Conformal transformations in metric-affine gravity and ghosts

Canan N. Karahan[1],*, Oktay Doğangün[2], and Durmuş A. Demir[1]

Conformal transformations play a widespread role in gravity theories in regard to their cosmological and other implications. In the pure metric theory of gravity, conformal transformations change the frame to a new one wherein one obtains a conformal-invariant scalar–tensor theory such that the scalar field, deriving from the conformal factor, is a ghost.

In this work, conformal transformations and ghosts will be analyzed in the framework of the metric-affine theory of gravity. Within this framework, metric and connection are independent variables, and, hence, transform independently under conformal transformations. It will be shown that, if affine connection is invariant under conformal transformations, then the scalar field of concern is a non-ghost, non-dynamical field. It is an auxiliary field at the classical level, and might develop a kinetic term at the quantum level.

Alternatively, if connection transforms additively with a structure similar to yet more general than that of the Levi-Civita connection, the resulting action describes the gravitational dynamics correctly, and, more importantly, the scalar field becomes a dynamical non-ghost field. The equations of motion, for generic geometrical and matter-sector variables, do not reduce connection to the Levi-Civita connection, and, hence, independence of connection from metric is maintained. Therefore, metric-affine gravity provides an arena in which ghosts arising from the conformal factor are avoided thanks to the independence of connection from the metric.

1 Introduction

Spacetime is a smooth manifold $\mathcal{M}(g, \Gamma)$ endowed with a metric g and connection Γ. Metric is responsible for measuring the distances while connection governs curving and twirling of the manifold. Connection specifies how vectors and tensors are to be differentiated in curved spacetime. We emphasize here that only the symmetric connections, i.e. torsion-free spacetimes, will be considered throughout in this work. These two geometrical structures, the metric and connection, are fundamentally independent geometrical variables, and they play completely different roles in spacetime dynamics. If they are to exhibit any relationship it derives from dynamical equations *a posteriori*. This fact, that the metric and connection are independent geometrical variables, gives rise to two alternative approaches to General Relativity (GR):

– GR with metricity only and
– GR with affinity and metricity.

The former is a purely metric theory of gravity since connection is completely determined by the metric and its partial derivatives, *a priori*. This determination is encoded in the Levi-Civita connection, $\check{\Gamma}$, as

$$\check{\Gamma}^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta} \right), \tag{1}$$

which defines a metric-compatible covariant derivative [1]. In this particular setup, the Einstein–Hilbert action produces gravitational field equations only by adding an extrinsic curvature term.

The metric-affine gravity (similar to the Palatini formalism [2, 34, 35] in philosophy), which considers an independence of metric tensor and connection [1, 3], encodes a more general approach to gravitation by breaking up the *a priori* relation (1). In this approach, gravitational field equations are successfully produced with no need for an extrinsic curvature term provided that the geometrical sector is minimal and the matter sector is independent of the connection [3]. Concerning the matter sector, the fermion kinetic term [4] and coupling be-

* Corresponding author   E-mail: cananduzturk@iyte.edu.tr
[1] Department of Physics, Izmir Institute of Technology, IZTECH, TR35430, Izmir, Turkey
[2] Department of Physical Sciences, University of Naples and INFN, Naples, Italy

tween scalar fields and curvature scalar [5] are just two examples that immediately come to mind.

This work is about yet another difference between the metrical and affine approaches to gravity. Conformal transformation is essentially a local change of scale. Since distances are measured by the metric, such transformations are executed by rescaling the metric by a smooth, non-vanishing and spacetime-dependent function $\Omega(x)$, called the conformal factor [1, 6]. Therefore, the transformation

$$\widetilde{g}_{\alpha\beta} = \Omega^2(x)\, g_{\alpha\beta} \tag{2}$$

shrinks or stretches the distances on the manifold locally.

Conformal transformations are particularly respectful to distinction between metric and connection. Indeed, transformation of the metric in (2) automatically induces a transformation of the Levi-Civita connection as

$$\widetilde{\check{\Gamma}}{}^{\lambda}_{\alpha\beta} = \check{\Gamma}{}^{\lambda}_{\alpha\beta} + \Delta^{\lambda}_{\alpha\beta} \tag{3}$$

with

$$\Delta^{\lambda}_{\alpha\beta} = \delta^{\lambda}_{\beta}\partial_{\alpha}\ln\Omega + \delta^{\lambda}_{\alpha}\partial_{\beta}\ln\Omega - g_{\alpha\beta}\partial^{\lambda}\ln\Omega. \tag{4}$$

However, this direct correlation is completely lost in the metric-affine gravity since there is no telling how the general connection

$$\Gamma^{\lambda}_{\alpha\beta} \neq \check{\Gamma}{}^{\lambda}_{\alpha\beta}(g) \tag{5}$$

should transform under a rescaling of distances. In fact, that connection has nothing to do with measuring the distances can be taken to imply that the connection $\Gamma^{\lambda}_{\alpha\beta}$ is completely inert under (2). However, it is still possible that connection still transforms in some way, not necessarily like (3). Stating in a clearer fashion, there arise two main categories to be explored:

– the connection Γ can be conformal-invariant: $\Gamma^{\lambda}_{\alpha\beta} \rightarrow \Gamma^{\lambda}_{\alpha\beta}$ despite (2) [6] and
– the connection Γ can transform in various ways, multiplicatively, additively or both, while the metric transforms as in (2).

Each of these two possibilities gives rise to novel effects not found in metrical GR, as indicated by the dependence of the Riemann curvature tensor on the connection:

$$\mathbb{R}^{\alpha}_{\mu\beta\nu}(\Gamma) = \partial_{\beta}\Gamma^{\alpha}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta}. \tag{6}$$

It is obvious that the two conformal transformation categories mentioned above will, in general, lead to completely new dynamics with no analogue in metrical GR.

This work is devoted to a comparative analysis of conformal transformations in GR with metricity and GR with affinity in the pathways described by these two categories.

In the body of the work below, we first give a discussion of metrical GR. Following this we turn to a detailed analysis of the metric-affine gravity by exploring plausible alternatives one by one. After this, in the last section, we discuss certain salient features of the model not covered in the text and draw conclusions.

2 Conformal transformations in GR with metricity

In metrical GR, conformal transformation of the metric (2) automatically leads to transformation of the connection (3), and, hence, of the Riemann curvature tensor (6). The transformed Riemann tensor reads as

$$\widetilde{R}^{\alpha}_{\mu\beta\nu}\left(\widetilde{\Gamma}\right) = R^{\alpha}_{\mu\beta\nu}\left(\check{\Gamma}\right) + \left[\delta^{\alpha}_{\beta}\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} - \delta^{\alpha}_{\nu}\delta^{\lambda}_{\beta}\delta^{\rho}_{\mu} + \delta^{\lambda}_{\beta}g_{\mu\nu}g^{\alpha\rho}\right.$$
$$\left. - \delta^{\lambda}_{\nu}g_{\mu\beta}g^{\alpha\rho}\right]\left(2\partial_{\lambda}\ln\Omega\,\partial_{\rho}\ln\Omega - \nabla_{\lambda}\partial_{\rho}\ln\Omega\right)$$
$$+ \left[\delta^{\alpha}_{\nu}g_{\mu\beta}g^{\lambda\rho} - \delta^{\alpha}_{\beta}g_{\mu\nu}g^{\lambda\rho}\right]\partial_{\lambda}\ln\Omega\,\partial_{\rho}\ln\Omega, \tag{7}$$

where use has been made of the definition $\mathbb{R}^{\alpha}_{\mu\beta\nu}\left(\Gamma = \check{\Gamma}\right) \equiv R^{\alpha}_{\mu\beta\nu}\left(\check{\Gamma}\right)$. Contraction of this rank (1,3) tensor gives the transformed Ricci tensor

$$\widetilde{R}_{\mu\nu}\left(\widetilde{\Gamma}\right) = R_{\mu\nu}\left(\check{\Gamma}\right) - \left[(D-2)\nabla_{\mu}\nabla_{\nu} + g_{\mu\nu}\Box\right]\ln\Omega$$
$$+ \left[2(D-2)\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - (D-3)g_{\mu\nu}g^{\alpha\beta}\right]\partial_{\alpha}\ln\Omega\,\partial_{\beta}\ln\Omega \tag{8}$$

so that the transformed Ricci scalar takes the form

$$\widetilde{R}\left(\widetilde{\Gamma}\right) = \widetilde{g}^{\mu\nu}\widetilde{R}_{\mu\nu}\left(\widetilde{\Gamma}\right)$$
$$= \Omega^{-2}\left[R\left(\check{\Gamma}\right) - 2(D-1)\Box\ln\Omega\right.$$
$$\left. - (D-1)(D-2)g^{\alpha\beta}\partial_{\alpha}\ln\Omega\partial_{\beta}\ln\Omega\right]. \tag{9}$$

These transformation properties dictate how gravitational action density transforms under conformal rescalings. The Einstein–Hilbert action reads in the $\left(g, \check{\Gamma}\right)$ frame as

$$S_{\text{EH}}[g] = \int d^D x\, \sqrt{-g}\left[\frac{1}{2}M_{\star}^{D-2}R - \Lambda_{\star} + \mathfrak{L}_{\text{m}}\left(g, \Psi\right)\right], \tag{10}$$

where $M_\star$ is the fundamental scale of gravity in D dimensions, $\Lambda_\star$ is the cosmological term, and $\mathfrak{L}_m$ is the Lagrangian of the matter and radiation fields, collectively denoted by Ψ. For the metric the $(-,+,\ldots,+)$ convention is adopted. Under the conformal transformation of the metric (2), this action becomes in the $\left(\widetilde{g},\widetilde{\Gamma}\right)$ frame

$$S_{\text{EH}}\left[g,\overline{\phi}\right] = \int d^D x \sqrt{-g}\left\{\frac{1}{2}M_\star^{D-2}\right.$$

$$\times\left[\Omega^{D-2}R - 2(D-1)\Omega^{D-2}\Box\ln\Omega\right.$$

$$\left. - (D-1)(D-2)\Omega^{D-4}g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega\right]$$

$$\left. - \Lambda_\star\Omega^D + \widetilde{\mathfrak{L}}_m\left(g,\widetilde{\Psi}\right)\right\}$$

$$\equiv \int d^D x \sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\overline{\phi}\right)\left(\partial_\nu\overline{\phi}\right)\right.$$

$$\left. + \frac{1}{2}\zeta_D\overline{\phi}^2 R - \lambda_D\left(\zeta_D\overline{\phi}^2\right)^{D/(D-2)} + \widetilde{\mathfrak{L}}_m\left(g,\widetilde{\Psi}\right)\right], \quad (11)$$

where the two dimensionless constants

$$\zeta_D = \frac{D-2}{4(D-1)}, \; \lambda_D = \frac{\Lambda_\star}{M_\star^D} \quad (12)$$

designate, respectively, the conformal coupling of $\overline{\phi}$ to R and the self-coupling of $\overline{\phi}$. The scalar field

$$\overline{\phi} = \frac{1}{\sqrt{\zeta_D}}\,(M_\star\Omega)^{(D-2)/2} \quad (13)$$

derives from the conformal factor Ω in order to have a canonical kinetic term. The quantity $\widetilde{\mathfrak{L}}_m\left(g,\widetilde{\Psi}\right)$ in (11) is the transformed matter Lagrangian, where each matter field Ψ transforms, together with the metric, by an appropriate conformal weight. The conformal weights of fields (charges of fields under scalings) are determined from (global) conformal invariance of their kinetic terms [7, 8].

There are certain salient features of the transformed action (11) that deserve a detailed discussion.

– First of all, this action executes local conformal invariance (the Weyl invariance) under the transformations

$$g_{\alpha\beta} \longrightarrow \psi^2 g_{\alpha\beta}, \; \overline{\phi} \longrightarrow \psi^{-(D-2)/2}\,\overline{\phi}, \quad (14)$$

where inhomogeneous terms generated by the kinetic term of $\overline{\phi}$ are neutralized by the terms generated by the transformation of the curvature scalar R. This happens because of the special, conformal value of ζ_D. Therefore, the transformed action (11) provides a locally conformal-invariant representation of the original Einstein–Hilbert action (10). Notably, the original action (10) exhibits no sign of conformal invariance

but the transformed one does, the reason behind it being the dressing of $M_\star$ and $\Lambda_\star$ by the transformation field Ω [9, 10].

– One can also notice that conformal transformation, like gauge transformations, adds a new degree of freedom to the system. This is a built-in property of the system; this is common to 'transformations' including the gauge transformations.

– Another point to notice about (11) is that the scalar field $\overline{\phi}$ (which is a function of the conformal factor Ω) is a ghost [11–13]. This is an unavoidable feature if gravity is to be an attractive force. Its ghost nature follows from its non-positive kinetic term, and it signals that the system has no lower bound for energy. Such systems are inherently non-physical, and there seems to be no way of avoiding it unless some nonlinearities are added as extra features [14].

– The transformed action (11), when $\widetilde{g}_{\alpha\beta} = \eta_{\alpha\beta}$, leads to a $\overline{\phi}^4$ theory in $D = 4$ dimensions. In this particular case spacetime is flat, and entire gravitational effects reduce to a conformal-invariant scalar field theory. This theory has been argued to exhibit an infrared fixed point at $\lambda_D = 0$, and this feature has been suggested to provide a solution to the cosmological constant problem (λ_D is proportional to the vacuum energy density $\Lambda_\star$ in D dimensions) [15].

3 Conformal transformations in GR with affinity

As mentioned in the introduction, GR with affinity treats metric and connection as independent geometric variables, as they indeed are. One of the most important consequences of this feature is that conformal transformation of the metric tensor gives rise to no direct change in connection, as happens in GR with metricity. Therefore, parallel to the classification made in the introduction, in this section we shall analyze conformal transformations in two separate cases in regard to the transformation properties of the connection. In the course of the analysis, the main goal will be to find appropriate transformation rules for $\Gamma^\lambda_{\alpha\beta}$ so that the resulting scalar field theory (in terms of the conformal factor Ω) assumes physically sensible properties like emergent conformal invariance and absence of ghosts. Indeed, the main problem with the metrical GR discussed above is the unavoidable presence of a ghost scalar in the spectrum. We will find that affine GR is capable of realizing conformal invariance and accommodating non-ghost scalar degrees of freedom.

In the metric-affine gravity, the Einstein–Hilbert action can be written as

$$S_{\text{EH}}[g,\Gamma] = \int d^D x \sqrt{-g}$$

$$\times \left\{ \frac{1}{2} M_\star^{D-2} g^{\mu\nu} \mathbb{R}_{\mu\nu}(\Gamma) - \Lambda_\star + \mathscr{L}(\Gamma - \check{\Gamma}, g, \Psi) \right\} \quad (15)$$

in a general (g,Γ) frame. Here, Ψ collectively denotes the matter fields, and $\mathscr{L}$ is composed of

$$\mathscr{L} = \mathscr{L}_{\text{geo}}(g,\mathscr{D}) + \mathfrak{L}_{\text{m}}(g,\mathscr{D},\Psi), \quad (16)$$

which, respectively, stand for the geometrical and matter sector contributions. The geometrical sector consists of the rank (1,2) tensor field

$$\mathscr{D}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \check{\Gamma}^\lambda_{\alpha\beta} \quad (17)$$

as an additional geometrodynamical tensorial quantity. This variable is very natural to consider since in the presence of the metric $g_{\alpha\beta}$ one naturally defines its compatible connection, i.e. the Levi-Civita connection. Then the difference between $\Gamma^\lambda_{\alpha\beta}$ and the Levi-Civita connection becomes a tensorial quantity to be taken into account.

Here it is useful to clarify the meaning of $\mathscr{L}_{\text{geo}}(g,\mathscr{D})$ in terms of the known dynamical quantities akin to non-Riemannian geometries. Non-Riemannian geometries are characterized by torsion tensor $\mathbb{S}^\lambda_{\alpha\beta} = \mathscr{D}^\lambda_{\alpha\beta} - \mathscr{D}^\lambda_{\beta\alpha}$, non-metricity $\mathbb{Q}^{\alpha\beta}_\lambda = \mathscr{D}^\alpha_{\lambda\rho} g^{\rho\beta} + \mathscr{D}^\beta_{\lambda\rho} g^{\alpha\rho}$, the Ricci curvature tensor $\mathbb{R}_{\mu\nu}(\Gamma) = \mathbb{R}^\alpha_{\mu\alpha\nu}(\Gamma)$, and the other Ricci curvature tensor $\mathbb{R}'_{\beta\nu} = \mathbb{R}^\alpha_{\alpha\beta\nu}(\Gamma)$. All these tensor fields make up the geometrical sector of the theory which obviously spans a much larger set compared to the purely metric formulation (GR). In GR connection and metric are put in direct relation from scratch. However, physically, it is more natural to induce a relation between them, if any, as a result of dynamical equations [16]. This is what is done by the Palatini formalism where metricity appears in the system automatically via the equations of motion. In the metric-affine gravity we explore here, metric and connection continue to be independent geometrical variables with no harm from their equations of motion (see Appendix A for further details, Supporting information online). One crucial aspect of non-Riemannian geometries (with non-vanishing torsion and/or non-metricity) is to provide a compact structuring of various tensor fields which can play important roles in cosmology [17].

Clearly, torsion vanishes for theories with symmetric connections, and this is also the case throughout the present work. Although the torsion-free cases are studied for simplicity, the non-metricity, which relaxes the restrictions on the theory, still holds. Moreover, $\mathbb{R}'_{\beta\nu}$ is an anti-symmetric tensor field whose curvature scalar vanishes identically. This tensor can give contributions to the Lagrangian at the quadratic and higher levels. The Lagrangian $\mathscr{L}_{\text{geo}}(g,\mathscr{D})$ includes all these tensorial contributions through the $\mathscr{D}^\lambda_{\alpha\beta}$ dependence

$$\mathscr{L}_{\text{geo}}(g,\mathscr{D}) = \mathscr{L}_{\text{geo}}(g,\mathbb{S},\mathbb{Q},\mathbb{R},\mathbb{R}') \quad (18)$$

throughout the text. It is clear that $\mathscr{L}_{\text{geo}}$ can involve arbitrary powers and derivatives of the tensorial connection $\mathscr{D}^\lambda_{\alpha\beta}$.

It is clear that the Lagrangian $\mathscr{L}$, through its Γ or $\mathscr{D}$ dependence, gives rise to important modifications in the equations of motion [18] so that the $\Gamma = \check{\Gamma}$ limit (which is precisely what is behind the Palatini formulation) does not necessarily hold. The contributions of $\mathscr{L}_{\text{geo}}(g,\mathscr{D})$ and $\mathfrak{L}_{\text{m}}(g,\mathscr{D},\Psi)$ generically avoid the limit $\Gamma = \check{\Gamma}$. We will discuss this point below. In the following section, however, we focus on the transformation properties of (15) without considering the contributions of $\mathscr{L}_{\text{geo}}(g,\mathscr{D})$ or $\mathfrak{L}_{\text{m}}(g,\mathscr{D},\Psi)$. This is done for the purpose of definiteness and simplicity. Nevertheless, in Appendix A, we come back to the effects of $\mathscr{L}$, especially $\mathscr{L}_{\text{geo}}(g,\mathscr{D})$, and give a detailed discussion of the equations of motion and other features.

3.1 Conformal-invariant connection

We start the analysis by first considering a conformal-invariant connection, by which we mean that connection is inert to rescalings of the metric. Therefore, along with the transformation of metric (2), the connection transforms as [6]

$$\widetilde{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} \quad (19)$$

and hence

$$\widetilde{\mathbb{R}}^\alpha_{\mu\beta\nu}(\widetilde{\Gamma}) = \mathbb{R}^\alpha_{\mu\beta\nu}(\Gamma), \quad \widetilde{\mathbb{R}}_{\mu\nu}(\widetilde{\Gamma}) = \mathbb{R}_{\mu\nu}(\Gamma) \quad (20)$$

since Riemann tensor (6) does not involve the metric tensor unless the connection does. The only non-trivial transformation occurs for the Ricci scalar

$$\widetilde{g}^{\mu\nu} \widetilde{\mathbb{R}}_{\mu\nu}(\widetilde{\Gamma}) = \Omega^{-2} g^{\mu\nu} \mathbb{R}_{\mu\nu}(\Gamma), \quad (21)$$

which is nothing but an overall dressing by Ω^{-2}. In particular, no derivatives of Ω are involved in the transformations of curvature tensors. This implies that Ω can develop no kinetic term. Indeed, under the transformation (21), the action (15) with conformal-invariant con-

nection goes over

$$S_{EH}\left[\widetilde{g},\widetilde{\Gamma},\overline{\phi}\right] = \int d^D x \sqrt{-g}$$

$$\times \left[\frac{1}{2}\zeta_D\overline{\phi}^2 g^{\mu\nu}\mathbb{R}_{\mu\nu}(\Gamma) - \lambda_D\left(\zeta_D\overline{\phi}^2\right)^{D/(D-2)}\right] \qquad (22)$$

in the $(\widetilde{g},\widetilde{\Gamma})$ frame and in the absence of the geometrical and matter parts $\mathscr{L}$. Obviously, this action is locally conformal-invariant under

$$g_{\alpha\beta} \longrightarrow \psi^2 g_{\alpha\beta}, \ \Gamma^\lambda_{\alpha\beta} \longrightarrow \Gamma^\lambda_{\alpha\beta}, \ \overline{\phi} \longrightarrow \psi^{-(D-2)/2}\overline{\phi} \qquad (23)$$

as was the case for metrical gravity, defined in (14). Therefore, although the original action (15) exhibits no sign of conformal invariance and hence the new action (22) arises, this transformed action exhibits manifest conformal invariance. The reason is as in the metrical gravity; the conformal factor Ω dresses the fixed scales ($M_\star$ and $\Lambda_\star$) in (15) to make them as effective fields transforming non-trivially under local rescalings of the fields [9].

Apart from this emergent conformal invariance, the action (22) possesses a highly important aspect not found in metrical GR: this is that $\overline{\phi}$ is not a ghost at all. It is a non-dynamical scalar field having vanishing kinetic energy, and, thus, the impasse caused by the ghost scalar field encountered in metrical GR is resolved. The non-dynamical nature of $\overline{\phi}$ continues to hold even if the matter sector is included. This result stems from the affine nature of the gravitational theory under consideration, and especially from the invariance of the connection under conformal transformations.

At this point it proves useful to discuss the 'non-dynamical' nature of the scalar field $\overline{\phi}$ in the action (22). At the level of the transformations employed and the Einstein–Hilbert action the non-dynamical nature of the conformal factor (and hence $\overline{\phi}$) is unavoidable. However, one immediately notices that this 'non-dynamical' structure depends sensitively on quantum fluctuations. Indeed, if quantum fluctuations are included into (22), the scalar field $\overline{\phi}$ is found to develop a kinetic term via graviton loops [19]. We shall keep the analysis at the classical level throughout this work. However, one is warned of such delicate effects which can arise from quantum corrections or higher order geometrical invariants.

3.2 Conformal-variant connection

As an alternative to conformal-invariant connection, in this subsection we investigate different scenarios where $\Gamma^\lambda_{\alpha\beta}$ exhibits non-trivial changes along with the transformation of the metric in (2).

As a possible transformation property, we first discuss the multiplicative transformation of connection. Namely, connection transforms similar to the metric itself:

$$\widetilde{\Gamma}^\lambda_{\alpha\beta} = f(\Omega)\Gamma^\lambda_{\alpha\beta}, \qquad (24)$$

where $f(\Omega)$ is a generic function of the conformal factor. Inserting this transformed connection into (6), one straightforwardly determines the transformed Riemann tensor

$$\widetilde{\mathbb{R}}^\alpha_{\mu\beta\nu}(\widetilde{\Gamma}) = f(\Omega)\mathbb{R}^\alpha_{\mu\beta\nu}(\Gamma) + \partial_\beta f(\Omega)\Gamma^\alpha_{\mu\nu} - \partial_\nu f(\Omega)\Gamma^\alpha_{\mu\beta}$$

$$+ f(\Omega)\left(f(\Omega) - 1\right)\left[\Gamma^\alpha_{\beta\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}\right] \qquad (25)$$

and hence the transformed Ricci scalar

$$\widetilde{g}^{\mu\nu}\widetilde{\mathbb{R}}_{\mu\nu}(\widetilde{\Gamma}) = \Omega^{-2}$$

$$\times \left\{f(\Omega)\mathbb{R}(\Gamma) + \partial_\alpha f(\Omega)g^{\mu\nu}\Gamma^\alpha_{\mu\nu} - \partial_\nu f(\Omega)g^{\mu\nu}\Gamma^\alpha_{\mu\alpha}\right.$$

$$\left. + f(\Omega)\left(f(\Omega) - 1\right)\left[\Gamma^\alpha_{\alpha\lambda}g^{\mu\nu}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\nu\lambda}g^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\right]\right\}. \qquad (26)$$

It is straightforward to check that the Γ-dependent terms on the right-hand side form a true scalar under general coordinate transformations (see Appendix B for details, Supporting information online). This conformal transformation rule for the Ricci scalar dictates what form the gravitational action (15) in the (g,Γ) frame takes in the $(\widetilde{g},\widetilde{\Gamma})$ frame. It is clear that the transformed action will involve Ω as well as its partial derivatives. Therefore, contrary to the previous case of conformal-invariant connection, Ω is a dynamical field. However, it does not possess a true kinetic term in the sense of a scalar field theory. Its derivative interactions are always accompanied by the connection $\Gamma^\lambda_{\alpha\beta}$.

As another transformation property of the connection, we now turn to analysis of additive transformation of $\Gamma^\lambda_{\alpha\beta}$. We thus consider the generic transformation rule

$$\widetilde{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + \Delta^\lambda_{\alpha\beta}(\Omega), \qquad (27)$$

where $\Delta^\lambda_{\alpha\beta}(\Omega)$, being the difference between $\widetilde{\Gamma}^\lambda_{\alpha\beta}$ and $\Gamma^\lambda_{\alpha\beta}$, is a rank (1,2) tensor field. It is a tensorial connection. This transformation of the connection is understood to run simultaneously with the transformation of the metric in (2). (One notes that $\Delta^\lambda_{\alpha\beta}$ here may be interpreted as containing a set of vector fields like the conformal factor itself is a scalar field. See [17] for details of such a reduction.) Then, as follows from (6), the Riemann tensor transforms as

$$\widetilde{\mathbb{R}}^\alpha_{\mu\beta\nu}(\widetilde{\Gamma})$$

$$= \mathbb{R}^\alpha_{\mu\beta\nu}(\Gamma) + \nabla_\beta\Delta^\alpha_{\mu\nu} - \nabla_\nu\Delta^\alpha_{\mu\beta} + \Delta^\alpha_{\lambda\beta}\Delta^\lambda_{\mu\nu} - \Delta^\alpha_{\lambda\nu}\Delta^\lambda_{\beta\mu}, \qquad (28)$$

where the Δ-dependent part on the right-hand side, although it seems so, is not a true curvature tensor; it is not generated by any of the covariant derivatives induced by $\Gamma_{\alpha\beta}^{\lambda}$ or $\check{\Gamma}_{\alpha\beta}^{\lambda}$. This extra Δ-dependent part is just a rank (1,3) tensor field induced by $\Delta_{\alpha\beta}^{\lambda}$ alone.

In accordance with the transformation of the Riemann tensor in (28), the Ricci scalar transforms as

$$\widetilde{g}^{\mu\nu}\widetilde{\mathbb{R}}_{\mu\nu}\left(\widetilde{\Gamma}\right) = \Omega^{-2}g^{\mu\nu}$$

$$\times \left\{ \mathbb{R}_{\mu\nu}\left(\Gamma\right) + \nabla_{\alpha}\Delta_{\mu\nu}^{\alpha} - \nabla_{\nu}\Delta_{\mu\alpha}^{\alpha} + \Delta_{\lambda\alpha}^{\alpha}\Delta_{\mu\nu}^{\lambda} - \Delta_{\lambda\nu}^{\alpha}\Delta_{\alpha\mu}^{\lambda} \right\}. \quad (29)$$

This transformation rule is rather generic for connections which transform additively [11]. Nevertheless, it is necessary to determine physically admissible forms of $\Delta_{\alpha\beta}^{\lambda}$ so that the conformal factor Ω assumes appropriate dynamics in regard to absence of ghosts and emergence of a new conformal invariance in the sense of (23).

At this stage, the right question to ask is this: 'How is $\Delta_{\alpha\beta}^{\lambda}$ related to Ω?' To answer this question, one has to check out a series of possibilities. Being a rank (1,2) tensor field, $\Delta_{\alpha\beta}^{\lambda}$ can assume a number of forms like $V^{\lambda}g_{\alpha\beta}$ or $\delta_{\alpha}^{\lambda}V_{\beta}$ or $V^{\lambda}T_{\alpha\beta}$, with V_{α} being a vector field and $T_{\alpha\beta}$ a symmetric tensor field. If the transformation of connection (27) is to coexist with that of the metric in (2), then V_{α}, $T_{\alpha\beta}$ or any other structure must be related to gradients of Ω so that $\Delta_{\alpha\beta}^{\lambda}$ vanishes when Ω is unity or, more precisely, constant. Therefore, one may identify V_{α} with $\partial_{\alpha}\Omega$ and $T_{\alpha\beta}$ with $\nabla_{\alpha}\partial_{\beta}\Omega$ or $\partial_{\alpha}\Omega\ \partial_{\beta}\Omega$. Consequently, $\Delta_{\alpha\beta}^{\lambda}$ should be composed of $\partial^{\lambda}\Omega\ g_{\alpha\beta}$, $\delta_{\alpha}^{\lambda}\partial_{\beta}\Omega$ or relevant higher derivatives of Ω or higher powers of $\partial_{\alpha}\Omega$. Hence, at the linear level, $\Delta_{\alpha\beta}^{\lambda}$ must be of the form

$$\Delta_{\alpha\beta}^{\lambda} = c_1\left(\delta_{\alpha}^{\lambda}\partial_{\beta}\ln\Omega + \delta_{\beta}^{\lambda}\partial_{\alpha}\ln\Omega\right) + c_2 g_{\alpha\beta}\partial^{\lambda}\ln\Omega, \quad (30)$$

where c_1 and c_2 are real constants. Here, one notices that a very similar form of this connection was also found in [20, 21] in spacetimes with non-vanishing torsion. In [21], prescription in (30) is obtained by requiring invariance of the Lorentz connection under conformal transformations. That work also points out the possibility of conformal-invariant gravitational action. In addition to this, by considering a similar prescription for torsion instead of connection, one can construct a conformally invariant theory. This option is studied in detail in [22].

That both metric and connection transform according to an assumed prescription (as given in (2) and (30), respectively) may lead one to conclude that metric and connection do actually depend on each other – not independent quantities as required by the metric-affine gravity. Actually, such a dependence does not need to ex-

ist. The situation can be clarified by considering, for example, scalar and fermion fields, comparatively. Indeed, they both transform non-trivially under conformal transformations yet they bear no relationship at all. In this sense, one concludes that their behaviors under conformal transformations do not need to impose an interdependence between metric and connection.

One readily notices that the tensorial structures involved in (30) are the same as the ones appearing in the transformation of the Levi-Civita connection under conformal transformations. This is seen from direct comparison of (30) with (4). The difference is the generality of (30) in terms of the constants c_1 and c_2 since $c_1 = -c_2 = 1$ in the transformation (4) of the Levi-Civita connection. Under the transformation (30), the metric-affine action (15) in the (g, Γ) frame takes the form

$$S_{\text{EH}}\left[\widetilde{g}, \widetilde{\Gamma}, \overline{\phi}\right] = \int d^D x\ \sqrt{-g}\left\{ \frac{1}{2}\Omega^{D-2}M_{\star}^{D-2}g^{\mu\nu}\mathbb{R}_{\mu\nu}\left(\Gamma\right)\right.$$

$$+ \frac{1}{2}(D-1)(D-2)\kappa_D\Omega^{D-4}M_{\star}^{D-2}g^{\mu\nu}\partial_{\mu}\Omega\partial_{\nu}\Omega$$

$$\left. - \Lambda_{\star}\Omega^D + \widetilde{\mathscr{L}} \right\}$$

$$= \int d^D x\sqrt{-g}\left\{ \frac{1}{2}\text{Sign}\left(\kappa_D\right)g^{\mu\nu}\partial_{\mu}\overline{\phi}\partial_{\nu}\overline{\phi}\right.$$

$$\left. + \frac{1}{2}\zeta_D'\overline{\phi}^2 g^{\mu\nu}\mathbb{R}_{\mu\nu}\left(\Gamma\right) - \lambda_D\left(\zeta_D'\overline{\phi}^2\right)^{D/(D-2)} \right\}, \quad (31)$$

where use has been made of the abbreviations

$$\kappa_D = \frac{(c_1+c_2)^2 + (D-2)c_1 c_2 + (D-2)(c_1-c_2)}{D-2} \quad (32)$$

$$\zeta_D' = \frac{\zeta_D}{|\kappa_D|} \quad (33)$$

along with the new canonical scalar field

$$\overline{\phi} = \frac{1}{\sqrt{\zeta_D'}}\left(M_{\star}\Omega\right)^{(D-2)/2}. \quad (34)$$

The action (31) is to be contrasted with the transformed action (11) in metrical gravity. The differences between the two are spectacular, and it could prove useful to discuss them here in detail.

– One first notes that the action (31) is invariant under the emergent conformal transformations

$$g_{\alpha\beta} \longrightarrow \psi^2\ g_{\alpha\beta}$$

$$\Gamma_{\alpha\beta}^{\lambda} \longrightarrow \Gamma_{\alpha\beta}^{\lambda} + \Delta_{\alpha\beta}^{\lambda}(\psi)$$

$$\overline{\phi} \longrightarrow \psi^{-(D-2)/2}\ \overline{\phi} \quad (35)$$

similar to what we have found in (14) for the metrical GR. This invariance guarantees that all the fixed scales in (15) are appropriately dressed by the conformal factor Ω [9].

– The conformal coupling ζ_D in (11) of the pure metric gravity changes to $\zeta_D / |\kappa_D|$ in the metric-affine action under consideration. The presence of κ_D reflects the generality of the transformation of the connection, as noted in (30). This is a highly important result since it generalizes the very concept of 'conformal coupling' between scalar fields and curvature scalar by changing ζ_D to ζ'_D. This modification can have noticeable consequences in cosmological [5, 23, 24] as well as collider observables [12, 25] of GR with affinity.

– In complete contrast to (11), the scalar field $\overline{\phi}$ in (31) obtains an indefinite kinetic term. The sign of the kinetic term is determined by the sign of κ_D. One here notes physically distinct cases:

1. If $\kappa_D > 0$ then $\overline{\phi}$ is a scalar ghost as in metrical GR. In (11) $\kappa_D = 1$ (since $c_1 = 1$ and $c_2 = -1$ for the change of Levi-Civita connection (4) under conformal transformations), and $\overline{\phi}$ is necessarily a ghost if gravity is to stay as an attractive force.

2. If, however, $\kappa_D < 0$ then $\overline{\phi}$ becomes a true scalar field with no problems like ghost behavior. One notices from (31) that this very regime is realized with no modification in the attractive nature of the gravitational force. Gravity is attractive and $\overline{\phi}$ is a non-ghost, true scalar field. This result follows from the generality of the transformation of $\Gamma^\lambda_{\alpha\beta}$ in (30) compared to that of the Levi-Civita connection. The real constants c_1 and c_2 give enough freedom to make κ_D negative for having a canonical scalar field theory, and this happens for

$$c_2 > -1 + \frac{1}{2}D(1 - c_1)$$
$$-\frac{1}{2}\sqrt{(D^2 - 4)c_1^2 - 2(D^2 - 4)c_1 + (D - 2)^2}$$

and

$$c_2 < -1 + \frac{1}{2}D(1 - c_1)$$
$$+\frac{1}{2}\sqrt{(D^2 - 4)c_1^2 - 2(D^2 - 4)c_1 + (D - 2)^2},$$

where c_1 is restricted to lie outside the interval

$$\left[\frac{D + 2 - 2\sqrt{D + 2}}{(D + 2)}, \frac{D + 2 + 2\sqrt{D + 2}}{(D + 2)}\right].$$

One can see that for any dimension $D \geq 4$ there exist wide ranges of values of c_1 for which c_2 takes on admissible negative or positive real values. In particular, if we consider one of the most likely cases in which the constants are equal but have opposite signs, we find $\kappa_D < 0$ for $c_1 = -c_2 \notin (0, 2]$ in $D > 2$ dimensions. Similar considerations pertaining to the metric–scalar–torsion system can be found in [26].

3. The fact that the metric-affine gravity offers a true scalar field $\overline{\phi}$ elevates the arguments on the cosmological constant problem in [15] to a more physical status since one then does not need to multiply the scalar field by the imaginary unit to make sense of the resulting scalar field theory. For $\kappa_D < 0$ and $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, the affine-gravitational action (31) can realize an infrared fixed point for $\overline{\phi}$ with no artificial changes in the sign of its kinetic term.

– The geometrical part of $\mathscr{L}(g, \mathscr{D}, \Psi)$, which only consists of the metric and $\mathscr{D}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \check{\Gamma}^\lambda_{\alpha\beta}$, will also transform under conformal transformation (2) with additively conformal-variant connection (27). Under the conformal transformations (35), $\mathscr{D}$ changes as

$$\widetilde{\mathscr{D}}^\lambda_{\alpha\beta} = \mathscr{D}^\lambda_{\alpha\beta} + (c_2 + 1)g_{\alpha\beta}\partial^\lambda \ln\psi + (c_1 - 1)$$
$$\times \left(\delta^\lambda_\alpha \partial_\beta \ln\psi + \delta^\lambda_\beta \partial_\alpha \ln\psi\right)$$

as expected from transformation properties of $\Gamma^\lambda_{\alpha\beta}$ and $\check{\Gamma}^\lambda_{\alpha\beta}$. This gives geometrodynamical terms and couplings of $\mathscr{D}$ with the emergent scalar field ψ.

– An important problem concerns the gravitational kinetic term. In metric formulation, a conformally invariant kinetic term is provided by the Weyl tensor [11, 27]. In the present case, however, such a conformal-invariant kinetic term might be difficult to induce. Actually, what is necessary is to construct a gravitational kinetic term which is invariant under the conformal transformations in (2) and (27) (with the specific form in (30)). With c_1 and c_2 differing from the Levi-Civita connection, construction of the kinetic term may not be straightforward.

The analysis above ensures that additively transforming connections, such as (30), gives rise to a physically sensible mechanism where the gravitational sector as well as the emergent scalar field from conformal transformation are both physical. Removal of the ghost degree of freedom in metrical GR is a highly important aspect of the metric-affine gravity. Essentially, freeing connection from metric enables one to reach a physically consistent picture in regard to conformal frame changes in the gravitational action.

4 Discussion and conclusion

In this work we have analyzed conformal transformations in metric-affine gravity (GR with affinity). The analysis is a comparative one between GR with affinity and metricity. The main result of the analysis is that metric-affine gravity admits, under general additive transformations of the connection, conformally related frames in which both gravitational and scalar sectors behave physically. The transformed frame consists of no ghost field, and exhibits emergent conformal invariance (sometimes called Weyl–Stückelberg invariance). The results can have far-reaching consequences for collider experiments [12, 25], cosmological evolution [5, 28, 33] as well as electroweak breaking [11].

We have also analyzed equations of motion under general circumstances allowed by general covariance, and concluded that general Lagrangians allow for generalized conformal transformations of the connection without spoiling the essence of the theory in the transformed frame.

The affine gravitational action (15) can give rise to novel effects not found in the minimal version (the Einstein–Hilbert action). The conformally reached frame can have various modifications in gravitational, matter and conformal factor (i.e. Ω related to $\overline{\phi}$) dynamics. The fact that the metric-affine gravity can accommodate correct gravitational dynamics plus non-ghost scalar degrees of freedom under conformal transformations is an important aspect. This feature can have important implications in cosmological and other settings since transformation of a system to a conformal frame now involves no ghost degree of freedom. Indeed, the appearance of ghost fields, as mentioned in the text, is the major problem of conformal GR [29]. Therefore, the ghost-free dynamics established in the present work can have significant applications in conformal field theory, cosmology and gravitation.

Key words. Metric-affine gravity, conformal transformations, ghosts.

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