Theoretical and Mathematical Physics, **152**(1): 991–1003 (2007)

SOLITONS OF THE RESONANT NONLINEAR SCHRÖDINGER EQUATION WITH NONTRIVIAL BOUNDARY CONDITIONS: HIROTA BILINEAR METHOD

J.-H. Lee* **and O. K. Pashaev†**

We use the Hirota bilinear approach to consider physically relevant soliton solutions of the resonant nonlinear Schrödinger equation with nontrivial boundary conditions, recently proposed for describing uniaxial waves in a cold collisionless plasma. By the Madelung representation, the model transforms into the reaction–diffusion analogue of the nonlinear Schrödinger equation, for which we study the bilinear representation, the soliton solutions, and their mutual interactions.

Keywords: resonant nonlinear Schrödinger equation, quantum potential, cold plasma, magnetoacoustic wave, soliton, Hirota method

1. Introduction

For describing low-dimensional gravity (the Jackiw–Teitelboim model) and the response of a medium to the action of a quasimonochromatic wave with a complex amplitude $\psi(x,t)$, which is a slowly varying function of the coordinate and the time, a novel integrable version of the nonlinear Schrödinger (NLS) equation was recently proposed [1]:

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\frac{\Lambda}{4}|\psi|^2\psi=s\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial x^2}\psi. \tag{1}$$

This has been called the *resonant NLS* (RNLS) equation. It can be considered a third version of the NLS equation, intermediate between the defocusing and focusing cases. Although the RNLS model is integrable for arbitrary values of the coefficient s, the critical value $s=1$ separates two distinct regions of behavior: for $s<1$, the model is reducible to the conventional NLS equation, but for $s>1$, it is reducible not to the usual NLS equation but to a reaction–diffusion (RD) system. In the latter case, the model exhibits resonance solitonic phenomena [1].

The RNLS equation can be interpreted as a particular realization of the NLS soliton propagating in the so-called quantum potential $U_{\mathrm{Q}}(x)=|\psi|_{xx}/|\psi|$. This potential, responsible for producing the quantum behavior, was introduced by de Broglie [2] and was subsequently used by Bohm [3] to develop a hidden-variable theory in quantum mechanics. It also appears in stochastic mechanics [4]. Connections between such nonclassical motions with the *internal* spin motion and the *zitterbewegung* were considered in a series of papers (see [5]). Quantum potentials also appear in proposed nonlinear extensions of quantum mechanics with regard both to stochastic quantization [6], [7] and to corrections from quantum gravity [8]. It is noted that the RNLS equation, like the conventional NLS equation, can also be derived in the context of capillarity models [9], [10].

*Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan.

†Department of Mathematics, Izmir Institute of Technology, Urla-Izmir 35430, Turkey, e-mail: oktaypashaev@iyte.edu.tr.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 152, No. 1, pp. 133–146, July, 2007.

0040-5779/07/1521-0991

It was very recently shown [11] that the RNLS equation appears in plasma physics, where it describes the propagation of one-dimensional long magnetoacoustic waves in a cold collisionless plasma subject to a transverse magnetic field. The complex wave function satisfying the RNLS equation is a combination of plasma density and the velocity fields as in the Madelung representation. The Bäcklund–Darboux transformations along with a novel associated nonlinear superposition principle were presented and used to generate solutions describing the interaction of solitonic magnetoacoustic waves. This application requires considering a solution of the RNLS equation with nontrivial boundary conditions at infinity. Our goal here is to derive such solutions in the Hirota bilinear approach and to study their mutual interactions.

2. Magnetoacoustic waves in cold plasma

The dynamics of a two-component cold collisionless plasma in the presence of an external magnetic field $\mathbf{B}$ [12], [13] for uniaxial plasma propagation

$$\mathbf{u} = u(x,t)\mathbf{e}_x, \qquad \mathbf{B} = B(x,t)\mathbf{e}_z \tag{2}$$

reduces to the form [14]

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{B}{\rho}\frac{\partial B}{\partial x} = 0, \\ &\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right) = B - \rho \end{aligned} \tag{3}$$

(where we set $B = 1$ and the plasma density $\rho = 1$ at infinity). This system is equivalent to the Whitham system and was also derived by Gurevich and Meshcherkin [15]. It describes the propagation of nonlinear magnetoacoustic waves in a cold plasma with a transverse magnetic field. El, Khodorovskii, and Tyurina recently showed [16] that a system of type (3) also occurs in the context of hypersonic flow past slender bodies.

3. A shallow-water approximation

Here, we consider a shallow-water approximation of magnetoacoustic system (3). Rescaling the space and time variables via $x' = \beta x$ and $t' = \beta t$, we have

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) = 0, \tag{4}$$

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} + \frac{B}{\rho}\frac{\partial B}{\partial x'} = 0, \tag{5}$$

$$\beta^2\frac{\partial}{\partial x'}\left(\frac{1}{\rho}\frac{\partial B}{\partial x'}\right) = B - \rho. \tag{6}$$

Expanding B as a power series in the parameter β^2 according to

$$B = \rho + \beta^2 b_2(\rho, \rho_{x'}, \rho_{x'x'}, \dots) + O(\beta^4) \tag{7}$$

and substituting it in (6) yields

$$b_2 = \frac{\partial}{\partial x'}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x'}\right). \tag{8}$$

Substituting (7) in (5) yields

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} + \frac{\partial \rho}{\partial x'} + \beta^2\left[\frac{1}{\rho}\frac{\partial^3\rho}{\partial x'^3} - \frac{2}{\rho^2}\frac{\partial\rho}{\partial x'}\frac{\partial^2\rho}{\partial x'^2} + \left(\frac{1}{\rho}\frac{\partial\rho}{\partial x'}\right)^3\right] = 0 \tag{9}$$

up to $O(\beta^2)$. Accordingly, we obtain the system

$$\frac{\partial\rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) = 0, \tag{10}$$

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x'} + \frac{\partial\rho}{\partial x'} + \beta^2\frac{\partial}{\partial x'}\left[\frac{1}{\rho}\frac{\partial^2\rho}{\partial x'^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x'}\right)^2\right] = 0. \tag{11}$$

This describes the propagation of long magnetoacoustic waves in a cold plasma of the density ρ moving with the velocity u across the magnetic field as given by (2) and (7) (see [17], [18]). The dispersion is negative in this system, i.e., the wave velocity decreases as the wave vector k increases.

4. Resonant nonlinear Schrödinger equation

Introducing the velocity potential

$$S(x,t) = -\frac{1}{2}\int^x u(x',t)\,dx'$$

such that $u = -2\partial S/\partial x$ and integrating Eq. (11) once, we obtain the system

$$\frac{\partial\rho}{\partial t} - 2\frac{\partial}{\partial x}\left(\rho\frac{\partial S}{\partial x}\right) = 0, \tag{12}$$

$$-\frac{\partial S}{\partial t} + \left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}\rho + \frac{\beta^2}{2}\left[\frac{1}{\rho}\frac{\partial^2\rho}{\partial x^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x}\right)^2\right] = 0 \tag{13}$$

(where we omit the primes). Combining ρ and S into one complex function

$$\psi = \sqrt{\rho}e^{-iS}, \tag{14}$$

we can represent system (12), (13) as the NLS equation with a quantum potential

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}|\psi|^2\psi = (1+\beta^2)\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial x^2}\psi, \tag{15}$$

i.e., as RNLS (1), where $\Lambda = -2$, $s = 1+\beta^2$, and the quantum potential is expressed as

$$\frac{1}{\rho}\frac{\partial^2\rho}{\partial x^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x}\right)^2 = 2\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}. \tag{16}$$

Because the parameter $s > 1$, Eq. (15) cannot be transformed into the NLS equation [1]. But by combining ρ and S into a couple of real functions

$$e^{(+)} = \sqrt{\rho}\,e^{S/\beta}, \qquad e^{(-)} = -\sqrt{\rho}\,e^{-S/\beta} \tag{17}$$

such that $e^{(+)} > 0$ and $e^{(-)} < 0$, we can write Eqs. (12) and (13) as the RD system

$$\mp \frac{\partial e^{(\pm)}}{\partial \tau} + \frac{\partial^2 e^{(\pm)}}{\partial x'^2} - \frac{1}{2\beta^2} e^{(+)} e^{(-)} e^{(\pm)} = 0, \tag{18}$$

where $\tau = \beta t'$.

Linearizing (15) near the "condensate" solution $\psi = \sqrt{\rho_0} e^{-i\rho_0 t/2}$ results in the dispersion $\omega = \sqrt{\rho_0}\, k \sqrt{1 - \beta^2 k^2/\rho_0}$. This dispersion is negative (i.e., the wave velocity decreases as the wave vector k increases) and is unstable for short waves with $k > k_{\text{cr}}$, where $k_{\text{cr}} = \sqrt{\rho_0}/\beta$. But this instability results from truncating the dispersion relation

$$\omega^2 = \rho_0 \frac{k^2}{1 + \beta^2 k^2/\rho_0} \tag{19}$$

for system (4)–(6) and does not correspond to any actual physical effect for shallow water waves [19]. Although this system is linearly stable for all wave numbers k, it is not known to be integrable. Therefore, it is not as suitable for studying wave interactions as the RNLS, which is completely integrable and admits a rich variety of exact solutions.

5. Steady-state flow and solitons

We now consider system (10), (11) for the steady-state flow. It describes motion with the fixed velocity $u(x,t) = u_0 = \text{const}$ for which continuity equation (10) implies that $\partial\rho/\partial t + u_0\, \partial\rho/\partial x = 0$ or that the fluid density has the traveling-wave form $\rho = \rho(x - u_0 t)$, where $x' = \beta x$ and $t' = \beta t$. Equation (11) in this case gives

$$\rho + \left[\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)^2\right] = a = \text{const}\,. \tag{20}$$

This has a simple physical interpretation. Motion with a fixed velocity implies that the sum of all forces acting on the system is zero. In our case, Eq. (20) describes compensation of the nonlinearity by the quantum potential. With (16) taken into account, this equation gives

$$\rho + 2\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2} = a. \tag{21}$$

In terms of $y(z) = \sqrt{\rho}$, $z = x - u_0 t$, we then obtain the nonlinear equation

$$\frac{d^2 y}{dz^2} - \frac{a}{2} y + \frac{1}{2} y^3 = 0 \tag{22}$$

or, multiplying by y' and integrating once,

$$\left(\frac{dy}{dz}\right)^2 - \frac{a}{2} y^2 + \frac{1}{4} y^4 = 2b = \text{const}\,. \tag{23}$$

This equation has the solution

$$y = 2p\, \text{dn}[p(x - u_0 t), \kappa], \tag{24}$$

where dn is the Jacobi elliptic function with the modulus κ and p is an arbitrary constant. It is related to the integration constants $a > 0$ and $b < 0$ by the equations

$$\frac{a}{2p^2} = 1 + \kappa'^2, \qquad \frac{b}{2p^4} = -\kappa'^2, \tag{25}$$

where $\kappa' = \sqrt{1-\kappa^2}$ is the complimentary modulus of the Jacobi elliptic function. These equations fix the relation between the integration constants

$$a = 2p^2\left(1 - \frac{b}{2p^4}\right) \tag{26}$$

and the modulus $\kappa = \sqrt{1 + b/(2p^4)}$. For the density ρ, we then have the traveling-wave solution

$$\rho(x,t) = 4p^2 \,\mathrm{dn}^2[p(x - u_0 t), \kappa]. \tag{27}$$

With fixed potential (21) from Eq. (13) for $u_0 = -(2/\beta)\partial S/\partial x$, we have the Hamilton–Jacobi equation

$$-\frac{1}{\beta}\frac{\partial S}{\partial t} + \frac{1}{\beta^2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{a}{2} = 0, \tag{28}$$

which has a solution in the form

$$S = \beta\left[S_0 - \frac{u_0}{2}x + \left(\frac{u_0^2}{4} + \frac{a}{2}\right)t\right] \tag{29}$$

or, with (26) taken into account,

$$S = \beta\left[S_0 - \frac{u_0}{2}x + \left(\frac{u_0^2}{4} + p^2(2-\kappa^2)\right)t\right]. \tag{30}$$

For the degenerate case $\kappa = 1$, which corresponds to $b = 0$ and hence $a = 2p^2$, elliptic solution (27), (30) becomes the soliton

$$\begin{gathered}\rho(x,t) = 4p^2\,\mathrm{sech}^2[p(x - u_0 t)],\\ S = \beta\left[S_0 - \frac{u_0}{2}x + \left(\frac{u_0^2}{4} + p^2\right)t\right].\end{gathered} \tag{31}$$

A more general form of the traveling wave appears if we consider a solution of system (10), (11) in the form $u(x,t) = u(x - u_0 t)$, $\rho(x,t) = \rho(x - u_0 t)$. Then the first equation in (10) implies that $d[(u-u_0)\rho]/dz = 0$, where $z = x - u_0 t$ and

$$u = u_0 + \frac{C}{\rho} \tag{32}$$

with $C = \text{const}$. Substituting (32) in Eq. (11) and integrating once, we obtain

$$\frac{1}{2}\frac{C^2}{\rho^2} + \rho + \left[\frac{1}{\rho}\frac{d^2\rho}{dz^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{d\rho}{dz}\right)^2\right] = A = \text{const}\,. \tag{33}$$

Multiplying by ρ^2 and differentiating once, we obtain (the traveling-wave form of the KdV equation)

$$\rho\left(\frac{d^3\rho}{dz^3} + 3\rho\frac{d\rho}{dz} - 2A\frac{d\rho}{dz}\right) = 0, \tag{34}$$

which, after one integration, implies that

$$\frac{d^2\rho}{dz^2} + \frac{3}{2}\rho^2 - 2A\rho = B = \text{const}\,. \tag{35}$$

This gives the equation

$$\left(\frac{d\rho}{dz}\right)^2 = -\rho^3 + 2A\rho^2 + 2B\rho + C^2 \tag{36}$$

and the solution

$$\rho(x-u_0t)=\alpha_1+(\alpha_3-\alpha_1)\,\mathrm{dn}^2\left[\frac{1}{2}(\alpha_3-\alpha_1)^{1/2}(x-u_0t),\kappa\right] \tag{37}$$

with the modulus $\kappa^2=(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)$ of the elliptic function, the constants α_1, α_2, and α_3, and

$$u(x-u_0t)=u_0-\frac{(\alpha_1\alpha_2\alpha_3)^{1/2}}{\rho}. \tag{38}$$

Solution (37) was first reported in [18]. In particular cases, we have the following reductions:

1. If $\alpha_1=\alpha_2=0$ and hence $\kappa^2=1$, then (37) reduces to (27) with the parameter values $\alpha_3=4k^2$ and $b=-\alpha_2k^2/2$ and the velocity $u=u_0$.

2. If $\alpha_1=\alpha_2\neq 0$, then again $\kappa^2=1$, but the solution

$$\rho(x-u_0t)=\alpha_1+(\alpha_3-\alpha_1)\,\mathrm{sech}^2\left[\frac{1}{2}(\alpha_3-\alpha_1)^{1/2}(x-u_0t)\right] \tag{39}$$

has a nontrivial asymptotic behavior. The physical value relevant for plasma physics is $\alpha_1=1$, which leads to $\lim_{|x|\to\infty}\rho=1$. Setting $\alpha_3\equiv\sigma^2$, we can write the solution in the form

$$\rho(x-u_0t)=1+\frac{(\sigma^2-1)}{\cosh^2[\sqrt{\sigma^2-1}(x-u_0t)/2]}. \tag{40}$$

Using these results, we can now construct solutions of RNLS (15). Substituting Eqs. (27) and (30) in Eq. (14) and changing the parameter $k\equiv p/\beta$, we obtain the quasiperiodic solution

$$\psi(x',t')=2\beta k\,\mathrm{dn}[k(x'-u_0t'),\kappa]\exp\left\{-i\left[\phi_0-\frac{u_0}{2}x'+\left[\frac{u_0^2}{4}+\beta^2k^2(2-\kappa^2)\right]t'\right]\right\}, \tag{41}$$

where $x'=\beta x$ and $t'=\beta t$. In the limit $\kappa=1$, it gives the envelope soliton solution

$$\psi(x',t')=2\beta k\frac{1}{\cosh k(x'-u_0t')}\exp\left\{-i\left[\phi_0-\frac{u_0}{2}x'+\left[\frac{u_0^2}{4}+\beta^2k^2\right]t'\right]\right\}. \tag{42}$$

For RD system (18), we correspondingly have the *dissipative* periodic solution

$$e^{(\pm)}(x',\tau)=\pm 2\beta k\,\mathrm{dn}[k(x'-v\tau),\kappa]\exp\left\{\pm\left[\phi_0-\frac{v}{2}x'+\left[\frac{v^2}{4}+k^2(2-\kappa^2)\right]\tau\right]\right\}, \tag{43}$$

where the velocity $v\equiv u_0/\beta$, $k\equiv p/\beta$, and the dissipative analogue of the envelope soliton

$$e^{(\pm)}(x',\tau)=\pm 2\beta k\frac{1}{\cosh k(x'-v\tau)}\exp\left\{\pm\left[\phi_0-\frac{v}{2}x'+\left[\frac{v^2}{4}+k^2\right]\tau\right]\right\} \tag{44}$$

is the so-called dissipaton solution [1].

6. Bilinear form and solitons

6.1. Trivial boundary conditions. The key for constructing multisoliton solutions is RD representation (15) of RNLS (18). Because the RD system is algebraically similar to the NLS equation, it is easy to write the bilinear representation for it. Representing the two real functions $e^{(+)}$ and $e^{(-)}$ in terms of three real functions,

$$e^{(\pm)} = 2\beta \frac{G^{\pm}}{F}, \tag{45}$$

we obtain the bilinear system of equations

$$(\pm D_\tau - D_x^2)(G^{(\pm)} \cdot F) = 0, \qquad D_x^2(F \cdot F) = -2G^{(+)}G^{(-)}. \tag{46}$$

The corresponding solution of RNLS (15) is

$$|\psi(x,t)|^2 = \rho = -e^{(+)}e^{(-)} = 2\beta^2 \frac{D_x^2(F\cdot F)}{F^2} = 4\beta^2 \frac{\partial^2 \log F}{\partial x^2}. \tag{47}$$

The one-dissipaton solution of system (46) is given by $G^{\pm} = \pm e^{\eta_1^{\pm}}$, $F = 1 + e^{\eta_1^+ + \eta_1^- + \phi_{1,1}}$, and $e^{\phi_{1,1}} = (k_1^+ + k_1^-)^{-2}$, where $\eta_1^{\pm} \equiv k_1^{\pm}x \pm (k_1^{\pm})^2\tau + \eta_1^{\pm(0)}$ and $k_1^{\pm}$ and $\eta_1^{\pm(0)}$ are constants. In terms of the redefined parameters $k \equiv (k_1^+ + k_1^-)/2$ and $v \equiv -(k_1^+ - k_1^-)$, it takes form (44). In the parameter space (v,k), there exists the critical value $v_{\text{crit}} = 2k$ such that for $v < v_{\text{crit}}$, we have $e^{\pm} \to 0$ at infinity, and the boundary conditions hence vanish for the dissipaton. At the critical value, the solution is a kink steady state in the moving frame $e^{\pm} = \pm k e^{\pm k\xi_0}(1 \mp \tanh k\xi)$ with the constant asymptotic behavior $e^{\pm} \to \pm 2ke^{\pm k\xi_0}$ for $x \to \mp\infty$ and $e^{\pm} \to \pm 0$ for $x \to \pm\infty$. In the supercritical case $v > v_{\text{crit}}$, $e^{\pm} \to \pm\infty$ for $x \to \mp\infty$ and $e^{\pm} \to \pm 0$ for $x \to \pm\infty$.

For the two-dissipaton solution, we have

$$G^{\pm} = \pm\left[e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + \left(\frac{\breve{k}_{12}^{\pm\pm}}{k_{21}^{\pm\mp}k_{11}^{+-}}\right)^2 e^{\eta_1^+ + \eta_1^- + \eta_2^{\pm}} + \left(\frac{\breve{k}_{12}^{\pm\pm}}{k_{12}^{\pm\mp}k_{22}^{+-}}\right)^2 e^{\eta_2^+ + \eta_2^- + \eta_1^{\pm}}\right], \tag{48}$$

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + \left(\frac{\breve{k}_{12}^{++}\breve{k}_{12}^{--}}{k_{12}^{+-}k_{21}^{+-}k_{11}^{+-}k_{22}^{+-}}\right)^2 e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \tag{49}$$

where $k_{ij}^{ab} \equiv k_i^a + k_j^b$, $\breve{k}_{ij}^{ab} \equiv k_i^a - k_j^b$, and $\eta_i^{\pm} \equiv k_i^{\pm}x \pm (k_i^{\pm})^2\tau + \eta^{\pm(0)}$. This solution shows the resonance character of the dissipaton interaction [1].

6.2. Nontrivial boundary conditions. As Hirota first noted, the bilinear form of the equations should be modified for the NLS equation of defocusing type with nonvanishing boundary conditions [20]. After substituting representation (45) in system (18), we choose the decoupling system in the form

$$(\pm D_\tau - D_x^2 + \lambda)(G^{\pm} \cdot F) = 0, \qquad (D_x^2 - \lambda)(F \cdot F) = -2G^+G^-, \tag{50}$$

where we introduce the constant λ to be determined. Equation (45) and the second equation in (50) imply that $-e^{(+)}e^{(-)} = -4\beta^2[\lambda/2 - (\log F)_{xx}]$. Expanding $G^{\pm}$ and F in Hirota's power series,

$$G^{\pm} = \pm g_0^{\pm}(1 + \epsilon g_1^{\pm} + \epsilon^2 g_2^{\pm} + \dots), \qquad F = 1 + \epsilon f_1 + \epsilon^2 f_2 + \dots, \tag{51}$$

and requiring that $\lim_{|x|\to\infty}(\log F)_{xx} = 0$, we obtain the boundary condition

$$\alpha_1 = \lim_{|x|\to\infty}[-e^{(+)}e^{(-)}] = \lim_{|x|\to\infty}\left\{-4\beta^2\left[\frac{\lambda}{2} - (\log F)_{xx}\right]\right\} = -2\beta^2\lambda, \tag{52}$$

which fixes the constant $\lambda = -\alpha_1/(2\beta^2)$. In the zeroth-order approximation, we have the system

$$(\pm D_\tau - D_x^2 + \lambda)(g_0^\pm \cdot 1) = 0, \qquad (D_x^2 - \lambda)(1 \cdot 1) = 2g_0^+ g_0^-. \tag{53}$$

It has a solution of the form $g_0^\pm = \beta^\pm e^{\theta^\pm}$, where $\beta_0^2 \equiv \beta^+\beta^- = -\lambda/2 = \alpha_1/(4\beta^2)$, $\theta^\pm = \pm kx \pm (k^2 - \lambda)t$, and $\beta^\pm = \beta_0 e^{\pm\gamma_0}$. Using the properties of Hirota's derivatives

$$\begin{aligned} D_x(fg \cdot h) &= \frac{\partial f}{\partial x} gh + f D_x(g \cdot h), \\ D_x^2(fg \cdot h) &= \frac{\partial^2 f}{\partial x^2} gh + 2\frac{\partial f}{\partial x} D_x(g \cdot h) + f D_x^2(g \cdot h), \end{aligned} \tag{54}$$

we rewrite the bilinear system as

$$\begin{aligned} &(\mp D_\tau \pm 2kD_x + D_x^2)((1 + \epsilon g_1^\pm + \epsilon^2 g_2^\pm + \cdots) \cdot (1 + \epsilon f_1 + \epsilon^2 f_2 + \cdots)) = 0, \\ &(D_x^2 + 2\beta_0^2)((1 + \epsilon f_1 + \cdots) \cdot (1 + \epsilon f_1 + \cdots)) = 2\beta_0^2(1 + \epsilon g_1^+ + \cdots)(1 + \epsilon g_1^- + \ldots). \end{aligned} \tag{55}$$

6.2.1. One-soliton solution. In the first order, we have the system

$$\begin{aligned} &(\mp\partial_\tau \pm 2k\partial_x + \partial_x^2) g_1^\pm + (\pm\partial_\tau \mp 2k\partial_x + \partial_x^2) f_1 = 0, \\ &(\partial_x^2 + 2\beta_0^2) f_1 = \beta_0^2 (g_1^+ + g_1^-). \end{aligned} \tag{56}$$

Considering a solution of the form $g_1^\pm = a_1^\pm e^{\eta_1}$ and $f_1 = b_1 e^{\eta_1}$, where $\eta_1 = k_1 x + \omega_1 \tau + \eta_1^0$, we obtain $a_1^+ = \gamma_1 b_1$, $a_1^- = b_1/\gamma_1$, and $\gamma_1 = (\omega_1 - 2kk_1 + k_1^2)/(\omega_1 - 2kk_1 - k_1^2)$, where dispersion formula is

$$\omega_1^\pm = k_1\big(2k \pm \sqrt{k_1^2 + 4\beta_0^2}\,\big). \tag{57}$$

We note that in contrast to the defocusing NLS equation, no restrictions on the values of k_1 appear in our case.

Truncating Hirota's expansion at this level gives one dissipative soliton

$$e^{(\pm)} = \pm 2\beta\beta^\pm e^{\pm[kx + (k^2 + 2\beta_0^2)\tau]} \frac{1 + \gamma_1^{\pm 1} e^{\tilde\eta_1}}{1 + e^{\tilde\eta_1}}, \tag{58}$$

where we absorb the constant b_1 in the exponential form $\tilde\eta_1 = k_1 x + \omega_1 \tau + \eta_1^0 + \log b_1$. We then have the one-soliton density

$$\rho = -e^{(+)} e^{(-)} = \alpha_1 \frac{(1 + \gamma_1 e^{\tilde\eta_1})(1 + \gamma_1^{-1} e^{\tilde\eta_1})}{(1 + e^{\tilde\eta_1})^2}. \tag{59}$$

This solution can be represented in the form

$$e^{(\pm)} = \pm\sqrt{\alpha_1}\mu^{\pm 1} e^{\pm[kx + (k^2 + 2\beta_0^2)\tau]} \left(\frac{\gamma^{\pm 1} + 1}{2} + \frac{\gamma^{\pm 1} - 1}{2} \tanh\frac{\tilde\eta}{2}\right) \tag{60}$$

or

$$\begin{aligned} e^{(+)} &= +\sqrt{\alpha_1}\,\frac{\mu}{2} e^{+[kx + (k^2 + 2\beta_0^2)\tau]} \left(\gamma + 1 + (\gamma - 1)\tanh\frac{\tilde\eta}{2}\right), \\ e^{(-)} &= -\sqrt{\alpha_1}\,\frac{1}{2\mu} e^{-[kx + (k^2 + 2\beta_0^2)\tau]} \left(\frac{1}{\gamma} + 1 + \left(\frac{1}{\gamma} - 1\right)\tanh\frac{\tilde\eta}{2}\right), \end{aligned} \tag{61}$$

and the product is

$$\rho = -e^{(+)}e^{(-)} = \alpha_1\left[1 + \frac{(\gamma-1)^2}{4\gamma\cosh^2(\tilde{\eta}/2)}\right] \tag{62}$$

with the asymptotic behavior $\lim_{|x|\to\infty}\rho \to \alpha_1$. Explicitly, it is

$$\rho = -e^{(+)}e^{(-)} = \alpha_1\left\{1 + \frac{k_1^2}{4\beta_0^2}\operatorname{sech}^2\left[\frac{k_1}{2}\left(x + \left(2k \pm \sqrt{k_1^2+4\beta_0^2}\,\right)\tau + x_0\right)\right]\right\}, \tag{63}$$

where $\beta_0^2 = 1/(4\beta^2\alpha_1)$. For the velocity field, we have

$$u = \frac{e_x^{(-)}}{e^{(-)}} - \frac{e_x^{(+)}}{e^{(+)}} = -2k - \frac{(\gamma^2-1)k_1}{(\gamma-1)^2 + 4\gamma\cosh^2(\tilde{\eta}/2)}. \tag{64}$$

We consider the particular solution for $k = 0$. It then follows that

$$\omega_1 = \pm k_1\sqrt{k_1^2+4\beta_0^2} \tag{65}$$

is the Bogoliubov dispersion from the superfluidity theory of a weakly nonideal Bose gas. For $k_1 \gg 2\beta_0$, it has the nonrelativistic free-particle form $\omega_1 \approx k_1^2$, while for $k_1 \ll 2\beta_0$, it has the relativistic collective form $\omega_1 \approx 2\beta_0 k_1$. The solution for the plus sign of the dispersion has the form

$$\begin{aligned} e^{(+)} &= \frac{\sqrt{\alpha_1}\mu}{v - \sqrt{v^2-4\beta_0^2}}e^{2\beta_0^2\tau}\left(v + \sqrt{v^2-4\beta_0^2}\tanh\frac{\sqrt{v^2-4\beta_0^2}}{2}(x+v\tau+x_0)\right),\\ e^{(-)} &= -\frac{\sqrt{\alpha_1}\,/\mu}{v + \sqrt{v^2-4\beta_0^2}}\,e^{-2\beta_0^2\tau}\left(v - \sqrt{v^2-4\beta_0^2}\tanh\frac{\sqrt{v^2-4\beta_0^2}}{2}(x+v\tau+x_0)\right), \end{aligned} \tag{66}$$

and the density is

$$\rho = -e^{(+)}e^{(-)} = \alpha_1\left\{1 + \frac{v^2-4\beta_0^2}{4\beta_0^2}\operatorname{sech}^2\left[\frac{\sqrt{v^2-4\beta_0^2}}{2}(x+v\tau+x_0)\right]\right\}. \tag{67}$$

This shows that soliton velocity is bounded below by the modulus $|v| > 2\beta_0$ and the soliton hence has the "tachyonic" character. These results show that in contrast to the defocusing NLS equation with the soliton velocity bounded above (subsonic type), the RNLS soliton has a velocity bounded below (supersonic type). Another difference is that the soliton of the defocusing (repulsive) NLS equation is a hole-like (bubble) excitation with $|\psi|^2 = \rho < 1$, while we have a wall-like form $\rho > 1$ for the RNLS soliton.

6.2.2. Two-soliton solution. To construct a two-soliton solution following Hirota [20], we consider

$$g_1^{(\pm)} = a_1^{\pm}e^{\eta_1} + a_2^{\pm}e^{\eta_2}, \qquad f_1 = e^{\eta_1} + e^{\eta_2}. \tag{68}$$

Substituting them in the bilinear equations

$$\begin{aligned} &(\mp D_\tau \pm 2kD_x + D_x^2)(a_1^{\pm}e^{\eta_1} + a_2^{\pm}e^{\eta_2})\cdot 1 + 1\cdot(e^{\eta_1}+e^{\eta_2}) = 0,\\ &2(D_x^2 + 2\beta_0^2)(1\cdot(e^{\eta_1}+e^{\eta_2})) = 2\beta_0^2(a_1^{+}e^{\eta_1} + a_2^{+}e^{\eta_2} + a_1^{-}e^{\eta_1} + a_2^{-}e^{\eta_2}), \end{aligned} \tag{69}$$

we obtain the system

$$\begin{aligned} &a_1^{\pm}(\mp\partial_\tau \pm 2k\partial_x + \partial_x^2)e^{\eta_1} + (\pm\partial_\tau \mp 2k\partial_x + \partial_x^2)e^{\eta_1} + \\ &\qquad + a_2^{\pm}(\mp\partial_\tau \pm 2k\partial_x + \partial_x^2)e^{\eta_2} + (\pm\partial_\tau \mp 2k\partial_x + \partial_x^2)e^{\eta_2} = 0, \\ &(k_1^2 + 2\beta_0^2)(e^{\eta_1}+e^{\eta_2}) = \beta_0^2[(a_1^{+}+a_1^{-})e^{\eta_1} + (a_2^{+}+a_2^{-})e^{\eta_2}]. \end{aligned} \tag{70}$$

Using the dispersion relations

$$\omega_i^{\pm} = k_i\big(2k \pm \sqrt{k_i^2 + 4\beta_0^2}\,\big), \quad i = 1, 2, \tag{71}$$

we obtain

$$a_i^{+} = \frac{(\omega_i - 2kk_i) + k_i^2}{(\omega_i - 2kk_i) - k_i^2} \equiv e^{\phi_i}, \qquad a_i^{-} = \frac{(\omega_i - 2kk_i) - k_i^2}{(\omega_i - 2kk_i) + k_i^2} = \frac{1}{a_i^{+}} e^{-\phi_i}. \tag{72}$$

The last relations imply that

$$\omega_i - 2kk_i = k_i^2 \coth\frac{\phi_i}{2} \tag{73}$$

and

$$k_i = 2\beta_0 \sinh\frac{\phi_i}{2}, \tag{74}$$

and hence

$$\omega_i - 2kk_i = 2\beta_0^2 \sinh\phi_i. \tag{75}$$

We note that the two signs in dispersion relations (71) correspond to the simple replacement $\phi_i \to -\phi_i$ in the above formulas. First, we restrict our consideration to the same sign for both frequencies. In the next order, we have the system

$$\begin{aligned} &(\mp D_\tau \pm 2kD_x + D_x^2)(g_2^{\pm} \cdot 1 + g_1^{\pm} \cdot f_1 + 1 \cdot f_2) = 0, \\ &(D_x^2 + 2\beta_0^2)(2 \cdot f_2 + f_1 \cdot f_1) = 2\beta_0^2(g_2^{+} + g_2^{-} + g_1^{+} g_1^{-}). \end{aligned} \tag{76}$$

We rewrite the first equation as

$$\begin{aligned} &(\mp\partial_\tau \pm 2k\partial_x + \partial_x^2)g_2^{\pm} + (\pm\partial_\tau \mp 2k\partial_x + \partial_x^2)f_2 + \\ &\quad + \{a_1^{\pm}[\mp(\omega_1 - \omega_2) \pm 2k(k_1 - k_2) + (k_1 - k_2)^2] + \\ &\quad + a_2^{\pm}[\mp(\omega_2 - \omega_1) \pm 2k(k_2 - k_1) + (k_1 - k_2)^2]\}e^{\eta_1 + \eta_2} = 0. \end{aligned} \tag{77}$$

This implies a solution of the form

$$g_2^{\pm} = a_{12}^{\pm} e^{\eta_1 + \eta_2}, \qquad f_2 = b_{12} e^{\eta_1 + \eta_2}. \tag{78}$$

Then the second equation in the system implies the relations

$$a_{12}^{+} = a_1^{+} a_2^{+} b_{12} = e^{\phi_1 + \phi_2} b_{12}, \qquad a_{12}^{-} = a_1^{-} a_2^{-} b_{12} = e^{-(\phi_1 + \phi_2)} b_{12}, \tag{79}$$

hence

$$b_{12} = \frac{\sinh^2((\phi_1 - \phi_2)/4)}{\sinh^2((\phi_1 + \phi_2)/4)}, \tag{80}$$

and the first equation in (76) is satisfied automatically. As a result, we have the solution

$$e^{(\pm)} = \pm 2\beta \frac{g_0^{\pm}(1 + g_1^{\pm} + g_2^{\pm})}{1 + f_1 + f_2} \tag{81}$$

or

$$e^{(\pm)} = \pm 2\beta \frac{g_0^{\pm}(1 + e^{\eta_1 \pm \phi_1} + e^{\eta_2 \pm \phi_2} + b_{12} e^{\eta_1 + \eta_2 \pm (\phi_1 + \phi_2)})}{1 + e^{\eta_1} + e^{\eta_2} + b_{12} e^{\eta_1 + \eta_2}}. \tag{82}$$

For the particular parameterization $\beta_0 = 1/2$ (which implies that $\alpha_1 = 1$), we have the two-soliton solution for the density

$$\rho = \frac{A_+ A_-}{[\sinh^2((\phi_1+\phi_2)/4)(1+e^{\eta_1}+e^{\eta_2}) + \sinh^2((\phi_1-\phi_2)/4)e^{\eta_1+\eta_2}]^2}, \tag{83}$$

where

$$\begin{aligned} A_\pm &= \sinh^2\frac{\phi_1+\phi_2}{4}(1+e^{\eta_1\pm\phi_1}+e^{\eta_2\pm\phi_2}) + \sinh^2\frac{\phi_1-\phi_2}{4}e^{\eta_1+\eta_2\pm(\phi_1+\phi_2)}, \\ \eta_i &= \sinh\frac{\phi_i}{2}x + \left[2k\sinh\frac{\phi_i}{2} + \frac{1}{2}\sinh\phi_i\right]\tau + \eta_i^{(0)}, \quad i=1,2. \end{aligned} \tag{84}$$

If one of the parameters ϕ_i vanishes or if $\phi_1 = \phi_2$, then the solution reduces to the one-soliton form. For example, if $\phi_2 = 0$, then

$$\rho = 1 + \frac{\sinh^2(\phi_1/2)}{\cosh^2(\eta_1/2)}. \tag{85}$$

Analyzing the two-soliton solution in the soliton moving frames, we can see that it describes a collision of two solitons of type (85) moving in the same direction with the initial position shifts

$$\Delta x_i = (-1)^{i-1}\frac{2}{\sinh(\phi_i/2)}\log\frac{\sinh((\phi_1-\phi_2)/4)}{\sinh((\phi_1+\phi_2)/4)}, \quad i=1,2, \tag{86}$$

and hence $\sinh(\phi_1/2)\Delta x_1 + \sinh(\phi_2/2)\Delta x_2 = 0$.

We obtain a different form of the two-soliton solution if we choose opposite signs for frequencies (71), and hence

$$\begin{aligned} \omega_1^+ &= 2\beta_0\left(2k\sinh\frac{\phi_1}{2} + \beta_0\sinh\phi_1\right), \\ \omega_2^- &= 2\beta_0\left(2k\sinh\frac{\phi_1}{2} - \beta_0\sinh\phi_1\right). \end{aligned} \tag{87}$$

Then $a_1^\pm = e^{\pm\phi_1}$, $a_2^\pm = e^{\mp\phi_2}$,

$$a_{12}^+ = a_1^+ a_2^+ b_{12} = e^{\phi_1-\phi_2}b_{12}, \qquad a_{12}^- = a_1^- a_2^- b_{12} = e^{-\phi_1+\phi_2}b_{12}, \tag{88}$$

and

$$b_{12} = \frac{\cosh^2((\phi_1+\phi_2)/4)}{\cosh^2((\phi_1-\phi_2)/4)}. \tag{89}$$

For $\beta_0 = 1/2$, the two-soliton solution is

$$\rho = \frac{B_+ B_-}{[\cosh^2((\phi_1-\phi_2)/4)(1+e^{\eta_1}+e^{\eta_2}) + \cosh^2((\phi_1+\phi_2)/4)e^{\eta_1+\eta_2}]^2}, \tag{90}$$

where

$$\begin{aligned} B_\pm &= \cosh^2\frac{\phi_1-\phi_2}{4}(1+e^{\eta_1\pm\phi_1}+e^{\eta_2\mp\phi_2}) + \cosh^2\frac{\phi_1+\phi_2}{4}e^{\eta_1+\eta_2\pm\phi_1\mp\phi_2}, \\ \eta_1 &= \sinh\frac{\phi_1}{2}(x-x_1) + \left[2k\sinh\frac{\phi_1}{2} + \frac{1}{2}\sinh\phi_1\right]\tau, \\ \eta_2 &= \sinh\frac{\phi_2}{2}(x-x_2) + \left[2k\sinh\frac{\phi_2}{2} - \frac{1}{2}\sinh\phi_2\right]\tau. \end{aligned} \tag{91}$$

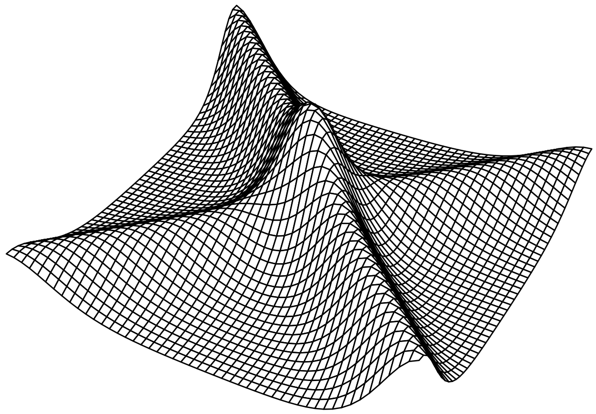

Fig. 1

It describes a collision of two solitons of form (85) moving in opposite directions with the initial position shifts

$$\Delta x_i = (-1)^{i-1}\frac{2}{\sinh(\phi_i/2)}\log\frac{\cosh((\phi_1+\phi_2)/4)}{\cosh((\phi_1-\phi_2)/4)}, \quad i=1,2. \tag{92}$$

In Fig. 1, we show a 3D plot of this solution.

For the velocity field, we have

$$\begin{aligned} u &= \frac{e_x^{(-)}}{e^{(-)}} - \frac{e_x^{(+)}}{e^{(+)}} = \\ &= -2k + + \frac{\cosh^2((\phi_1-\phi_2)/4)a_- + \cosh^2((\phi_1-\phi_2)/4)b_-}{\cosh^2((\phi_1-\phi_2)/4)(1+e^{\eta_1-\phi_1}+e^{\eta_2+\phi_2}) + \cosh^2((\phi_1+\phi_2)/4)e^{\eta_1+\eta_2-\phi_1+\phi_2}} - \\ &\quad - \frac{\cosh^2((\phi_1-\phi_2)/4)a_+ + \cosh^2((\phi_1-\phi_2)/4)b_+}{\cosh^2((\phi_1-\phi_2)/4)(1+e^{\eta_1+\phi_1}+e^{\eta_2-\phi_2}) + \cosh^2((\phi_1+\phi_2)/4)e^{\eta_1+\eta_2+\phi_1-\phi_2}}, \end{aligned} \tag{93}$$

where

$$a_\mp = \sinh\frac{\phi_1}{2}e^{\eta_1\mp\phi_1} + \sinh\frac{\phi_2}{2}e^{\eta_2\pm\phi_2}, \qquad b_\mp = \left(\sinh\frac{\phi_1}{2} + \sinh\frac{\phi_2}{2}\right)e^{\eta_1+\eta_2\mp\phi_1\pm\phi_2}. \tag{94}$$

It has the same phase shift (92) and describes a collision of two solitons of the form

$$u = -2k - \frac{k_i\sinh\phi_i}{2\cosh((\eta_i+\phi_i)/2)\cosh((\eta_i-\phi_i)/2)}, \quad i=1,2. \tag{95}$$

Acknowledgments. This work was supported in part by the Institute of Mathematics, Academia Sinica, Taipei, Taiwan, and the Izmir Institute of Technology, Izmir, Turkey (BAP Project No. 2005 IYTE 08).

REFERENCES

1. O. K. Pashaev and J.-H. Lee, *Modern Phys. Lett. A*, **17**, 1601–1619 (2002).
2. L. de Broglie, *C. R. Acad. Sci. Paris*, **183**, 447–448 (1926).
3. D. Bohm, *Phys. Rev.*, **85**, 166–179, 180–193 (1952).

4. E. Nelson, *Phys. Rev.*, **150**, 1079–1085 (1966).
5. G. Salesi, *Modern Phys. Lett. A*, **11**, 1815–1823 (1996).
6. F. Guerra and M. Pusterla, *Lett. Nuovo Cimento (2)*, **34**, 351–356 (1982).
7. L. Smolin, *Phys. Lett. A*, **113**, 408–412 (1986).
8. O. Bertolami, *Phys. Lett. A*, **154**, 225–229 (1991).
9. L. K. Antanovskii, C. Rogers, and W. K. Schief, *J. Phys. A*, **30**, L555–L557 (1997).
10. C. Rogers and W. K. Schief, *Nuovo Cimento B*, **114**, 1409–1412 (1999).
11. J.-H. Lee, O. K. Pashaev, C. Rogers, and W. K. Schief, *J. Plasma Phys.*, **73**, 257–272 (2007).
12. V. I. Karpman, *Nonlinear Waves in Dispersive Media* [in Russian], Nauka, Moscow (1973); English transl., Pergamon, Oxford (1975).
13. A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Plasma Electrodynamics* [in Russian], Nauka, Moscow (1974); English transl., Pergamon, Oxford (1975).
14. J. H. Adlam and J. E. Allen, *Philos. Mag.*, **3**, 448–455 (1958).
15. A. V. Gurevich and A. P. Meshcherkin, *Soviet Phys. JETP*, **60**, 732–740 (1984).
16. G. A. El, V. V. Khodorovskii, and A. V. Tyurina, *Phys. Lett. A*, **333**, 334–340 (2004).
17. A. V. Gurevich and A. L. Krylov, *Soviet Phys. JETP*, **65**, 944–948 (1987).
18. A. V. Gurevich and A. L. Krylov, *Soviet Phys. Dokl.*, **33**, 73–74 (1988).
19. G. A. El, R. H. J. Grimshaw, and M. V. Pavlov, *Stud. Appl. Math.*, **106**, 157–186 (2001).
20. R. Hirota, "Direct method of finding exact solutions of nonlinear evolution equations," in: *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications* (Lect. Notes Math., Vol. 515, R. M. Miura, ed.), Springer, Berlin (1976), p. 40–68.