)-\Phi(E_2)+b(R_0(E_1)-R_0(E_2))b^\dagger+E_1-E_2\right]\Phi^{-1}(E_2)=0, \tag{141}$$

due to (139). Similarly, the other terms can be put into the following forms:

$$\Phi^{-1}(E_1)\left[\Phi(E_1)-\Phi(E_2)+b(R_0(E_1)-R_0(E_2))b^\dagger+E_1-E_2\right]\Phi^{-1}(E_2)bR_0(E_2)=0$$

$$\Phi^{-1}(E_1)\left[\Phi(E_1)-\Phi(E_2)+b(R_0(E_1)-R_0(E_2))b^\dagger+E_1-E_2\right]\Phi^{-1}(E_2)=0, \tag{142}$$

and they are all satisfied thanks to the equality (139). Hence, we prove that the resolvent identity is satisfied.

We will now prove the second part of the theorem for the existence of Hamiltonian. Recall that the resolvent for the Lee model is defined in the following Fock space $\mathcal{F_B}^{(n+1)}(\mathcal{H})\otimes\chi_+\oplus\mathcal{F_B}^{(n)}(\mathcal{H})\otimes\chi_-$, for any given $n\in\mathbb{N}$, and $\chi_\pm$ is the spin states. In matrix form, we have $R(E):\mathcal{F_B}^{(n+1)}(\mathcal{H})\oplus\mathcal{F_B}^{(n)}(\mathcal{H})\to\mathcal{F_B}^{(n+1)}(\mathcal{H})\oplus\mathcal{F_B}^{(n)}(\mathcal{H})$. Then we must show that

$$||E_kR(E_k)|f\rangle+|f\rangle||=|||E_k|R(E_k)|f\rangle-|f\rangle||\to 0, \tag{143}$$

as $k\to\infty$. Here $|f\rangle\in\mathcal{F_B}^{(n+1)}(\mathcal{H})\oplus\mathcal{F_B}^{(n)}(\mathcal{H})$ and the norm is taken with respect to $\mathcal{F_B}^{(n+1)}(\mathcal{H})\oplus\mathcal{F_B}^{(n)}(\mathcal{H})$. Let us decompose the vector $|f\rangle$ as

$$\begin{pmatrix}|f^{(n+1)}\rangle\\ |f^{(n)}\rangle\end{pmatrix}, \tag{144}$$

where

$$|f^{(n)}\rangle=\int_{\mathcal{M}^n}\mathrm{d}_g^Dx_1\dots\mathrm{d}_g^Dx_n\ f(x_1,x_2,\dots,x_n)|x_1,x_2,\dots,x_n\rangle. \tag{145}$$

So we have

$$\begin{aligned}
&\left|\left|\begin{pmatrix}|E_k|\alpha(E_k) & |E_k|\gamma(E_k)\\ |E_k|\beta(E_k) & |E_k|\delta(E_k)\end{pmatrix}\begin{pmatrix}|f^{(n+1)}\rangle\\ |f^{(n)}\rangle\end{pmatrix}-\begin{pmatrix}|f^{(n+1)}\rangle\\ |f^{(n)}\rangle\end{pmatrix}\right|\right|\\
&=\Bigg[|||E_k|\alpha(E_k)|f^{(n+1)}\rangle-|f^{(n+1)}\rangle+|E_k|\gamma(E_k)|f^{(n)}\rangle||^2\\
&+|||E_k|\beta(E_k)|f^{(n+1)}\rangle+|E_k|\delta(E_k)|f^{(n)}\rangle-|f^{(n)}\rangle||^2\Bigg]^{1/2} \qquad (146)\\
&\le\Bigg[\Bigg(|||E_k|\alpha(E_k)|f^{(n+1)}\rangle-|f^{(n+1)}\rangle||+|||E_k|\gamma(E_k)|f^{(n)}\rangle||\Bigg)^2\\
&+\Bigg(|||E_k|\beta(E_k)|f^{(n+1)}\rangle||+|||E_k|\delta(E_k)|f^{(n)}\rangle-|f^{(n)}\rangle||\Bigg)^2\Bigg]^{1/2},
\end{aligned}$$

since $||A+B||\le||A||+||B||$. We shall investigate each norm separately. Let us first consider the term $|||E_k|\beta(E_k)|f^{(n+1)}\rangle||$

$$||\lambda|E_k|\Phi^{-1}(E_k)\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle||\le\lambda|E_k|||\Phi^{-1}(E_k)||||\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle||. \tag{147}$$

Using the formula (137) for $E=-|E_k|$ and $x=a$, we get

$$\begin{aligned}
||\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle||&\le||\int_{\mathcal{M}}\mathrm{d}_g^Dx\int_0^\infty\mathrm{d}t\ e^{-t|E_k|}K_t(x,a;g)\phi_g(x)|f^{(n+1)}\rangle||\\
&\le\left[\int_{\mathcal{M}}\mathrm{d}_g^Dx\left(\int_0^\infty\mathrm{d}t\ e^{-t|E_k|}K_t(x,a;g)\right)^2\right]^{1/2}\sqrt{n+1}||\,|f^{(n+1)}\rangle||.
\end{aligned}$$

(148)

Let us first consider the compact manifolds. Then, one can take the integral over the variable t by the help of the upper bound of the heat kernel (19) for compact manifolds and obtain

$$||\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle|| \times\left[\int_{\mathcal{M}} \mathrm{d}_g^D x \left(\frac{md^2(x,a)}{|E_k|}\right)\left(\frac{C_{33}}{V(\mathcal{M})}K_1\left(2d(x,a)\sqrt{m|E_k|/C_3}\right) + C_{34}\left(\frac{m|E_k|}{d^2(x,a)}\right)^{D/4} K_{\frac{D}{2}-1}\left(2d(x,a)\sqrt{m|E_k|/C_3}\right)\right)^2\right]^{1/2}. \tag{149}$$

We now choose Riemann normal coordinates around the point a, assuming that $\delta < \mathrm{inj}(a)$. Then, we split the integration region into the two parts as $\int_{\mathcal{M}} = \int_{B_\delta(a)} + \int_{\mathcal{M}\setminus B_\delta(a)}$. Expressing the first integral in the Gaussian spherical coordinates, we get

$$\begin{aligned} ||\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle|| &\leq \sqrt{n+1}|||f^{(n+1)}\rangle|| \\ &\times\left[\int_{\mathbb{S}^{D-1}} \mathrm{d}\Omega \int_0^\delta \mathrm{d}r \;\; r^{D+1}\left(\frac{mA_+^{D-1}(K_1,0)}{|E_k|}\right)\left[\frac{C_{33}}{V(\mathcal{M})}K_1\left(2r\sqrt{m|E_k|/C_3}\right)\right.\right. \\ &\left. + C_{34}\left(\frac{m|E_k|}{r^2}\right)^{D/4} K_{\frac{D}{2}-1}\left(2r\sqrt{m|E_k|/C_3}\right)\right]^2 \\ &+ \int_{\mathcal{M}\setminus B_\delta(a)} \mathrm{d}_g^D x \left(\frac{md^2(x,a)}{|E_k|}\right)\left[\frac{C_{33}}{V(\mathcal{M})}K_1\left(2d(x,a)\sqrt{m|E_k|/C_3}\right)\right. \\ &\left.\left. + C_{34}\left(\frac{m|E_k|}{d^2(x,a)}\right)^{D/4} K_{\frac{D}{2}-1}\left(2d(x,a)\sqrt{m|E_k|/C_3}\right)\right]^2\right]^{1/2}, \end{aligned} \tag{150}$$

where we have used Eqs. (46) and (51). Let us now consider the first integral. It is smaller than the following expression:

$$\int_{\mathbb{S}^{D-1}} \mathrm{d}\Omega \int_0^\infty \mathrm{d}r \;\; r^{D+1}\left(\frac{mA_+^{D-1}(K_1,0)}{|E_k|}\right)\left[\frac{C_{33}}{V(\mathcal{M})}K_1\left(2r\sqrt{m|E_k|/C_3}\right) + C_{34}\left(\frac{m|E_k|}{r^2}\right)^{D/4} K_{\frac{D}{2}-1}\left(2r\sqrt{m|E_k|/C_3}\right)\right]^2. \tag{151}$$

One can evaluate the integrals[32]

$$\begin{aligned} \int_0^\infty \mathrm{d}r \;\; r^{D+1}K_1^2(ar) &= \frac{\sqrt{\pi}\,\Gamma(1+\frac{D}{2})\Gamma(2+\frac{D}{2})\Gamma(\frac{D}{2})}{4a^{D+2}\Gamma(\frac{3+D}{2})} \\ \int_0^\infty \mathrm{d}r \;\; r\,K_{\frac{D}{2}-1}^2(ar) &= \frac{\pi(D-2)\csc(\pi D/2)}{4a^2} \\ \int_0^\infty \mathrm{d}r \;\; r^{\frac{D}{2}+1}K_1(ar)K_{\frac{D}{2}-1}(ar) &= \frac{2^{\frac{D}{2}}\Gamma(\frac{D}{2})}{(D+2)a^{\frac{D}{2}+2}}, \end{aligned} \tag{152}$$

where $a \in \mathbb{R}^+$ and $D = 2, 3$. Then the upper bound of the first integral in (150) becomes

$$\frac{mA_+^{D-1}(K_1,0)}{|E_k|}\left(\frac{C_{35}}{V^2(\mathcal{M})}(m|E_k|)^{-\frac{(D+2)}{2}} + C_{36}(m|E_k|)^{\frac{D}{2}-1} + \frac{C_{37}}{V(\mathcal{M})}(m|E_k|)^{-1}\right). \tag{153}$$

For Cartan-Hadamard manifolds, we do not repeat the analysis above because the upper bound of the heat kernel for Cartan-Hadamard manifolds given in Eq. (19) corresponds to removing the volume term from the one for the compact manifolds. As a result, we get the upper bound of the first term in Eq. (150) for Cartan-Hadamard manifolds

$$\frac{m C_{38}}{|E_k|} (m|E_k|)^{\frac{D}{2}-1}. \tag{154}$$

Let us now consider the second term in the Eq. (150) for compact and Cartan-Hadamard manifolds. Due to the upper bounds of the Bessel functions used in Ref. 8, we find for compact manifolds

$$\begin{aligned}
&\int_{\mathcal{M}\setminus B_\delta(a)} \mathrm{d}_g^D x \left(\frac{m d^2(x,a)}{|E_k|}\right)\Bigg[\frac{C_{33}}{V(\mathcal{M})} K_1\left(2d(x,a)\sqrt{m|E_k|/C_3}\right) \\
&\qquad + C_{34}\left(\frac{m|E_k|}{d^2(x,a)}\right)^{D/4} K_{\frac{D}{2}-1}\left(2d(x,a)\sqrt{m|E_k|/C_3}\right)\Bigg]^2 \\
&\leq \int_{\mathcal{M}\setminus B_\delta(a)} \mathrm{d}_g^D x \left(\frac{m d^2(x,a)}{|E_k|}\right)\Bigg[\frac{C_{33}}{V(\mathcal{M})} \exp\left(-d(x,a)\sqrt{m|E_k|/C_3}\right) \\
&\times\left(\frac{1}{2d(x,a)\sqrt{m|E_k|/C_3}}+\frac{1}{2}\right) + C_{34}\left(\frac{m|E_k|}{d^2(x,a)}\right)^{D/4} \frac{2\exp\left(-\frac{2d(x,a)}{(4-D)}\sqrt{m|E_k|/C_3}\right)}{(2d(x,a)\sqrt{m|E_k|/C_3})^{(4-D)/2}}\Bigg]^2.
\end{aligned} \tag{155}$$

Since $d(x, a) \geq \delta$ for all $x \in \mathcal{M}\setminus B_\delta(a)$, the upper bound of the above equation is

$$\begin{aligned}
&\exp\left(-\delta\sqrt{m|E_k|/C_3}\right)\int_{\mathcal{M}\setminus B_\delta(a)} \mathrm{d}_g^D x \left(\frac{m d^2(x,a)}{|E_k|}\right)\Bigg[\frac{C_{33}}{V(\mathcal{M})} \exp\left(-\frac{d(x,a)}{2}\sqrt{m|E_k|/C_3}\right) \\
&\times\left(\frac{1}{2\delta\sqrt{m|E_k|/C_3}}+\frac{1}{2}\right) + C_{34}\left(\frac{m|E_k|}{\delta^2}\right)^{D/4} \frac{2}{(2\delta\sqrt{m|E_k|/C_3})^{(4-D)/2}} \\
&\qquad \times\exp\left(\left(\frac{d(x,a)}{2}-\frac{2d(x,a)}{(4-D)}\right)\sqrt{m|E_k|/C_3}\right)\Bigg]^2.
\end{aligned} \tag{156}$$

For compact manifolds, we have a simplification. This upper bound above is smaller than

$$\begin{aligned}
&\exp\left(-\delta\sqrt{m|E_k|/C_3}\right)\int_{\mathcal{M}} \mathrm{d}_g^D x \left(\frac{m d^2(x,a)}{|E_k|}\right)\Bigg[\frac{C_{33}}{V(\mathcal{M})}\left(\frac{1}{2\delta\sqrt{m|E_k|/C_3}}+\frac{1}{2}\right) \\
&\qquad + C_{34}\left(\frac{m|E_k|}{\delta^2}\right)^{D/4} \frac{2}{(2\delta\sqrt{m|E_k|/C_3})^{(4-D)/2}}\Bigg]^2.
\end{aligned} \tag{157}$$

Due to the fact the geodesic distance between any two points on the manifold and the volume of the manifold is finite, that is, $d(x, a) \leq d_{\max}(a) = \max_x d(x, a)$, the upper bound to the above integral can easily be found as

$$\begin{aligned}
&\left(\frac{m d_{\max}^2(a)}{|E_k|}\right)\exp\left(-\delta\sqrt{m|E_k|/C_3}\right)V(\mathcal{M})\Bigg[\frac{C_{33}}{V(\mathcal{M})}\left(\frac{1}{2\delta\sqrt{m|E_k|/C_3}}+\frac{1}{2}\right) \\
&\qquad + C_{34}\left(\frac{m|E_k|}{\delta^2}\right)^{D/4} \frac{2}{(2\delta\sqrt{m|E_k|/C_3})^{(4-D)/2}}\Bigg]^2.
\end{aligned} \tag{158}$$

For Cartan-Hadamard manifolds, we similarly find

$$\leq C_{39} m^{D-1} |E_k|^{D-3} \exp\left(-\delta\,(m|E_k|/C_5)^{1/2}\right) \int_{\mathcal{M}\setminus B_\delta(a)} \mathrm{d}_g^D x \frac{\exp\left[2(\frac{d(x,a)}{2} - \frac{2d(x,a)}{(4-D)})\sqrt{m|E_k|/C_5}\right]}{d^2(x,a)}$$
$$\leq \frac{C_{39} m^{D-1} |E_k|^{D-3} \exp\left(-\delta\sqrt{m|E_k|/C_5}\right)}{\delta^2} \int_{\mathcal{M}} \mathrm{d}_g^D x \exp\left[-d(x,a)\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_k|/C_5}\right], \tag{159}$$

where we have used $d(x, a) \geq \delta$ for all $x \in \mathcal{M}\setminus B_\delta(a)$. Let us write the above integral in Gaussian spherical coordinates as we did in Sec. II,

$$\int_{\mathbb{S}^{D-1}} \mathrm{d}\Omega \int_0^{\rho_\Omega} \mathrm{d}r\; r^{D-1} J(r,\theta) \exp\left[-r\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_k|/C_5}\right]. \tag{160}$$

To proceed further we assume that $\mathcal{M}$ has *Ricci tensor bounded from below by* K_1. As a result of this and using Eqs. (46) and (47), the upper bound to Eq. (159) becomes

$$\leq \frac{C_{40} m^{D-1} |E_k|^{D-3} \exp\left(-\delta\sqrt{m|E_k|/C_5}\right)}{\delta^2 (-K_1)^{(D-1)/2}} \int_0^\infty \mathrm{d}r\;\; \sinh^{D-1}(\sqrt{-K_1}r)$$
$$\times \exp\left[-r\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_k|/C_5}\right]. \tag{161}$$

Since $\sinh^{D-1}(x) \leq e^{(D-1)x}/2^{D-1}$, we can take the integral and get

$$\frac{C_{41} m^{D-1} |E_k|^{D-3} e^{-\delta\sqrt{m|E_k|/C_5}}}{(-K_1)^{(D-1)/2}\delta^2 \left[\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_k|/C_5} - (D-1)\sqrt{-K_1}\right]}, \tag{162}$$

as long as $\left[\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_k|/C_5} - (D-1)\sqrt{-K_1}\right] \geq 0$. But this is always satisfied for sufficiently large values of $|E_k|$.

Therefore, if we combine the results (153) and (158) we obtain for compact manifolds that

$$||\phi_g(a)\frac{1}{H_0 + |E_k|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle||\left[\frac{m A_+^{D-1}(K_1,0)}{|E_k|}\left(\frac{C_{33}}{V^2(\mathcal{M})}(m|E_k|)^{-\frac{(D+2)}{2}}\right.\right.$$
$$\left. + C_{34}(m|E_k|)^{\frac{D}{2}-1} + \frac{C_{35}}{V(\mathcal{M})}(m|E_k|)^{-1}\right) + \left(\frac{m d_{\max}^2(a)}{|E_k|}\right)\exp\left(-\delta\sqrt{m|E_k|/C_3}\right)$$
$$\left.\times V(\mathcal{M})\left(\frac{C_{33}}{V(\mathcal{M})}\left(\frac{1}{2\delta\sqrt{m|E_k|/C_3}} + \frac{1}{2}\right) + C_{34}\left(\frac{m|E_k|}{\delta^2}\right)^{D/4}\frac{2}{(2\delta\sqrt{m|E_k|/C_3})^{(4-D)/2}}\right)^2\right]^{1/2}, \tag{163}$$

and the results (154) and (162) for Cartan-Hadamard manifolds give

$$||\phi_g(a)\frac{1}{H_0 + |E_k|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle||\left[\frac{m C_{38}}{|E_k|}(m|E_k|)^{\frac{D}{2}-1}\right.$$
$$\left. + \frac{C_{41} m^{D-1} |E_k|^{D-3} e^{-\delta\sqrt{m|E_k|/C_5}}}{(-K_1)^{(D-1)/2}\delta^2\left(\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_k|/C_5} - (D-1)\sqrt{-K_1}\right)}\right]^{1/2}. \tag{164}$$

We are now going to find an upper bound of the inverse norm of the principal operator. In order to do this, let us recall that we split the principal operator when we try find the lower bound of the ground state energy. We now split the principal operator in the following way: $\Phi = (K - U_1) - U_2$, where U_1 and U_2 are defined exactly as before. Then, we have

$$\Phi^{-1} = (K - U_1)^{-1/2} \left[1 - (K - U_1)^{-1/2} U_2 (K - U_1)^{-1/2}\right]^{-1} (K - U_1)^{-1/2} . \tag{165}$$

Let us substitute the identity operator $K^{1/2}K^{-1/2}$ between the operators $(K - U_1)^{-1/2}$ and U_2. Hence

$$\Phi^{-1} = (K - U_1)^{-1/2} [1 - X]^{-1} (K - U_1)^{-1/2} , \tag{166}$$

where we have defined $X = (K - U_1)^{-1/2} K^{1/2} \tilde{U}_2 K^{1/2} (K - U_1)^{-1/2}$ for simplicity. Here the following operator can be written as an infinite geometric sum

$$[1 - X]^{-1} = \sum_{l=0}^{\infty} X^l , \tag{167}$$

as long as $||X|| < 1$. This leads to

$$||[1 - X]^{-1}|| \leq [1 - ||X||]^{-1} . \tag{168}$$

Since $-U_1$ is a positive operator, $(K - U_1)^{-1/2} \leq K^{-1/2}$. Then, we have

$$||X|| \leq ||\tilde{U}_2|| . \tag{169}$$

If we make $|E_k|$ sufficiently large, then $||\tilde{U}_2|| \leq 1/2$ and

$$[1 - ||X||]^{-1} \leq 2 . \tag{170}$$

As a result of this, we get

$$||\Phi^{-1}(E_k)|| \leq 2||K^{-1/2}(E_k)||^2 \leq \frac{2}{|E_k|} , \tag{171}$$

where we have used

$$K^{-1/2}(E_k) = (H_0 + \mu + |E_k|)^{-1/2} \leq \frac{1}{|E_k|^{1/2}} . \tag{172}$$

Then, substituting Eqs. (153) and (158) for compact or substituting Eqs. (154) and (162) for Cartan-Hadamard manifolds into Eq. (150) and using the above upper bound for the inverse principal operator, and taking the limit as $k \to \infty$, we eventually obtain

$$|E_k| \; ||\Phi^{-1}(E_k)|| \; ||\phi_g(a) \frac{1}{H_0 + |E_k|} |f^{(n+1)}\rangle|| \to 0 . \tag{173}$$

Let us consider the other terms in Eq. (146) now:

$$\begin{aligned} |||E_k|\alpha(E_k)|f^{(n+1)}\rangle - |f^{(n+1)}\rangle|| &\leq || \left(\frac{|E_k|}{H_0 + |E_k|} - 1 \right) |f^{(n+1)}\rangle|| \\ &+ \lambda^2 |E_k| \; || \frac{1}{H_0 + |E_k|} \phi_g^{\dagger}(a)|| \; ||\Phi^{-1}(E_k)|| \; ||\phi_g(a) \frac{1}{H_0 + |E_k|} |f^{(n+1)}\rangle|| . \end{aligned} \tag{174}$$

The *upper* bound for the norm $||\frac{1}{H_0+|E_k|}\phi_g^{\dagger}(a)||$ can be similarly found, and comes out to be the same as the one for $||\phi_g(a)\frac{1}{H_0+|E_k|}||$(the norms are different in general). As a result of this, we obtain

$$|||E_k|\alpha(E_k)|f^{(n+1)}\rangle - |f^{(n+1)}\rangle|| \to 0 , \tag{175}$$

as $k \to \infty$. Similarly, the following term

$$|||E_k|\gamma(E_k)|f^{(n)}\rangle|| \leq \lambda |E_k| \; || \frac{1}{H_0 + |E_k|} \phi_g^{\dagger}(a) |f^{(n)}\rangle|| \; ||\Phi^{-1}(E_k)|| \tag{176}$$

and

$$|||E_k|\beta(E_k)|f^{(n+1)}\rangle|| \leq \lambda |E_k| \, ||\phi_g(a) \frac{1}{H_0+|E_k|} |f^{(n+1)}\rangle|| \, ||\Phi^{-1}(E_k)|| \tag{177}$$

both vanishes as $k \to \infty$. Moreover, we have

$$\begin{aligned}
||\,(|E_k|\delta(E_k)-1)\,|f^{(n)}\rangle|| &= ||\left[|E_k| \; \Phi^{-1}(E_k)-1\right]|f^{(n)}\rangle|| \\
&= ||\left[|E_k| \; K^{-1/2}(E_k)\left[1+(1-\tilde{U}(E_k))^{-1}-1\right]K^{-1/2}(E_k)-1\right]|f^{(n)}\rangle|| \\
&\leq ||\left(|E_k|K^{-1}(E_k)-1\right)|f^{(n)}\rangle|| + |||E_k|K^{-1/2}(E_k)\left[(1-\tilde{U}(E_k))^{-1}-1\right]K^{-1/2}(E_k)|f^{(n)}\rangle|| \\
&= ||\left(|E_k|K^{-1}(E_k)-1\right)|f^{(n)}\rangle|| + |||E_k|K^{-1/2}(E_k)(1-\tilde{U}(E_k))^{-1}\tilde{U}(E_k)K^{-1/2}(E_k)|f^{(n)}\rangle||,
\end{aligned} \tag{178}$$

where we have used the fact that the factor $(1-\tilde{U}(E_k))^{-1}$ can be considered as an infinite geometric sum. The first term goes to zero as $k \to \infty$ since

$$\begin{aligned}
||\left(\frac{|E_k|}{H_0+|E_k|+\mu}-1\right)|f^{(n)}\rangle|| &= ||\left(\frac{|E_k|+\mu-\mu}{H_0+|E_k|+\mu}-1\right)|f^{(n)}\rangle|| \\
&\leq ||\left(\frac{|E_k|+\mu}{H_0+|E_k|+\mu}-1\right)|f^{(n)}\rangle|| + ||\left(\frac{\mu}{H_0+|E_k|+\mu}\right)|f^{(n)}\rangle|| \\
&\leq ||\left(\frac{|E_k|+\mu}{H_0+|E_k|+\mu}-1\right)|f^{(n)}\rangle|| + \frac{\mu}{|E_k|}|||f^{(n)}\rangle||,
\end{aligned} \tag{179}$$

where the term containing H_0 vanishes as $k \to \infty$ because it is the free resolvent and the second part clearly goes to zero. This shows that the first term in Eq. (178) vanishes in the limit. As for the second term, it is smaller than

$$|E_k|||K^{-1/2}(E_k)|| \; \left[(1-||\tilde{U}(E_k)||)^{-1}\right] \; \left[||\tilde{U}_1(E_k)||+||\tilde{U}_2(E_k)||\right]||K^{-1/2}(E_k)|| \; |||f^{(n)}\rangle||. \tag{180}$$

Since $m > \mu$ for bound states, one can easily see that

$$\begin{aligned}
||\tilde{U}_1(E_k)|| &\leq C_{42}\lambda^2||(H_0+m+|E_k|)^{-1/2} \int_0^\infty \mathrm{d}t \; K_t(a,a;g)\left[e^{-t(m-\mu)}-e^{-t(H_0+m+|E_k|)}\right] \\
&\qquad \times (H_0+m+|E_k|)^{-1/2}|| \\
&\leq \begin{cases} C_{43}\lambda^2||(H_0+m+|E_k|)^{-1}\ln\left(\frac{H_0+m+|E_k|}{m-\mu}\right)|| & \text{for } D=2 \\ C_{44}\lambda^2||(H_0+m+|E_k|)^{-1/2}|| & \text{for } D=3 \end{cases}.
\end{aligned} \tag{181}$$

Here we use the fact that the operator in the parenthesis, which we call $A(s)$ is positive, and for a positive family, if two integrable functions satisfy $0 \leq f(s) \leq g(s)$, then $\int \mathrm{d}s\, f(s)A(s) \leq \int \mathrm{d}s\, g(s)A(s)$. Moreover, for positive operators, order relation implies the same ordering for the norms of the operators. For simplicity, we have also disregarded the term, which is more convergent in $|E_k|$, coming from the volume terms of upper bound of the heat kernel for compact manifolds. We now note that

$$\begin{aligned}
||\frac{1}{H_0+|E_k|+m}\ln\left(\frac{H_0+|E_k|+m}{m-\mu}\right)|| &\leq ||\int_0^1 \frac{\mathrm{d}t}{[H_0+|E_k|+\mu]t+m-\mu}|| \\
&\leq ||\int_0^1 \frac{\mathrm{d}t}{[H_0+|E_k|+\mu]t^2+m-\mu}|| \\
&\leq ||\frac{1}{\sqrt{m-\mu}[H_0+|E_k|+\mu]^{1/2}}||\int_0^\infty \frac{\mathrm{d}s}{s^2+1} \\
&\leq \frac{C_{45}}{\sqrt{m-\mu}|E_k|^{1/2}}.
\end{aligned} \tag{182}$$

Hence, we get

$$||\tilde{U}_1(E_k)|| \leq \begin{cases} \frac{C_{46}\lambda^2}{\sqrt{m-\mu}|E_k|^{1/2}} & \text{for } D=2 \\ \frac{C_{47}\lambda^2}{|E_k|^{1/2}} & \text{for } D=3 \end{cases}. \tag{183}$$

Using the results (125) and (128) for $E = -|E_k|$ with the above analysis, we finally obtain

$$|E_k|||K^{-1}(E_k)||\ ||\tilde{U}(E_k)||\ ||(1-\tilde{U}(E_k))^{-1}||\ |||f^{(n)}\rangle|| \to 0, \tag{184}$$

as $k \to \infty$ and this completes the proof that our renormalized formula corresponds to the resolvent of a densely defined closed operator.

We will further show that $\Phi(E)$ is a holomorphic self-adjoint family of type (A) in the sense of Kato.[13] This will in turn justify the claim that the resolvent corresponds to a self-adjoint operator. To prove this, we will use the theorem given by Wüst.[33] First, we define a holomorphic family of type (A) as follows: Let $G \subset \mathbb{C}$ be domain and $L(z)$ be a family of *closed* linear operators, acting on a Hilbert space $\mathcal{H}$, $\{L(z)|z \in G\}$. If

$$\begin{aligned} &(1)\ \text{the domain } \mathcal{D}(L(z)) = \mathcal{D} \text{ is independent of } z \in G, \\ &(2)\ \text{for any } f \in \mathcal{D}, \text{ and } g \in \mathcal{H} \text{ then } \langle g|L(z)|f\rangle \text{ is holomorphic in } G, \end{aligned} \tag{185}$$

then this is a *holomorphic family of type (A)*.

An operator which is a holomorphic family of type (A) is a *self-adjoint* holomorphic family of type (A) if

$$\begin{aligned} &(1)\ G \text{ is a } \textit{symmetric domain} \text{ of the complex plane relative to the real axis,} \\ &(2)\ \mathcal{D} \text{ is } \textit{dense} \text{ in } \mathcal{H}, \\ &(3)\ \mathcal{D}(L(z)^\dagger) = \mathcal{D}(L(z^*)) \text{ for all } z \in G. \end{aligned} \tag{186}$$

Theorem(Wüst): Let G be a symmetric domain of complex plane relative to the real axis, and $L(z)$ is a holomorphic family of type (A) defined on G. Assume

$$\begin{aligned} &(1)\ \mathcal{D} \text{ is dense in } \mathcal{H}, \\ &(2)\ \mathcal{D}(L(z)^\dagger) \supset \mathcal{D}(L(z^*)). \end{aligned} \tag{187}$$

Let

$$M = \{z \in G | L(z)^\dagger = L(z^*)\}. \tag{188}$$

If *M is not the empty set then it is the whole domain G*. This implies that the family is a self-adjoint holomorphic family of type (A).

Let us consider our case. We choose the domain G as

$$G = \{E \in \mathbb{C} | \Re(E) < \mu\}, \tag{189}$$

which is symmetric with respect to the real axis. The principal operator $\Phi(E)$ given explicitly in (116) formally satisfies the relation $\Phi(E)^\dagger = \Phi(E^*)$ so this implies $\mathcal{D}(\Phi(E)^\dagger) \supset \mathcal{D}(\Phi(E^*))$. Let us assume that the family is holomorphic for now. Note that a densely defined holomorphic family of operators satisfying the formal relation $\Phi(E)^\dagger = \Phi(E^*)$ is closable. Proof: Let us consider the common domain $\mathcal{D}$, and choose $|g_l\rangle \in \mathcal{D} \to 0$ as $l \to \infty$. We further assume that $\Phi(E)|g_l\rangle$ converges to some $|g(E)\rangle$. Then we have

$$\langle f|\Phi(E)g_l\rangle = \langle \Phi^\dagger(E)f|g_l\rangle = \langle \Phi(E^*)f|g_l\rangle \to 0, \tag{190}$$

for any $|f\rangle \in \mathcal{D}$ as $l \to \infty$. This implies that

$$\Phi(E)|g_l\rangle \to |g(E)\rangle = 0. \tag{191}$$

This is the requirement of closure. Of course, we must establish this closure uniformly, that is, we need to show that for every sequence $|g_l\rangle$ converging to some element, if $\Phi(E_0)|g_l\rangle$ converges for one E_0 inside the region G, then it converges for all $E \in G$. Hence, we can define a unique closure over G, the closures having a common domain $\mathcal{D}^-$. Once we determine a common domain for the family $\Phi(E)$, we will prove that there is indeed a closure over a common domain.

For the family $\Phi(E)$, we choose $\mathcal{D} = \mathcal{D}(H_0)$. It is well known that if $\mathcal{M}$ is a geodesically complete manifold then Laplacian defined on $\mathcal{M}$ is a closed, densely defined self-adjoint operator.[14,15] Then, the operator $H_0 = \int_{\mathcal{M}} \mathrm{d}_g^D x\ \phi_g^\dagger(x)(-\frac{1}{2m}\nabla_g^2)\phi_g(x)$ defined over the direct products of n copies of the Hilbert space $L^2(\mathcal{M})$ is also a densely defined (essentially) self-adjoint operator. Moreover, the finite direct sum of such operators will preserve this property.

The first term $H_0 - E + \mu$ is obviously defined over this domain $\mathcal{D}(H_0)$ and it is a closed operator. Note that the other term can be defined by the spectral theorem,

$$\int_0^\infty \mathrm{d}s\ K_s(a, a; g)[e^{-s(m-\mu)} - e^{-s(H_0+m-E)}], \tag{192}$$

and it is a positive operator when E is *real* and $\Re(E) < \mu$. Its domain of definition includes $\mathcal{D}(H_0)$, when $\Re(E) < \mu$. To see this we will use a different integral representation,

$$\begin{aligned}&\int_0^\infty \mathrm{d}s\ K_s(a, a; g)[e^{-s(m-\mu)} - e^{-s(H_0-E+m)}]\\ &\qquad = \int_0^\infty \mathrm{d}s\ s\ K_s(a, a; g) \int_0^1 \mathrm{d}u\ e^{-su(H_0+\mu-E)-s(m-\mu)}(H_0 - E + \mu).\end{aligned} \tag{193}$$

When the operator acts on an element $|f^{(n)}\rangle$ in the domain of H_0, the norm of the resulting vector is smaller than

$$\int_0^\infty \mathrm{d}s\ s\ K_s(a, a; g) \int_0^1 \mathrm{d}u\ ||e^{-su(H_0+\mu-E)-s(m-\mu)}||\,||(H_0 - E + \mu)|f^{(n)}\rangle||. \tag{194}$$

Now we can estimate the following factor using the bounds on the heat kernels given in (19),

$$\begin{aligned}&\int_0^\infty \mathrm{d}s\ s K_s(a, a; g) \int_0^1 \mathrm{d}u||e^{-su(H_0+\mu-E)-s(m-\mu)}||\\ &\leq \int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}s\ s\left[\frac{C_1}{V(\mathcal{M})} + \frac{C_2}{(s/2m)^{D/2}}\right] e^{-sunm} e^{-su(\mu-\Re(E))-s(m-\mu)}.\end{aligned} \tag{195}$$

Thus we show that

$$|| \int_0^\infty \mathrm{d}s\ K_s(a, a; g)[e^{-s(H_0-E+m)} - e^{-s(m-\mu)}]|| \leq F(\mu - \Re(E))\Big(||H_0|f^{(n)}\rangle|| + |\mu - E|\,||f^{(n)}\rangle||\Big), \tag{196}$$

where

$$\begin{aligned}F(\mu - \Re(E)) &= \frac{C_1}{(nm + \mu - \Re(E))V(\mathcal{M})}\left(\frac{1}{m-\mu} + \frac{1}{(n+1)m - \Re(E)}\right)\\ &+ \frac{C_2(2m)^{D/2}\Gamma(2-\frac{D}{2})}{(nm + \mu - \Re(E)(\frac{D}{2}-1)}\Big[((n+1)m - \Re(E))^{\frac{D}{2}-1} - (m-\mu)^{\frac{D}{2}-1}\Big].\end{aligned} \tag{197}$$

As a result the domain of this operator family includes $\mathcal{D}(H_0)$. In fact, by the spectral theorem the operators so defined are closed, when we restrict them to a smaller domain, i.e., to $\mathcal{D}(H_0)$ they remain closed. So the sum of the two pieces, $H_0 + \mu - E$ and the term above, defined over $\mathcal{D}(H_0)$ is closed, since they were already closed operators defined over a common domain.

The last part requires more work, for this we will first show that $U(E)$ is relatively bounded with respect to H_0 hence its domain includes $\mathcal{D}(H_0)$. Moreover, if we have a holomorphic family of operators defined over a dense domain, then they are preclosed, that is we can define the closure of this family, as we have shown. It is easy to see that

$$||U(E)H_0^{-1}H_0|f^{(n)}\rangle|| \leq ||U(E)H_0^{-1}||\ ||H_0|f^{(n)}\rangle||, \tag{198}$$

where the first norm can be estimated by exactly the same method developed in Ref. 7. So we are giving the result in order not to repeat the similar calculations, for $n \geq 1$,

$$||U(E)H_0^{-1}|| \leq C_{48} n \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}u \, s \, K_{us}^{1/2}(a,a;g) K_s^{1/2}(a,a;g) e^{-snm} e^{su\Re(E)}. \tag{199}$$

After using the upper bound of the heat kernel given in (19) and defining new variables $p = C_1/V(\mathcal{M})$ and $q(s) = C_2/(s/2m)^{D/2}$, we get

$$\begin{aligned} ||U(E)H_0^{-1}|| &\leq C_{48} n \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}u \, s \, e^{-snm} e^{su\Re(E)} \left[p^2 + pq(s) + \frac{pq(s)}{u^{3/2}} + \frac{q(s)^2}{u^{3/2}} \right. \\ &\qquad \left. + \frac{2pq(s)}{u^{3/4}} - \frac{2pq(s)}{u^{3/4}} \right]^{1/2} \\ &\leq C_{48} n \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}u \, s \, e^{-snm} e^{su\Re(E)} \left[\left(p + \frac{q(s)}{u^{3/4}} \right) + \sqrt{pq(s)} \left(1 + \frac{1}{u^{3/2}} - \frac{2}{u^{3/4}} \right) \right]. \end{aligned} \tag{200}$$

Taking the s and u integral, we obtain

$$||U(E)H_0^{-1}|| \leq \frac{nC_{49}}{V(\mathcal{M})(nm-\mu)^2} + \frac{n(2m)^{D/2} C_{50}}{(nm-\mu)^{2-D/2}} + \frac{n(2m)^{D/4} C_{51}}{\sqrt{V(\mathcal{M})}(nm-\mu)^{2-D/4}}, \tag{201}$$

since $\Re E < \mu$. Thus we choose the domain of $U(E)$ as $\mathcal{D}(H_0)$, and now the family is closable over this domain. However, as a result of the closure, the domains for different values of E may become different. In fact, this does not happen, as we will see.

Now we show that we can perform the closure *uniformly*, as a result of the following: for any $E_1, E_2 \in G$, $\Phi(E_1) - \Phi(E_2)$ becomes a bounded operator. A short computation shows that

$$||\Phi(E_1) - \Phi(E_2)|| \leq |E_1 - E_2| \left[1 + (n+1)\lambda^2 \int_0^\infty \mathrm{d}s \, s K_s(a,a;g) e^{-snm} e^{-s(m-\mu)} \right]. \tag{202}$$

If $|g_l\rangle \in \mathcal{D}$ is convergent to a vector $|f\rangle$, and assume that $\Phi(E_1)|g_l\rangle$ converges to $|g(E_1)\rangle$ for one E_1, then we set $\Phi(E_1)|f\rangle = |g(E_1)\rangle$ to define the closure at point E_1. Then, for any E_2, we have

$$\begin{aligned} ||\Phi(E_2)|g_l\rangle - \Phi(E_2)|f\rangle|| &= ||\Phi(E_2)|g_l\rangle - |g(E_1)\rangle - [\Phi(E_2) - \Phi(E_1)]|f\rangle|| \\ &= ||[\Phi(E_2) - \Phi(E_1)]|g_l > + \Phi(E_1)|g_l\rangle - |g(E_1)\rangle - [\Phi(E_2) - \Phi(E_1)]|f\rangle|| \\ &< ||[\Phi(E_2) - \Phi(E_1)]|| \, |||g_l\rangle - |f\rangle|| + ||\Phi(E_1)|g_l\rangle - |g(E_1)\rangle|| \mapsto 0, \end{aligned} \tag{203}$$

and this shows that whenever $|g_l\rangle$ converges to $|f\rangle$ and $\Phi(E_1)|g_l\rangle$ converges to $|g(E_1)\rangle$, $\Phi(E_2)|g_l\rangle$ becomes convergent and the resulting vector is exactly equal to $\Phi(E_2)|f\rangle$ as it should be for the requirements of the closure. Hence the sum of all these three parts will make a holomorphic family $\Phi(E)$ with a dense common domain $\mathcal{D}(H_0)$. Moreover, the sum is *closable over a dense common domain* which we call $\mathcal{D}(H_0)^-$.

We would now make holomorphicity more precise, up to now we have not actually made use of it. To prove that the family is holomorphic we will refer to the following theorem, which is stated in a slightly simplified form according to our needs and the proof of which can be found in Ref. 34. Assume X is a measure space with a σ-finite measure ν defined on it, let I be a measurable subset of X. Let G be a open domain of the complex plane. Consider a function $\gamma : I \times G \mapsto \mathbb{C}$ such that

$$\begin{aligned} &(1) \;\; \gamma(x,.) \in L^1(X, |\nu|), \\ &(2) \;\; \gamma(.,z) \;\; \text{is holomorphic in } G, \\ &(3) \;\; \int_I |\mathrm{d}\nu||\gamma(x,z)| \;\; \text{is bounded on all compact subsets of } G. \end{aligned} \tag{204}$$

Then the function

$$\Gamma(z) = \int_I \mathrm{d}\nu \, \gamma(x,z) \;\; \text{is holomorphic in } \; G. \tag{205}$$

To use the above theorem, let us write our family in the following form $\Phi(E) = V(E)(H_0 - E + \mu) = [1 + V_1(E) + V_2(E)](H_0 - E + \mu)$, where

$$V_1(E) = \lambda^2 (H_0 - E + \mu)^{-1} \int_0^\infty \mathrm{d}s \, K_s(a, a; g)[e^{-s(m-\mu)} - e^{-s(H_0 - E + m)}]$$

$$V_2(E) = \lambda^2 \int_0^\infty \mathrm{d}s \, \phi_g^\dagger(x) K_s(x, a; g) e^{-s(H_0 - E + 2m)} K_s(a, y; g) \phi_g(y)(H_0 - E + \mu)^{-1}. \quad (206)$$

By using our previous estimate in (166) and (167), we see that the first term $V_1(E)$ is indeed uniformly bounded in $\Re E < \mu$. Hence, the integrals $\langle f^{(n)}|V_1(E)|g^{(n)}\rangle$ for all functions f, g in the Fock space are absolutely convergent. Moreover, any such matrix element satisfies all the other conditions on holomorphicity and integrability. The second part, again using ideas very similar to the previous estimates, can be written as

$$V_2(E) = \lambda^2 \int_0^\infty \mathrm{d}s \, s \int_0^1 \mathrm{d}u \, \phi_g^\dagger(K_{su}(., a; g)) e^{-s(H_0 - E + m + (1-u)\mu + mu)} \phi_g(K_s(., a; g)). \quad (207)$$

This integrand as a function of E is holomorphic in E for $\Re E < \mu$ and it is absolutely integrable for any $\Re E < \mu$ on $[0, \infty) \times [0, 1]$. Similarly, it can be shown that the following bound holds

$$|\langle f^{(n)}|V_2(E)|g^{(n)}\rangle| \leq n \frac{C_{52}}{(nm - \Re E)^{D/2-1}} |||f^{(n)})\rangle||||g^{(n)}\rangle||, \quad (208)$$

which clearly shows that for $\Re E < \mu$ is uniformly bounded everywhere, hence on compact subsets as well. Hence, applying the theorem stated above we see that the resulting function $V_2(E)$ is holomorphic for $\Re E < \mu$. There is one subtle point about the closure operation, but this is also solved by the following observation. Let us consider the limit of $\Phi(E)|g_l\rangle$ as $l \to \infty$ as a function of E for any convergent $|g_l\rangle$ sequence in the closure operation. If this sequence is uniformly convergent on compact subsets, the limit is a holomorphic function (by an application of Morera's theorem). Note that this family $\Phi(E)|g_l\rangle$ is norm bounded by a constant multiple of a simple function given in Eq. (171), $|E_1 - E_2|$. This function itself is uniformly bounded on compact sets centered around any given point E_1, hence the sequence of functions $\Phi(E)|g_l\rangle$ is a uniformly convergent sequence. This shows that the closure remains a holomorphic function of E for $\Re E < \mu$ as required. Thus we complete the proof that the closure of the family $\Phi(E)$ over $D(\mathcal{H}_0)^-$ is holomorphic for $\Re(E) < \mu$.

Now we are ready to apply the theorem of Wüst. If we choose $E \in \mathbb{R}$ and sufficiently small $E < E_*$, then $U(E)$ has relative bound with respect to $K(E)$ which is less than 1. Hence, by Kato-Rellich theorem[13] of perturbations of self-adjoint operators, $\Phi(E)$ will be self-adjoint for $E < E_*$. By the theorem of Wüst, the family is self-adjoint everywhere as desired. This result is important to establish that the spectrum only lies along the real axis, and justifies our search for the lower bound of energy and shows that the resulting operator is self-adjoint as it should be.

V. CONCLUSION

In this paper, we have proven that for the three models that we have constructed, namely non-relativistic point interactions in two and three dimensional Riemannian manifolds, relativistic point interactions in two dimensional Riemannian manifolds and non-relativistic Lee model in two and three dimensional Riemannian manifolds, the Hamiltonian after renormalization is a densely defined self-adjoint operator.

ACKNOWLEDGMENTS

Ç. Dogan and O.T. Turgut would like to thank Professor M. Znojil and Professor P. Exner for the kind invitation to the Doppler Institute in Prague and for various discussions related to the subject under consideration. O.T. Turgut has two times visited the Department of Mathematics of KTH, Stockholm during the completion of this work and would like to thank Professor J. Hoppe for his kind invitations and his continuous support. The authors also would like to thank the anonymous referee for his/her suggestion on the discussion of the resonances for our problem.

[1] F. A. Berezin and L. D. Faddeev, Soviet mathematics Doklady, **2**, 372–375 (1961).
[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics, 2nd ed.* (AMS Chelsea, RI, 2004).
[3] S. Albeverio and P. Kurasov, *Singular Perturbations of Differential Operators Solvable Schrödinger-Type Operators* (Cambridge University Press, Cambridge, 2000).
[4] J. Hoppe, "*Quantum theory of a massless relativistic surface and a two-dimensional bound state problem*," Ph.D. dissertation, Massachusetts Institute of Technology, 1983.
[5] B. T. Kaynak and O. T. Turgut, J. Phys. A: Math. Theor. **42**(22), 225402 (2009).
[6] B. İ. Altunkaynak, F. Erman, and O. T. Turgut, J. Math. Phys. **47**(8), 082110 (2006).
[7] F. Erman and O. T. Turgut, J. Math. Phys. **48**(12), 122103 (2007).
[8] F. Erman and O. T. Turgut, J. Phys. A: Math. Theor. **43**, 335204 (2010).
[9] Ç. Dogan and O. T. Turgut, J. Math. Phys. **51**, 082305 (2010).
[10] F. Erman and O. T. Turgut, "Non-relativistic Lee model in two dimensional Riemannian manifolds," J. Math. Phys. (to be published); e-print arXiv:1110.5817v1.
[11] S. G. Rajeev, "Bound states in models of asymptotic freedom," (submitted); e-print arXiv:hep-th/9902025v1.
[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer-Verlag, New York,1983).
[13] T. Kato, *Perturbation Theory for Linear Operators, Classics in Mathematics*, 2nd ed. (Springer-Verlag, Berlin, 1995).
[14] M. P. Gaffney, Ann. Math. **60**, 140–145 (1954).
[15] M. P. Gaffney, Trans. Am. Math. Soc. **78**(2), 426–444 (1955).
[16] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, AMS/IP Studies in Advanced Mathematics Vol. 47, edited by S.-T. Yau (American Mathematical Society, Rhode Island, 2009).
[17] A. Grigor'yan, *Spectral Theory and Geometry*, London Mathematical Society Lecture Notes Vol. 273, edited by E. B. Davies and Y. Safarov (Cambridge University Press, Cambridge, 1999), pp. 140–225.
[18] S. Gallot, D. Hulin, and J. Lafontain, *Riemannian Geometry*, 3rd ed. (Springer-Verlag, New York, 2004).
[19] I. Chavel, *Eigenvalues in Riemannian Geometry, Pure and Applied Mathematics* (Academic, Orlando, 1984), Vol. 115.
[20] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University, Princeton, NJ, 1970).
[21] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. (Springer, New York, 1966).
[22] N. N. Lebedev, *Special Functions and Their Applications* (Printice-Hall, Englewood Cliffs, NJ, 1965).
[23] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[24] E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill, New York, 1962), Chap. 13–14.
[25] S. L. Trubatch, Am. J. Phys. **38**, 3 (1970).
[26] H. Nickle, Phys. Rev. **178**, 2382 (1969).
[27] W. Dittrich, Phys. Rev. D **10**, 1902 (1974).
[28] M. G. Fuda, Phys. Rev. C **25**, 1972 (1982).
[29] C. M. Bender and C. Nash, Phys. Rev. D **10**, 1753 (1974).
[30] J. R. Morris, R. Acharya, and B. P. Nigam, Phys. Rev. D **21**, 2429 (1980).
[31] J. T. Marshall and J. L. Pell, Phys. Rev. D **24**, 394 (1981).
[32] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Academic, 2007).
[33] R. Wüst, Math Z. **125**, pp. 349–358 (1972).
[34] W. N. Everitt, W. K. Hayman, and G. Nauri-Roudsavi, Appl. Anal. **65**, 95–102 (1997).