U.P.B. Sci. Bull., Series A, Vol. 74, Iss. 1, 2012 ISSN 1223-7027

RATIONAL AND MULTI–WAVE SOLUTIONS TO NONLINEAR EVOLUTION EQUATIONS BY MEANS OF THE EXP–FUNCTION METHOD

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In this paper, we present a new application of the Exp–function method to carry out the integration of nonlinear evolution equations in terms of multi–wave and rational solutions. To elucidate the solution procedure, we analytically investigate the Sharma–Tasso–Olver equation and the fifth–order Korteweg de Vries equation. Unlike Hirota's method, our procedure does not require the bilinear formalism of the equations studied.

Keywords: Exp–function method; Multi–wave solution; Rational solution; Sharma–Tasso–Olver equation; Fifth–order Korteweg de Vries equation.

1. Introduction

Nonlinear evolution equations (NEEs) have been of fundamental importance in the study of applied physical and mathematical sciences. They are crucial for obtaining an understanding of the physical sciences, as well as the biological and social sciences. Thus, exactly solving NEEs has gained increasing importance. Nowadays, some modern analytic methods are available for tackling NEEs. To make mention of some; tanh function method [1], Adomian decomposition method [2], variational iteration method [3], first integral method [4], Exp–function method [5], homotopy perturbation method [6], (G'/G)–expansion method [7], multiple–exp function method [8] and so forth.

It is an important fact that one should be aware of the limitations of these methods and there is no guarantee that any of these techniques will succeed for a specific nonlinear problem. Among the others, the Exp–function method has received more attention and consequently it has been adapted, extended, and generalized to various kinds of nonlinear problems; for example, NEEs with variable coefficients [9], multi–dimensional equations [10–12], differential–difference equations [13, 14], coupled NEEs [15], stochastic equations [16], n–soliton solutions [17–19], rational solutions [20–22]. Hence, the Exp–function method provide a valuable addition to the wave theory.

On the other hand, traveling waves of NEEs may be coupled with different frequencies and different velocities. Multi–wave solutions are crucial in the sense

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that they may sometimes be converted into a single soliton of very high energy that propagates over large domains of space without dispersion. Therefore, an extremely destructive wave may be produced. The *tsunami* is a good example for this kind of phenomena. As is well known, Hirota's method [23] can be used to construct multi–wave solutions if the equation considered can be transformed into a bilinear form. However, the existence of the bilinear form cannot be guaranteed or it may not be known.

The main objective of this work is to show the applicability of the Exp–function method to two important equations of mathematical physics having distinct physical structures (namely, the Sharma–Tasso–Olver equation and the fifth–order Korteweg de Vries equation) for rational and travelling wave solutions with distinct velocities and distinct frequencies. The rest of this paper is organized as follows. In Section 2, we briefly describe our method. In Sections 3 and 4, we analyze our problems. In Section 5, we give some concluding remarks.

2. The Exp–function method

For a given NEE, say, in two variables x and t,

$$P\left(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots\right)=0\,, \tag{1}$$

where P is a polynomial in its arguments, the Exp–function method is based on the assumption that its solutions can be expressed as

$$u\left(x,t\right)=\frac{\sum_{i=0}^{m}a_i\exp\left(i\xi\right)}{\sum_{j=0}^{n}b_j\exp\left(j\xi\right)},\ \xi=kx+wt+\delta\,, \tag{2}$$

where m and n are positive integers to be determined; a_i, b_j, k and w are arbitrary constants to be specified; δ is the phase shift. Substituting the function (2) into Eq. (1) and balancing the highest–order terms, one can determine the constants m and n. To seek for N – wave solutions to Eq. (1), the function (2) can be generalized as

$$u\left(x,t\right)=\frac{\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}a_{i_1i_2}\exp\left(i_1\xi_1+i_2\xi_2\right)}{\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}b_{j_1j_2}\exp\left(j_1\xi_1+j_2\xi_2\right)},\ \xi_l=k_lx+w_lt+\delta_l,\ l=1,2\,, \tag{3}$$

which corresponds to the case $N=2$; and

$$u\left(x,t\right)=\frac{\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\sum_{i_3=0}^{m_3}a_{i_1i_2i_3}\exp\left(i_1\xi_1+i_2\xi_2+i_3\xi_3\right)}{\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3}b_{j_1j_2j_3}\exp\left(j_1\xi_1+j_2\xi_2+j_3\xi_3\right)},\ \xi_l=k_lx+w_lt+\delta_l,\ l=1,2,3\,, \tag{4}$$

which corresponds to the case $N=3$; and so on.

To obtain a rational solution for Eq. (1), we modify the function (2) as

$$u(x,t)=\frac{\sum_{i=0}^{m}a_i\left(\mu_1\exp(\xi)+\mu_2\xi\right)^i}{\sum_{j=0}^{n}b_j\left(\mu_1\exp(\xi)+\mu_2\xi\right)^j},\ \xi=kx+wt+\delta\,, \tag{5}$$

where μ_1 and μ_2 are two embedded constants. It is obvious that when $\mu_1=1$ and $\mu_2=0$, the function (5) turns out to be the function (2).

Remark 1. The positive integers m and n in the function (2) are determined by balancing the linear and nonlinear terms of highest order in Eq. (1). However, as far as we could verify through the research literature, performing this procedure takes a lot of effort and time consuming. Recently, Ali [24] proved that the balancing step in the Exp–function method is redundant. Thus, taking the function (2) into account directly and assigning arbitrary values to the constants m and n will make laborious calculations unnecessary.

3. The Sharma–Tasso–Olver equation

Let us consider the so–called Sharma–Tasso–Olver (STO) equation which reads

$$u_t+3\alpha u_x^2+3\alpha u^2u_x+3\alpha uu_{xx}+\alpha u_{xxx}=0\,, \tag{6}$$

where α is a nonzero constant, and $u=u(x,t)$. We suppose that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_1\exp(\xi)}{1+b_1\exp(\xi)},\ \xi=kx+wt+\delta\,, \tag{7}$$

which is embedded in (2). Substituting (7) into Eq. (6), we get the relation

$$\left(1+b_1\exp(\xi)\right)^{-4}\sum_{n=1}^{3}C_n\exp(n\xi)=0\,, \tag{8}$$

where

$$C_1=\alpha a_1k^3+a_1w\,,$$
$$C_2=6\alpha a_1^2k^2-4\alpha a_1b_1k^3+2a_1b_1w\,,$$
$$C_3=3\alpha a_1^3k-3\alpha a_1^2b_1k^2+\alpha a_1b_1^2k^3+a_1b_1^2w\,.$$

Then, solving the system $C_n=0$ $\left(n=1,2,3\right)$ simultaneously, we obtain the solution set

$$w=-\alpha k^3,\ a_1=kb_1\,, \tag{9}$$

which yields a one–wave solution to Eq. (6) as

$$u_1(x,t)=\frac{b_1k\exp\left(kx-\alpha k^3t+\delta\right)}{1+b_1\exp\left(kx-\alpha k^3t+\delta\right)}\,, \tag{10}$$

where k, b_1, and δ remain arbitrary.

3.1. Two–wave solutions

Assume that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_{10}\exp(\xi_1)+a_{01}\exp(\xi_2)+a_{11}\exp(\xi_1+\xi_2)}{1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+b_{11}\exp(\xi_1+\xi_2)},\ \xi_l=k_lx+w_lt+\delta_l,\ l=1,2. \quad (11)$$

It is clear that the function (11) is embedded in (3). Substituting (11) into Eq. (6), we obtain the relation

$$\left(1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+b_{11}\exp(\xi_1+\xi_2)\right)^{-4}\sum_{i=0}^{4}\sum_{j=0}^{4}C_{ij}\exp(i\xi_1+j\xi_2)=0, \quad (12)$$

where $C_{00}=C_{04}=C_{40}=C_{44}=0$. Hence, solving the system $C_{ij}=0\ (0\le i,j\le 4)$ simultaneously, we obtain the solution set

$$w_1=-\alpha k_1^3,\ w_2=-\alpha k_2^3,\ b_{11}=0,\ a_{11}=0,\ a_{01}=k_2b_{01},\ a_{10}=k_1b_{10}, \quad (13)$$

which gives a two–wave solution to Eq. (6) as

$$u_2(x,t)=\frac{k_1b_{10}\exp\left(k_1x-\alpha k_1^3t+\delta_1\right)+k_2b_{01}\exp\left(k_2x-\alpha k_2^3t+\delta_2\right)}{1+b_{10}\exp\left(k_1x-\alpha k_1^3t+\delta_1\right)+b_{01}\exp\left(k_2x-\alpha k_2^3t+\delta_2\right)}, \quad (14)$$

where b_{01}, b_{10}, k_1, k_2, δ_1, and δ_2 remain arbitrary.

3.2. Three–wave solutions

Assume that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{\begin{array}{c}a_{100}\exp(\xi_1)+a_{010}\exp(\xi_2)+a_{001}\exp(\xi_3)+a_{110}\exp(\xi_1+\xi_2)+a_{101}\exp(\xi_1+\xi_3)\\+a_{011}\exp(\xi_2+\xi_3)+a_{111}\exp(\xi_1+\xi_2+\xi_3)\end{array}}{\begin{array}{c}1+b_{100}\exp(\xi_1)+b_{010}\exp(\xi_2)+b_{001}\exp(\xi_3)+b_{110}\exp(\xi_1+\xi_2)+b_{101}\exp(\xi_1+\xi_3)\\+b_{011}\exp(\xi_2+\xi_3)+b_{111}\exp(\xi_1+\xi_2+\xi_3)\end{array}}, \quad (15)$$

where $\xi_l=k_lx+w_lt+\delta_l\ (l=1,2,3)$.

Obviously, the function (15) is embedded in (4). After substituting (15) into Eq. (6) and making similar manipulations, we get the solution set of the resultant algebraic system as

$$\begin{array}{l}w_1=-\alpha k_1^3,\ w_2=-\alpha k_2^3,\ w_3=-\alpha k_3^3,\ a_{100}=k_1b_{100},\ a_{010}=k_2b_{010},\ a_{001}=k_3b_{001},\\ b_{101}=0,\ b_{011}=0,\ a_{011}=0,\ a_{101}=0,\ b_{111}=0,\ a_{110}=0,\ b_{110}=0,\ a_{111}=0,\end{array} \quad (16)$$

which leads to a three–wave solution to Eq. (6) as

$$u_3(x,t)=\frac{k_1b_{100}\exp\left(k_1x-\alpha k_1^3t+\delta_1\right)+k_2b_{010}\exp\left(k_2x-\alpha k_2^3t+\delta_2\right)+k_3b_{001}\exp\left(k_3x-\alpha k_3^3t+\delta_3\right)}{1+b_{100}\exp\left(k_1x-\alpha k_1^3t+\delta_1\right)+b_{010}\exp\left(k_2x-\alpha k_2^3t+\delta_2\right)+b_{001}\exp\left(k_3x-\alpha k_3^3t+\delta_3\right)}, \quad (17)$$

where b_{001}, b_{010}, b_{100}, k_1, k_2, k_3, δ_1, δ_2, and δ_3 remain arbitrary.

3.3. Rational solutions

Suppose that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_1\left(\mu_1\exp(\xi)+\mu_2\xi\right)+a_0+a_{-1}\left(\mu_1\exp(\xi)+\mu_2\xi\right)^{-1}}{b_1\left(\mu_1\exp(\xi)+\mu_2\xi\right)+b_0+b_{-1}\left(\mu_1\exp(\xi)+\mu_2\xi\right)^{-1}},\ \xi=kx+wt\,. \tag{18}$$

By the same procedure, we obtain the solution set of the resultant algebraic system as

$$a_0=\frac{a_1b_0}{b_1}+kb_1,\ a_{-1}=\frac{kb_0}{2}+\frac{a_1b_{-1}}{b_1}\mp\frac{1}{2}k\sqrt{b_0^2-4b_{-1}b_1},\ \ w=-\frac{3k\alpha a_1^2}{b_1^2},\ \mu_1=0,\ \mu_2=1, \tag{19}$$

which leads to a rational solution to Eq. (6) as

$$u_4^{\mp}(x,t)=\frac{a_1\left(kx-\frac{3k\alpha a_1^2}{b_1^2}t\right)^2+\left(\frac{a_1b_0}{b_1}+kb_1\right)\left(kx-\frac{3k\alpha a_1^2}{b_1^2}t\right)+\frac{kb_0}{2}+\frac{a_1b_{-1}}{b_1}\mp\frac{1}{2}k\sqrt{b_0^2-4b_{-1}b_1}}{b_1\left(kx-\frac{3k\alpha a_1^2}{b_1^2}t\right)^2+b_0\left(kx-\frac{3k\alpha a_1^2}{b_1^2}t\right)+b_{-1}}, \tag{20}$$

where a_1, b_{-1}, b_0, b_1, and k remain arbitrary.

4. The fifth–order Korteweg de Vries equation

Let us consider the so–called fifth–order Korteweg de Vries equation which reads

$$u_t+10uu_{xxx}+30u^2u_x+20u_xu_{xx}+u_{xxxxx}=0\,. \tag{21}$$

First, we assume that Eq. (21) admits a solution in the form

$$u(x,t)=\frac{a_1\exp(\xi)}{\left(1+b_1\exp(\xi)\right)^2},\ \xi=kx+wt+\delta\,, \tag{22}$$

which is embedded in (2). Substituting (22) into Eq. (21) and solving the resultant algebraic system for the unknowns a_1, b_1, k, and w, we obtain the solution set

$$w=-k^5,\ a_1=2b_1k^2, \tag{23}$$

which leads a one–wave solution to Eq. (6) as

$$u(x,t)=\frac{2b_1k^2\exp\left(kx-k^5t+\delta\right)}{\left(1+b_1\exp\left(kx-k^5t+\delta\right)\right)^2}, \tag{24}$$

where k, b_1, and δ remain arbitrary.

4.1. Two–wave solutions

Second, we suppose that Eq. (21) admits a solution of the form

$$u(x,t)=\frac{a_{10}\exp(\xi_1)+a_{01}\exp(\xi_2)+a_{11}\exp(\xi_1+\xi_2)+a_{21}\exp(2\xi_1+\xi_2)+a_{12}\exp(\xi_1+2\xi_2)}{\left(1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+b_{11}\exp(\xi_1+\xi_2)\right)^2}, \quad (25)$$

where $\xi_l=k_lx+w_lt+\delta_l$, $l=1,2$.

One can see that the function (25) is embedded in (3). Substituting (25) into Eq. (21) and solving the resultant algebraic system for the unknowns a_{10}, a_{01}, a_{11}, a_{21}, a_{12}, b_{10}, b_{01}, b_{11}, k_1, k_2, w_1, and w_2, we get the solution set

$$\begin{aligned}&a_{10}=2b_{10}k_1^2,\ a_{01}=2b_{01}k_2^2,\ a_{11}=4b_{01}b_{10}(k_1-k_2)^2,\ a_{21}=2b_{01}b_{10}^2(k_1-k_2)^2k_2^2/(k_1+k_2)^2,\\&a_{12}=2b_{01}^2b_{10}k_1^2(k_1-k_2)^2/(k_1+k_2)^2,\ b_{11}=b_{01}b_{10}(k_1-k_2)^2/(k_1+k_2)^2,\ w_1=-k_1^5,\ w_2=-k_2^5,\end{aligned} \quad (26)$$

which provides a two–wave solution to Eq. (6) as

$$u(x,t)=\frac{\begin{array}{c}2b_{10}k_1^2\exp(\xi_1)+2b_1k_2^2\exp(\xi_2)+4b_1b_{10}(k_1-k_2)^2\exp(\xi_1+\xi_2)\\+\dfrac{2b_1b_{10}^2(k_1-k_2)^2k_2^2}{(k_1+k_2)^2}\exp(2\xi_1+\xi_2)+\dfrac{2b_1^2b_{10}k_1^2(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\xi_1+2\xi_2)\end{array}}{\left(1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+\dfrac{b_{01}b_{10}(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\xi_1+\xi_2)\right)^2}, \quad (27)$$

where $\xi_1=k_1x-k_1^5t+\delta_1$, $\xi_2=k_2x-k_2^5t+\delta_2$, and b_{01}, b_{10}, k_1, k_2, δ_1, δ_2 remain arbitrary.

4.2. Three–wave solutions

Third, we assume that Eq. (21) admits a solution of the form

$$u(x,t)=\frac{v_1(\xi_1,\xi_2,\xi_3)}{v_2(\xi_1,\xi_2,\xi_3)}, \quad (28)$$

where $\xi_l=k_lx+w_lt+\delta_l$, $l=1,2,3$, and

$$\begin{aligned}v_1(\xi_1,\xi_2,\xi_3)=&\,a_{100}\exp(\xi_1)+a_{010}\exp(\xi_2)+a_{001}\exp(\xi_3)+a_{110}\exp(\xi_1+\xi_2)+a_{101}\exp(\xi_1+\xi_3)\\&+a_{011}\exp(\xi_2+\xi_3)+a_{120}\exp(\xi_1+2\xi_2)+a_{102}\exp(\xi_1+2\xi_3)+a_{012}\exp(\xi_2+2\xi_3)\\&+a_{210}\exp(2\xi_1+\xi_2)+a_{201}\exp(2\xi_1+\xi_3)+a_{021}\exp(2\xi_2+\xi_3)+a_{111}\exp(\xi_1+\xi_2+\xi_3)\\&+a_{211}\exp(2\xi_1+\xi_2+\xi_3)+a_{121}\exp(\xi_1+2\xi_2+\xi_3)+a_{112}\exp(\xi_1+\xi_2+2\xi_3)\\&+a_{221}\exp(2\xi_1+2\xi_2+\xi_3)+a_{122}\exp(\xi_1+2\xi_2+2\xi_3)+a_{212}\exp(2\xi_1+\xi_2+2\xi_3),\end{aligned}$$

$$v_2(\xi_1,\xi_2,\xi_3)=\left(\begin{array}{c}1+b_{100}\exp(\xi_1)+b_{010}\exp(\xi_2)+b_{001}\exp(\xi_3)+b_{110}\exp(\xi_1+\xi_2)+b_{101}\exp(\xi_1+\xi_3)\\+b_{011}\exp(\xi_2+\xi_3)+b_{111}\exp(\xi_1+\xi_2+\xi_3)\end{array}\right)^2.$$

It is obvious that the function (28) is embedded in (4). After substituting (28) into Eq. (21) and proceeding as before, we get the solution set of the resultant algebraic system as

$$w_1 = -k_1^5,\ w_2 = -k_2^5,\ w_3 = -k_3^5,\ a_{100} = 2b_{100}k_1^2,\ a_{010} = 2b_{010}k_2^2,\ a_{001} = 2b_{001}k_3^2, \tag{29}$$

$$a_{011} = 4b_{001}b_{010}\left(k_2 - k_3\right)^2,\ a_{101} = 4b_{001}b_{100}\left(k_1 - k_3\right)^2,\ a_{110} = 4b_{010}b_{100}\left(k_1 - k_2\right)^2, \tag{30}$$

$$a_{102} = \frac{2b_{001}^2 b_{100} k_1^2 \left(k_1 - k_3\right)^2}{\left(k_1 + k_3\right)^2},\ a_{012} = \frac{2b_{001}^2 b_{010} k_2^2 \left(k_2 - k_3\right)^2}{\left(k_2 + k_3\right)^2},\ b_{011} = \frac{b_{001}b_{010}\left(k_2 - k_3\right)^2}{\left(k_2 + k_3\right)^2}, \tag{31}$$

$$a_{021} = \frac{2b_{001}b_{010}^2\left(k_2 - k_3\right)^2 k_3^2}{\left(k_2 + k_3\right)^2},\ b_{101} = \frac{b_{001}b_{100}\left(k_1 - k_3\right)^2}{\left(k_1 + k_3\right)^2},\ a_{201} = \frac{2b_{001}b_{100}^2\left(k_1 - k_3\right)^2 k_3^2}{\left(k_1 + k_3\right)^2}, \tag{32}$$

$$a_{120} = \frac{2b_{010}^2 b_{100} k_1^2 \left(k_1 - k_2\right)^2}{\left(k_1 + k_2\right)^2},\ a_{112} = \frac{4b_{001}^2 b_{010} b_{100}\left(k_1 - k_2\right)^2\left(k_1 - k_3\right)^2\left(k_2 - k_3\right)^2}{\left(k_1 + k_3\right)^2\left(k_2 + k_3\right)^2}, \tag{33}$$

$$b_{110} = \frac{b_{010}b_{100}\left(k_1 - k_2\right)^2}{\left(k_1 + k_2\right)^2},\ a_{221} = \frac{2b_{001}b_{010}^2 b_{100}^2\left(k_1 - k_2\right)^4\left(k_1 - k_3\right)^2\left(k_2 - k_3\right)^2 k_3^2}{\left(k_1 + k_2\right)^4\left(k_1 + k_3\right)^2\left(k_2 + k_3\right)^2}, \tag{34}$$

$$a_{210} = \frac{2b_{010}b_{100}^2\left(k_1 - k_2\right)^2 k_2^2}{\left(k_1 + k_2\right)^2},\ a_{122} = \frac{2b_{001}^2 b_{010}^2 b_{100} k_1^2\left(k_1 - k_2\right)^2\left(k_1 - k_3\right)^2\left(k_2 - k_3\right)^4}{\left(k_1 + k_2\right)^2\left(k_1 + k_3\right)^2\left(k_2 + k_3\right)^4}, \tag{35}$$

$$a_{121} = \frac{4b_{001}b_{010}^2 b_{100}\left(k_1 - k_2\right)^2\left(k_1 - k_3\right)^2\left(k_2 - k_3\right)^2}{\left(k_1 + k_2\right)^2\left(k_2 + k_3\right)^2},\ b_{111} = \frac{b_{001}b_{010}b_{100}\left(k_1 - k_2\right)^2\left(k_1 - k_3\right)^2\left(k_2 - k_3\right)^2}{\left(k_1 + k_2\right)^2\left(k_1 + k_3\right)^2\left(k_2 + k_3\right)^2}, \tag{36}$$

$$a_{111} = \frac{8b_{001}b_{010}b_{100}\left(k_2^2k_3^2\left(k_2^2 - k_3^2\right)^2 + k_1^6\left(k_2^2 + k_3^2\right) - 2k_1^4\left(k_2^4 + k_3^4\right) + k_1^2\left(k_2^6 + k_3^6\right)\right)}{\left(k_1 + k_2\right)^2\left(k_1 + k_3\right)^2\left(k_2 + k_3\right)^2}, \tag{37}$$

$$a_{212} = \frac{2b_{001}^2 b_{010} b_{100}^2\left(k_1 - k_2\right)^2 k_2^2\left(k_1 - k_3\right)^4\left(k_2 - k_3\right)^2}{\left(k_1 + k_2\right)^2\left(k_1 + k_3\right)^4\left(k_2 + k_3\right)^2}, \tag{38}$$

$$a_{211} = \frac{4b_{001}b_{010}b_{100}^2\left(k_1 - k_2\right)^2\left(k_1 - k_3\right)^2\left(k_2 - k_3\right)^2}{\left(k_1 + k_2\right)^2\left(k_1 + k_3\right)^2}. \tag{39}$$

Finally, employing the determined coefficients (29)–(39) to (28), we derive a three–wave solution to Eq. (21), where b_{100} , b_{010} , b_{001} , k_1 , k_2 , k_3 , δ_1 , δ_2 , and δ_3 remain arbitrary.

4.3. Rational solutions

For a rational solution, we suppose that Eq. (21) admits the function (18) as a solution. Then, following the same procedure, we obtain the solution set of the resultant algebraic system as

$$w = -30ka_1^2 / b_1^2,\ a_0 = a_1 b_0 / b_1,\ a_{-1} = \left(a_1 b_0^2 - 8k^2 b_1^3\right) / 4b_1^2,\ b_{-1} = b_0^2 / 4b_1,\ \mu_1 = 0,\ \mu_2 = 1, \quad (40)$$

which leads to a rational solution to Eq. (21) as

$$u(x,t) = \frac{4a_1 b_1^2 \left(kx - \left(30ka_1^2 / b_1^2\right)t + \delta\right)^2 + 4a_1 b_0 b_1 \left(kx - \left(30ka_1^2 / b_1^2\right)t + \delta\right) + a_1 b_0^2 - 8k^2 b_1^3}{4b_1^3 \left(kx - \left(30ka_1^2 / b_1^2\right)t + \delta\right)^2 + 4b_0 b_1^2 \left(kx - \left(30ka_1^2 / b_1^2\right)t + \delta\right) + b_0^2 b_1}, \quad (41)$$

where a_1 , b_1 , b_0 , k , and δ remain arbitrary.

Remark 2. The existence of *N*–wave solutions often implies the integrability of the equation considered. The determination of three–wave solutions for our equations confirms the fact that $N(\geq 4)$–wave solutions exist and can be obtained in a parallel manner. However, we observed that the computation becomes tedious and much more complicated.

5. Conclusions

Seeking exact and explicit solutions with multi–velocities and multi–frequencies for NEEs is an important and active research area in the applied mathematical and physical sciences. In this study, we implemented the Exp–function method to two completely integrable NEEs for explicitly constructing one–wave, two–wave, and three–wave solutions, as well as rational solutions. We successfully obtained such solutions involving more arbitrary parameters. Our results indicate that the Exp–function method can be used as a simplified form of the Hirota's bilinear method. We conclude that the Exp–function method posses powerful features that make it practical for the determination of multi–wave solutions for a wide class of NEEs; this will be our future task.

REFERENCES

[1]. *W. Malfliet*, *W. Hereman*, "The tanh method I: Exact solutions of nonlinear evolution and wave equations", in Phys. Scr., **vol. 54**, no. 6, 1996, pp. 563–568

[2] *G. Adomian*, Solving frontier problems of physics: the decomposition method, Kluwer, Boston, 1994.
[3] *J.H. He*, “A new approach to nonlinear partial differential equations”, in Commun. Nonlinear Sci. Numer. Simul., **vol. 2**, no.4, December. 1997, pp. 230–235
[4] *Z.S. Feng*, “The first–integral method to study the Burgers–Korteweg–de Vries equation”, in J. Phys. A: Math. Gen., **vol. 35**, no.2, January. 2002, pp. 343–349
[5] *J.H. He, X.H. Wu*, “Exp–function method for nonlinear wave equations”, in Chaos Solitons Fract., **vol. 30**, no.3, November. 2006, pp. 700–708
[6] *J.H. He*, “An approximate solution technique depending on an artificial parameter: A special example”, in Commun. Nonlinear Sci. Numer. Simul., **vol.** 3, no.2, June. 1998, pp. 92–97
[7] *M. Wang, X. Li, and J. Zhang*, “The (G'/G)–expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics”, in Phys. Lett. A., **vol. 372**, no.4, January. 2008, pp. 417–423
[8] *W.X. Ma, T. Huang, and Y. Zhang*, “A multiple exp–function method for nonlinear differential equations and its application”, in Phys. Scr., **vol. 82**, no.6, December. 2010, 065003 (8pp)
[9] *S. Zhang*, “Application of Exp–function method to a KdV equation with variable coefficients”, in Phys. Lett. A., **vol. 365**, no.5-6, June. 2007, pp. 448–453
[10] *T. Öziş, İ. Aslan*, “Exact and explicit solutions to the (3+1)–dimensional Jimbo–Miwa equation via the Exp–function method”, in Phys. Lett. A., **vol. 372**, no.47, November. 2008, pp. 7011–7015
[11] *İ. Aslan*, “Generalized solitary and periodic wave solutions to a (2 + 1)–dimensional Zakharov–Kuznetsov equation”, in Appl. Math. Comput., **vol. 217**, no.4, October. 2009, pp. 1421–1429.
[12] *İ. Aslan*, “Application of the Exp–function method to the (2+1)–dimensional Boiti–Leon-Pempinelli equation using symbolic computation”, in Int. J. Comput. Math., **vol. 88**, no.4, 2011, pp. 747–761
[13] *S.D. Zhu*, “Exp–function Method for the Hybrid–Lattice System”, in Int. J. Nonlinear Sci. Numer. Simul., **vol. 8**, no.3, September. 2007, pp. 461–464
[14] *İ. Aslan*, “Application of the Exp-function method to nonlinear lattice differential equations for multi–wave and rational solutions”, in Math. Methods Appl. Sci., **vol. 34**, no.14, September. 2011, pp. 1707-1710
[15] *E. Yomba*, “Application of Exp–function method for a system of three component–Schrödinger equations”, in Phys. Lett. A., **vol. 373**, no.44, October. 2009, pp. 4001–4011
[16] *C.Q. Dai, J.F. Zhang*, “Application of He's Exp–function method to the stochastic mKdV equation”, in Int. J. Nonlinear Sci. Numer. Simul., **vol. 10**, no.5, May. 2009, pp. 675–680
[17] *V. Marinakis*, “The Exp–function method and $n-$soliton solutions”, in Z. Naturforsch., **vol. 63a,** no.10/11, 2008, pp. 653–656
[18] *S. Zhang, H.Q. Zhang*, “Exp–function method for N–soliton solutions of nonlinear evolution equations in mathematical physics”, in Phys. Lett. A., **vol. 373**, no.30, July. 2009, pp. 2501–2505
[19] *İ. Aslan*, “Constructing rational and multi–wave solutions to higher order NEEs via the Exp–function method”, in Math. Methods Appl. Sci., **vol. 34**, no.8, May. 2011, pp. 990–995
[20] *S. Zhang*, “Exp–function method: Solitary, periodic and rational wave solutions of nonlinear evolution equations”, in Nonl. Sci. Lett. A., **vol. 1**, no.3, 2010, pp. 143–146
[21] *İ. Aslan*, “The Exp–function approach to the Schwarzian Korteweg–de Vries equation”, in Comput. Math. Appl., **vol. 59**, no.8, April. 2010, pp. 2896–2900
[22] *İ. Aslan*, “Multi–wave and rational solutions for nonlinear evolution equations”, in Int. J. Nonlinear Sci. Numer. Simul., **vol. 11**, no.8, 2010, pp. 619–623

[23] *R. Hirota*, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.

[24] *A.T. Ali*, "A note on the Exp–function method and its application to nonlinear equations", in Phys.Scr., **vol. 79**, no.2, February. 2009, 025006, 6pp.