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Asymptotic behavior of linear impulsive integro-differential equations

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Abstract

Asymptotic equilibria of linear integro-differential equations and asymptotic relations between solutions of linear homogeneous impulsive differential equations and those of linear integro-differential equations are established. A new Gronwall–Bellman type lemma for integro-differential inequalities is proved. An example is given to demonstrate the validity of one of the results.

Keywords: Impulsive integro-differential equations; Linear systems; Asymptotic equilibria; Asymptotic equivalence

1. Introduction and preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be sets of all natural numbers, integers and real numbers, respectively. Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $\mathbb{R}_+ = [0, \infty]$, and θ_i, $i = 1, 2, \ldots$, be a sequence from $\mathbb{R}_+$ such that $0 < \theta_1 < \theta_2 < \cdots$ and that $\theta_k \to \infty$ as $k \to \infty$. In what follows, $PC(\mathbb{R}_+, \mathbb{Y})$ denotes the set of all functions $\varphi : \mathbb{R}_+ \to \mathbb{Y}$, which are piecewise continuous and continuous on the left with discontinuities of the first kind at points θ_i, $i = 1, 2, \ldots$. A function $\varphi(t)$ is said to be from a space $PC^1(\mathbb{R}_+, \mathbb{Y})$ if $\varphi' \in PC(\mathbb{R}_+, \mathbb{Y})$.

The main object of investigation in this work are the following two systems of impulsive integro-differential equations:

$$\begin{aligned} \dot{x}(t) &= [A(t) + B(t)]x(t) + \int_0^t K(t,\tau)x(\tau)\mathrm{d}\tau, \quad t \neq \theta_i,\ t \geq 0, \\ \Delta x\,|_{t=\theta_i} &= [C_i + D_i]x(\theta_i) + \sum_{0<\theta_j \leq \theta_i} L_{ij}x(\theta_j), \end{aligned} \tag{1}$$

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where, $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $\Delta x|_{t=\theta} = x(\theta+) - x(\theta)$, $x(\theta+) = \lim_{t\to\theta+} x(t)$, and

$$\begin{aligned} \dot{u}(t) &= P(t)u(t) + \int_0^t M(t,s)u(s)\mathrm{d}s, \quad t \neq \theta_i, \quad t \geq 0 \\ \Delta u|_{t=\theta_i} &= Q_i u(\theta_i) + \sum_{0<\theta_j\leq\theta_i} N_{ij} u(\theta_j), \end{aligned} \tag{2}$$

where, $u(t) \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. The problem of asymptotic properties of solutions of the systems is investigated.

Throughout this paper we need the following conditions on system (1):

(C1) $A(t), B(t) \in PC(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $K(t,s)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ in both variables t and s;
(C2) C_i, D_i and L_{ij}, $i, j = 1, 2, \ldots$ are real constant $n \times n$ matrices, and $\det[C_i + D_i] \neq 0$, $\det C_i \neq 0$, $i \in \mathbb{N}$.

Similar conditions are imposed on system (2):

(P1) $P(t) \in PC(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $M(t,s)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ in both variables t and s;
(P2) Q_i and N_{ij}, $i, j = 1, 2, \ldots$ are real constant $n \times n$ matrices and $\det Q_i \neq 0$, $i \in \mathbb{N}$.

The results of [1,2] imply that solutions for systems (1) and (2) exist and are unique. They are from $PC^1(\mathbb{R}_+, \mathbb{R}^n)$, if conditions (C1), (C2) and (P1), (P2), respectively, are valid. Both systems have the zero solutions.

Theory of impulsive differential equations has been investigated intensively for the last several decades [3–13]. They are of great interest for applications [14–20]. The asymptotic behavior of solutions and integro-differential equations have an important place in the theory [21–23]. There are many research articles which deal with asymptotic properties of solutions of differential equations. These properties are basic subjects of investigation for qualitative theory of differential equations [24–31].

In this paper, by making use of the results of [30] for integro-differential equations, and investigation technique of impulsive integro-differential equations [1,2,23], new asymptotic properties of solutions of (1) and (2) are established. The first problem we try to tackle is the asymptotic relations between solutions of (1) and solutions of the associated linear homogeneous system

$$\begin{aligned} \dot{y}(t) &= A(t)y(t), \quad t \neq \theta_i, \\ \Delta y|_{t=\theta_i} &= C_i y(\theta_i). \end{aligned} \tag{3}$$

We shall prove the following equations under some conditions that will be specified later,

$$x(t) = Y(t)[b + o(1)], \tag{4}$$
$$x(t) = Y(t)c + o(1), \tag{5}$$

where $Y(t)$ is a fundamental matrix of solutions of (3) such that $Y(0) = I$ (I is the identity matrix), $x(t)$ is a solution of (1), $c, b \in \mathbb{R}^n$ are vectors, and $o(1) \to 0$ as $t \to \infty$. The last equation accompanied with a one-to-one correspondence condition implies asymptotic equivalence of the systems. Another problem to be considered is the existence of the asymptotic equilibria for system (2) which is secondary to the first problem in our paper, but it certainly has its own interest for the theory of impulsive integro-differential equations.

Remark 1.1. If the integral terms in linear perturbation (1) of (3), and in (2) vanish, then from our results one can easily obtain assertions on asymptotic behavior of solutions for linear impulsive differential equations.

The following assertion is needed in the proof of the main results of the paper.

Lemma 1.2. *Assume that*

$$u(t) \leq C + \int_0^t \left[\nu(s)u(s) + \int_0^s \omega(s,\tau)u(\tau)\mathrm{d}\tau\right]\mathrm{d}s + \sum_{0<\theta_i<t}\left[\beta_i u(\theta_i) + \sum_{0<\theta_j\leq\theta_i} \gamma_{ij} u(\theta_j)\right], \tag{6}$$

where $0 \in \mathbb{R}_+$, *constants* C, $\omega(s,\tau)$, β_i *and* γ_{ij} *are nonnegative and* $\nu(s) > 0$.

Then, the following inequality holds

$$u(t) \le C \exp\left[\int_0^t \left[\nu(s) + \int_0^s \omega(s,\tau)\mathrm{d}\tau\right]\mathrm{d}s\right] \cdot \prod_{0<\theta_i<t}\left[1 + \beta_i + \sum_{\theta_i \le \theta_j < t} \gamma_{ij}\right]. \quad (7)$$

Proof. By applying Lemma 1 of [2], we can write the following

$$\sum_{0<\theta_i<t}\sum_{0<\theta_j\le\theta_i} \gamma_{ij} u(\theta_j) = \sum_{0<\theta_i<t}\left[\sum_{\theta_i\le\theta_j<t}\gamma_{ij}\right] u(\theta_i). \quad (8)$$

Hence,

$$u(t) \le C + \int_0^t \left[\nu(s) + \int_s^t \omega(s,\tau)\mathrm{d}\tau\right] u(s)\mathrm{d}s + \sum_{0<\theta_i<t}\left[\beta_i + \sum_{\theta_i\le\theta_j<t}\gamma_{ij}\right] u(\theta_i).$$

By applying Lemma 1 of [13], it is easily seen that (7) holds. □

2. Asymptotic equilibria

Let us list the following conditions, which we shall use to formulate several theorems on asymptotic equilibria for system (2).

(P3)

$$\int_0^\infty \left[\|P(t)\| + \int_0^t \|M(t,s)\|\mathrm{d}s\right]\mathrm{d}t + \sum_{0<\theta_i<t} \log\left(1 + \|Q_i\| + \sum_{0<\theta_j\le\theta_i}\|N_{ij}\|\right) < \infty;$$

(P4)

$$\mu_1 = \int_0^\infty \left[\|P(s)\| \exp[H(s)] + \int_0^s \|M(s,\tau)\| \exp[H(\tau)]\mathrm{d}\tau\right]\mathrm{d}s$$
$$+ \sum_{0<\theta_i<\infty}\left[\|Q_i\|\exp[H(\theta_i)] + \sum_{0<\theta_j\le\theta_i}\|N_{ij}\|\exp[H(\theta_j)]\right] < 1.$$

(P5)

$$|Sp(Q_i)| < 1, \quad i = 1, 2, \ldots,$$

where $Sp(Q_i)$ means the sum of diagonal elements of a square matrix Q_i;

(P6)

$$\mu_2 = n! \int_0^\infty \int_0^t [\|M(t,\tau)\| \exp[(n-1)H(t) + \mu_+(t) + H(\tau)]]\mathrm{d}\tau$$
$$+ n! \sum_{0<\theta_i<\infty}\sum_{0<\theta_j\le\theta_i} (1 + n\|Q_i\|)\|N_{ij}\| \exp[(n-1)H(\theta_i) + \mu_+(\theta_i) + H(\theta_j)] < 1,$$

where

$$\mu_+(t) = \int_0^t [Sp(P(s))]\mathrm{d}s + \sum_{0<\theta_i<t}\log(1 + Sp(Q_i)), \quad t \ge 0,$$

$$H(t) = \phi(t) + \sum_{0<\theta_i<t}\log(1+\psi_i),$$

$$\phi(t) = \int_0^t \left[\|P(s)\| + \int_0^s \|M(s,\tau)\|\mathrm{d}\tau\right]\mathrm{d}s$$

and

$$\psi_i = \left[\|Q_i\| + \sum_{\theta_i < \theta_j \le t} \|N_{ij}\| \right]$$

we shall need also the following function

$$\mu_-(t) = \int_0^t [Sp(P(s))]\mathrm{d}s - \sum_{0<\theta_i<t} \log(1 + Sp(Q_i)), \quad t < 0.$$

Theorem 2.1. *If conditions* (P1)–(P3) *hold, then every solution $u(t)$ of* (2) *approaches c_u as $t \to \infty$, where c_u is a constant vector in $\mathbb{R}^n$.*

Proof. It is easy to see that a solution $u(t)$ of (2) with $u(0) = u_0$ satisfies the following integral equation,

$$u(t) = u(0) + \int_0^t \left[P(s)u(s) + \int_0^s M(s,\tau)u(\tau)\mathrm{d}\tau \right] \mathrm{d}s + \sum_{0<\theta_i<t} \left[Q_i u(\theta_i) + \sum_{0<\theta_j \le \theta_i} N_{ij} u(\theta_j) \right]. \tag{9}$$

By taking the norms of both sides of (9), we get,

$$\|u(t)\| \le \|u(0)\| + \int_0^t \left[\|P(s)\| \, \|u(s)\| + \int_0^s \|M(s,\tau)\| \, \|u(\tau)\| \mathrm{d}\tau \right] \mathrm{d}s$$
$$+ \sum_{0<\theta_i<t} \left[\|Q_i\| \, \|u(\theta_i)\| + \sum_{0<\theta_j \le \theta_i} \|N_{ij}\| \, \|u(\theta_j)\| \right]. \tag{10}$$

By applying Lemma 1.2, we obtain,

$$\|u(t)\| \le \|u(0)\| \exp[\phi(t)] \prod_{0<\theta_i<t} [1 + \psi_i]. \tag{11}$$

Then (11) can be written as

$$\|u(t)\| \le \|u(0)\| \exp[H(t)]. \tag{12}$$

Condition (P3) implies that every solution $u(t)$ of (2) is bounded. Let

$$m_u = \sup_{t\in\mathbb{R}_+} \|u(t)\| < \infty.$$

Using (9) one can find that for fixed $t_1, t_2 \in \mathbb{R}_+$, $t_2 < t_1$

$$\|u(t_1) - u(t_2)\| \le m_u \left[\int_{t_1}^{t_2} \left[\|P(s)\| + \int_0^s \|M(s,\tau)\|\mathrm{d}\tau \right] \mathrm{d}s + \sum_{t_1<\theta_i<t_2} \left[\|Q_i\| + \sum_{0<\theta_j\le\theta_i} \|N_{ij}\| \right] \right].$$

Condition $(P3)$ implies that for arbitrary $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that if $t_1 > t_2 > T(\epsilon)$ then

$$\int_{t_1}^{t_2} \left[\|P(s)\| + \int_0^s \|M(s,\tau)\|\mathrm{d}\tau \right] \mathrm{d}s + \sum_{t_1<\theta_i<t_2} \left[\|Q_i\| + \sum_{0<\theta_j\le\theta_i} \|N_{ij}\| \right] < \frac{\epsilon}{m_u}.$$

That is,

$$\|u(t_1) - u(t_2)\| \le \epsilon \quad \text{if } t_1 > t_2 > T(\epsilon). \quad \square$$

Theorem 2.2. *If conditions* (P1), (P2) *and* (P4) *are valid, then every nontrivial solution of* (2) *has the nonzero limit*

$$\lim_{t\to\infty} u(t) = c_u.$$

Proof. It is easily seen that condition $(P4)$ implies the condition of Theorem 2.1. So, every solution $u(t)$ of (2) approaches a limit, i.e. $\lim_{t\to\infty} u(t) = c_u$. Let us show that $c_u \neq 0$. By applying triangle inequality to (9), we have the following:

$$\|u(t)\| \geq \|u(0)\| \left[1 - \int_0^\infty \left[\|P(s)\| \exp[H(s)] + \int_0^s \|M(s,\tau)\| \exp[H(\tau)]\mathrm{d}\tau\right]\mathrm{d}s - \sum_{0<\theta_i<\infty}\left[\|Q_i\| \exp[H(\theta_i)] + \sum_{0<\theta_j\leq\theta_i} \|N_{ij}\| \exp[H(\theta_j)]\right]\right].$$

That is, in the limit,

$$c_u \geq \|u_0\|(1-\mu_1). \quad \square$$

Theorem 2.3. *If conditions* (P1), (P2), (P5) *and* (P6) *are valid, then for every vector* $c \in \mathbb{R}^{\ltimes}$ *there exists a unique solution* $u_c(t)$ *of* (2) *such that*

$$\lim_{t\to\infty} u_c(t) = c. \tag{13}$$

Proof. Denote $D(t) = \det U(t)$, where $U(t) = (u_{ij}(t))$, $U(0) = I$ is the fundamental matrix of solutions of (2), such that

$$\frac{\mathrm{d}U}{\mathrm{d}t} = P(t)U(t) + \int_0^t M(t,s)U(s)\mathrm{d}s, \quad t \neq \theta_i$$
$$\Delta U|_{t=\theta_i} = Q_i U(\theta_i) + \sum_{0<\theta_j\leq\theta_i} N_{ij} U(\theta_j) \tag{14}$$

and by (12)

$$\|u_{ij}(t)\| \leq \exp[H(t)]. \tag{15}$$

Denote $\mathfrak{A} = \lim_{t\to\infty} U(t)$. By Theorem 2.1 matrix $\mathfrak{A}$ exists and is constant. Every solution of (2) can be written as

$$u(t) = U(t)d \tag{16}$$

where $d \in \mathbb{R}^n$ and $d = u(0)$. To prove the theorem it is sufficient to show that

$$\det \mathfrak{A} \neq 0. \tag{17}$$

Indeed, if $c \in \mathbb{R}$ is given, then a solution of the equation

$$\mathfrak{A}d = c \tag{18}$$

defines a solution of (16) which we are looking for.

In what follows we shall show that (17) is valid under conditions of the theorem. Let $D_{ij}(t)$ be the minor corresponding to element $u_{ij}(t)$ of $U(t)$. Then for $t \neq \theta_i$, we have that

$$\begin{aligned} D'(t) &= \sum_{i,j=1}^n D_{ij}(t)u'_{ij}(t) \\ &= \sum_{i,j=1}^n D_{ij}(t)\sum_{k=1}^n \left[P_{ik}(t)u_{kj}(t) + \int_0^t M_{ik}(t,\tau)u_{kj}(\tau)\mathrm{d}\tau\right] \\ &= [Sp(P(t))]D(t) + R(t) \end{aligned} \tag{19}$$

where

$$R(t) = \sum_{i,j,k=1}^n D_{ij}(t)\int_0^t M_{ik}(t,\tau)u_{kj}(\tau)\mathrm{d}\tau.$$

Similarly one can find that

$$\Delta D\,|_{t=\theta_i} = [Sp(Q_i)]D(\theta_i) + S_i, \tag{20}$$

where number S_i is a product of $n!$ elements of matrix,

$$S = (I + Q_i)U(\theta_i) \sum_{0<\theta_j\le\theta_i} N_{ij}U(\theta_j).$$

Next, we consider the equation,

$$\begin{aligned} &D'(t) = [Sp(P(t))]D(t) + R(t)\\ &\Delta D\,|_{t=\theta_i} = [Sp(Q_i)]D(\theta_i) + S_i, \end{aligned} \tag{21}$$

and its associated equation,

$$\begin{aligned} &Z'(t) = [Sp(P(t))]Z(t)\\ &\Delta Z\,|_{t=\theta_i} = [Sp(Q_i)]Z(\theta_i). \end{aligned} \tag{22}$$

The solution $\mathfrak{D}(t)$, $\mathfrak{D}(0) = 1$ of (22) is equal to

$$\mathfrak{D}(t) = \begin{cases} \exp[\mu_+(t)], & t \ge 0\\ \exp[\mu_-(t)], & t < 0, \end{cases}$$

and the solution $D(t)$, $D(0) = 1$ of (21) is

$$D(t) = \mathfrak{D}(t)\left[1 + \int_0^t \mathfrak{D}(-s)R(s)\mathrm{d}s + \sum_{0<\theta_i<t} \mathfrak{D}(-\theta_i)S_i\right]. \tag{23}$$

Since D_{ij} is $(n-1)$ order determinant of elements of $D(t)$, then

$$|D_{ij}(t)| \le (n-1)!\exp[(n-1)H(t)], \quad i, j = 1, \dots, n,\ t \in \mathbb{R}_+$$

and

$$|R(t)| \le n! \int_0^t \|M(t,\tau)\| \exp[(n-1)H(t) + H(\tau)]\mathrm{d}\tau, \quad t \in \mathbb{R}_+.$$

Similarly one can find that

$$|S_i| \le n!(1 + n\|Q_i\|)\exp[(n-1)H(\theta_i)] \sum_{0<\theta_j\le\theta_i} \|N_{ij}\| \exp[H(\theta_j)].$$

Hence

$$\det\mathfrak{A} = \exp[\mu_+(\infty)]\left[1 + \int_0^\infty \mathfrak{D}(-s)R(s)\mathrm{d}s + \sum_{0<\theta_i} \mathfrak{D}(-\theta_i)S_i\right] \ge \exp[\mu_+(\infty)][1 - \mu_2],$$

and using (P6) theorem is proved. □

In following two examples we illustrate Theorem 2.1.

Example 2.4. Consider the system

$$\begin{aligned} &u_1'(t) = \frac{t}{t-1},\\ &u_2'(t) = \int_2^t \frac{5(s-1)}{4t^5}u_1(s)\mathrm{d}s, \quad t \ne i,\\ &\Delta u_1|_{t=i} = \frac{1}{i(i-1)}u(i),\\ &\Delta u_2|_{t=i} = \sum_{2<j\le t} \frac{1}{i^3}\frac{1}{j(j-1)}u(j), \end{aligned} \tag{24}$$

where $t \in [2, \infty)$, and $i > 2$ are integers. It is obvious that conditions (P1) and (P2) are satisfied by the system (24). Condition (P3) is also satisfied by the system, that is,

$$\int_2^\infty \left[\frac{t}{t-1} + \int_1^t \frac{5(s-1)}{4t^5}\mathrm{d}s\right]\mathrm{d}t + \sum_{2<i}\log\left[1 + \frac{1}{i(i-1)} + \sum_{2<j\le i}\frac{1}{i^3}\frac{1}{j(j-1)}\right] < \infty.$$

Indeed, the convergence of the improper integrals is obvious. Moreover, we have that

$$\exp\left(\sum_{2<i}\log\left[1 + \frac{1}{i(i-1)} + \sum_{2<j\le i}\frac{1}{i^3}\frac{1}{j(j-1)}\right)\right] = \prod_{i>1}\left(1 + \frac{1}{i(i-1)} + \sum_{2<j\le i}\frac{1}{i^3}\frac{1}{j(j-1)}\right)\right]$$
$$= \prod_{i>2}\left(1 + \frac{1}{i(i-1)} + \frac{i+1}{i^4}\right).$$

Hence, the sum is convergent by a theorem on the infinite product convergence [32]. Thus, Theorem 2.1 implies that for each solution $u(t) = (u_1(t), u_2(t))$ of (24) there exists a vector $c_u \in \mathbb{R}^2$, such that $u(t) \to c_u$ as $t \to \infty$. It is difficult to obtain the exact solution in closed form in this example. However, we simplify the system in the next example to get an explicit solution.

Example 2.5. Consider a slightly different version of the last system

$$\begin{aligned}
u_1'(t) &= \frac{t}{t-1},\\
u_2'(t) &= \int_2^t \frac{5(s-1)}{4t^5}u_1(s)\mathrm{d}s, \quad t \ne i,\\
\Delta u_2|_{t=i} &= \sum_{2<j\le i}\frac{1}{i^3}\frac{1}{(j-1)^2}u_1(j).
\end{aligned} \tag{25}$$

One can verify that condition (P3) is valid in this case too, and the solution $u(t)$, $u(2) = (u_1(2), u_2(2))$, has a form

$$u_1(t) = 2u_1(2)\frac{t-1}{t}, \qquad u_2(t) = u_2(2) + 5\frac{t^2-4}{t^5} + 2u_1(2)\sum_{2<i<t}\sum_{2<j\le i}\frac{1}{i^3}\frac{1}{(j-1)^2}\frac{j-1}{j}$$
$$= u_2(2) + 5\frac{t^2-4}{t^5} + 2u_1(2)\sum_{2<i<t}\frac{i+1}{i^4}.$$

In Fig. 1 the graphs of the solution, when $u_1(2) = 2$, and $u_2(2) = 3$ are provided. One can observe that both coordinates of the solution approach finite values as $t \to \infty$.

3. Asymptotic equivalence

Denote

$$m_+(t) = \int_0^t [Sp(Y^{-1}(s)B(s)Y(s))]\mathrm{d}s + \sum_{0<\theta_i<t}\log(1 + Sp(Y^{-1}(\theta_i^+)D_iY(\theta_i))), \quad t \ge 0,$$

$$\mathcal{H}(t) = \int_0^t\left[\|Y^{-1}(t)B(t)Y(t)\| + \int_0^s \|Y^{-1}(s)K(s,\tau)Y(\tau)\|\mathrm{d}\tau\right]\mathrm{d}s$$
$$+ \sum_{0<\theta_i<t}\log\left(1 + \|Y^{-1}(\theta_i)D_iY(\theta_i)\| + \sum_{\theta_i<\theta_j\le t}\|Y^{-1}(\theta_i)L_{ij}Y(\theta_j)\|\right).$$

Let us formulate the following conditions which we shall need to prove some of the theorems given below:

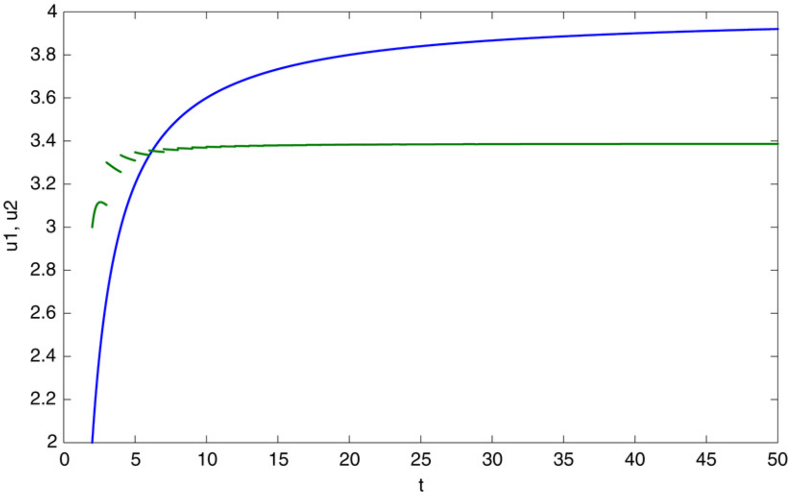

Fig. 1. The graph of the coordinate $u_1(t)$ of the solution is shown in blue, and of the coordinate $u_2(t)$ is green.

(C4)

$$\int_0^\infty \left[\|Y^{-1}(t)B(t)Y(t)\| + \int_0^t \|Y^{-1}(t)K(t,s)Y(s)\|\mathrm{d}s\right]\mathrm{d}t + \sum_{0<\theta_i<t} \log\left(1 + \|Y^{-1}(\theta_i^+)D_iY(\theta_i)\| + \sum_{0<\theta_j\le\theta_i} \|Y^{-1}(\theta_i^+)L_{ij}Y(\theta_j)\|\right) < \infty;$$

(C5)

$$\int_0^\infty \left[\|Y^{-1}(t)B(t)Y(t)\| \exp[\mathcal{H}(t)] + \int_0^t \|Y^{-1}(t)K(t,s)Y(s)\| \exp[\mathcal{H}(s)]\mathrm{d}s\right]\mathrm{d}t + \sum_{0<\theta_i<t} \left[\|Y^{-1}(\theta_i^+)D_iY(\theta_i)\| \exp[\mathcal{H}(\theta_i)] + \sum_{0<\theta_j\le\theta_i} \|Y^{-1}(\theta_i^+)L_{ij}Y(\theta_j)\| \exp[\mathcal{H}(\theta_j)]\right] < 1;$$

(C6)

$$|Sp(Y^{-1}(\theta_i)D_iY(\theta_i))| < 1;$$

(C7)

$$n! \int_0^\infty \int_0^t \|Y^{-1}(t)K(t,\tau)Y(\tau)\| \exp[(n-1)\mathcal{H}(t) + m_+(t) + \mathcal{H}(\tau)]\mathrm{d}\tau + n! \sum_{0<\theta_i<\infty} \sum_{0<\theta_j\le\theta_i} (1 + n\|Q_i\|)\|N_{ij}\| \exp[(n-1)\mathcal{H}(\theta_i) + m_+(\theta_i) + \mathcal{H}(\theta_j)] < 1;$$

(C8)

$$\int_t^\infty \left[\|Y(t)Y^{-1}(s)B(s)Y(s)\| \exp[\mathcal{H}(s)] + \int_0^s \|Y(t)Y^{-1}(s)K(s,\tau)Y(\tau)\| \exp[\mathcal{H}(\tau)]\mathrm{d}\tau\right]\mathrm{d}s + \sum_{t<\theta_i} \left[\|Y(t)Y^{-1}(\theta_i)D_iY(\theta_i)\| \exp[\mathcal{H}(\theta_i)] + \sum_{0<\theta_j\le\theta_i} \|Y(t)Y^{-1}(\theta_i)L_{ij}Y(\theta_j)\| \exp[\mathcal{H}(\theta_j)]\right] \to 0, \quad \text{as } t\to\infty.$$

If the substitution

$$x(t) = Y(t)u(t) \tag{26}$$

is applied to system (1) then one obtains system (2) with the following coefficients

$$\begin{aligned} &P(t) = Y^{-1}(t)B(t)Y(t), \qquad M(t,s) = Y^{-1}(t)K(t,s)Y(s) \\ &Q_i = Y^{-1}(\theta_i^+)D_iY(\theta_i), \quad N_{ij}Y^{-1}(\theta_i^+)L_{ij}Y(\theta_j). \end{aligned} \tag{27}$$

One can easily see that conditions (C1)–(C7) are transformed to the conditions (P1)–(P7).

On the basis of Theorems 2.1–2.3, we can formulate, using the transformation $x(t) = Y(t)u$, the following assertions.

Theorem 3.1. *Assume that conditions* (C1)–(C4) *are valid, then, every solution $x(t)$ of* (1) *satisfies $x(t) = Y(t)[c_x + o(1)]$, where $c_x \in \mathbb{R}^n$ is a constant vector.*

Theorem 3.2. *Assume that conditions* (C1)–(C3) *and* (C5) *are valid, then, every nontrivial solution $x(t)$ of* (1) *can be represented as $x(t) = Y(t)[c_x + o(1)]$, where $c_x \neq 0$.*

Theorem 3.3. *Assume that conditions* (C1)–(C4) *and* (C6), (C7) *are valid, then, for every vector $c \in \mathbb{R}^n$ there exists a unique solution $x_c(t)$ of* (1) *such that*

$$x_c(t) = Y(t)[c + o(1)].$$

Let us prove the following important assertion.

Theorem 3.4. *Assume that conditions* (C1)–(C4) *and* (C8) *are valid, then, every solution of* (1) *can be written as*

$$x(t) = Y(t)b + o(1) \tag{28}$$

where $b \in \mathbb{R}^n$ is a constant vector.

If conditions (C1)–(C3), (C5), (C8) *are valid then for every nontrivial solution $x(t)$ of* (1), *there exists a nonzero vector b such that* (28) *is valid.*

If conditions (C1)–(C4), (C7), (C8) *are valid then for every vector $c \in \mathbb{R}^n$ there exists a unique solution $x_c(t)$ of* (1) *such that*

$$x_c(t) = Y(t)c + o(1).$$

Proof. On the base of condition (C4) and Theorem 3.1 we have that

$$x(t) = Y(t)b + \alpha(t), \tag{29}$$

where according to the proof of Theorem 2.1, (26) and (27)

$$\alpha(t) \equiv -Y(t)\int_t^\infty \left[P(s)u(s) + \int_t^s M(s,\tau)u(\tau)\mathrm{d}\tau\right]\mathrm{d}s + \sum_{t<\theta_i}\left[Q_iu(\theta_i) + \sum_{t<\theta_j\le\theta_i} N_{ij}u(\theta_j)\right] \tag{30}$$

and

$$\|u(t)\| \le \|u(0)\|\exp[\mathcal{H}(t)]. \tag{31}$$

By the last inequality and condition $(C8)$ we have that $\|\alpha(t)\| \to 0$ as $t \to \infty$, and, hence, (29) has the form of (28). Similarly one can prove other assertions of the theorem. □

The notion of asymptotic equivalence for differential equations is very interesting from the theoretical point of view as well as from the point of view of applications. In this part of the paper we shall consider the equivalence of systems (1) and (3) on $\mathbb{R}_+$.

Let us introduce, following [24], the next definition.

Definition 3.5. Systems (1) and (3) are asymptotically equivalent on $\mathbb{R}_+$ if there exists a one-to-one correspondence between solutions $x(t) : \mathbb{R}_+ \to \infty$ and $y(t) : \mathbb{R}_+ \to \infty$ of the systems, respectively, such that $x(t) - y(t) \to 0$ as $t \to \infty$.

The last theorem implies the following assertion:

Theorem 3.6. *Assume that conditions* (C1)–(C4), (C5), (C7) *and* (C8) *are valid, then systems* (1) *and* (3) *on* $\mathbb{R}_+$ *are asymptotically equivalent on* $\mathbb{R}_+$.

Proof. Indeed, (29) implies the one-to-one correspondence

$$x(0) = y(0) + \alpha(0),$$

where $y(0) = c$, between solutions of (1) and (3) on $\mathbb{R}_+$. □

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