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Rad-supplements in injective modules

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Abstract. We introduce and study the notion of Rad-s-injective modules (i.e. modules which are Rad-supplements in their injective hulls). We compare this notion with another generalization of injective modules. We show that the class of Rad-s-injective modules is closed under finite direct sums. We characterize Rad-s-injective modules over several type of rings, including semilocal rings, left hereditary rings and left Harada rings.

1. Introduction

Throughout this paper all rings are associative with an identity element. Unless otherwise stated R denotes an arbitrary ring and all modules are unital left R-modules. Let M be a module. We use $N \leqslant M$ to denote that N is a submodule M. The radical of M and the injective hull of M are denoted by $\mathrm{Rad}(M)$ and $E(M)$, respectively. Let $N \leqslant M$. We say that N is *small* in M (written $N \ll M$) if the fact that $N + L = M$ for some submodule L of M implies $L = M$. Let N and K be submodules of M. Then K is said to be a *supplement* of N in M if $N + K = M$ and $N \cap K \ll K$. We say that K is a *Rad-supplement* of N in M if $N + K = M$ and $N \cap K \subseteq \mathrm{Rad}(K)$. It is clear that the following implications of conditions on submodules of any module hold:

$$\text{Direct summands} \Rightarrow \text{supplements} \Rightarrow \text{Rad-supplements.}$$

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It is well known that a module M is injective if and only if M is a direct summand of every module containing M. These modules play an important role in both module theory and homological algebra. Recently, a new generalization of injective modules has been studied in [4]. Namely, a module M is called *almost injective* if M is a supplement submodule of every module containing M. So a module is almost injective if and only if it is a supplement submodule of an injective module containing it (see [4, Proposition 3.2]). It is of interest to investigate the analogue of this notion by replacing “supplement”with “Rad-supplement”. We call a module M Rad-*s-injective*, if M is a Rad-supplement in every module containing M.

Proposition 1.1. *The following are equivalent for a module M:*

(i) *M is a* Rad-*s-injective module;*

(ii) *There exists an injective module E containing M such that M is a* Rad-*supplement in E;*

(iii) *M is a* Rad-*supplement in its injective hull $E(M)$.*

Proof. This follows from [2, Proposition 4.3(i),(iii)]. □

Clearly every almost injective module is Rad-s-injective. We begin with an example which shows a Rad-s-injective module which is not almost injective. Then we investigate the structure of Rad-s-injective modules over some type of rings. We show that the Rad-s-injective property always transfer from a module to each of its direct summands. It is shown that the class of Rad-s-injective modules is closed under finite direct sums, but it is not closed under factor modules. We prove that over a left hereditary ring R, an R-module M is Rad-s-injective if and only if $M/\mathrm{Rad}(M)$ is injective. Another characterization of Rad-s-injective modules over semilocal rings is provided.

2. Results

Let M be an R-module with $\mathrm{Rad}(M) = M$. Note that M is a Rad-supplement of $E(M)$ in $E(M)$. So, M is a Rad-s-injective module.

It is obvious that every almost injective (in particular, every injective) module is Rad-s-injective. The following example shows that there are Rad-s-injective modules which are not almost injective.

Example 2.1. Let M be a module such that $\mathrm{Rad}(M) = M$ and $M \ll E(M)$. Then M is a Rad-s-injective module, but is not an almost injective module. To construct a module satisfying these conditions, let F be a field and let R be the ring of polynomials in countably many commuting variables $x_1, x_2, \ldots$, over F subject to the relations $x_1^2 = 0$ and $x_n^2 = x_{n-1}$ for $n \geqslant 2$. This ring appears in [13] in another context. The ring R is local with maximal ideal J generated by the x_i. Moreover, we have $J^2 = J$. Therefore $\mathrm{Rad}({}_RJ) = {}_RJ \ll {}_RR$. It follows that the R-module ${}_RJ$ is Rad-s-injective, while ${}_RJ$ is not almost injective.

In the following result, we provide a condition under which a Rad-s-injective module is almost injective.

Proposition 2.2. *Let M be an R-module with $Rad(M) \ll M$. Then M is* Rad-*s-injective if and only if M is an almost injective module.*

Proof. Assume that M is Rad-s-injective. Then there exists a submodule K of $E(M)$ such that $K + M = E(M)$ and $K \cap M \subseteq \mathrm{Rad}(M)$. Since $\mathrm{Rad}(M) \ll M$, we have $K \cap M \ll M$. It follows that M is a supplement of K in $E(M)$. Therefore M is almost injective. The converse is immediate. □

Corollary 2.3. *Let M be a module over a left perfect ring. Then M is* Rad-*s-injective if and only if M is almost injective.*

Proof. This follows from Proposition 2.2 and [1, Remark 28.5(3)]. □

Let R be a commutative ring. Then an element $a \in R$ is said to be a *zero-divisor* in case there is an element $b \neq 0$ in R with $ba = 0$.

The next result shows that [4, Proposition 3.16] still holds if it is assumed that R is a commutative ring.

Proposition 2.4. *Let R be a commutative ring. Assume that there exists an index set I such that the R-module $R^{(\mathbf{I})}$ is* Rad-*s-injective and $Rad(R^{(\mathbf{I})}) \ll R^{(\mathbf{I})}$. Then, for any $x \in R$, x is invertible if and only if x is not a zero-divisor in R.*

Proof. Let E be the injective hull of $R^{(\mathbf{I})}$. By Proposition 2.2, there exists a submodule M of E such that $R^{(\mathbf{I})}$ is a supplement of M in E. Let r be an element of R which is not a right zero-divisor. Then $rM + rR^{(\mathbf{I})} = rE$. By [12, Proposition 2.6], we have $rE = E$. Thus $M + (rR)^{(\mathbf{I})} = E$. By the minimality of $R^{(\mathbf{I})}$, we have $(rR)^{(\mathbf{I})} = R^{(\mathbf{I})}$. So $rR = R$. This implies that r is invertible. The converse is obvious. □

Corollary 2.5. *Let R be a commutative domain. Then the following conditions are equivalent:*

(i) $_RR$ *is a* Rad-*s-injective module;*

(ii) $_RR$ *is an almost injective module;*

(iii) R *is a division ring.*

Proof. This follows from Propositions 2.2 and 2.4. □

Next, we investigate the structure of Rad-s-injective modules over some type of rings. We first state the following lemma.

Lemma 2.6 (see [2, Corollary 4.2]). *Let N be a submodule of an R-module M. If N is a* Rad-*supplement in M, then $Rad(N) = N \cap Rad(M)$.*

Recall that a ring R is called *left small* if the R-module $_RR$ is small in its injective hull $E(_RR)$. By [9, Proposition 2.4], a ring R is left small if and only if $\mathrm{Rad}(E) = E$ for any injective R-module E.

Proposition 2.7. *The following statements are equivalent for a ring R:*

(i) *R is a left small ring;*

(ii) *For any R-module M, M is* Rad-*s-injective if and only if $Rad(M) = M$.*

Proof. (i) ⇒ (ii) Let M be a Rad-s-injective R-module. Then M is a Rad-supplement in its injective hull $E(M)$. Since R is left small, $\mathrm{Rad}(E(M)) = E(M)$ by [9, Proposition 2.4]. So $\mathrm{Rad}(M) = M \cap \mathrm{Rad}(E(M)) = M \cap E(M) = M$ by Lemma 2.6. This is the desired conclusion.

(ii) ⇒ (i) This follows from [9, Proposition 2.4] and the fact that every injective module is Rad-s-injective. □

Combining Proposition 2.7 and [8, Theorem 2], we obtain the following corollaries.

Corollary 2.8. *Let R be a commutative domain which is not a field. If M is an R-module, then M is* Rad-*s-injective if and only if $Rad(M) = M$.*

Corollary 2.9. *Let M be a module over a Dedekind domain R. Then M is* Rad-*s-injective if and only if M is injective.*

A ring R is called a *left max* (or *left Bass*) ring if $Rad(M) \neq M$ for every nonzero left R-module M. It is well known that a ring R is left max if and only if $Rad(M) \ll M$ for every nonzero left R-module M. From Proposition 2.2, it follows that for a module M over a left max ring, M is Rad-s-injective if and only if M is almost injective.

Proposition 2.10. *The following conditions are equivalent for a ring R:*

(i) *R is a left max ring;*

(ii) *$Rad(M) \neq M$ for any* Rad-*s-injective R-module M.*

Proof. This follows from the fact that every R-module M with $\mathrm{Rad}(M) = M$, is Rad-s-injective. □

A ring R is called a *left Harada ring* if R is left artinian and every nonsmall R-module contains a nonzero injective direct summand (see, for example, [5, 28.10]). Note that quasi-Frobenius rings and artinian serial rings are left Harada rings. Recall that a module M is called *small* if $M \ll L$ for some R-module L. By [9, Proposition 2.1], a module M is a small module if and only if $M \ll E(M)$.

Proposition 2.11. *Let R be a left Harada ring. Then every* Rad-*s-injective R-module is injective.*

Proof. Note that R is a perfect ring by [5, 28.10]. Let M be a Rad-s-injective R-module. Therefore M is almost injective by Corollary 2.3. Using again [5, 28.10], $M = N \oplus K$ such that N is an injective module and K is a small module. But K is almost injective by [4, Corollary 3.3]. Then $K = 0$ and M is an injective module. This completes the proof. □

Corollary 2.12. *The following conditions are equivalent for a ring R:*

(i) *R is a quasi-Frobenius ring;*

(ii) *Every* Rad-*s-injective R-module M is projective.*

Proof. This follows from Proposition 2.11 and [1, Theorem 31.9]. □

It is of natural interest to investigate if or not the class of Rad-s-injective modules is closed under submodules, direct summands, direct sums and factor modules. The next example shows that the Rad-s-injective property is not inherited by submodules, in general.

Example 2.13. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$ and the submodule $N = \mathbb{Z}$ of M. Note that M is an injective module, $N \ll M$ and $Rad(N) = 0$. Then M is Rad-s-injective, but N is not a Rad-s-injective module.

Proposition 2.14. *Every* Rad-*supplement submodule of a* Rad-*s-injective module is* Rad-*s-injective.*

Proof. Let M be a Rad-s-injective module and let K be a Rad-supplement in M. Since M is a Rad-supplement in $E(M)$, K is a Rad-supplement in $E(M)$ by [2, Proposition 4.3(iii)]. The result follows from Proposition 1.1. □

From the last result, it follows that a direct summand of a Rad-s-injective module is Rad-s-injective. The next example exhibits that the direct sum of Rad-s-injective modules does not always inherit the property.

Example 2.15. Let F be a field and let the commutative ring R be the direct product $\prod_{n=1}^{\infty} F_n$, where $F_n = F$ for each $n \in \mathbb{N}$. Then R is a von Neumann regular ring. It is easy to see that the ideal $I = \bigoplus_{n=1}^{\infty} F_n$ is not a direct summand of ${}_R R$. Since $\mathrm{Rad}({}_R I) = 0$, the R-module ${}_R I$ is not a Rad-supplement submodule of ${}_R R$. Therefore ${}_R I$ is not a Rad-s-injective module. On the other hand, it is clear that the R-module ${}_R F_n$ is injective for each $n \in \mathbb{N}$.

Next, in order to show that a finite direct sum of Rad-s-injective modules is also Rad-s-injective, we need the following lemma.

Lemma 2.16 (see [6, Lemma 2.13]). *Let X, Y and Z be submodules of a module M such that $M = X + Y + Z$. If X is a* Rad-*supplement of $Y + Z$ in M and Y is a* Rad-*supplement of $X + Z$ in M, then $X + Y$ is a* Rad-*supplement of Z in M.*

Proof. By assumption, we have $X \cap (Y+Z) \subseteq Rad(X)$ and $Y \cap (X+Z) \subseteq Rad(Y)$. Since $Z \cap (X+Y) \subseteq [X \cap (Y+Z)] + [Y \cap (X+Z)]$, we have $Z \cap (X+Y) \subseteq Rad(X) + Rad(Y)$. Thus $Z \cap (X+Y) \subseteq Rad(X+Y)$. This proves the lemma. □

Proposition 2.17. *A finite direct sum $M_1 \oplus \cdots \oplus M_n$ is* Rad-*s-injective if and only if M_i is* Rad-*s-injective for each $i = 1, \ldots, n$.*

Proof. Without loss of generality we can assume that $n = 2$. Let $M = M_1 \oplus M_2$. Then $E(M) = E(M_1) \oplus E(M_2)$. Assume that M_1 and M_2 are Rad-s-injective. Therefore there exist submodules $N_1 \leqslant E(M_1)$ and $N_2 \leqslant E(M_2)$ such that M_i is a Rad-supplement of N_i in $E(M_i)$ ($i = 1, 2$). Note that $M_1 \cap (N_1 + N_2 + M_2) = M_1 \cap N_1 \subseteq Rad(M_1)$ and $M_2 \cap (N_1 + M_1 + N_2) = M_2 \cap N_2 \subseteq Rad(M_2)$. Then M_1 is a Rad-supplement of $N_1 + N_2 + M_2$ in $E(M)$ and M_2 is a Rad-supplement of $N_1 + M_1 + N_2$ in $E(M)$. Therefore $M_1 + M_2$ is a Rad-supplement of $N_1 + N_2$ in $E(M)$ by Lemma 2.16. Consequently, $M_1 \oplus M_2$ is a Rad-s-injective module. The converse follows from Proposition 2.14. □

The next result will be of interest.

Proposition 2.18. *The following statements are equivalent for an R-module M:*

(i) M *is* Rad-*s-injective;*

(ii) $M/Rad(M)$ *is a direct summand of* $E(M)/Rad(M)$.

Proof. (i) $\Rightarrow$ (ii) By assumption, there exists a submodule N of $E(M)$ such that $M + N = E(M)$ and $M \cap N \subseteq \text{Rad}(M)$. Therefore,

$$[M/\text{Rad}(M)] + [(N + \text{Rad}(M))/\text{Rad}(M)] = E(M)/\text{Rad}(M)$$

and

$$M \cap (N + \text{Rad}(M)) = \text{Rad}(M) + (M \cap N) = \text{Rad}(M).$$

Thus,

$$[M/\text{Rad}(M)] \oplus [(N + \text{Rad}(M))/\text{Rad}(M)] = E(M)/\text{Rad}(M).$$

This is our claim.

(ii) $\Rightarrow$ (i) Let N be a submodule of $E(M)$ such that $\text{Rad}(M) \subseteq N$ and $(M/\text{Rad}(M)) \oplus (N/\text{Rad}(M)) = E(M)/\text{Rad}(M)$. Therefore $M + N = E(M)$ and $M \cap N \subseteq \text{Rad}(M)$, i.e. M is a Rad-supplement of N in $E(M)$. This completes the proof. □

The next result is a direct consequence of Proposition 2.18.

Corollary 2.19. (i) *Let M be an R-module with $Rad(M) = 0$. Then M is* Rad-*s-injective if and only if M is an injective module.*

(ii) *A semisimple R-module M is* Rad-*s-injective if and only if M is an injective module.*

(iii) *Let M be an R-module such that $M/Rad(M)$ is injective. Then M is* Rad-*s-injective.*

Recall that a ring R is called a *left V-ring* if every simple left R-module is injective or, equivalently, $Rad(M) = 0$ for every left R-module M. The next result is a direct consequence of Corollary 2.19(i).

Corollary 2.20. *Let R be a left V-ring. For any R-module M, M is* Rad-*s-injective if and only if M is injective.*

Proposition 2.21. *The following are equivalent for a ring R:*

(i) *Every R-module is* Rad-*s-injective;*

(ii) *Every factor module of ${}_RR$ is* Rad-*s-injective;*

(iii) *The ring R is semisimple.*

Proof. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are immediate.

(ii) $\Rightarrow$ (iii) By hypothesis, every simple R-module is Rad-s-injective. Therefore every simple R-module is injective (Corollary 2.19(ii)). This implies that R is a left V-ring. So $Rad(M) = 0$ for any R-module M. By (ii) and Corollary 2.19(i), it follows that every cyclic R-module is injective. Hence R is semisimple by [11, Theorem on p. 649]. □

The next example shows that the class of Rad-s-injective modules is not always closed under factor modules.

Example 2.22. Let R be a ring such that ${}_RR$ is a Rad-s-injective module, but R is not semisimple. For example, we can take a perfect local ring R which is not semisimple by [4, Example 2.3]). Then the module ${}_RR$ has a factor module which is not Rad-s-injective by Proposition 2.21. Note that some examples of this type of rings are cited in [4, Examples 2.7].

Proposition 2.23. *Let L be a submodule of a module M such that $L \subseteq Rad(M)$. Assume that M/L is* Rad-*s-injective. Then M is* Rad-*s-injective.*

Proof. By assumption, there exists a submodule N of $E(M)$ such that $L \subseteq N$ and M/L is a Rad-supplement of N/L in $E(M)/L$. Then

$$(M \cap N)/L = (M/L) \cap (N/L) \subseteq \text{Rad}(M/L) = \text{Rad}(M)/L.$$

Therefore $M \cap N \subseteq \text{Rad}(M)$. Since $M + N = E(M)$, it follows that M is a Rad-s-injective module. □

The following example illustrates that the condition "$L \subseteq Rad(M)$"in the hypothesis of Proposition 2.23 is not superfluous.

Example 2.24. It is well known that the $\mathbb{Z}$-module $\mathbb{Q} \cong \mathbb{Z}^{(I)}/L$ for some index set I and a submodule L of the $\mathbb{Z}$-module $\mathbb{Z}^{(I)}$. Since $\mathbb{Q}$ is injective, $\mathbb{Z}^{(I)}/L$ is a Rad-s-injective $\mathbb{Z}$-module. On the other hand, $\mathbb{Z}^{(I)}$ is not Rad-s-injective, since otherwise the $\mathbb{Z}$-module ${}_\mathbb{Z}\mathbb{Z}$ will be Rad-s-injective by Proposition 2.14. Then ${}_\mathbb{Z}\mathbb{Z}$ will be injective as $\text{Rad}({}_\mathbb{Z}\mathbb{Z}) = 0$ (see Corollary 2.19(i)), a contradiction.

Proposition 2.25. *Let M_1 and M_2 be two submodules of a module M such that $M = M_1 + M_2$ and $M_1 \cap M_2 \subseteq Rad(M)$. Suppose that every factor module of M_i is* Rad-*s-injective for each $i = 1, 2$. Then M is a* Rad-*s-injective module.*

Proof. Note that $M/M_1 \cap M_2 = (M_1/M_1 \cap M_2) \oplus (M_2/M_1 \cap M_2)$. By hypothesis, $M_1/M_1 \cap M_2$ and $M_2/M_1 \cap M_2$ are Rad-s-injective. Thus $M/M_1 \cap M_2$ is Rad-s-injective by Proposition 2.17. The result follows from Proposition 2.23. □

Recall that a ring R is said to be *left hereditary* if every left ideal of R is a projective R-module. The next result characterizes Rad-s-injective modules over left hereditary rings.

Theorem 2.26. *Let R be a left hereditary ring. Then the following conditions are equivalent for an R-module M:*

(i) *M is a* Rad-*s-injective R-module;*

(ii) *Every factor module of M is a* Rad-*s-injective R-module;*

(iii) *$M/Rad(M)$ is an injective R-module.*

Proof. (i) ⇒ (ii) Let N be a submodule of M. By hypothesis, there exists a submodule L of $E(M)$ such that $L + M = E(M)$ and $L \cap M \subseteq \mathrm{Rad}(M)$. Therefore $((L + N)/N) + (M/N) = E(M)/N$. Moreover, by [1, Proposition 9.14], we have

$$\begin{aligned}&((L + N)/N) \cap (M/N)\\ &\quad = [N + (L \cap M)]/N \subseteq [N + \mathrm{Rad}(M)]/N \subseteq \mathrm{Rad}(M/N).\end{aligned}$$

This implies that M/N is a Rad-supplement of $(L + N)/N$ in $E(M)/N$. Since R is left hereditary, $E(M)/N$ is an injective R-module by [14, 39.16]. Thus M/N is a Rad-s-injective R-module by Proposition 1.1.

(ii) ⇒ (iii) This follows from Corollary 2.19(i) and [1, Proposition 9.15].

(iii) ⇒ (i) This follows from Corollary 2.19(iii). □

Matlis showed that a ring R is left hereditary if and only if the sum of two injective submodules of any left R-module is injective (see, for example, [10, Exercise 10 on p. 114] or [12, Exercise 2.11]). Next, we examine the Rad-s-injective analogue of this characterization.

Proposition 2.27. *If R is a left hereditary ring, then the sum of two* Rad-*s-injective submodules of any R-module is* Rad-*s-injective.*

Proof. Assume that R is a left hereditary ring. Let $M = M_1 + M_2$ such that M_1 and M_2 are Rad-s-injective. Then

$$M/\mathrm{Rad}(M) = [(M_1 + \mathrm{Rad}(M))/\mathrm{Rad}(M)] + [(M_2 + \mathrm{Rad}(M))/\mathrm{Rad}(M)].$$

Let $i \in \{1, 2\}$. Note that

$$(M_i + \mathrm{Rad}(M))/\mathrm{Rad}(M) \cong M_i/(M_i \cap \mathrm{Rad}(M)).$$

Since $\mathrm{Rad}(M_i) \subseteq M_i \cap \mathrm{Rad}(M)$, $M_i/(M_i \cap \mathrm{Rad}(M))$ is a factor module of $M_i/\mathrm{Rad}(M_i)$. Since M_i is Rad-s-injective, $M_i/\mathrm{Rad}(M_i)$ is injective by Theorem 2.26. Therefore $M_i/(M_i \cap \mathrm{Rad}(M))$ is also injective as R is left hereditary. It follows that $M/\mathrm{Rad}(M)$ is an injective module. Applying again Theorem 2.26, we conclude that M is a Rad-s-injective module. □

It is shown in [10, Theorem (Bass, Papp) 3.46] that the left noetherian rings are exactly the rings over which every direct sum of injective modules is injective (see also [1, Proposition 18.13]). One may ask whether this is still true if we replace "injective"with "Rad-s-injective"in this result. Next, we investigate this question.

Proposition 2.28. *If R is a left noetherian ring, then every direct sum of* Rad*-s-injective R-modules is* Rad*-s-injective.*

Proof. Assume that R is a left noetherian ring. Let $(M_i)_{i \in I}$ be an indexed set of submodules of an R-module M such that $M = \bigoplus_{i \in I} M_i$ and M_i is a Rad-s-injective module for each $i \in I$. Then $E(M) = \bigoplus_{i \in I} E(M_i)$ and $\mathrm{Rad}(M) = \bigoplus_{i \in I} \mathrm{Rad}(M_i)$ by [1, Propositions 9.19 and 18.13]. So there exists an isomorphism $\psi : E(M)/\mathrm{Rad}(M) \to \bigoplus_{i \in I}(E(M_i)/\mathrm{Rad}(M_i))$ such that $\psi(M/\mathrm{Rad}(M)) = \bigoplus_{i \in I}(M_i/\mathrm{Rad}(M_i))$. By Proposition 2.18, $M_i/\mathrm{Rad}(M_i)$ is a direct summand of $E(M_i)/\mathrm{Rad}(M_i)$ for each $i \in I$. It follows that $M/\mathrm{Rad}(M)$ is a direct summand of $E(M)/\mathrm{Rad}(M)$. Again by Proposition 2.18, M is a Rad-s-injective module. This proves the proposition. □

The following example shows that both the converses of Propositions 2.27 and 2.28 are not true, in general.

Example 2.29. Let R be a commutative domain which is not noetherian (e.g., we can take the polynomial ring $R = F[X_1, \ldots, X_n, \ldots]$ in infinitely many indeterminates over a field F or we can find other examples in [7, Examples 1.13 and 1.14 on p. 8]). Thus R is not a Dedekind domain. So the ring R is not hereditary. By [8, Theorem 2], the R-module ${}_RR$ is small in its injective hull $E({}_RR)$. Therefore the class of Rad-s-injective R-modules is exactly the class of modules M with $\mathrm{Rad}(M) = M$ (see Proposition 2.7). It follows easily that the sum of two Rad-s-injective submodules of any R-module is Rad-s-injective. Also, note that the factor modules of any

Rad-s-injective R-module are Rad-s-injective and every direct sum of Rad-s-injective modules is Rad-s-injective (see [1, Proposition 9.19]).

Remark 2.30. Example 2.29 illustrates that if R is a ring such that every factor module of a Rad-s-injective module is Rad-s-injective, then the ring R need not be left hereditary. This should be contrasted with [10, Theorem 3.22].

Recall that a ring R is called *semilocal* if $R/\mathrm{Rad}(R)$ is semisimple. Recall that if M is a module over a semilocal ring R with $J = \mathrm{Rad}(R)$, then $\mathrm{Rad}(M) = JM$ and the module M/JM is semisimple. In the next proposition, we provide a characterization of Rad-s-injective modules over semilocal rings.

Proposition 2.31. *Let R be a semilocal ring with Jacobson radical J and let M be an R-module. Then M is* Rad*-s-injective if and only if $JM = M \cap JE(M)$.*

Proof. The necessity follows from Lemma 2.6 and [1, Corollary 15.18]. Conversely, note that the module $E(M)/\mathrm{Rad}(E(M))$ is semisimple as R is a semilocal ring. Therefore $(M + \mathrm{Rad}(E(M)))/\mathrm{Rad}(E(M))$ is a direct summand of $E(M)/\mathrm{Rad}(E(M))$. Let K be a submodule of $E(M)$ such that $\mathrm{Rad}(E(M)) \subseteq K$ and $[(M + \mathrm{Rad}(E(M)))/\mathrm{Rad}(E(M))] \oplus [K/\mathrm{Rad}(E(M))] = E(M)/\mathrm{Rad}(E(M))$. So $M+K = E(M)$ and $M\cap K \subseteq \mathrm{Rad}(E(M))$. Moreover, we have $M \cap K \subseteq \mathrm{Rad}(M)$ since $\mathrm{Rad}(M) = M \cap \mathrm{Rad}(E(M))$. Thus M is a Rad-supplement of K in $E(M)$. Hence M is Rad-s-injective. □

Recall that if N is a submodule of a module M, then N is said to be a *pure submodule* of M if $IN = N \cap IM$ for each right ideal I of R (see [1, p. 232]).

Corollary 2.32. *Let R be a semilocal ring and E be an injective R-module. Then every pure submodule of E is a* Rad*-s-injective module.*

Proof. Let J be the Jacobson radical of R and let M be a pure submodule of E. Then $JM = M \cap JE$. This gives $JM = M \cap JE(M)$. Thus M is a Rad-s-injective module by Proposition 2.31. □

A module M is said to be *w-local* if it has a unique maximal submodule (see [3]). Clearly, every local module is w-local. The next result characterizes w-local Rad-s-injective modules.

Proposition 2.33. *Let M be a w-local module with maximal submodule K. Then the following conditions are equivalent:*

(i) *M is a* Rad-*s-injective module;*

(ii) *$M \not\subseteq Rad(E(M))$.*

Proof. Note that if X is a submodule of $E(M)$ such that $M+X=E(M)$, then $M/(X \cap M) \cong E(M)/X$. Using this fact, we see that
"M is a Rad-s-injective module."

$\Updownarrow$

"There is a submodule N of $E(M)$ such that $M+N=E(M)$ and $M \cap N \subseteq K$."

$\Updownarrow$

"$E(M)$ has a maximal submodule L such that $M+L=E(M)$."

$\Updownarrow$

"$E(M)$ has a maximal submodule L such that $M \not\subseteq L$."

$\Updownarrow$

"$M \not\subseteq Rad(E(M))$." □

Corollary 2.34. *Let $M=\oplus_{i=1}^n M_i$ such that each M_i is a w-local submodule of M. Then M is* Rad-*s-injective if and only if $M_i \not\subseteq Rad(E(M))$ for each $i=1,\ldots,n$.*

Proof. This follows from Lemma 2.6 and Propositions 2.17 and 2.33. □

The next corollary is a direct consequence of Proposition 2.33.

Corollary 2.35. *Let R be a left small ring (e.g. R is a Dedekind domain which is not a field). Then R has no w-local* Rad-*s-injective modules.*

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